MAURIZIO GASPERINI,

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PAPERS ON PRE-BIG BANG COSMOLOGY

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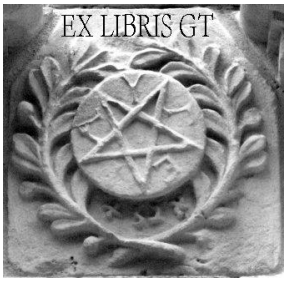

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The Pre-Big Bang Scenario in String Cosmology

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Abstract

We review physical motivations, phenomenological consequences, and open problems of the so-called pre-big bang scenario in superstring cosmology.

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1 Introduction

During the past thirty years, mainly thanks to accelerator experiments of higher and higher energy and precision, the standard model of particle physics has established itself as the uncontested winner in the race for a consistent description of electroweak and strong interaction phenomena at distances above 10^{-15} cm or so. There are, nonetheless, good reasons (in particular the increasing evidence for non-vanishing neutrino masses [388, 568, 569]) to believe that the standard model is not the end of the story. The surprising validity of this model at energies below 100 GeV, as well as the (in)famous Higgs mass fine-tuning problem, suggest some supersymmetric extension of the standard model (for a review see [501]) as the most likely improved description of non-gravitational phenomena over a few more decades in the ladder of scales. It is however quite likely that other questions that are left unanswered by the standard model, such as the peculiarities of fermionic masses and mixings, the family pattern, C, P, CP, B violation, etc., will only find their answers at –or around– the much higher energies at which all gauge interactions appear to unify [21]. This energy scale appears to be embarrassingly close (on a logarithmic scale) to the so-called Planck mass, $M_{\rm P} \sim 10^{19}$ GeV, the scale at which gravity becomes strong and needs to be quantized.

The situation with gravitational phenomena is completely different. Even the good old Newton law is known to be valid only down to the 1 mm scale [380], so that much interest has been devoted to the possibility of large modifications of gravity below that distance, either from new forces mediated by light scalars such as the dilaton of string theory [577], or from the existence of large extra dimensions felt exclusively by gravity [35, 530]. General relativity is well tested at large scales; nevertheless; present evidence for a (small) vacuum energy density [535, 516] suggests that, even on cosmologically large distances, the strict Einstein theory might turn out to be inadequate. Evidently, the construction of a standard model for gravity and cosmology lags much behind its particle physics counterpart.

The hot big bang model (see for instance [618]), originally thought of as another great success of general relativity, was later discovered to suffer from huge fine-tuning problems. Some of these conceptual problems are solved by the standard inflationary paradigm (see [441, 420] for a review), yet inflation remains a generic idea in search of a theory that will embody it naturally. Furthermore, the classical theory of inflation does not really address the problem of how the initial conditions needed for a successful inflation came about. The answer to this question is certainly related to even more fundamental issues, such as: How did it all start? What caused the big bang? Has there been a singularity at $t = 0$? Unfortunately, these questions lie deeply inside the short-distance, high-curvature regime of gravity where quantum corrections cannot be neglected. Attempts at answering these questions using quantum cosmology based on Einstein's theory has resulted in a lot of heated discussions [443, 592], with no firm conclusions.

It is very likely that both a standard model for gravity and cosmology and a full under-

lambda P = 10 ^(-33) cm

standing of the standard model of particle physics will require our understanding of physics down to the shortest scale, the Planck length $\lambda_P \sim 10^{-33}$ cm. Until the Green–Schwarz revolution of 1984 [346], the above conclusion would have meant postponing indefinitely those kinds of questions. Since then, however, particle theorists have studied and developed superstring theory (see [524] for a recent review, as well as [347] for a non-specialized introduction), which appears to represent a consistent framework not only for addressing (and possibly answering) those questions, but even for unifying our understanding of gravitational and non-gravitational phenomena, and therefore for relating the two classes of questions.

The so-called "pre-big bang" scenario described in this report has to be seen in the above perspective as a possible example, even just as a toy model, of what cosmology can look like if we assume that the sought for standard model of gravity and cosmology is based on (some particular version of) superstring theory. Although most string theorists would certainly agree on the importance of studying the cosmological consequences of string theory, it is a priori far from obvious that the "state of the art" in this field can provide an unambiguous answer to this question. Indeed, most of our understanding of superstring theory is still based on perturbative expansions, while most of the recent progress in non-perturbative string theory has been achieved in the context of "vacua" (i.e. classical solutions to the field equations) that respect a large number of supersymmetries [524]. By contrast, our understanding of string theory at large curvatures and couplings, especially in the absence of supersymmetry, is still largely incomplete. A cosmological background, and a fortiori one that evolves rapidly in time, breaks (albeit spontaneously) supersymmetry. This is why the Planckian regime of cosmology appears to be intractable for the time being.

???????

It is very fortunate, in this respect, that in the pre-big bang scenario the Universe is supposed to emerge from a highly perturbative initial state *preceding* the big bang. Therefore, early enough before (and late enough after) the big bang, we may presume to know the effective theory to be solved. The difficult part to be dealt with non-perturbatively remains the transition from the pre- to the post-big bang regime, through a high-curvature (and/or possibly a large-coupling) phase. Thus, from a more phenomenological standpoint, the relevant question becomes: Are the predictions of the pre-big bang scenario robust with respect to the details of the non-perturbative phase?

It is difficult of course to give a clear-cut answer to this question, but an analogy with QCD and the physics of strong interactions may be helpful. Because of asymptotic freedom, QCD can be treated perturbatively at short distance (high momentum transfers). However, even "hard" processes such as $e^+e^- \to$ hadrons are not fully within perturbative control. Some soft non-perturbative physics always gets mixed in at some level, e.g. when partons eventually turn into hadrons. The reason why certain sufficiently inclusive quantities are believed to be calculable is that large- and short-distance physics "decouple", so that, for instance, the hadronization process does not affect certain "infrared-safe" quantities, computed at the quark–gluon level.

In the case of string cosmology the situation should be similar, although somehow reversed [600]. For gravity, in fact, the large-distance, small-curvature regime is easy to deal with, while the short-distance, high-curvature is hard. Yet, we shall argue that some consequences of string cosmology, those concerning length scales that were very large with respect to the string scale (or the horizon) in the high-curvature regime, should not be affected (other than by a trivial kinematical redshift) by the details of the pre- to post-big bang transition. The above reasoning does not imply, of course, that string theorists should not address the hard, non-perturbative questions *now*. On the contrary, the "easy" part of the game will provide precious information about what the relevant hard questions are, and on how to formulate them.

Finally, possible reservations on a "top–down" string cosmology approach may naturally arise from a cosmology community accustomed to a data-driven, "bottom–up" approach. We do believe ourselves that a good model of cosmology is unlikely to emerge from theoretical considerations alone. Input from the data will be essential in the selection among various theoretical alternatives. We also believe, however, that a balanced combination of theoretical and experimental imput should be the best guarantee for an eventual success.

Insisting on the soundness of the underlying theory (e.g. on its renormalizability) was indeed essential in the progressive construction of the standard model, just as were the quantity and the quality of experimental data. Cosmology today resembles the particle physics of the sixties: there is no shortage of data, and these are becoming more and more precise but also more and more challenging while, theoretically, we are still playing with very phenomenological (even if undoubtedly successful) models, lacking a clear connection to other branches of fundamental physics, and therefore remaining largely unconstrained.

1.1 Coping with a beginning of time

Both the standard Friedmann–Robertson–Walker (FRW) cosmological scenario [618] and the standard inflationary scenario [360, 420, 441] assume that time had a beginning. Many of the problems with the former model simply stem from the fact that, at the start of the classical era, so little time had elapsed since the beginning. Indeed, in the FRW framework, the proper size of the (now observable) Universe was about 10^{-3} cm across at the start of the classical era, say at a time of the order of a few Planck times, $t_{\rm P} \sim 10^{-43}$ s. This is of course a very tiny Universe with respect to its present size ($\sim 10^{28}$ cm), yet it is huge with respect to the horizon (the distance travelled by light) at that time, $\lambda_{\rm P} = ct_{\rm P} \sim 10^{-33}$ cm.

In other words, a few Planck times after the big bang, our observable Universe consisted of about $(10^{30})^3 = 10^{90}$ Planckian-size, causally disconnected regions. Simply not enough time had elapsed since the beginning for the Universe to become homogeneous (e.g. to thermalize) over its entire size. Furthermore, soon after $t = t_{\rm P}$, the Universe must have been characterized by a huge hierarchy between its Hubble radius, on the one hand, and

10^30 hierarchy pb

10^90 disconnected!!!!!!!!!!!!!!!!!

its spatial-curvature radius, on the other. The relative factor of (at least) 10^{30} appears as an incredible amount of fine-tuning on the initial state of the Universe, corresponding to a huge asymmetry between space and time derivatives, or, in more abstract terms, between intrinsic and extrinsic curvature. Was this asymmetry really there? And, if so, can it be explained in any, more natural way?

The conventional answer to the difficulties of the standard scenario is to wash out inhomogeneities and spatial curvature by introducing, in the history of the Universe, a long period of accelerated expansion, called inflation [360, 420, 441]. It has been pointed out, however, that standard inflation cannot be "past-eternal" [94] (and cannot avoid the initial singularity [612, 95]), so that the question of what preceded inflation is very relevant. Insisting on the assumption that the Universe (and time itself) started at the big bang leaves only the possibility of having post-big bang inflation mend an insufficiently smooth and flat Universe arising from the big bang.

inflation solution

Unfortunately, that solution has its own problems, for instance those of fine-tuned initial conditions for the inflaton field and its potential. A consistent quantum cosmology approach giving birth to a Universe in the "right" initial state is still much under debate [370, 609, 440, 633, 541]. Furthermore, the inflaton is introduced *ad hoc* and inflation is not part of a grander theory of elementary particles and fundamental interactions such as superstring theory. In spite of its possible importance, and of repeated motivated attempts [236, 461, 89], a conventional realization of an inflationary phase in a string theory context is in fact problematic [141], in particular because the dilaton –the fundamental string theory scalar– cannot be (at least trivially) identified with the inflaton –the fundamental scalar of the standard inflationary scenario [125].

Here we shall argue that, instead, superstring theory gives strong hints in favour of a totally different approach to solving the problems of the standard cosmological scenario. This new possibility arises if we assume that, in string theory, the big bang singularity is fictitious and that it makes therefore sense to "continue" time to the past of the big bang itself.

1.2 Inflation before the big bang

If the history of the Universe can be continued backward in time past the big bang, new possibilities arise for a causal evolution to have *produced* a big bang with the desired characteristics. The actual pre-big bang scenario presented in this report is just one possible realization of the above general idea. Since, as we shall see, it is easy to generate a phase of pre-big bang inflation driven by the kinetic energy of the dilaton (somewhat in analogy with kinetic-inflation ideas [429]), we will discuss, as the simplest possibility, a minimal cosmological scenario, which avoids making use of standard (i.e. potential-energy-driven) post-big bang inflation.

PBB inflation
driven by DILATON

PBB evolution TOwards HIGH curvature

mean

This does not, though, that pre- and post-big bang inflation are mutually exclusive or incompatible. Should near-future high-precision experiments definitely indicate that an inflation that is exclusively of the pre-big bang type is disfavoured with respect to conventional, post-big bang, "slow-roll" inflation, one should ask whether a pre-big bang phase can naturally lead to "initial" conditions suitable for igniting an inflationary epoch of the slow-roll type, rather than a standard, non-inflationary, FRW cosmology.

One model-independent feature of pre-big bang cosmology is clear: by its very definition, the pre-big bang phase should be an evolution towards –rather than away from– a high-curvature regime. As we shall see in Section 2, this is precisely what the symmetries of the string cosmology equations suggest, an unconventional realization of the inflationary scenario, in which the phase of accelerated cosmological evolution occurs while the Universe is approaching –rather than getting away from– the high-curvature, Planckian regime.

The main difference between the string cosmology and the standard inflationary scenarios can therefore be underlined through the opposite behaviour of the curvature scale as a function of time, as shown in Fig. 1.1. As we go backward in time, instead of a monotonic growth (predicted by the standard scenario), or of a "de-Sitter-like" phase of nearly constant curvature (as in the conventional inflationary picture), the curvature grows, reaches a maximum controlled by the string scale $M_s = \lambda_s^{-1}$, and then starts decreasing towards an asymptotically flat state, the string perturbative vacuum. The big bang singularity is regularized by a "stringy" phase of high but finite curvature, occurring at the end of the initial inflationary evolution.

We should warn the reader, from the very beginning of this review, that this scenario is far from being complete and understood in all of its aspects, and that many important problems are to be solved still. Nevertheless, the results obtained up to now have been encouraging, in the sense that it now seems possible to formulate models for the pre-big bang evolution of our Universe that fit consistently in a string theory context, and which are compatible with various phenomenological and theoretical bounds. Not only: the parameter space of such models seems to be accessible to direct observations in a relativiely near future and, at present, it is already indirectly constrained by various astrophysical, cosmological, and particle physics data.

To close this subsection we should mention, as a historical note, that the idea of a phase of growing curvature preceding that of standard decelerated expansion, is neither new in cosmology, nor peculiar to string theory. Indeed, if the growth of the curvature corresponds to a contraction, it is reminiscent of Tolman's cyclic Universe [578], in which the birth of our present Universe is preceded by a phase of gravitational collapse (see also [225, 92, 503]). Also, and more conventionally, the growth of the curvature may be implemented as a phase of Kaluza–Klein superinflation [563, 1, 419], in which the accelerated expansion of our three-dimensional space is sustained by the contraction of the internal dimensions and/or by some exotic source, with the appropriate equation of state (in particular, strings [315]

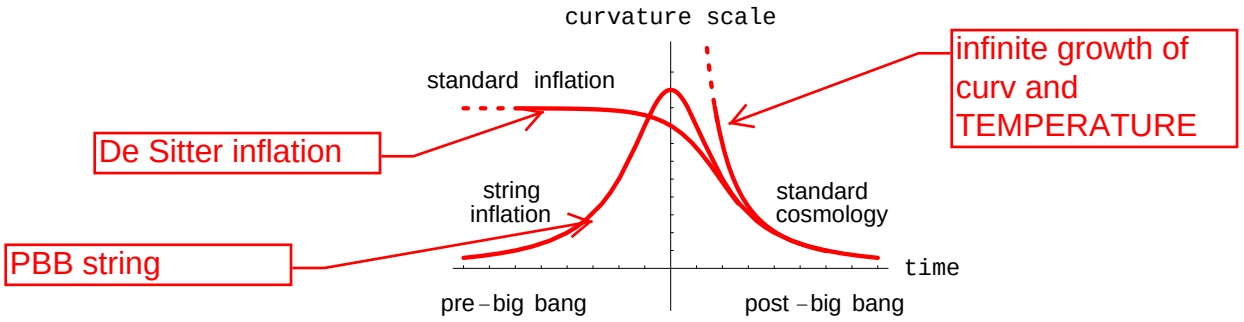


Figure 1.1: *Qualitative evolution of the curvature scale in the standard cosmological model, in conventional inflationary models and in string-cosmology models.*

and extended objects).

In the context of general relativity, however, the problem is how to avoid the curvature singularity appearing at the end of the phase of growing curvature. This is in general impossible, for both contraction and superinflationary expansion, unless one accepts rather drastic modifications of the classical gravitational theory. In the contracting case, for instance, the damping of the curvature and a smooth transition to the phase of decreasing curvature can be arranged through the introduction of a non-minimal and gauge-non-invariant coupling of gravity to a cosmic vector [504] or scalar [555, 56] field, with a (phenomenological) modification of the equation of state in the Planckian curvature regime [540, 621], or with the use of a non-metric, Weyl-integrable connection [503]. In the case of superinflation, a smooth transition can be arranged through a breaking of the local Lorentz symmetry of general relativity [266, 281], a geometric contribution of the spin of the fermionic sources [267], or the embedding of the space-time geometry into a more fundamental quantum phase-space dynamics [136, 269]. In the more exotic context of topological transitions, a smooth evolution from contraction to expansion, through a state of minimal size, is also obtained with the adiabatic compression and the dimensional transmutation of the de Sitter vacuum [316].

In the context of string theory, on the contrary, the growth of the curvature is naturally associated to the growth of the dilaton and of the coupling constants (see for instance Section 2). This effect, on the one hand, sustains the phase of superinflationary expansion, with no need of matter sources or extra dimensions. On the other hand, it necessarily leads the Universe to a regime in which not only the curvature but also the couplings become strong, so that typical "stringy" effects become important and are expected to smooth out the curvature singularity. This means that there is no need to look for more or less *ad hoc* modifications of the theory, as string theory itself is expected to provide the appropriate tools for a complete and self-consistent cosmological scenario.

r, spatial curv term

1.3 Pre-big bang inflation and conformal frames

While postponing to the next section the issue of physical motivations, it is important to classify the various inflationary possibilities just from their kinematical properties. To be more precise, let us consider the so-called flatness problem (the arguments are similar, and the conclusions the same, for the horizon problem mentioned in Subsection 1.1). We shall assume, on the basis of the approximate isotropy observed at large scale, that our present cosmological phase can be correctly described by the ordinary Einstein–Friedmann equations. In that case, the gravitational part of the equations contains two contributions from the metric: k/a^2, coming from the spatial (or intrinsic) curvature, and H^2, coming from the gravitational kinetic energy (or extrinsic curvature). Present observations imply that the spatial curvature term, if not negligible, is at least non-dominant, i.e.

$$r^2 = \frac{k}{a^2 H^2} \lesssim 1. \tag{1.1}$$

On the other hand, during a phase of standard, decelerated expansion, the ratio r grows with time. Indeed, if $a \sim t^\beta$,

$$r \sim \dot{a}^{-1} \sim t^{1-\beta}, \tag{1.2}$$

so that r is growing both in the matter-dominated ($\beta = 2/3$) and in the radiation-dominated ($\beta = 1/2$) era. Thus, as we go back in time, r becomes smaller and smaller. If, for instance, we wish to impose initial conditions at the Planck scale, we must require a fine-tuning suppressing by 30 orders of magnitude the spatial curvature term with respect to the other terms of the cosmological equations. Even if initial conditions are given at a lower scale (say the GUT scale) the amount of fine-tuning is still nearly as bad.

!!!!!!!!!!!!!

This problem can be solved by introducing an early phase during which the value of r, initially of order 1, decreases so much in time that its subsequent growth during FRW evolution keeps it still below 1 today. It is evident that, on pure kinematic grounds, this requirement can be implemented in two classes of backgrounds.

(I) : $a \sim t^\beta,\ \beta > 1,\ t \to +\infty$. This class of background corresponds to what is conventionally called "power inflation" [447], describing accelerated expansion and decreasing curvature scale, $\dot{a} > 0$, $\ddot{a} > 0$, $\dot{H} < 0$. It contains, as the limiting case ($\beta \to \infty$), exponential de Sitter inflation, $a \sim e^{Ht}$, $\dot{H} = 0$, describing accelerated expansion with constant curvature.

(II) : $a \sim (-t)^\beta,\ \beta < 1,\ t \to 0_-$. This case contains two subclasses.

- **(IIa)** : $\beta < 0$, corresponding to "superinflation" or "pole inflation" [563, 1, 419], and describing accelerated expansion with growing curvature scale, $\dot{a} > 0$, $\ddot{a} > 0$, $\dot{H} > 0$;
- **(IIb)** : $0 < \beta < 1$, describing accelerated contraction and growing curvature scale [320], $\dot{a} < 0$, $\ddot{a} < 0$, $\dot{H} < 0$.

STRING AND EINSTEIN FRAMES

epsilon inflation AFTER BB !!!!!!!!!!!!!!!!!!!!!

In the first class of backgrounds, corresponding to post-big bang inflation, the Universe is driven *away* from the singularity/high-curvature regime, while in the second class inflation drives the Universe *towards* it, with the typical pre-big bang behaviour illustrated in Fig. 1.1. We may thus immediately note a very important "phenomenological" difference between post- and pre-big bang inflation. In the former case the Planck era lies very far in the past, and its physics remains screened from present observations, since the scales that probed Planckian physics are still far from re-entering. By contrast, in the pre-big bang case, the Planck/string regimes are closer to us (assuming that no or little inflation occurs after the big bang itself). Scales that probe Planckian physics are now the first to re-enter, and to leave an imprint for our observations (see, for instance, the case of a stochastic background of relic gravitational waves, discussed in Section 5).

the grand trick !!

The inflationary character of Class IIa backgrounds is well known, and recognized since the earlier studies of the inflationary scenario [447]. The inflationary character of Class IIb is more unconventional –a sort of "inflation without inflation" [273], if we insist on looking at inflation as accelerated expansion– and was first pointed out only much later [320]. It is amusing to observe that, in the pre-big bang scenario, both subclasses IIa and IIb occur. However, as discussed in detail in Subsection 2.5, they do not correspond to different models of pre-big bang inflation, but simply to different kinematical representations of the *same* scenario in two *different* conformal frames.

In order to illustrate this point, which is important also for our subsequent arguments, we shall proceed in two steps. First we will show that, through a field redefinition $g = g(\widetilde{g}, \widetilde{\phi})$, $\phi = \phi(\widetilde{\phi})$, it is always possible to move from the string frame (S-frame), in which the lowest order gravidilaton effective action takes the form

conformal factor

p15

1.

$$S(g,\phi) = -\int d^{d+1}x\sqrt{|g|}\, e^{-\phi}\left[R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \tag{1.3}$$

to the Einstein frame (E-frame), in which the dilaton ϕ is minimally coupled to the metric and has a canonical kinetic term:

$$S(\widetilde{g},\widetilde{\phi}) = -\int d^{d+1}x\sqrt{|\widetilde{g}|}\,\left[\widetilde{R} - \frac{1}{2}\widetilde{g}^{\mu\nu}\partial_\mu\widetilde{\phi}\partial_\nu\widetilde{\phi}\right] \tag{1.4}$$

(see Subsection 1.4 for notations and conventions). Secondly, we will show that, by applying such a redefinition, a superinflationary solution obtained in the S-frame becomes an accelerated contraction in the E-frame, and vice versa.

2.

We shall consider, for simplicity, an isotropic, spatially flat background with d spatial dimensions, and set:

$$g_{\mu\nu} = \mathrm{diag}\left(N^2, -a^2\delta_{ij}\right), \qquad \phi = \phi(t), \tag{1.5}$$

where $g_{00} = N^2$ is to be fixed by an arbitrary gauge choice. For this background the S-frame action (1.3) becomes, modulo a total derivative,

$$S(g,\phi) = -\int d^{d+1}x\frac{a^d e^{-\phi}}{N}\left[\dot{\phi}^2 - 2dH\dot{\phi} + d(d-1)H^2\right], \tag{1.6}$$

where, as expected, N has no kinetic term and plays the role of a Lagrange multiplier. In the E-frame the variables are $\widetilde{N}, \widetilde{a}, \widetilde{\phi}$, and the action(1.4), after integration by parts, takes the canonical form

$$S(\widetilde{g}, \widetilde{\phi}) = -\int d^{d+1}x \frac{\widetilde{a}^d}{\widetilde{N}} \left[-\frac{1}{2}\dot{\widetilde{\phi}}^2 + d(d-1)H^2 \right]. \tag{1.7}$$

A quick comparison with Eq. (1.6) finally leads to the field redefinition (not a coordinate transformation!) connecting the Einstein and String frames:

E and S frames

$$\widetilde{a} = a e^{-\phi/(d-1)}, \qquad \widetilde{N} = N e^{-\phi/(d-1)}, \qquad \widetilde{\phi} = \phi \sqrt{\frac{2}{d-1}}. \tag{1.8}$$

Consider now an isotropic, d-dimensional vacuum solution of the action (1.6), describing a superinflationary, pre-big bang expansion driven by the dilaton (see Section 2) [489, 599], of Class IIa, with $\dot{a} > 0, \ddot{a} > 0, \dot{H} > 0, \dot{\phi} > 0$:

$$a = (-t)^{-1/\sqrt{d}}, \qquad e^{\phi} = (-t)^{-(\sqrt{d}+1)}, \qquad t < 0, \qquad t \to 0_-, \tag{1.9}$$

E:

and look for the corresponding E-frame solution. Since the above solution is valid in the synchronous gauge, $N = 1$, we can choose, for instance, the synchronous gauge also in the E-frame, and fix $\widetilde{N}$ by the condition:

$$\widetilde{N} dt \equiv N e^{-\phi/(d-1)} dt = d\widetilde{t}, \tag{1.10}$$

which defines the E-frame cosmic time $\widetilde{t}$ as:

$$d\widetilde{t} = e^{-\phi/(d-1)} dt. \tag{1.11}$$

After integration

$$t \sim \widetilde{t}^{\frac{d-1}{d+\sqrt{d}}}; \tag{1.12}$$

the transformed solution takes the form:

$$\widetilde{a} = (-\widetilde{t})^{1/d}, \qquad e^{\widetilde{\phi}} = (-\widetilde{t})^{-\sqrt{\frac{2(d-1)}{d}}}, \qquad \widetilde{t} < 0, \qquad \widetilde{t} \to 0_-. \tag{1.13}$$

It can easily be checked that this solution describes accelerated contraction of Class IIb, with growing dilaton and growing curvature scale:

$$\frac{d\widetilde{a}}{d\widetilde{t}} < 0, \qquad \frac{d^2\widetilde{a}}{d\widetilde{t}^2} < 0, \qquad \frac{d\widetilde{H}}{d\widetilde{t}} < 0, \qquad \frac{d\widetilde{\phi}}{d\widetilde{t}} > 0. \tag{1.14}$$

The same result applies if we transform other isotropic solutions from the String to the Einstein frame, for instance the superinflationary solutions with perfect fluid sources [320], presented in Section 2.

To conclude this section, and for later use, let us stress that the main dynamical difference between post-big bang inflation, Class I metrics, and pre-big bang inflation, Class II

IIa and IIb are related by transfo E/S due to dilaton

metrics, can also be conveniently illustrated in terms of the proper size of the event horizon, relative to a given comoving observer.

Consider in fact the proper distance $d_e(t)$ of the event horizon from a comoving observer, at rest in an isotropic, conformally flat background [536]:

$$d_e(t) = a(t) \int_t^{t_M} dt' a^{-1}(t'). \tag{1.15}$$

Here t_M is the maximal allowed extension, towards the future, of the cosmic time coordinate for the given background manifold. The above integral converges for all the above classes of accelerated (expanding or contracting) scale factors. In the case of Class I metrics we have, in particular,

$$d_e(t) = t^\beta \int_t^\infty dt' t'^{-\beta} = \frac{t}{\beta - 1} = \frac{\beta}{\beta - 1} H^{-1}(t) \tag{1.16}$$

for power-law inflation ($\beta > 1, t > 0$), and

$$d_e(t) = e^{Ht} \int_t^\infty dt' e^{-Ht'} = H^{-1} \tag{1.17}$$

for de Sitter inflation. For *Class II* metrics ($\beta < 1, t < 0$) we have instead

sign error!!!!!

$$d_e(t) = (-t)^\beta \int_t^0 dt' (-t')^{-\beta} = \frac{(-t)}{1 - \beta} = \frac{\beta}{\beta - 1} H^{-1}(t). \tag{1.18}$$

In all cases the proper size $d_e(t)$ evolves in time like the so-called Hubble horizon (i.e. the inverse of the modulus of the Hubble parameter), and then like the inverse of the curvature scale. The size of the horizon is thus constant or growing in standard inflation (Class I), decreasing in pre-big bang inflation (Class II), both in the S-frame and in the E-frame.

Such an important difference is clearly illustrated in Figs. 1.2 and 1.3, where the dashed lines represent the evolution of the horizon and the solid curves the evolution of the scale factor. The shaded area at time t_0 represents the portion of Universe inside our present Hubble radius. As we go back in time, according to the standard scenario, the horizon shrinks linearly ($H^{-1} \sim t$); however, the decrease of the scale factor is slower, so that, at the beginning of the phase of standard evolution ($t = t_1$), we end up with a causal horizon much smaller than the portion of Universe that we presently observe. This is the "horizon problem" already mentioned at the beginning of this section.

horizon problem

In Fig. 1.2 the phase of standard evolution is preceded in time by a phase of standard, post-big bang (in particular de Sitter) inflation. Going back in time, for $t < t_1$, the scale factor keeps shrinking, and our portion of Universe "re-enters" the Hubble radius during a phase of constant (or slightly growing in time) horizon.

In Fig. 1.3 the standard evolution is preceded in time by a phase of pre-big bang inflation, with growing curvature. The Universe "re-enters" the Hubble radius during a

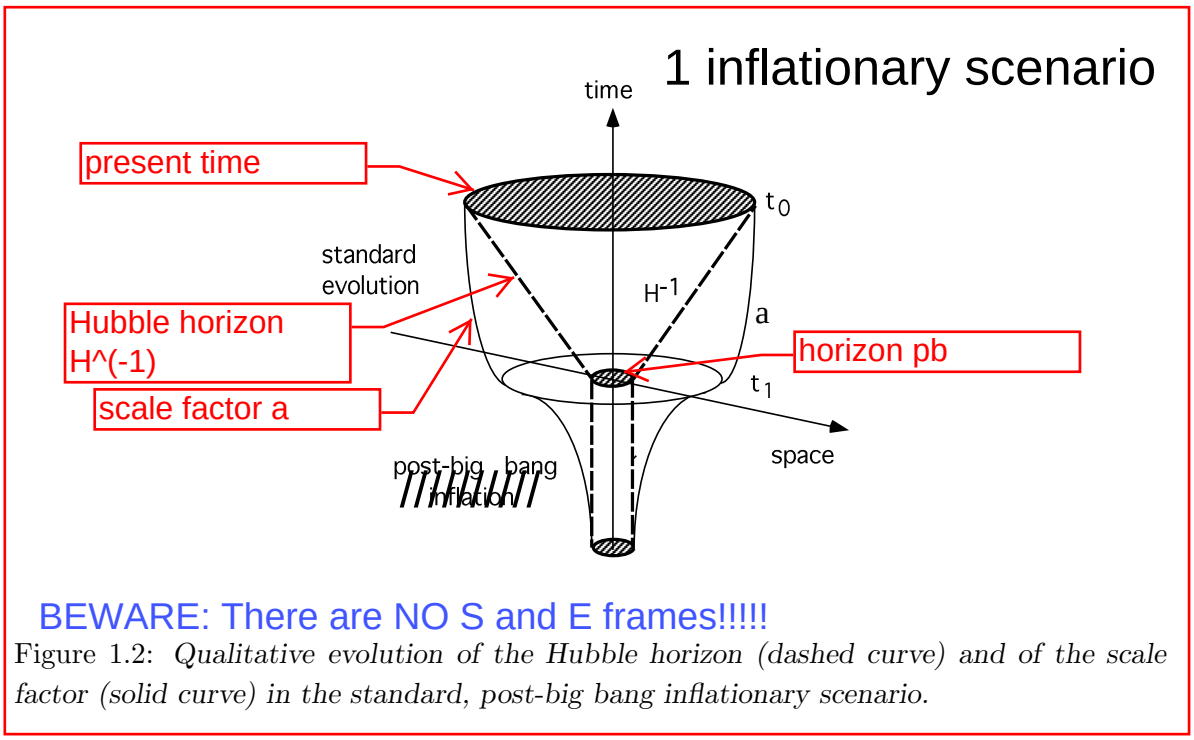


Figure 1.2: *Qualitative evolution of the Hubble horizon (dashed curve) and of the scale factor (solid curve) in the standard, post-big bang inflationary scenario.*

phase of shrinking horizon. To emphasize the difference, we have plotted the evolution of the scale factor both as expanding in the S-frame, $a(t)$, and as contracting in the E-frame, $\tilde{a}(t)$. Unlike in post-big bang inflation, the proper size of the initial portion of the Universe may be very large in strings (or Planck) units, *but not larger than the initial horizon* itself [286], as emphasized in the picture, and as will be discussed in a more quantitative way in Section 2.4. The initial horizon H_i^{-1} is large because the initial curvature scale is small (in string cosmology, in particular, $H_i \ll 1/\lambda_s$).

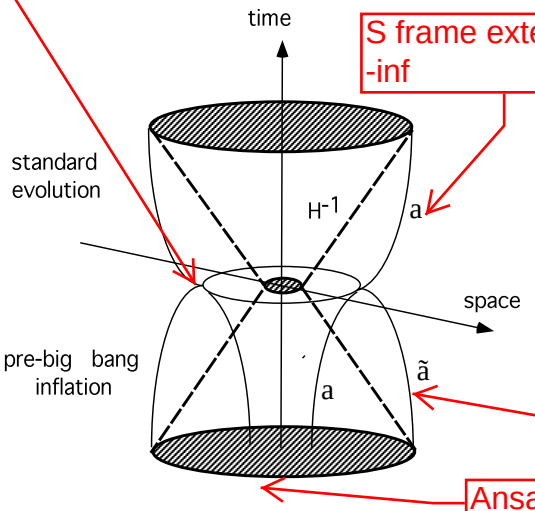
This is a basic consequence of the choice of the initial state; in the context of the string cosmology scenario, this approaches the flat, cold and empty string perturbative vacuum, (see the discussion of Section 3). This initial state has to be contrasted with the extremely curved, hot and dense initial state of the standard scenario, characterizing a Universe that starts inflating at (or soon after) the Planck scale, $H_i \sim 1/\lambda_P$ (see also [288] for a more detailed comparison and discussion of pre-big bang versus post-big bang inflation).

1.4 Outline, notations and conventions

We give here a general overlook at the material presented in the various sections of this report. Furthermore, each section will begin with an outline of the content of each of its subsections, and will try to be as self-contained as possible, in order to help the reader interested only in some particular aspects of this field.

In Section 2, after a very quick reminder of some relevant properties of superstring theory, we review the string-theoretic motivations behind the pre-big bang scenario, and

CAUTION: S and E are NOT symmetric!!! See a and ã curves for that



S frame extends to -inf

E frame stops to t = 0

Ansatz: asymptotic past triviality

Figure 1.3: *Qualitative evolution of the Hubble horizon (dashed curve) and of the scale factor (solid curve) in the pre-big bang inflationary scenario, in the S-frame, $a(t)$, and in the E-frame, $\tilde{a}(t)$.*

outline the main ideas. In Section 3, after formulating on the basis of the previous discussion a postulate of " asymptotic past triviality", we discuss, within that framework, the problem of initial conditions and fine-tuning.

As a preliminary to the discussion of the observational consequences of the pre-big bang scenario, Section 4 presents some general results on the evolution of quantum fluctuations. Section 5 will deal with the specific case of tensor (metric) perturbations and Section 6 with scalar (dilatonic) ones, while Section 7 will consider gauge (in particular electromagnetic) and axionic perturbations, and their possible physical relevance to galactic magnetic fields and large scale structure, respectively.

The last part of this report is devoted to various open problems, in particular to the possibility of a smooth transition from the pre- to the post-big bang regime, the so-called "exit problem" (Section 8). Several lines of approach to (and partial solutions of) this problem are presented, including a discussion of heat and entropy production in pre-big bang cosmology. A possible minisuperspace approach to the quantum regime will be illustrated in Section 9. Finally, Section 10 contains the discussion of a possible dilatonic interpretation of the quintessence, a short presentation of other open problems, and an outlook.

EXIT PROBLEM

For further study of string cosmology and of the pre-big bang scenario, we also refer the interested reader to other review papers [437, 288, 63, 244, 102], as well as to two recent introductory lectures [289, 605]. A regularly updated collection of selected papers on the pre-big bang scenario is also available at [265].

We finally report here our conventions for the metric and the curvature tensor, together

with the definitions of some variables frequently used in the paper.

We shall always use natural units $\hbar = c = k_B = 1$. Unless otherwise stated, the metric signature is fixed by $g_{00} > 0$; the Riemann and Ricci tensors are defined by

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} + \Gamma_{\mu\rho}{}^{\beta}\Gamma_{\nu\alpha}{}^{\rho} - (\mu \leftrightarrow \nu), \qquad R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu}. \tag{1.19}$$

In particular, for a Bianchi-I-type metric, and in the synchronous gauge,

$$g_{\mu\nu} = \mathrm{diag}\left(1, -a_i^2(t)\delta_{ij}\right), \tag{1.20}$$

our conventions lead to

$$R_0{}^0 = -\sum_i \left(\dot{H}_i + H_i^2\right), \qquad R_i{}^j = -\dot{H}_i\delta_i^j - H_i\delta_i^j \sum_k H_k,$$
$$R = -\sum_i \left(2\dot{H}_i + H_i^2\right) - \left(\sum_i H_i\right)^2, \tag{1.21}$$

where $H_i = d\ln a_i/dt$. In a $D = d+1$ space-time manifold, Greek indices run from 0 to d, while Latin indices run from 1 to d.

phibar

The duality invariant dilatonic variable, the "shifted dilaton" $\overline{\phi}$, is referred to a d-dimensional spatial section of finite volume, $(\int d^dx\sqrt{|g|})_{t=\mathrm{const}} < \infty$, and is defined by

$$e^{-\overline{\phi}} = \int \frac{d^dx}{\lambda_{\mathrm{s}}^d}\sqrt{|g_d|}e^{-\phi}. \tag{1.22}$$

In a Bianchi-I-type metric background, in particular, we shall absorb into ϕ the constant shift $-\ln(\lambda_{\mathrm{s}}^{-d}\int d^dx)$ (required to secure the scalar behaviour of $\overline{\phi}$ under coordinate reparametrizations), and we shall set

$$\boxed{\overline{\phi} = \phi - \sum_i \ln a_i.} \tag{1.23}$$

lambda s

Finally, λ_{s} is the fundamental length scale of string theory, related to the string mass M_{s} and to the string tension T (the mass per unit length) by

$$\lambda_{\mathrm{s}}^2 = M_{\mathrm{s}}^{-2}T^{-1} \equiv 2\pi\alpha'. \tag{1.24}$$

At the tree level (i.e. at lowest order) in the string coupling g_{s}, the string length is related to the Planck length λ_{P}, and to the gravitational constant G_D in $d+1$ dimensions, by

G D

$$8\pi G_D = \lambda_{\mathrm{P}}^{d-1} = \lambda_{\mathrm{s}}^{d-1}e^{\phi}. \tag{1.25}$$

In $d = 3$, in particular, the relation between the string and the Planck mass $M_{\mathrm{P}} = \lambda_{\mathrm{P}}^{-1}$ reads

$$(\lambda_{\mathrm{P}}/\lambda_{\mathrm{s}})^2 = (M_{\mathrm{s}}/M_{\mathrm{P}})^2 = e^{\phi}. \tag{1.26}$$

We shall often work in units such that $2\lambda_{\mathrm{s}}^{d-1} = 1$, i.e. $16\pi G_D = 1$, in which $\exp(\phi)$ parametrizes, in the String frame, the (dimensionless) strength of the gravitational coupling.

2 String theory motivations

A very important concept in string theory, as well as in field theory, is that of moduli space, the space of vacua. In field theory, and in the absence of gravity, the coordinates of moduli space label the possible ground states of the theory. It is very important to immediately distinguish classical moduli space from its exact, quantum counterpart. The two are generally different, since a classical-level ground state can fail to be a true ground state when perturbative or non-perturbative quantum corrections are added (consider for instance dynamical symmetry breaking à la Coleman–Weinberg, or the double-well potential in quantum mechanics).

moduli space

When gravity is added to the picture (and this is always the case in s tring theory) the concept of a lowest-energy state becomes less well defined, since total energy is always zero in a general-covariant theory. It is therefore better, in string theory, to extend the definition of moduli space to include all string backgrounds that allow a consistent string propagation, i.e. those consistent with world-sheet conformal invariance [346]. Such backgrounds correspond to the vanishing of the two-dimensional σ-model β-functions [445] and, at the same time, they can also be shown to satisfy the field equations of an effective action living in ordinary space-time. The most famous example of a consistent background is, for superstrings, $D = 10$ Minkowski space-time with trivial (i.e. constant) dilaton and antisymmetric tensor potentials. Unfortunately, even if quantum (i.e. string-loop) corrections are neglected, our knowledge of moduli space is very limited. Basically, apart from a handful of exact conformal field theories, such as the Wess–Zumino–Witten models, only low-energy solutions (i.e. classical solutions of the effective low-energy field theory) are known.

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The known solutions are, at the same time, too many and too few. They are too many because they typically leave the vacuum expectation values of a few scalar fields completely undetermined. Such fields correspond to gravitationally coupled massless scalar particles that mediate dangerous long-range forces, badly violating the well-tested equivalence principle. The way out of this problem is clear: these flat directions should be lifted by quantum corrections, typically (in the supersymmetric case) at the non-perturbative level. The known solutions are also too few, because some of them, which we would like to see appearing, are missing: notably those describing gravitational collapse or cosmological backgrounds, which evolve as a function of time from a regime of low curvature and/or coupling to one of high curvature and/or coupling (and vice versa). These solutions are very hard to analyse for two reasons: first, because time evolution spontaneously breaks supersymmetry, rendering the solutions unstable to radiative corrections; secondly, because the solutions go out of theoretical control, as they enter the non-perturbative regime.

For the purpose of this section, the important property of superstring's moduli space is that it exhibits duality symmetries. Generically, this means that points in moduli space that seem to describe different theories actually describe the same theory (up to some irrelevant relabelling of the fields). Let us illustrate this in the simplest example of T-duality (for a

DUALITY OF MODULI SPACE

physical lower limit to dim of compact

review see, for instance, [340]).

Consider a theory of closed strings moving in a space endowed with some compact dimensions, say, for the sake of simplicity, with one extra dimension having the topology of a circle. Let us denote by R_c the radius of this circle. In point-particle theory, momentum along the compact dimension is quantized, in units of $\hbar/R_c$. This is also true in string theory, as far as the motion of the string's centre of mass (a non-oscillatory "zero mode") is concerned. However, for closed strings moving on a compact space, like our circle, there is a second zero mode: the string can simply wind around the circle an integer number of times. By doing so it acquires winding energy, which is quantized in units of $(2\pi T)R_c$, if $T = (2\pi\alpha')^{-1}$ is the string tension.

Something remarkable does happen if we replace R_c by $\widetilde{R_c} \equiv \hbar/TR_c \equiv \lambda_s^2/R_c$. A point particle would certainly notice the difference between R_c and $\widetilde{R_c}$ (unless $R_c = \widetilde{R_c} = \lambda_s$), since the new momenta will be different from the old ones. A closed string, instead, does not feel the change of R since the role of the momenta in the original theory will mow be played by the winding modes, and vice versa. This symmetry of closed string theory has been called T-duality and is believed to be exact, at least to all orders of perturbation theory, provided a suitable transformation of the dilaton accompanies the one on the radius (it also has interesting extensions to discrete groups of the $O(d,d;Z)$ type [340]). It is important to stress, in our context, that T-duality actually implies that there is a physical lower limit to the dimensions of a compact space, controlled by the string length λ_s itself.

When applied to open strings, T-duality leads to the concept of Dirichlet strings, or D-strings. In other words, while closed strings are self-dual, open strings with Neumann boundary conditions are dual to open strings with Dirichlet boundary conditions, and vice versa. These developments [523] have led to the study of D-branes, the manifolds on which the end-points of open D-strings are confined; they play a major role in establishing the basic unity of all five known types of 10-dimensional superstring theories (Type I, Type IIA, Type IIB, HETSO(32), HETE8) as different limits of a single, more fundamental "M-theory" [626].

1 2 ???

In order to briefly illustrate this point we start recalling that all superstring theories are actually defined through a *double* perturbative expansion in *two* dimensionless parameters: the first, the string coupling expansion, can be seen as the analogue of the loop expansion in quantum field theory, except that the coupling constant gets promoted to a scalar field, the dilaton. Consequently, the range of validity of the loop expansion depends on the value of the dilaton, and can break down in certain regions of space-time if the dilaton is not constant. The second expansion, which has no quantum field theory analogue, is an expansion in derivatives, the dimensionless parameter being $\lambda_s^2\partial^2$, with λ_s the fundamental length scale of string theory (see Subsection 2.1). Obviously, the validity of this second expansion breaks down when curvature or field space-time derivatives become of order 1 in string-length units.

String 2 perturb expansion!!!!! + dilaton variable

D = 11 SUGRA

Witten

One of the most amazing recent developments in string theory [626] is the recognition that the above five theories, rather than forming isolated islands in moduli space, are connected to one another via a web of duality transformations. In the huge moduli space, they represent "corners" where the above-mentioned perturbative expansions are, qualitatively at least, correct. A sixth corner actually should be added, corresponding to 11-dimensional supergravity [579]. The mysterious theory approaching these six known theories in appropriate limits was given the name of M-theory. It might seem curious, at first sight, that one is able to flow continuously from 10 to 11 dimensions, within a single theory. The puzzle was solved after it was realized that the $D = 11$ supergravity theory has no dilaton, hence no free coupling. In other words, the dilaton of the five 10-dimensional superstring theories becomes (at large coupling) an extra dimension of space [626, 376]. At weak coupling, this extra dimension is so small that one may safely describe physics in 10 dimensions.

Townsend

In spite of the beauty of all this, the previous discussion shows that the moduli-space diagram connecting the five superstring theories can be quite misleading. Since each point in the diagram is supposed to represent a possible solution of the β-function constraints, and the diagram itself is supposed to show how apparently different theories are actually connected by moving in coupling constant (or other moduli-) space, it necessarily includes the flat directions we have been arguing against. At the same time, cosmological solutions of the above-mentioned type, i.e. evolving in different regions of coupling constant/curvature, cannot be "localized" in the diagram. A single cosmology, for instance, may indeed correspond to an initial Universe, well described by heterotic string theory, ending in another Universe better described by perturbative Type I theory. Yet, such a cosmological solution should be only a point, not a curve, in moduli space. As we shall argue below, thanks to some gravitational instability, solutions of the above type are rather the rule than the exception.

DIFFICULT POINT

It seems appropriate, at this point, to comment on the fact that modern research in string/M-theory looks to be strongly biased towards analysing supersymmetric vacua, or at least solutions that preserve a large number of supersymmetries [524]. While the mathematical motivation for that is quite obvious, and physically interesting results can be rigorously derived in special cases (concerning, for instance, BPS states and black holes), it is quite clear that cosmology, especially inflationary or rapidly evolving solutions, requires extensive breaking of supersymmetries. Given their phenomenological importance, we think that more effort should be devoted to developing new techniques dealing with non-supersymmetric solutions and, in particular, with their high-curvature and/or large-coupling regimes.

In this section we will emphasize the role of the duality symmetry for the pre-big bang scenario, starting from the short- and large-distance properties of string theory (Subsection 2.1), and from the related invariance of the classical cosmological equations under a large group of non-compact transformations (Subsections 2.2, 2.3). We will discuss the inflationary aspects of families of solutions related by such groups of transformations in Subsection 2.4, and will end up with a discussion of the relative merits of the so-called Einstein and

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String frames (in Subsection 2.5), in spite of their ultimate equivalence.

2.1 Short- and large-distance motivations

Since the classical (Nambu–Goto) action S of a string is proportional to the area A of the space-time surface it sweeps, its quantization requires the introduction of a fundamental "quantum" of length λ_s, through the rlation:

$$S/\hbar = A/\lambda_s^2 \ . \tag{2.1}$$

As discussed in [597, 598], the appearance of this length in string theory is so much tied to quantum mechanics that, after introducing λ_s, we are actually dispensed from introducing $\hbar$ itself, provided we use the natural units of energy of string theory, i.e. provided we replace E by $\alpha' E$, the classical length of a string of energy E. For instance, the Regge trajectory relation between angular momentum and mass, $J = \alpha' M^2 + \alpha(0)\hbar$ ($c = 1$ throughout), is rewritten in string units as $J = M^2 + \alpha(0)\lambda_s^2$, where $\hbar$ has neatly disappeared.

One of the most celebrated virtues of string theory is its soft behaviour at short distances. This property, which is deeply related to the extended nature of fundamental particles in the theory, makes gravity and gauge interactions not just renormalizable, but finite, at least order by order in the loop expansion [524]. At the same time, the presence of an effective short-distance cut-off in loop diagrams should have a counterpart in a modification of the theory, when the external momenta of the diagram approach the cut-off itself.

An analogy with the standard electroweak theory may be of order here: the Glashow–Weinberg–Salam model makes Fermi's model softer at short distances, turning a non-renormalizable effective theory into a fully-fledged renormalizable gauge theory. At the same time, the naive predictions of Fermi's model of electroweak interactions become largely modified when the W/Z-mass region is approached. Note that this mass scale, and not $G_F^{-1/2}$, is where new physics comes into play.

In string theory, something similar is expected to occur, this time in the gravitational sector, where Einstein's effective theory should be recovered at low energy, as soon as the string scale (and *not* $G_N^{-1/2} \sim M_P$) is reached. At present, there are not so many available tests of this idea: one comes from the study of trans-Planckian energy collisions in string theory [22, 23]. When one prepares the initial state in such a way as to expect the formation of tiny black holes, having radius (or curvature) respectively smaller (or larger) than the fundamental scale of string theory, one finds that such objects are simply not formed. Arguments based on entropy considerations [98] also suggest that the late stage of black hole evaporation ends, in string theory, into a normal non-collapsed string state of radius equal to the string length, rather than into a singularity. Finally, we recall the arguments from T-duality for a minimal compactification scale, given at the beginning of this section.

This considerable amount of circumstantial evidence leads to the conclusion that λ_s

An area law ??
black hole evaporation??
alpha'
Polchinski string set
NO BH of size<lambda s
string/BH correspondence

plays, in string theory, the role of a minimal observable length, i.e. of an ultraviolet cut-off. Physical quantities are expected to be bounded (in natural units) by the appropriate powers of λ_s, e.g. the curvature scale $R \sim H^2 \sim G\rho \leq \lambda_s^{-2}$, the temperature $T \leq \lambda_s^{-1}$, the compactification radius $R_c \geq \lambda_s$, and so on. It follows, in particular, that space-time singularities are expected to be avoided (or at least reinterpreted) in any geometric model of gravity which is compatible with string theory.

In other words, in quantum string theory, relativistic quantum mechanics should solve the singularity problems in much the same way as non-relativistic quantum mechanics solves the singularity problem of the hydrogen atom, by keeping the electron and the proton a finite distance apart. By the same token, string theory gives us a rationale for asking daring questions such as: What was there before the big bang? Even if we do not know the answer to this question in string theory, in no other currently available framework can such a question be meaningfully asked.

To answer it, we should keep in mind, however, that even at large distance (i.e. low energy, small curvatures), superstring theory does *not* automatically give Einstein's general relativity. Rather, it leads to a scalar–tensor theory (which in some limit reduces to a theory of the Jordan–Brans–Dicke variety). In fact, the conformal invariance of string theory (see for instance [346]) unavoidably requires a new scalar particle/field ϕ, the dilaton, which, as already mentioned, gets reinterpreted as the radius of a new dimension of space in M-theory [626, 376].

Witten, Horava/ Witten

dilaton required for Cinvar

By supersymmetry, the dilaton is massless to all orders in perturbation theory (i.e. as long as supersymmetry remains unbroken) andcan thus give rise to dangerous violations of the equivalence principle. This is just one example of the general problem with classical moduli space that we mentioned at the beginning of this section. Furthermore, ϕ controls the strength of all forces [624], gravitational and gauge alike, by fixing the grand-unification coupling through the (tree-level) relation [403]:

?????

"mass breaks SUSY"

$$\alpha_{\rm GUT} \simeq (\lambda_{\rm P}/\lambda_s)^2 \simeq \exp(\phi) \ , \qquad (2.2)$$

showing the basic unification of all forces in string theory and the fact that, in our conventions, the weak-coupling region coincides with $\phi \ll -1$.

The bottom line is that, in order not to contradict precision tests of the equivalence principle and of the constancy (today and in the "recent" past) of the gauge and gravitational couplings, we require [577] the dilaton to have a mass and to be frozen at the bottom of its own potential *today* (see, however, [197, 198] for an ingenious alternative, whose possible consequences will be discussed in Sections. 6 and 10). This does not exclude, however, the possibility of the dilaton having evolved cosmologically in the past (after all, the metric did!), within the weak coupling region ($\phi \to -\infty$) where it was practically massless. The amazing (yet simple) observation [599] is that, by so doing, the dilaton may have inflated the Universe.

GV scale factor duality

1 breaks SUSY
2 No violations!!

DILATON: 1 has a mass

2 is frozen

" Dilaton driven cosmo"

DUALITY

"slow roll" dilaton

A simplified argument, which, although not completely accurate, captures the essential physical point, consists in writing the Friedmann equation $3H^2 = 8\pi G\rho$, and in noticing that a growing dilaton (meaning through Eq. (2.2) a growing G) can drive the growth of H even if the energy density of standard matter decreases (as typically expected in an expanding Universe). This particular type of superinflation, characterized by growing H and ϕ, has been termed dilaton-driven inflation.

The basic idea of pre-big bang cosmology [599, 319, 320, 321] can then be illustrated as in Fig. 2.1, where the dilaton starts at very large negative values, rolling up a potential which, at the beginning, is practically zero (at weak coupling, the potential is known to be instantonically suppressed [89], $V(\phi) \sim \exp[-\exp(-\phi)]$). The dilaton grows and inflates the Universe, until the potential develops some non-perturbative structure, which can eventually damp and trap the dilaton (possibly after some oscillations). The whole process may be seen as the slow, but eventually explosive, decay of the string perturbative vacuum (the flat and interaction-free asymptotic initial state of the pre-big bang scenario), into a final, radiation-dominated state typical of standard cosmology. Incidentally, as shown in Fig. 2.1, the dilaton of string theory can easily roll-up –rather than down– potential hills, as a consequence of its non-standard coupling to gravity.

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BEWARE!!!!!!!!!

A phase of accelerated evolution, sustained by the kinetic energy of a growing dilaton [599] (and possibly by other antisymmetric tensor fields [308, 177], in more complicated backgrounds) is not just possible: it does necessarily occur in a class of (lowest-order) cosmological solutions based on a cosmological variant of the (previously mentioned) T-duality symmetry [599, 580, 477, 478, 560, 367, 581, 582, 318]. In such a way duality provides a strong motivation for (and becomes a basic ingredient of) the pre-big bang scenario first introduced in [599] and whose developments are the subject of this report.

The importance of duality for a string-motivated cosmology was indeed pointed out already in some pioneer papers [18, 428, 20, 109] based on superstring thermodynamics; the fact that our standard Universe could emerge after a phase of inflation with "dual" cosmological properties was also independently suggested by the study of string motion in curved backgrounds [315]. For future applications (see Section 8) we wish to recall here, in particular, the T-duality approach to a superstring Universe discussed in [109] (see also [378]), in which all nine spatial dimensions are compact (with similar radius), and the presence of winding strings wrapped around the tori prevents the expansion of the primordial Universe, unless such winding modes disappear by mutual annihilation.

The probability of annihilation, however, depends on the number of dimensions. In $d = 9$ it is so small that strings prevent the nine dimensions from expanding. If $n < 6$ dimensions contract to the string size, string annihilation in the other $9 - n$ is still so small that even they cannot expand. Only for $n \geq 6$ is the annihilation probability in the remaining dimensions large enough, and the expansion becomes possible: this would give 3 as the maximal number of large space dimensions (such a mechanism has recently been

d = 3

windind modes disappearance

D=3 space generated in 109

DILATON DRIVEN COSMOLOGY

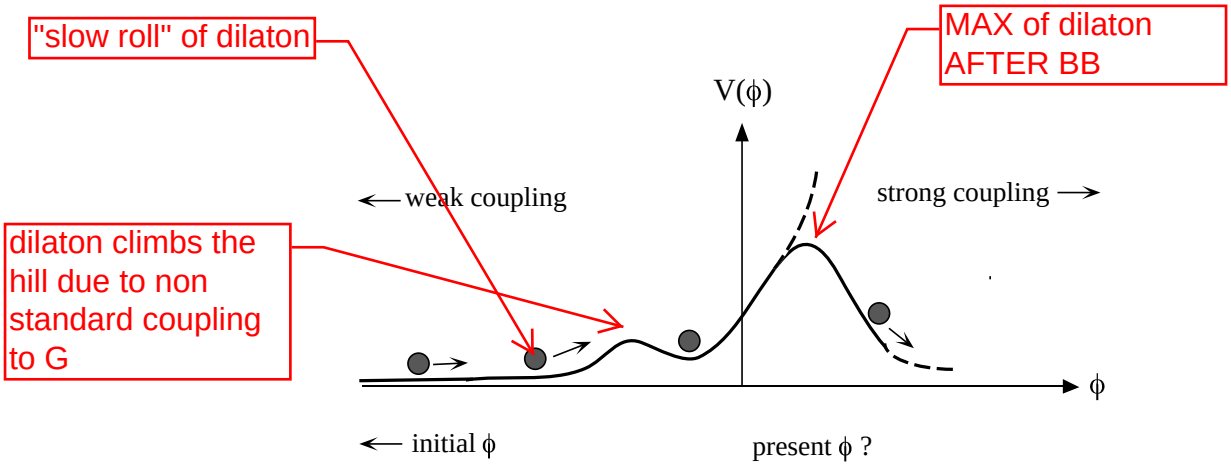


Figure 2.1: *According to the pre-big bang scenario, the dilaton starts in the asymptotic past of our Universe at very large negative values, and grows through a flat potential towards the strong-coupling regime. At present, it is either trapped at a minimum of the potential, or keeps growing monotonically towards $+\infty$ (see Section 10).*

extended also to a more general brane–gas context, see Subsection 8.5).

This as well as the other, early attempts were based on Einstein's cosmological equations, i.e. on gravitational equations at fixed dilaton. Taking into account the large-distance modifications of general relativity required by string theory, and including a dynamical dilaton, the target-space duality typical of closed strings moving in compact spaces can be extended (in a somewhat modified version) even to non-compact cosmological backgrounds [599, 478, 580, 581, 582]. Consider in fact a generic solution of the field equations of string theory (hence a point in our moduli space), which possesses a certain number n of Abelian isometries (the generalization to non-Abelian isometries is subtle, see [206]). Working in an adapted coordinate system, in which the fields appearing in the solution are independent of n-coordinates, it can then be argued [477] that there is an $O(n,n;R)$ group that, acting on the solution, generates new ones (in other words, this group has a representation in that part of moduli space that possesses the said isometries).

ok

Note that, unlike strict T-duality, this continuous $O(d,d;R)$ extension is not a true symmetry of the theory, but only a symmetry of the classical field equations. The corresponding transformations can be used to generate, from a given solution, other, generally inequivalent ones, and this is possible even in the absence of compactification. In the next subsections we will show in detail that such transformations, applied to a decelerated cosmological solution (and combined with a time-reversal transformation) lead in general to inflation, and we shall present various (low-energy) exact inflationary solutions, with and without sources, which may represent possible models of pre-big bang evolution. We shall consider, in particular, both scale-factor [599, 580, 582] and $O(d,d)$ [477, 478, 560, 367, 318] duality tranformations, and we will discuss some peculiar kinematic aspects of such pre-big

scale factor / O(d,d) Dualities

1st trick: time reversal of E eqs

a, H

1 reparam invar
2 gauge invar +
3 conformal invar at q level

a scale factor

bang solutions.

2.2 Scale-factor duality without and with sources

We start by recalling that, in general relativity, the Einstein action is invariant under time reflections. It follows that, if we consider an isotropic, spatially flat metric parametrized by the scale factor $a(t)$,

$$ds^2 = dt^2 - a^2(t)dx_i^2, \tag{2.3}$$

and if $a(t)$ is a solution of the Einstein equations, then $a(-t)$ is also a solution. On the other hand, when $t \to -t$ the Hubble parameter flips sign;

$$a(t) \to a(-t), \qquad H = \dot{a}/a \to -H. \tag{2.4}$$

Thus, to any standard cosmological solution $H(t)$, describing decelerated expansion and decreasing curvature ($H > 0$, $\dot{H} < 0$), time reversal associates a "reflected" solution, $H(-t)$, describing a contracting Universe.

In string theory, the reparametrization and gauge invariance of conventional field theories are expected to be only a tiny subset of a much larger symmetry group, which should characterize the effective action even at lowest order. The string effective action that we shall use in this section, in particular, is determined by the usual requirement that the string motion is conformally invariant at the quantum level [346]. The starting point is the (non-linear) sigma model describing the coupling of a closed string to external metric ($g_{\mu\nu}$), scalar (ϕ), and antisymmetric tensor ($B_{\mu\nu}$) fields. In the bosonic sector the action reads:

GSW

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi \left[\sqrt{-\gamma}\gamma^{ij}\partial_i x^\mu \partial_j x^\nu g_{\mu\nu}(x) + \epsilon^{ij}\partial_i x^\mu \partial_j x^\nu B_{\mu\nu}(x) + \frac{\alpha'}{2}\sqrt{-\gamma}R^{(2)}\phi(x)\right]. \tag{2.5}$$

Here $2\pi\alpha' = \lambda_s^2$, $\partial_i \equiv \partial/\partial\xi^i$, and ξ^i are the coordinates spanning the two-dimensional string world-sheet ($i, j = 1, 2$), whose induced metric is $\gamma_{ik}(\xi)$. The coordinates $x^\mu = x^\mu(\xi)$ are the fields determining the embedding of the string world-sheet in the external (also called "target") space, ϵ_{ij} is the two-dimensional Levi-Civita tensor density, and $R^{(2)}(\gamma)$ is the scalar curvature for the world-sheet metric γ. We have included the interaction of the string with all three massless states (the graviton, the dilaton and the antisymmetric tensor) appearing in the lowest energy level of the spectrum of quantum string excitations (the unphysical tachyon is removed by supersymmetry [346]). We note, for further use, that the antisymmetric field $B_{\mu\nu}$ is often called the Neveu–Schwarz/Neveu–Schwarz (NS-NS) two-form.

If we quantize the above action for the self-coupled fields $x^\mu(\xi)$, we can expand the loop corrections of the corresponding non-linear quantum field theory in powers of the curvature (i.e. in higher derivatives of the metric and of the other background fields) [346]. Such a higher derivative expansion is typical of an extended object like a string, and is indeed

1

Kalb-Ramond

alpha'

controlled by the powers of α', i.e. of the string length parameter λ_s. At each order in α', however, there is an additional higher genus expansion in the topology of the world-sheet metric, which corresponds, in the quantum field theory limit, to the usual loop expansion, controlled by the effective coupling parameter $g_s^2 = \exp(\phi)$.

2

The conformal invariance of the classical string motion in an external background can then be imposed, at the quantum level, at any loop order: we obtain, in this way, a set of differential conditions to be satisfied by the background fields for the absence of conformal anomalies, order by order. At tree level in g_s, and to lowest order in α', such differential equations (in a critical number of dimensions) are [346]:

$$R_{\mu\nu} + \nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta} = 0, \tag{2.6}$$

$$R + 2\nabla^2 \phi - (\nabla_\mu \phi)^2 - \frac{1}{12} H^2_{\mu\nu\alpha} = 0, \tag{2.7}$$

where $H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} +$ cyclic permutations. By introducing the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}(R/2)$, the first equation can be rewritten in a more "Einsteinian form" as

$$G_{\mu\nu} + \nabla_\mu \nabla_\nu \phi + \frac{1}{2} g_{\mu\nu} \left[(\nabla\phi)^2 - 2\nabla^2 \phi + \frac{1}{12} H^2_{\alpha\beta\gamma} \right] - \frac{1}{4} H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta} = 0, \tag{2.8}$$

and it can be easily checked that Eqs. (2.7), (2.8) can be derived by the $(d+1)$-dimensional effective action

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|}\, e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{1}{12} H^2_{\mu\nu\alpha} \right]. \tag{2.9}$$

This action (possibly supplemented by a non-perturbative dilaton potential, and/or by a cosmological term in non-critical dimensions [346]) is the starting point for the formulation of a string-theory-compatible cosmology in the small-curvature and weak-coupling regime, $\alpha' R \ll 1$, $g_s^2 \ll 1$ (see Section 8 for higher-order corrections).

For an illustration of scale factor duality it will now be sufficient to restrict our attention to the gravidilaton sector of the action (2.9) (setting e.g. $H_{\mu\nu\alpha} = 0$). We will consider an anisotropic Bianchi-I-type metric background, with homogeneous dilaton $\phi = \phi(t)$, and we will parametrize the action in terms of the "shifted" scalar field $\overline{\phi}$ (see Section 1.4 for the notation). The field equations (2.7), (2.8) then provide a system of $d+2$ equations for the $d+1$ variables $\{a_i, \overline{\phi}\}$:

$$\dot{\overline{\phi}}^2 - \sum_i H_i^2 = 0, \tag{2.10}$$

$$\dot{H}_i - H_i \dot{\overline{\phi}} = 0, \tag{2.11}$$

$$2\ddot{\overline{\phi}} - \dot{\overline{\phi}}^2 - \sum_i H_i^2 = 0. \tag{2.12}$$

In the absence of sources only $d+1$ equations are independent (see for instance [307]; Eq. (2.10), in particular, represents a constraint on the set of initial data, which is preserved by the evolution).

1 tree level g s
2 lwst order alpha'
3 small curvature
4 weak coupling

scale factor duality see Veneziano paper here

The above string cosmology equations are invariant not only under a time-reversal transformation,

$$t \to -t, \qquad H \to -H, \qquad \dot{\overline{\phi}} \to -\dot{\overline{\phi}}, \tag{2.13}$$

1 time reversal

but also under a transformation that inverts any one of the scale factors, preserving the shifted dilaton,

$$a_i \to \tilde{a}_i = a_i^{-1}, \qquad \overline{\phi} \to \overline{\phi}, \tag{2.14}$$

2 scale factor inversion

and which represents a so-called scale-factor duality transformation [599, 580]. Note that the dilaton ϕ is not invariant under this transformation: if we invert, for instance, the first $k \leq d$ scale factors, the transformed dilaton $\tilde{\phi}$ is determined by the condition

$$\overline{\phi} = \phi - \sum_{i=1}^{d} \ln a_i = \tilde{\phi} - \sum_{i=1}^{k} \ln \tilde{a}_i - \sum_{i=k+1}^{d} \ln a_i. \tag{2.15}$$

Thanks to scale-factor duality, given any exact solution of Eqs. (2.10)–(2.12), represented by the set of variables

$$\{a_1, ..., a_d, \ \phi\}, \tag{2.16}$$

the inversion of $k \leq d$ scale factors then defines a new exact solution, represented by the set of variables

$$\left\{ a_1^{-1}, ..., a_k^{-1}, a_{k+1}, ..., a_d, \ \phi - 2\sum_{i=1}^{k} \ln a_i \right\}. \tag{2.17}$$

Consider in particular the isotropic case $a_i = a$, where all the scale factors get inverted, and the duality transformation takes the form:

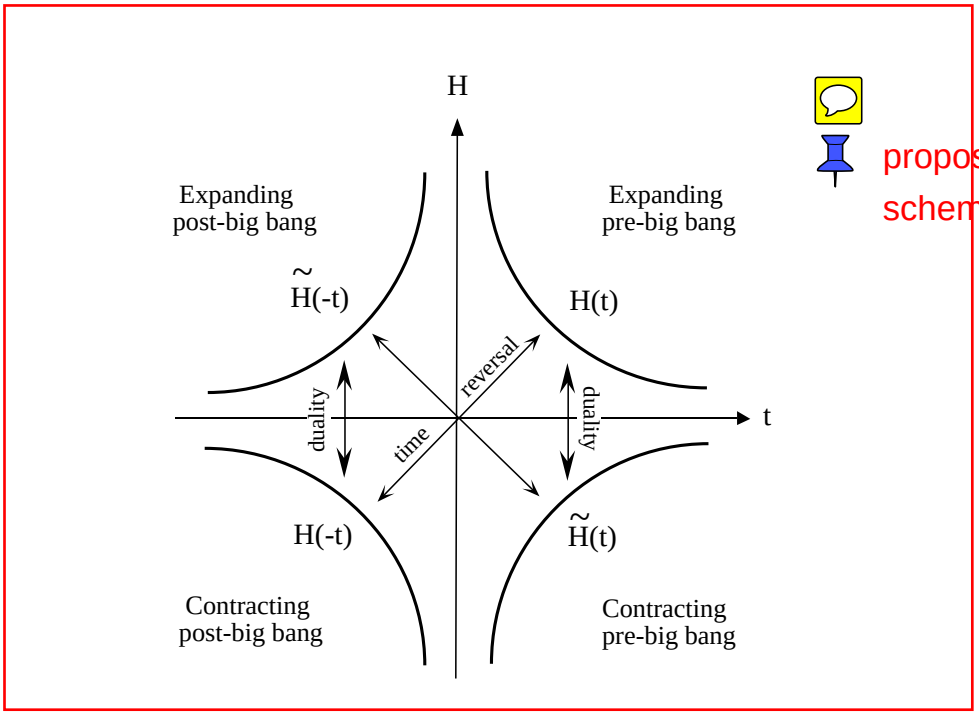
2.13, 2.14:

$$a \to \tilde{a} = a^{-1}, \qquad \phi \to \tilde{\phi} = \phi - 2d \ln a. \tag{2.18}$$

When $a \to a^{-1}$ the Hubble parameter $H = d(\ln a)/dt$ goes into $-H$ so that, to each of the two solutions related by time reversal, $H(t)$ and $H(-t)$, is also associated a dual solution, $\tilde{H}(t)$ and $\tilde{H}(-t)$, respectively (see Fig. 2.2). The space of solutions, in a string cosmology context, is thus richer than in the standard Einstein cosmology, because of the combined invariance under duality and time-reversal transformations. In string cosmology, a solution has in general four branches: $a(t), a(-t), a^{-1}(t), a^{-1}(-t)$. Two branches describe expansion ($H > 0$), the other two branches describe contraction ($H < 0$). Also, as illustrated in Fig. 2.2, for two branches the curvature scale ($\sim H^2$) grows in time, so that they describe a Universe that evolves *towards* a singularity, with a typical "pre-big bang" behaviour; for the other two branches the curvature scale decreases, so that the corresponding Universe emerges *from* a singularity, with a typical "post-big bang" behaviour.

What is important, in our context, is that to any given decelerated, expanding solution, $H(t) > 0$, with decreasing curvature, $\dot{H}(t) < 0$ (typical of the standard cosmological scenario), is always associated an inflationary "dual partner" describing accelerated expansion, $\tilde{H}(-t) > 0$, and growing curvature, $\dot{\tilde{H}}(-t) > 0$. This pairing of solutions (which has

inflation not in "classical' meaning



proposed scheme, GT

Figure 2.2: *The four branches of the low-energy string cosmology backgrounds.*

no analogue in the context of the Einstein cosmology, where there is no dilaton, and the duality symmetry cannot be implemented) naturally suggests a "self-dual" completion of standard cosmology, in which the Universe smoothly evolves from the inflationary pre-big bang branch $\widetilde{H}(-t)$ to the post-big bang branch $H(t)$ (after an appropriate regularization of the curvature singularity appearing in the lowest-order solutions).

As a simple example of the four cosmological branches, we may consider here the particular isotropic solution defined in the positive range of the time coordinate,

$$a = (t/t_0)^{1/\sqrt{d}}, \qquad \overline{\phi} = -\ln(t/t_0), \qquad t > 0, \tag{2.19}$$

which is singular at $t = 0$ and satisfies identically the set of equations (2.10) – (2.12). By applying a duality *and* a time-reversal transformation we obtain the four inequivalent solutions

$$\begin{aligned} a_\pm(t) &= t^{\pm 1/\sqrt{d}}, & \overline{\phi}(t) &= -\ln t, & t &> 0, \\ a_\pm(-t) &= (-t)^{\pm 1/\sqrt{d}}, & \overline{\phi}(-t) &= -\ln(-t), & t &< 0, \end{aligned} \tag{2.20}$$

corresponding to the four branches illustrated in Fig. 2.2. These solutions are separated by a curvature singularity at $t = 0$; they describe decelerated expansion $a_+(t)$, decelerated contraction $a_-(t)$, accelerated contraction $a_+(-t)$, accelerated expansion $a_-(-t)$ (the solution is accelerated or decelerated according to whether $\dot{a}$ and $\ddot{a}$ have the same or opposite signs, respectively). The curvature is growing for $a_\pm(-t)$, decreasing for $a_\pm(t)$. Note that the

expanding metric: a- (-t)

1 isotropy =====>

2 expanding inflationary sol

3 dilaton away from VAC

4 growing S coupling

transformation connecting two different branches represents in this case not really a symmetry, but rather a group acting on the space of solutions, transforming non-equivalent conformal backgrounds into each other, like in the case of the Narain transformations [498, 499].

For further applications, it is also important to consider the dilaton evolution in the various branches of Eq. (2.20). Using the definition (1.23),

$$\phi_\pm(\pm t) = \overline{\phi}(\pm t) + d \ln a_\pm(\pm t) = \left(\pm\sqrt{d} - 1\right) \ln(\pm t). \tag{2.21}$$

It follows that, in a phase of growing curvature ($t < 0, t \to 0_-$), the dilaton is growing only for an expanding metric, $a_-(-t)$. This means that, in the isotropic case, the expanding inflationary solutions describe a cosmological evolution away from the string perturbative vacuum ($H = 0, \phi = -\infty$), i.e. are solutions characterized by a growing string coupling, $\dot{g}_s = (\exp \phi/2)\dot{} > 0$. The string perturbative vacuum thus naturally emerges as the initial state for a state of pre-big bang inflationary evolution. This is to be contrasted with the recently proposed "ekpyrotic" scenario [412], where the "pre-big bang" configuration (i.e. the phase of growing curvature preceding the brane collision that simulates the big bang explosion) is contracting even in the S-frame [413] and indeed corresponds to a phase of decreasing dilaton.

expyrotic= brane collision NOT Vene!!!!

Note that in a more general, anisotropic case, and in the presence of contracting dimensions, a growing-curvature solution is associated to a growing dilaton only for a large enough number of expanding dimensions. To make this point more precise, consider the particular, exact solution of Eqs. (2.10)–(2.12), with d expanding and n contracting dimensions, and scale factors $a(t)$ and $b(t)$, respectively:

$$a = (-t)^{-1/\sqrt{d+n}}, \qquad b = (-t)^{1/\sqrt{d+n}}, \qquad \overline{\phi} = -\ln(-t), \qquad t \to 0_-. \tag{2.22}$$

This gives

$$\phi = \overline{\phi} + d \ln a + n \ln b = \frac{n - d - \sqrt{d+n}}{\sqrt{d+n}} \ln(-t), \tag{2.23}$$

so that the dilaton is growing if

$$d + \sqrt{d+n} > n. \tag{2.24}$$

We note, incidentally, that this result may have interesting implications for a possible "hierarchy" of the present size of extra dimensions, if we assume that our Universe starts evolving from the string perturbative vacuum (i.e. with initial $\dot{\phi} > 0$). Indeed, in a superstring theory context ($d + n = 10$), it follows that the initial number of expanding dimensions is $d > 3$, while only $n < 6$ may be contracting. A subsequent freezing, or late-time contraction (possibly induced by quantum effects [130]), of $d - 3$ dimensions will eventually leave only three expanding dimensions, but with a possible huge asymmetry in the spatial sections of our Universe, even in the sector of the "internal" dimensions (see Refs. [109, 10], and the discussion of Section 8.5, for a possible mechanism selecting $d = 3$ as the maximal, final number of expanding dimensions in a string- or brane-dominated Universe).

brandenberger vafa here

d = 3

The invariance of the gravidilaton equations (2.10)–(2.12) is in general broken by a dilaton potential $V(\phi)$, unless V is just a function of $\overline{\phi}$. The invariance, however, is still valid in the presence of matter sources, provided they transform in a way that is compatible with the string equations of motion in the given background [318] (see the next subsection). In the perfect-fluid approximation, in particular, a scale-factor duality transformation is associated to a "reflection" of the equation of state [599].

Consider in fact the addition to the action (2.9) of a matter action S_m, minimally coupled to the S-frame metric (but uncoupled to the dilaton), and describing an anisotropic fluid with diagonal stress tensor,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}}, \qquad T_\mu{}^\nu = \mathrm{diag}(\rho, -p_i\ \delta_i^j), \qquad p_i/\rho = \gamma_i = \mathrm{const}, \qquad \rho = \rho(t). \tag{2.25}$$

The field equations (2.8) are now completed by a source term

$$G_{\mu\nu} + \nabla_\mu\nabla_\nu\phi + \frac{1}{2}g_{\mu\nu}\left[(\nabla\phi)^2 - 2\nabla^2\phi + \frac{1}{12}H^2_{\alpha\beta\gamma}\right] - \frac{1}{4}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} = \frac{1}{2}e^\phi T_{\mu\nu} \tag{2.26}$$

(in units in which $2\lambda_{\rm s}^{d-1} = 1$), and the cosmological equations (2.10)–(2.12) become

$$\dot{\overline{\phi}}^2 - \sum_i H_i^2 = e^{\overline{\phi}}\overline{\rho}, \tag{2.27}$$

$$\dot{H}_i - H_i\dot{\overline{\phi}} = \frac{1}{2}e^{\overline{\phi}}\overline{p}_i, \tag{2.28}$$

$$2\ddot{\overline{\phi}} - \dot{\overline{\phi}}^2 - \sum_i H_i^2 = 0, \tag{2.29}$$

where we have introduced the "shifted" density and pressure

$$\overline{\rho} = \rho\sqrt{-g} = \rho\prod_i a_i, \qquad\qquad \overline{p} = p\sqrt{-g} = p\prod_i a_i. \tag{2.30}$$

They are a system of $d+2$ independent equations for the $d+2$ variables $\{a_i, \phi, \rho\}$. Their combination gives

$$\dot{\overline{\rho}} + \sum_i H_i\overline{p}_i = 0, \tag{2.31}$$

which represents the usual covariant conservation of the source energy density. The above equations with sources are invariant under time reflection and under the duality transformation [599]

$$a_i \to a_i^{-1}, \qquad \overline{\phi} \to \overline{\phi}, \qquad \overline{\rho} \to \overline{\rho}, \qquad \overline{p}_i \to -\overline{p}_i, \tag{2.32}$$

which preserves $\overline{\rho}$ but changes ρ in a non-trivial way, and "reflects" the barotropic equation of state, $\gamma_i \to -\gamma_i$. Thus, a cosmological solution is still characterized by four distinct branches.

We will present here a simple isotropic example, corresponding to the power-law evolution

$$a \sim t^\alpha, \qquad \overline{\phi} \sim -\beta\ln t, \qquad p = \gamma\rho \tag{2.33}$$

(see the next subsection for more general solutions). We use (2.27), (2.29), (2.31) as independent equations. The integration of Eq. (2.31) immediately gives

$$\overline{\rho} = \rho_0 a^{-d\gamma}; \tag{2.34}$$

Eq. (2.27) is then satisfied, provided

$$d\gamma\alpha + \beta = 2. \tag{2.35}$$

Finally, Eq. (2.29) leads to the constraint

$$2\beta - \beta^2 - d\alpha^2 = 0. \tag{2.36}$$

We then have a (quadratic) system of two equations for the two parameters α, β (note that, if α is a solution for a given γ, then also $-\alpha$ is a solution, associated to $-\gamma$). We have in general two solutions. The flat-space solution, $\beta = 2, \alpha = 0$, corresponds to a non-trivial dilaton evolving in a frozen pseudo-Euclidean background, sustained by the energy density of dust matter ($\gamma = 0$), according to Eq. (2.28). For $\gamma \neq 0$ we obtain instead

$$\alpha = \frac{2\gamma}{1 + d\gamma^2}, \qquad \beta = \frac{2}{1 + d\gamma^2}, \tag{2.37}$$

which fixes the time evolution of a and $\overline{\phi}$:

$$a \sim t^{\frac{2\gamma}{1+d\gamma^2}}, \qquad \overline{\phi} = -\frac{2}{1 + d\gamma^2} \ln t, \tag{2.38}$$

and also of the more conventional variables ρ, ϕ:

$$\rho = \overline{\rho} a^{-d} = \rho_0 a^{-d(1+\gamma)}, \qquad \phi = \overline{\phi} + d \ln a = \frac{2(d\gamma - 1)}{1 + d\gamma^2} \ln t. \tag{2.39}$$

This particular solution reproduces the small-curvature limit of the general solution with perfect fluid sources sufficiently far from the singularity, as we shall see in the next subsection. As in the vacuum solution (2.19), there are four branches, related by time-reversal and by the duality transformation (2.32), and characterized by the scale factors

$$a_\pm(\pm t) \sim (\pm t)^{\pm 2\gamma/(1+d\gamma^2)}. \tag{2.40}$$

For $d = 3$, $\gamma = 1/3$ and $t > 0$ we recover in particular the standard, radiation-dominated solution with constant dilaton:

$$a \sim t^{1/2}, \qquad \rho = 3p \sim a^{-4}, \qquad \phi = \text{const}, \qquad t \to +\infty, \tag{2.41}$$

describing decelerated expansion and decreasing curvature:

$$\dot{a} > 0, \qquad \ddot{a} < 0, \qquad \dot{H} < 0, \qquad \dot{\phi} = 0, \tag{2.42}$$

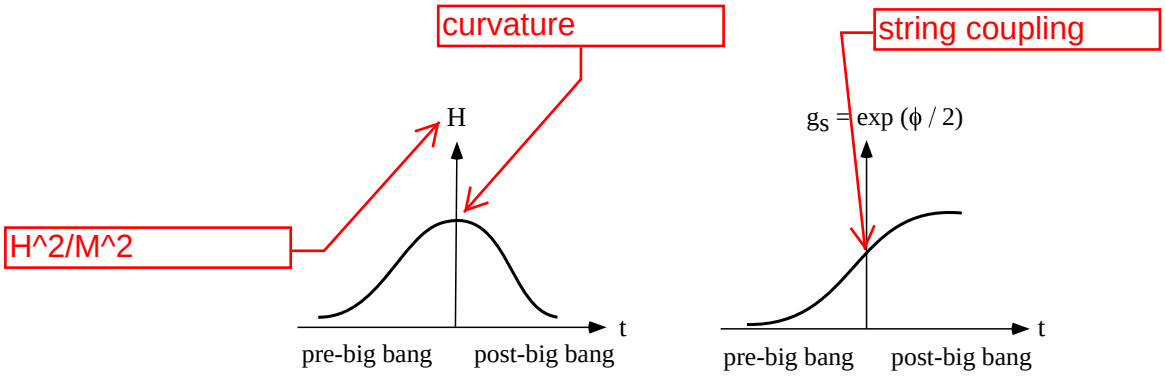


Figure 2.3: *Qualitative time evolution of the curvature scale and of the string coupling for a typical self-dual solution of the string-cosmology equations.*

typical of the post-big bang radiation era. Through a duality and time-reversal transformation we obtain the "dual" complement:

$$a \sim (-t)^{-1/2}, \qquad \phi \sim -3\ln(-t), \qquad \rho = -3p \sim a^{-2}, \qquad t \to -\infty, \tag{2.43}$$

which is still an exact solution of the string cosmology equations, and describes accelerated (i.e. inflationary) expansion, with growing dilaton and growing curvature:

$$\dot{a} > 0, \qquad \ddot{a} > 0, \qquad \dot{H} > 0, \qquad \dot{\phi} > 0 \tag{2.44}$$

(the unconventional equation of state, $\gamma = -1/3$, is typical of a gas of stretched strings, see [314, 315] and the next subsection).

This confirms that, if we start with our present cosmological phase in which the dilaton is constant (in order to guarantee a constant strength of gravitational and gauge interactions), and we postulate for the early Universe a dual complement preceding the big bang explosion, then string theory requires for the pre-big bang phase not only *growing curvature*, but also *growing dilaton*.

In other words, string theory naturally suggests to identify the initial configuration of our Universe with a state asymptotically approaching the flat, cold and empty string perturbative vacuum, $H^2 \to 0$, $\exp(\phi) \to 0$. As a consequence, the initial cosmological evolution occurs in the small curvature ($H^2/M_s^2 \ll 1$) and weak coupling ($g_s \ll 1$) regime, and can be appropriately described by the lowest order effective action (2.9) (see Fig. 2.3). The solutions (2.41) and (2.43) provide a particular, explicit representation of the scenario represented in Fig. 2.3, for the two asymptotic regimes of t large and positive, Eq. (2.41), and t large and negative, Eq. (2.43).

It should be mentioned, to conclude this subsection, that the invariance under the discrete symmetry group Z_2^d, generated by the inversion of d scale factors, can be generalized so as to be extended to spatially flat solutions of more general scalar–tensor theories [142, 434], with the generic Brans–Dicke parameter ω, described by the action

$$S = -\int d^{d+1}x\sqrt{-g}\; e^{-\phi}\; \left[R - \omega(\nabla\phi)^2 - 2\Lambda\right]. \tag{2.45}$$

present state initial conditions

The case $\omega = -1$ corresponds to the string effective action. For $\omega \neq -(d+1)/d$, the equations of an isotropic and spatially flat background (with scale factor $a(t)$) are invariant under the transformation $\alpha \to \tilde{\alpha}$, $\phi \to \tilde{\phi}$, where $\alpha = \ln a$, and [434]

$$\alpha = \frac{d-1+d\omega}{d+1+d\omega}\,\tilde{\alpha} - \frac{2(1+\omega)}{d+1+d\omega}\,\tilde{\phi}, \qquad \phi = -\frac{2d}{d+1+d\omega}\,\tilde{\alpha} + \frac{d-1+d\omega}{d+1+d\omega}\,\tilde{\phi} \tag{2.46}$$

(when $\omega = -1$ one recovers the transformation (2.18)). Similar symmetries are also present in a restricted class of homogeneous, Bianchi-type models [169] (but the presence of spatial curvature tends to break the scale-factor duality symmetry); for the Bianchi-I-type metrics, in addition, the discrete transformation (2.46) can be embedded in a continuous $O(3)$ symmetry group.

It should be stressed, however, that when $\omega \neq -1$ such generalized transformations *do not* necessarily associate, to any decelerated solution of the standard scenario, an inflationary solution with growing dilaton. In this sense, a self-dual cosmological scenario in which inflation emerges naturally from the perturbative vacuum seems just to be a peculiar prediction of string theory, in its low energy limit.

In the next subsection we will extend the discussion of this section to more general models of background and sources.

2.3 $O(d,d)$-covariance of the cosmological equations

Narain O(d,d)

The target-space duality introduced in the previous subsection is not restricted to the gravidilaton sector and to cosmological backgrounds, but is expected to be a general property of the solutions of the string effective action (possibly valid at all orders [560, 367], with the appropriate generalizations [476, 397]).

Already at the lowest order, in fact, the inversion of the scale factor is only a special case of a more general transformation of the global $O(d,d)$ group which leaves invariant the action (2.9) for all background characterized by d Abelian isometries, and which mixes in a non-trivial way the components of the metric and of the antisymmetric tensor $B_{\mu\nu}$ (see [340] for a general review).

Such an invariance property of the action can also be extended to non-Abelian isometries [206], but then there are problems for "non-semisimple" isometry groups [313]. Here we shall restrict ourselves to the Abelian case, and we shall consider (for cosmological applications) a set of background fields $\{\phi, g_{\mu\nu}, B_{\mu\nu}\}$ which is isometric with respect to d spatial translations and for which there exists a synchronous frame where $g_{00} = 1$, $g_{0i} = 0$, $B_{0\mu} = 0$, and all the non-zero components of g_{ij}, B_{ij} (as well as the dilaton itself) are only dependent on time.

In order to illustrate the invariance properties of such a class of backgrounds under global $O(d,d)$ transformations, we shall first rewrite the action (2.9) directly in the synchronous gauge (since, for the moment, we are not interested in the field equations, but only in the

gauge choice!!!

gamma defined

symmetries of the action). We set $g_{ij} = -\gamma_{ij}$ and find, in this gauge,

$$\Gamma_{ij}{}^{0} = \frac{1}{2}\dot{\gamma}_{ij}, \qquad \Gamma_{0i}{}^{j} = \frac{1}{2}g^{jk}\dot{g}_{ik} = \frac{1}{2}\left(g^{-1}\dot{g}\right)_{i}{}^{j} = \left(\gamma^{-1}\dot{\gamma}\right)_{i}{}^{j}$$
$$R_{0}{}^{0} = -\frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)^{2} - \frac{1}{2}\mathrm{Tr}\left(\gamma^{-1}\ddot{\gamma}\right) - \frac{1}{2}\mathrm{Tr}\left(\dot{\gamma}^{-1}\dot{\gamma}\right),$$
$$R_{i}{}^{j} = -\frac{1}{2}\left(\gamma^{-1}\ddot{\gamma}\right)_{i}{}^{j} - \frac{1}{4}\left(\gamma^{-1}\dot{\gamma}\right)_{i}{}^{j}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right) + \frac{1}{2}\left(\gamma^{-1}\dot{\gamma}\gamma^{-1}\dot{\gamma}\right)_{i}{}^{j}, \qquad (2.47)$$

where

$$\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right) = \left(\gamma^{-1}\right)^{ij}\dot{\gamma}_{ji} = g^{ij}\dot{g}_{ji}, \qquad (2.48)$$

and so on [note also that $\dot{\gamma}^{-1}$ means $d\left(\gamma^{-1}\right)/dt$]. Similarly we find, for the antisymmetric tensor,

$$H_{0ij} = \dot{B}_{ij}, \qquad H^{0ij} = g^{ik}g^{jl}\dot{B}_{kl} = \left(\gamma^{-1}\dot{B}\gamma^{-1}\right)^{ij},$$
$$H_{\mu\nu\alpha}H^{\mu\nu\alpha} = 3H_{0ij}H^{0ij} = -3\mathrm{Tr}\left(\gamma^{-1}\dot{B}\right)^{2}. \qquad (2.49)$$

Let us introduce the shifted dilaton, by absorbing the spatial volume into ϕ, as in Section 1.4:

spatial volume

$$\sqrt{|\det g_{ij}|}e^{-\phi} = e^{-\overline{\phi}}, \qquad (2.50)$$

from which

$$\dot{\overline{\phi}} = \dot{\phi} - \frac{1}{2}\frac{d}{dt}\ln\left(\det\gamma\right) = \dot{\phi} - \frac{1}{2}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right). \qquad (2.51)$$

By collecting the various contributions from ϕ, R and $H^{2}_{\mu\nu\alpha}$, the action (2.9) can be rewritten as:

$$S = -\frac{\lambda_{s}}{2}\int dt e^{-\overline{\phi}}\left[\dot{\overline{\phi}}^{2} + \frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)^{2} - \mathrm{Tr}\left(\gamma^{-1}\ddot{\gamma}\right)\right.$$
$$\left. -\frac{1}{2}\mathrm{Tr}\left(\dot{\gamma}^{-1}\dot{\gamma}\right) + \dot{\overline{\phi}}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right) + \frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{B}\right)^{2}\right]. \qquad (2.52)$$

We can now eliminate the second derivatives, and the mixed terms containing $\dot{\overline{\phi}}\dot{\gamma}$, by noting that

$$\frac{d}{dt}\left[e^{-\overline{\phi}}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)\right] = e^{-\overline{\phi}}\left[\mathrm{Tr}\left(\gamma^{-1}\ddot{\gamma}\right) + \mathrm{Tr}\left(\dot{\gamma}^{-1}\dot{\gamma}\right) - \dot{\overline{\phi}}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)\right]. \qquad (2.53)$$

Finally, by using the identity $\left(\gamma^{-1}\right)^{\cdot} = -\gamma^{-1}\dot{\gamma}\gamma^{-1}$, we can rewrite the action in quadratic form, modulo a total derivative, as

$$S = -\frac{\lambda_{s}}{2}\int dt e^{-\overline{\phi}}\left[\dot{\overline{\phi}}^{2} - \frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)^{2} + \frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{B}\right)^{2}\right]. \qquad (2.54)$$

This action can be set into a more compact form by using the $2d\times 2d$ matrix M, defined in terms of the spatial components of the metric and of the antisymmetric tensor,

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix},$$
$$G = g_{ij} \equiv -\gamma_{ij}, \qquad G^{-1} \equiv g^{ij}, \qquad B \equiv B_{ij}, \qquad (2.55)$$

M

and using also the matrix η, representing the invariant metric of the $O(d,d)$ group in the off-diagonal representation:

$$\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{2.56}$$

eta

(I is the unit d-dimensional matrix). By computing $M\eta, \dot{M}\eta$ and $(\dot{M}\eta)^2$ we find, in fact,

$$\mathrm{Tr}\left(\dot{M}\eta\right)^2 = 2\mathrm{Tr}\left[\dot{\gamma}^{-1}\dot{\gamma} + \left(\gamma^{-1}\dot{B}\right)^2\right] = 2\mathrm{Tr}\left[-\left(\gamma^{-1}\dot{\gamma}\right)^2 + \left(\gamma^{-1}\dot{B}\right)^2\right], \tag{2.57}$$

so that the action can be rewritten as [477, 478]

$$S = -\frac{\lambda_s}{2}\int dt e^{-\overline{\phi}}\left[\dot{\overline{\phi}}^2 + \frac{1}{8}\mathrm{Tr}\left(\dot{M}\eta\right)^2\right]. \tag{2.58}$$

We may note, at this point, that M itself is a (symmetric) matrix element of the pseudo-orthogonal $O(d,d)$ group, since

$$M^T\eta M = \eta, \qquad M = M^T, \tag{2.59}$$

for any B and G. Therefore:

$$M\eta = \eta M^{-1}, \qquad \left(\dot{M}\eta\right)^2 = \eta\left(M^{-1}\right)^{\cdot}M\eta, \tag{2.60}$$

and the action can be finally rewritten in the form

$$S = -\frac{\lambda_s}{2}\int dt e^{-\overline{\phi}}\left[\dot{\overline{\phi}}^2 + \frac{1}{8}\mathrm{Tr}\,\dot{M}\left(M^{-1}\right)^{\cdot}\right], \tag{2.61}$$

which is explicitly invariant under global $O(d,d)$ transformations preserving the shifted dilaton $\overline{\phi}$:

$$\overline{\phi} \to \overline{\phi}, \qquad M \to \widetilde{M} = \Lambda^T M\Lambda, \qquad \Lambda^T\eta\Lambda = \eta. \tag{2.62}$$

When $B = 0$, the matrix M is block-diagonal, and the special $O(d,d)$ transformation represented by $\Lambda = \eta$ corresponds to an inversion of the metric tensor:

$$M = \mathrm{diag}(G^{-1}, G), \qquad \widetilde{M} = \Lambda^T M\Lambda = \eta M\eta = \mathrm{diag}(G, G^{-1}), \tag{2.63}$$

so that $\widetilde{G} = G^{-1}$. For a diagonal metric, in particular, $G = a^2 I$, and the invariance under the scale factor duality transformation (2.18) is recovered as a particular case of the global $O(d,d)$ symmetry (as already anticipated).

This $O(d,d)$ invariance holds even in the presence of sources representing bulk string matter [318], namely sources evolving consistently with the solutions of the string equations of motion in the background we are considering. A distribution of non-interacting strings, minimally coupled to the metric and the antisymmetric tensor of an $O(d,d)$-covariant background, is in fact characterized by a stress tensor (source of $g_{\mu\nu}$) and by an antisymmetric current (source of $B_{\mu\nu}$), which are automatically $O(d,d)$-covariant).

In order to illustrate this important point we add to Eq. (2.9) an action S_m describing matter sources coupled to g and to B, and we define

$$\frac{1}{2}\sqrt{-g}T_{\mu\nu} = \frac{\delta S_m}{\delta g^{\mu\nu}}, \qquad \frac{1}{2}\sqrt{-g}J^{\mu\nu} = \frac{\delta S_m}{\delta B_{\mu\nu}}. \tag{2.64}$$

The variation with respect to the dilaton, to $g_{\mu\nu}$ and $B_{\mu\nu}$ gives then, respectively, Eqs. (2.7), Eq. (2.26) and the additional equation

$$\nabla_\mu \left(e^{-\phi} H^{\mu\alpha\beta}\right) = J^{\alpha\beta}. \tag{2.65}$$

For a background with d spatial isometries, and in the synchronous gauge, such field equations can be written in matrix form using M, $\overline{\phi}$, and a new set of "shifted" variables defined as follows:

$$\overline{\rho} = \sqrt{-g}T_0{}^0, \qquad \overline{\theta} = \sqrt{-g}T^{ij}, \qquad \overline{J} = \sqrt{-g}J^{ij}, \tag{2.66}$$

where $\overline{\theta}$ and $\overline{J}$ are $d \times d$ matrices. In particular, the dilaton equation (2.7) takes the form

$$\dot{\overline{\phi}}^2 - 2\ddot{\overline{\phi}} - \frac{1}{8}\,\mathrm{Tr}\,(\dot{M}\eta)^2 = 0, \tag{2.67}$$

the $(0,0)$ component of Eq. (2.26) gives

$$\dot{\overline{\phi}}^2 + \frac{1}{8}\,\mathrm{Tr}\,(\dot{M}\eta)^2 = \overline{\rho}e^{\overline{\phi}}, \tag{2.68}$$

while the spatial part of Eq. (2.26), combined with Eq. (2.65), can be written in the form

$$\frac{d}{dt}(e^{-\overline{\phi}} M\eta\dot{M}) = \overline{T}, \tag{2.69}$$

where $\overline{T}$ is a $2d \times 2d$ matrix composed with $\overline{\theta}$ and $\overline{J}$:

$$\overline{T} = \begin{pmatrix} -\overline{J}, & -\overline{\theta}G + \overline{J}B \\ G\overline{\theta} - B\overline{J}, & G\overline{J}G + B\overline{J}B - G\overline{\theta}B - B\overline{\theta}G \end{pmatrix} \tag{2.70}$$

(see [318] for an explicit computation). By differentiating Eq. (2.68), using Eqs. (2.67), (2.69), and the identity

$$(M\eta\dot{M}\eta)^2 = -(\dot{M}\eta)^2, \tag{2.71}$$

we obtain the generalized energy conservation equation, written in matrix form as

$$\dot{\overline{\rho}} + \frac{1}{4}\,\mathrm{Tr}\,(\overline{T}\eta M\eta\dot{M}\eta) = 0. \tag{2.72}$$

The $O(d,d)$ covariance of the string cosmology equations is thus preserved even in the presence of matter sources, provided $\overline{T}$ transforms in the same way as M. In that case, the whole set of equations (2.67)–(2.69) is left invariant by the generalized transformation

$$\overline{\phi} \to \overline{\phi}, \qquad \overline{\rho} \to \overline{\rho}, \qquad M \to \Lambda^T M\Lambda, \qquad \overline{T} \to \Lambda^T\overline{T}\Lambda, \tag{2.73}$$

where $\Lambda^T \eta \Lambda = \eta$. This is indeed what happens if the sources are represented by a gas of non-self-interacting strings.

Suppose in fact that the matter action is given by the sum over all components of the string distribution, $S_m = \sum_i S^i_{\rm strings}$, where we use the action (2.5), and we choose the conformally flat gauge for the world-sheet metric (i.e. $\gamma_{ij} = 0$, $R^{(2)} = 0$):

$$S_{\rm string} = -\frac{1}{4\pi\alpha'} \int d^{d+1}x\ \delta^{d+1}(x - X(\sigma,\tau))\ d\sigma d\tau \times \left[\left(\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu \right) g_{\mu\nu}(x) + \left(\dot{X}^\mu X'^\nu - X'^\mu \dot{X}^\nu \right) B_{\mu\nu}(x) \right]. \quad (2.74)$$

Here σ and τ are the world-sheet coordinates, $\dot{X} = dX/d\tau$ and $X' = dX/d\sigma$. The variation with respect to g and B gives the tensors (2.66), which depend on the world-sheet integral of a form bilinear in $X^\mu(\sigma,\tau)$. The variation with respect to X^μ gives the string equations of motion, the variation with respect to γ_{ij} (before imposing the gauge) gives the constraints. If we have a solution X^μ of the equations of motion, in a given background $M(t)$, and we perform an $O(d,d)$ transormation $M \to \widetilde{M} = \Lambda^T M \Lambda$, the new solution $\widetilde{X}^\mu$ is found to correspond to transformed matrices $\widetilde{\overline{\theta}}$ and $\widetilde{\overline{J}}$, which combine to give $\widetilde{\overline{T}} = \Lambda^T \overline{T} \Lambda$ [318]. The $O(d,d)$ covariance is thus preserved, provided the matter sources transform according to the string equations of motion, which can themselves be written in a fully covariant form.

Let us now exploit this $O(d,d)$ covariance to find more general solutions of the string cosmology equations, and more general examples of duality-related backgrounds corresponding to possible models for the pre-big bang scenario. Let us introduce a convenient (dimensionless) time-coordinate x, such that

$$dx = L\overline{\rho}\ dt \quad (2.75)$$

(L is a constant length). Assuming that the equation of state can be written in terms of a given $2d \times 2d$ matrix $\Gamma(x)$, such that

$$\overline{\rho}\ d\Gamma = \overline{T} dx, \quad (2.76)$$

we can integrate a first time Eqs. (2.67)–(2.69), also with the help of the identity (2.71). The result is [321]

$$\overline{\phi}\,' = -\frac{2}{D(x)}(x + x_0), \quad (2.77)$$

$$M \eta M' = \frac{4\Gamma(x)}{D(x)}, \quad (2.78)$$

where

$$D(x) \equiv (x + x_0)^2 - \frac{1}{2} {\rm Tr}\,(\Gamma\eta)^2 = 4L^2 \overline{\rho} e^{-\overline{\phi}} \quad (2.79)$$

(a prime denotes differentiation with respect to x, and x_0 is an integration constant).

By exploiting the fact that M is a symmetric $O(d,d)$ matrix, $M\eta M = \eta$, and that $M\eta\Gamma = -\Gamma\eta M$, because of the definition of $\overline{T}$ (see [318]), the above equations can be formally integrated to give

$$\overline{\phi}(x) = \phi_0 - 2\int \frac{dx}{D}(x + x_0), \tag{2.80}$$

$$M(x) = P_x \exp\left[-4\int \frac{dx}{D}\Gamma\eta\right] M_0, \tag{2.81}$$

where ϕ_0 is a constant, M_0 a constant symmetric $O(d,d)$ matrix, and P_x denotes the "x-ordering" of the exponential. For any given "equation of state" $\overline{T} = \overline{T}(\overline{\rho})$, providing an integrable relation for $\Gamma(x)$, according to Eq. (2.76), the general exact solution of the low-energy string cosmology equations (for space-independent fields and vanishing dilaton potential) is then represented by Eqs. (2.79)–(2.81).

Such solutions contain in general singularities for the curvature and the string coupling, in correspondence with the zero of $D(x)$. Near the singularity (i.e., for $D(x) \to 0$) the contribution of the matter sources becomes negligible with respect to the curvature terms in the field equations (as,in general relativity, for the Kasner anisotropic solutions [438]), and one recovers the general vacuum solutions presented in [477]. This can be shown in general for any background $M(x)$, and any kind of matter distribution $\Gamma(x)$ [321]; for further applications, however, it will be sufficient to report here the case of anisotropic but torsionless ($B = 0$) backgrounds, with diagonal metric $g_{ij} = -a_i^2\delta_{ij}$, and fluid sources with a barotropic equation of state, as in Eq. (2.25).

In this case we obtain, from the previous definitions,

$$M\eta M' = 2\begin{pmatrix} 0 & \frac{a_i'}{a_i}\delta_{ij} \\ -\frac{a_i'}{a_i}\delta_{ij} & 0 \end{pmatrix}, \quad \overline{T} = \begin{pmatrix} 0 & \overline{p}_i\delta_{ij} \\ -\overline{p}_i\delta_{ij} & 0 \end{pmatrix},$$
$$\Gamma = \begin{pmatrix} 0 & \Gamma_i\delta_{ij} \\ -\Gamma_i\delta_{ij} & 0 \end{pmatrix}, \qquad \Gamma_i = \gamma_i x + x_i,$$
$$D = (x + x_0)^2 - \sum_i (\gamma_i x + x_i)^2 = \alpha(x - x_+)(x - x_-), \tag{2.82}$$

where

$$\overline{p}_i = p_i\sqrt{|g|}, \qquad \alpha = 1 - \sum_i \gamma_i^2,$$
$$x_\pm = \frac{1}{\alpha}\left\{\sum_i \gamma_i x_i - x_0 \pm \left[\left(\sum_i \gamma_i x_i - x_0\right)^2 + \alpha\left(\sum_i x_i^2 - x_0^2\right)\right]^{1/2}\right\}; \tag{2.83}$$

$x_\pm$ are the zeros of D, and x_i, x_0 are integration constants. The general solution (2.79)–(2.81) reduces to [320]

$$a_i = a_{0i}|(x - x_+)(x - x_-)|^{\gamma_i/\alpha}\left|\frac{x - x_+}{x - x_-}\right|^{\alpha_i}, \tag{2.84}$$

$$e^{\overline{\phi}} = e^{\phi_0}|(x-x_+)(x-x_-)|^{-1/\alpha}|\frac{x-x_+}{x-x_-}|^{-\sigma}, \tag{2.85}$$

$$\overline{\rho} = \frac{\alpha}{4L^2}e^{\phi_0}|(x-x_+)(x-x_-)|^{(\alpha-1)/\alpha}|\frac{x-x_+}{x-x_-}|^{-\sigma}, \tag{2.86}$$

where

$$\sigma = \sum_i \alpha_i\gamma_i, \qquad \alpha_i = \frac{\alpha x_i + \gamma_i(\sum_i \gamma_i x_i - x_0)}{\alpha\left[(\sum_i \gamma_i x_i - x_0)^2 + \alpha(\sum_i x_i^2 - x_0^2)\right]^{1/2}}, \tag{2.87}$$

and a_{i0}, ϕ_0 are additional integration constants.

This solution has two curvature singularities at $x_\pm$. Near the singularity the sources are negligible, and one recovers the anisotropic vacuum solution of string cosmology in critical dimensions [599, 489]. Indeed, for $x \to x_\pm$, one finds from Eqs. (2.75) and (2.86) that $|x| \sim |t|^{\alpha/(1\pm\alpha\sum\alpha_i\gamma_i)}$, and the solution reduces to

$$a_i(t) \sim |t-t_\pm|^{\beta_i^\pm}, \qquad \overline{\phi} \sim -\ln|t-t_\pm|, \tag{2.88}$$

where

$$\beta_i^\pm = \frac{x_i \pm \gamma_i x_\pm}{x_0 + x_\pm}, \qquad \sum_i (\beta_i^\pm)^2 = 1. \tag{2.89}$$

In the large-$|x|$ (small-curvature) limit, on the contrary, the relation between x and cosmic time is $|x| \simeq |t|^{\alpha/(2-\alpha)}$, and the solution (2.84)–(2.86) behaves like (for $|x| \to \pm\infty$)

$$\begin{aligned} a_i(t) &\sim |t|^{2\gamma_i/(1+\sum\gamma_i^2)}, & \overline{\phi} &\sim -\frac{2}{1+\sum\gamma_i^2}\ln|t|, \\ \phi &\sim 2\frac{\sum\gamma_i - 1}{1+\sum\gamma_i^2}\ln|t|, & \overline{\rho} &\sim |t|^{-2\sum\gamma_i^2/(1+\sum\gamma_i^2)}. \end{aligned} \tag{2.90}$$

We may note, in general, that the relative importance of the matter sources with respect to the curvature is measured by the ratio (see Eq. (2.68),

$$\Omega(x) = -\frac{8\overline{\rho}e^{\overline{\phi}}}{(d-1)\,\mathrm{Tr}\,(\dot{M}\eta)^2} \equiv \frac{2}{d-1}\frac{D}{\mathrm{Tr}\,(\Gamma\eta)^2} = \frac{\rho e^{\phi}}{(d-1)\sum_i H_i^2} \tag{2.91}$$

(normalized in such a way that one recovers the usual critical energy–density ratio, $\Omega = \rho/\rho_c$, when ϕ is constant and the metric is isotropic). It follows that $\Omega \to 0$ when $x \to x_\pm$, while in the opposite limit $|x| \to \infty$, dominated by the matter sources, Ω goes to a constant,

$$\Omega \to \Omega_\infty = \frac{1-\sum_i\gamma_i^2}{(d-1)\sum_i\gamma_i^2}, \tag{2.92}$$

which is obviously $\Omega_\infty = 1$ for the isotropic, radiation-dominated background with $\gamma_i = 1/d$ and constant dilaton.

Now it can easily be checked that, in the isotropic case $a_i = a$, $\gamma_i = \gamma$, the vacuum solution (2.20) and the matter-dominated solutions (2.38), (2.39) presented in the previous

subsection can be directly recovered from Eqs. (2.88) and (2.90), respectively. As before, we have invariance under time reflections and under the duality transformation $a_i \to a_i^{-1}$, $\gamma_i \to -\gamma_i$. In addition, we can now perform more complicated $O(d,d)$ transformations to find anisotropic and non-diagonal pre-big bang backgrounds [308] (the possibility of singularity-free solutions, in this context, will be discussed in Section 8). It should be stressed, however, that the "$O(d,d)$ complement" of a standard cosmological solution is in general associated to the introduction of an effective viscosity in the matter stress tensor, as pointed out in [318].

C time & cf time

To conclude this subsection let us finally report the cosmic-time and conformal-time parametrization of the isotropic solutions, in the asymptotic limit of large and small curvature, both in the S-frame and in the E-frame.

The small-curvature limit, dominated by a perfect fluid with $p/\rho = \gamma$, has been given in cosmic time in Eqs. (2.38), (2.39). In the conformal time gauge, defined by $dt = a d\eta$, we obtain 1

$$a(\eta) \sim |\eta|^{2\gamma/(1-2\gamma+d\gamma^2)}, \qquad \phi \sim \frac{d\gamma - 1}{\gamma} \ln a, \qquad \rho \sim a^{-d(\gamma+1)}. \tag{2.93}$$

Instead, the vacuum, dilaton-dominated limit (2.20) becomes, in conformal time,

$$a_\pm(\eta) \sim |\eta|^{\pm 1/(\sqrt{d}\mp 1)}, \qquad \phi_\pm \sim \sqrt{d}(\sqrt{d} \mp 1) \ln a_\pm. \tag{2.94}$$

In the E-frame, the scale-factor and the curvature parameters are obtained through the transformation (1.8), (1.11) (the conformal time is obviously the same in both frames). The small-curvature limit (2.38), (2.39) becomes, in cosmic time, 2

$$\widetilde{a}(\widetilde{t}) \sim |\widetilde{t}|^\beta, \qquad \widetilde{\phi} \sim \sqrt{\frac{2}{d-1}} \frac{(d-1)(1-d\gamma)}{\gamma - 1} \ln \widetilde{a}, \qquad \widetilde{\rho} \sim \widetilde{a}^{-2/\beta}, \tag{2.95}$$

where

$$\widetilde{\rho} = \rho \frac{\sqrt{|g|}}{\sqrt{|\widetilde{g}|}} = \rho e^{\phi(d+1)/(d-1)}, \qquad \beta = \frac{2(1-\gamma)}{(d-1)(1+d\gamma^2) - 2(d\gamma - 1)}. \tag{2.96}$$

In conformal time,

$$a_E(\eta) \sim |\eta|^{-2(\gamma-1)/(d-1)(1-2\gamma+d\gamma^2)}. \tag{2.97}$$

The vacuum, dilaton-dominated limit (2.20) becomes, in cosmic time,

$$\widetilde{a}_\pm(\widetilde{t}) \sim |\widetilde{t}|^{1/d}, \qquad \widetilde{\phi}_\pm \sim \pm\sqrt{2d(d-1)} \ln \widetilde{a}_\pm, \tag{2.98}$$

and

$$\widetilde{a}_\pm(\eta) \sim |\eta|^{1/(d-1)} \tag{2.99}$$

in conformal time.

The above solutions will be used to discuss the kinematics of the phase of pre-big bang inflation (in the next subsection), and to compute the corresponding amplification of the

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quantum fluctuations (see Section 4). For the discussion of more general homogeneous solutions (see for instance [60, 57, 59]), we refer the reader to the existing literature, and in particular to the detailed phase-space analysis of the string cosmology equations performed including non-trivial two-form potentials ($B_{\mu\nu} \neq 0$), moduli fields, spatial curvature, anisotropy, and a cosmological constant in both the matter and the gravidilaton parts of the action [84, 85, 86].

2.4 Kinematics of pre-big bang inflation

The symmetry properties of the string effective action suggest a "self-dual" cosmological scenario, in which the Universe evolves from an initial configuration approaching (asymptotically) the string perturbative vacuum ($H \to 0$, $\phi \to -\infty$). Such a state, on the other hand, is particularly appropriate to provide the initial conditions, because it is unstable with respect to the decay into an inflationary phase eventually driven by the dilaton kinetic energy. A small inhomogeneity of the perturbative vacuum is indeed sufficient to trigger inflation, as we shall discuss in Section 3. Or, keeping homogeneity, a small perturbation represented by a constant non-zero energy density is also sufficient.

In the first models of pre-big bang evolution [319, 321], such a perturbation was generated by a (sufficiently diluted) gas of fundamental strings, as its presence tends to enhance the instability of the full system (background fields plus mattere sources). Suppose in fact that we look for self-consistent, simultaneous solutions of the gravidilaton background equations (2.27)–(2.29) and of the string equations of motion (in the same background), obtained from Eq. (2.74) with $B_{\mu\nu} = 0$:

$$\ddot{X}^\mu - X''^\mu + \Gamma^\mu_{\alpha\beta}\left(\dot{X}^\alpha + X'^\alpha\right)\left(\dot{X}^\beta - X'^\beta\right) = 0,$$
$$g_{\mu\nu}\left(\dot{X}^\mu\dot{X}^\nu + X'^\mu X'^\nu\right) = 0, \qquad g_{\mu\nu}\dot{X}^\mu X'^\nu = 0. \tag{2.100}$$

Given a consistent embedding $X^\mu(\sigma,\tau)$, the sum over all the stress tensors of each individual string,

$$\sqrt{-g}T^{\mu\nu}(x) = \frac{1}{2\pi\alpha'}\int d\sigma d\tau\left(\dot{X}^\mu\dot{X}^\nu - X'^\mu X'^\nu\right)\delta^{d+1}(x - X(\sigma,\tau)) \tag{2.101}$$

(after an appropriate averaging procedure) provides the effective sources to be inserted into the background equations.

The above system of string and background equations can be solved in a perturbative way [302, 303], by imposing the string perturbative vacuum as asymptotic initial conditions at $t \to -\infty$. To zeroth order, the background manifold is flat, and the string solutions are characterized by $|\dot{X}^0| \gg |X'^0|$, $|\dot{X}^i| = |X'^i|$, corresponding to a pressureless effective stress tensor, $T^0_0 = \text{const}$, $T^i_i = 0$ (dust matter, in the perfect-fluid approximation).

To first order, the background solution thus corresponds to the case $\gamma_i = 0$ and to the

constant string energy condition

negative branch ($x < x_-$) of Eqs. (2.84)–(2.86), namely

$$a_i(t) = a_{i0}\left|\frac{t-2t_c}{t}\right|^{-\beta_i}, \qquad e^{\overline{\phi}} = \frac{16L^2e^{-\phi_0}}{|t(t-2t_c)|}, \qquad \overline{\rho} = \frac{e^{\phi_0}}{4L^2} = \text{const},$$
$$-\beta_i = \frac{t_i}{t_c}, \qquad t_c = \left(\sum_i t_i^2\right)^{1/2}, \qquad t \leq 0 \tag{2.102}$$

(t_i are integration constants). Note that $t \sim x$ since $\overline{\rho} = \text{const}$, and that we have performed a time translation to shift the singularity from $x = x_-$ to the origin. Note also that, at $t \to -\infty$, the metric is flat and only the dilaton is rolling, $\phi \sim -2\ln(-t)$, sustained by a constant (pressureless) string energy density $\rho = \text{const}$, $p = 0$ (consistently with the chosen initial conditions).

1 The string equation of state keeps unchanged until $a \simeq \text{const}$, namely for $|t| \gg t_c$. When t approaches the critical scale $-t_c$, however, the scale factors start rolling and evolve
2 towards a final (expanding or contracting) accelerated configuration

$$a_i(t) \sim (-t)^{\beta_i}, \qquad |\beta_i| < 1, \qquad \sum_i \beta_i^2 = 1, \tag{2.103}$$

3 reached asymptotically for $|t| \ll t_c$. When $t \sim -t_c$, therefore, the horizon $|H|^{-1}$ starts shrinking (see Section 1.3) and we know, in that regime, that extended objects become "unstable" [551, 268, 270]. In the case of string matter, in particular, we know that the asymptotic solutions of the string equations of motion, outside the horizon [314, 315], are characterized by $|\dot{X}^0| \gg |X'^0|$, $|\dot{X}^i| \ll |X'^i|$ for accelerated expansion ($\beta_i < 0$), and by $|\dot{X}^0| \gg |X'^0|$, $|\dot{X}^i| \gg |X'^i|$ for accelerated contraction ($0 < \beta_i < 1$). As a consequence, the string stress tensor (2.101) satisfies, respectively, the conditions

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$$T_0^0 \simeq T_i^i, \qquad \beta_i < 0,$$
$$T_0^0 \simeq -T_i^i, \qquad 0 < \beta_i < 1, \tag{2.104}$$

corresponding to the radiation equation of state ($\gamma = 1/d$) for accelerated contraction, and to its dual ($\gamma = -1/d$) for accelerated expansion. In both cases, the string pressure triggers a regime of positive feedback, which enhances the instabilitys and pushes the system (background fields plus sources) even further away from the string perturbative vacuum [315, 319, 302, 303].

4 When $|t| \ll t_c$, however, the contribution of the matter sources becomes negligible in the field equations: a first differentiation of Eq. (2.102) gives in fact

$$H_i = \frac{2t_i}{t(t-2t_c)}, \qquad \dot{\overline{\phi}} = -\frac{2(t-t_c)}{t(t-2t_c)}, \qquad \overline{\rho}e^{\overline{\phi}} = \frac{4}{t(t-2t_c)} = \rho e^{\phi} \tag{2.105}$$

and then $\rho e^{\phi} \ll H^2 \sim \dot{\overline{\phi}}^2$ for $t \to 0_-$. So, even if the string pressure tends to deviate from its initial vanishing value, its dynamical effects eventually become negligible when

the background enters the dilaton-dominated regime. The solution (2.102) thus represents a reliable description of the coupled system background plus string sources, both in the $|t| \gg t_c$ and $|t| \ll t_c$ limits, namely in the initial and final regimes, sufficiently far from the critical time scale $t = t_c$ [303].

What is important, for the purpose of this section, is that the phase of pre-big bang inflation, at low energy, is in general characterized by two possible (connected) branches: an initial (model-dependent) phase in which matter (like string, or other sources) are possibly important, and a second "vacuum" phase dominated by the dilaton kinetic energy. The validity of the low energy solutions stops of course at the string scale, $|t| \sim \lambda_s$. At that point the higher order (loop and α') quantum corrections come into play, but inflation can nonetheless continue during a phase of constant curvature and linearly growing dilaton [307], appearing as late-time attractor in the high-curvature regime (see Section 8).

So, in spite of the fact that the inflationary epoch of the pre-big bang scenario is often referred to as the vacuum, dilaton-driven phase appearing in the low energy solutions, it is important to stress that the *total* pre-big bang inflationary period possibly contains also a phase of "late inflation" at high curvature (as also stressed in [466]), and a phase of "early inflation" driven by matter sources (as also stressed in [324]). To discuss the main kinematic aspects of pre-big bang inflation, however, it will be sufficient to concentrate our attention on the dilaton-driven, isotropic vacuum solutions given by Eq. (2.20) (or by Eq. (2.94), in the conformal-time parametrization).

For a quantitative estimate of the duration of the inflationary period we observe that the ratio r, determined by the flatness problem as in Eq. (1.2), also controls the solution of the horizon problem through the ratio

$$\frac{\text{proper size horizon scale}}{\text{proper size homogeneous region}} \sim \frac{H^{-1}(t)}{a(t)} \sim r(t), \tag{2.106}$$

and that, for a power-law metric $a \sim t^\beta$, such a ratio scales linearly with respect to the conformal time coordinate,

$$r \sim t^{1-\beta} \sim \int a^{-1} dt \sim \eta. \tag{2.107}$$

Also, we shall assume that the standard radiation era can be extended back in time down to the Planck scale, and that it is immediately preceded by a phase of accelerated expansion. At the beginning of the radiation era the horizon size is then controlled by the Planck length λ_P, while the proper size of the homogeneous and causally connected region inside our present Hubble radius, rescaled down according to the decelerated evolution of the standard scenario, is unnaturally larger than the horizon by the factor $\sim 10^{30}\lambda_P$ (as already remarked in Subsection 1.1). During the inflationary epoch the ratio r must thus increase by at least a factor 10^{30}, so as to push the homogeneous region outside the horizon, by he amount required by the subsequent decelerated evolution. This determines the condition

$$|\eta_f|/|\eta_i| \lesssim 10^{-30}, \tag{2.108}$$

freezing of dilaton??

where η_i and η_f mark, respectively, the beginning and the end of the inflationary period.

Let us now suppose, for simplicity (and for a more direct comparison between the pre-big bang and the standard inflationary scenarios), that at the end of inflation the dilaton immediately freezes out, and that the string and Planck lengths have already comparable sizes at $\eta = \eta_f$ (which means $\exp(\phi_f) \sim 1$, according to Eq. (2.2)).

If we go back in time during a dilaton-dominated phase of isotropic superinflation,

$$a \sim (-t)^{-1/\sqrt{d}} \sim (-\eta)^{-1/(\sqrt{d}+1)}, \tag{2.109}$$

starting from the string/Planck scale ($t_f \sim \lambda_{\rm s}$), it follows from (2.108) that the scale factor is *at most* reduced by the factor $a_i/a_f \sim 10^{-30/(1+\sqrt{d})}$, corresponding to a homogeneous region, which, at the beginning of inflation, is still very large in string units, i.e. at least

$$10^{30\sqrt{d}/(1+\sqrt{d})}\lambda_{\rm s} \tag{2.110}$$

(i.e. $\sim 10^{19}\lambda_{\rm s}$ for $d = 3$). This is to be contrasted with the case of standard (although not fully realistic) de Sitter inflation, $a \sim (-\eta)^{-1}$, where, going back in time, the scale factor during inflation is reduced by the factor 10^{-30}, so that the size of the initial homogeneous region is just $\lambda_{\rm P}$, like the horizon, which stays frozen during inflation.

The contrast is even more striking if the initial size (2.110) is expressed in Planck units, since, at the beginning of inflation, the string coupling $g_{\rm s} = \exp(\phi/2)$ –and thus the Planck length $\lambda_{\rm P}$– is reduced with respect to its final value ($\lambda_{\rm P} \sim \lambda_{\rm s}$) by the factor

$$\frac{g_{\rm s}(\eta_i)}{g_{\rm s}(\eta_f)} = \frac{\lambda_{\rm P}(\eta_i)}{\lambda_{\rm P}(\eta_f)} = \left(\frac{\eta_f}{\eta_i}\right)^{\sqrt{d}/2} \lesssim 10^{-15\sqrt{d}}, \tag{2.111}$$

so that the lower bound (2.110) for the size of the initial region can be written as

$$10^{\sqrt{d}(45+15\sqrt{d})/(1+\sqrt{d})}\lambda_{\rm P} \tag{2.112}$$

(i.e. $\sim 10^{45}\lambda_{\rm P}$ for $d = 3$). This, by the way, is exactly the initial size evalued in the E-frame where $\lambda_{\rm P}$ is a constant, and the pre-big bang phase is represented as a contraction, according to Eq. (2.99).

One might think that, in such a case, the situation is not much better than in the case of non-inflationary cosmology, where one finds that the initial size of the homogeneous part of our Universe is greater than $10^{30}\lambda_{\rm P}$ [394]. It should be noted, however, that without inflation the initial homogeneous region has to be much bigger than the horizon. In the pre-big bang scenario, on the contrary, the initial homogeneous region is large in Planck or string units, but not larger than the horizon itself [286]. Indeed, during superinflation, the horizon scale shrinks linearly in cosmic time. As we go backwards in time, in the particular example that we are considering, the horizon increases by the factor $H_i^{-1}/H_f^{-1} = (t_i/t_f) = (\eta_i/\eta_f)^{\sqrt{d}/(1+\sqrt{d})}$, so that, at the beginning of inflation, $H_i^{-1} \sim 10^{30\sqrt{d}/(1+\sqrt{d})}\lambda_{\rm P}$, i.e. the

THE question:

initial horizon size is just the same as that of the homogeneous region (2.110) (as also illustrated in Fig. 1.3).

In this sense, the initial state of pre-big bang inflation is similar to the one of the standard inflationary scenario. In one case, however, the initial horizon is large in Planck units; in the other case it is of order 1, as an obvious consequence of the different curvature scales at the beginning of inflation. The question about the naturalness of the initial conditions (see the next section) thus seems to concern the initial unit of length used to measure the size of the initial homogeneous domain: Should we use the Planck (or string) length, or the classical radius of the causal horizon?

The Planck length certainly provides natural units for the size of the initial homogeneous patches when initial conditions are imposed on a cosmological state approaching the high-curvature, quantum-gravity regime. In the pre-big bang scenario, however, initial conditions are to be imposed when the Universe is deeply inside the low-curvature, weak-coupling, classical regime. In that regime, the Universe does not know about the Planck length, and the causal horizon H^{-1} seems to provide a natural candidate for the initial homogeneity scale. It should be noted, to this purpose, that the formation of a large homogeneous domain is not prevented by the amplification of quantum fluctuations [343]. However, classical inhomogeneities, not normalized to a vacuum fluctuation spectrum, may require a constraint on the duration of inflation stronger than Eq. (2.108), as recently pointed out in [128].

causal horizon

Note also that a large (homogeneous) Hubble horizon, if we assume the saturation of the holographic bound [375] applied in a cosmological context [604], implies a large initial entropy $S \sim$ (horizon area in Planck units) and corresponds, in a quantum context, to a small probability that such configuration be obtained through a process of quantum tunnelling (proportional to $\exp[-S]$). In the context of the pre-big bang scenario, however, quantum cosmology effects, such as tunnelling or reflection of the Wheeler–De Witt wave function, are expected to be important towards *the end* of inflation, and *not the beginning*, as they may be effective *to exit*, eventually, from the inflationary regime (see Section 9), *not to enter* it, and to explain the origin of the initial state. A large entropy of the initial state, in the weakly coupled, highly classical regime, can only correspond to a large probability of such a configuration (proportional to $\exp[+S]$), as expected for classical and macroscopic configurations.

exit problem

The naturalness of the initial conditions will be discussed in Section 3. Here we note, to conclude this subsection, that an initial state characterized by a set of large (or small) dimensionless parameters, determined by a large initial horizon scale H^{-1}, is an unavoidable aspect of all models in which inflation starts at scales smaller than Planckian. In the standard scenario, where observational tracks of what happened before the Planck epoch are washed out, one may regard as unnatural [394] having an initial homogeneity scale of order H^{-1} whenever H is very small in Planck or string units. In the context of the pre-big

bang scenario, however, the phenomenological imprint of the Planck epoch is not necessarily washed out by a long and subsequent inflationary phase. The pre-Planckian history may become visible, and the sub-Planckian initial conditions are accessible, in principle, to observational tests, so that their naturalness can also be analysed with a Bayesan approach, in terms of "a posteriori" probabilities, as in [129] (see also [170, 171, 168] for a discussion of "generic" initial conditions in the pre-big bang scenario).

2.5 Frame-independence: Which is the "right" metric?

As already stressed in the previous discussion, and explicitly computed in Section 1.3, an S-frame inflationary solution of the string cosmology equations, when transformed to the E-frame, tends to describe a phase of accelerated contraction. This is strictly true only for isotropic backgrounds because, if the metric is anisotropic, and there is a large enough number of (S-frame) contracting dimensions, the other dimensions may keep expanding (and accelerated) in both frames.

Consider, for instance, the S-frame vacuum solutions (2.22), corresponding to d expanding and n contracting dimensions. When expressed in terms of the E-frame metric $\widetilde{g}$, and in the cosmic time gauge, through the transformation

$$\widetilde{g}_{\mu\nu} = g_{\mu\nu}\, e^{-2\phi/(d+n-1)}, \qquad d\widetilde{t} = dt\, e^{-\phi/(d+n-1)}, \qquad \widetilde{\phi} = \phi \left(\frac{2}{d+n-1}\right)^{1/2} \tag{2.113}$$

(which generalizes to the anisotropic case the transformations of Eqs. (1.8), (1.11)), the solution becomes

$$\begin{aligned} \widetilde{a}(\widetilde{t}) &= \left(-\widetilde{t}\right)^{-\frac{2n-1-\sqrt{d+n}}{\sqrt{d+n}(d+n)+d-n}}, \qquad \widetilde{b}(\widetilde{t}) = \left(-\widetilde{t}\right)^{\frac{2d-1+\sqrt{d+n}}{\sqrt{d+n}(d+n)+d-n}}, \\ \widetilde{\phi}(\widetilde{t}) &= \left(\frac{2}{d+n-1}\right)^{1/2} \frac{(n-d-\sqrt{d+n})(d+n-1)}{\sqrt{d+n}(d+n)+d-n} \ln(-\widetilde{t}). \end{aligned} \tag{2.114}$$

The expanding part of the solution thus remains expanding (and inflationary $\dot{\widetilde{a}} > 0$, $\ddot{\widetilde{a}} > 0$) even in the E-frame, provided $2n > 1 + \sqrt{d+n}$. Note that, for $d = 3$, this condition is compatible with the condition of growing dilaton, $n < d + \sqrt{d+n}$, only for $n = 1$.

However, isotropic ($n = 0$) metrics in the E-frame are contracting. This does not represent a difficulty for the pre-big bang scenario because the contraction is accelerated, $\dot{\widetilde{a}} < 0$, $\ddot{\widetilde{a}} < 0$, and thus corresponds to a phase in which the ratio r decreases in time, as required for the solution of the horizon/flatness problems (see Sections 1.3 and 2.4). In particular, if the phase of accelerated expansion is long enough to solve the problems in the S-frame, according to Eq. (2.108), then the problems are also solved in the E-frame [320], since the conformal time is the same in both:

$$d\widetilde{\eta} = \frac{d\widetilde{t}}{\widetilde{a}(\widetilde{t})} = \frac{dt}{a(t)} = d\eta. \tag{2.115}$$

Furthermore, as we will see in Section 4.1, the spectral distribution of the metric fluctuations amplified by inflation is also the same in both frames [320, 321].

Such a frame independence of the inflationary aspects of the pre-big bang solutions also extends to the string-driven (and, more generally, matter-dominated) solutions presented in the previous subsection.

Suppose, in fact, that we start, at some initial time t_i, with a distribution of strings characterized in the S-frame by a packing factor (average distance/average size) of the order of unity, and with sufficiently small interactions to represent a perfect fluid source for the superinflationary solution (2.38), (2.39), with $\gamma = -1/d$. In the S-frame, where the metric a is expanding, the perfect-fluid approximation is also valid at any subsequent time $t > t_i$, as the number of strings *per unit of string volume* $n(t)$ is diluted in time by the factor $n/n_i = (a_i/a)^d < 1$. In the E-frame the metric $\widetilde{a}$ is contracting, but the string's proper size $\widetilde{L}(t)$ shrinks with time as $\widetilde{L} = (\widetilde{a}/a)\lambda_s$. As a consequence, the number of strings per unit of string volume scales as $\widetilde{n}(t) = (\widetilde{L}/\lambda_s)^d\ \widetilde{a}^{-d} = a^{-d}$, and it is again diluted as time goes on, exactly by the same amount as in the S-frame [321].

With similar arguments it can be shown that the heating up of the string gas with respect to the radiation, which is easy to understand in the S-frame where the metric is expanding and the radiation is redshifted, also occurs in the E-frame, in spite of the fact that the radiation is blueshifted because of the contraction [320].

Using the standard thermodynamical arguments (see for instance [618]) we find indeed, for the perfect-fluid sources with $p/\rho = \gamma$, generating the solution (2.38), that their temperature T_γ satisfies $T_\gamma \sim a^{-d\gamma}$. On the other hand, $\rho_\gamma \sim a^{-d(1+\gamma)}$. In the S-frame the radiation ($\gamma = 1/d$) is thus supercooled and diluted with respect to the (string) sources that drive inflation ($\gamma < 1/d$),

$$\frac{T_\gamma}{T_r} = \frac{\rho_\gamma}{\rho_r} \sim a^{1-d\gamma}, \tag{2.116}$$

since the metric is expanding.

In the E-frame the fluid sources satisfy a modified conservation equation,

$$\dot{\widetilde{\rho}} + d\widetilde{H}(\widetilde{\rho} + \widetilde{p}) - \frac{1}{\sqrt{2(d-1)}}\dot{\widetilde{\phi}}(\widetilde{\rho} - d\widetilde{p}) = 0 \tag{2.117}$$

(the dots refer to the E-frame cosmic time $\widetilde{t}$), obtained from Eq. (2.31), with $\widetilde{\rho}$ defined in Eq. (2.96). The radiation, which has a traceless stress tensor, still evolves adiabatically, but now its temperature is blueshifted because of the contraction: $\widetilde{T}_r \sim \widetilde{a}^{-1}$. The effective temperature of the other matter sources is also blueshifted, however, and using thermodynamical arguments we obtain [320], from Eq. (2.117):

$$\frac{\widetilde{T}_\gamma}{\widetilde{T}_r} = \frac{\widetilde{\rho}_\gamma}{\widetilde{\rho}_r} \sim \widetilde{a}^{\gamma(d-1)(d\gamma-1)/(\gamma-1)}. \tag{2.118}$$

conformal transfos of metric/frame

For the sources of the inflationary S-frame expansion we have $\gamma < 0$, so that even in the E-frame the above ratio is growing, with a corresponding cooling down and dilution of the radiation density (with respect to the density of the inflationary sources).

There is thus a complete frame-independence for the physical effects of the inflationary solutions of the pre-big bang scenario. This can also be understood as an invariance of physics under a local field redefinition. Different frames are just related by a local (field-dependent) conformal transformation of the metric. No physical observables should depend on it, and this is what we shall find whenever we push a calculation to its very end (see also [19] for a recent discussion of this point). Further help is provided by the fact that, at late times, when the dilaton is frozen, all frames actually coincide. ????????????

Yet it is true that, in developing some physical intuition, there are important kinematical differences in different frames, as also stressed by the examples illustrated before. Inflation in the S-frame, for instance, can be represented as gravitational collapse in the E-frame (see the next section). We think that the S-frame (whose metric coincides with the sigma-model metric to which fundamental strings are directly coupled, see Section 2.2) is the one offering the simplest intuitive picture of how things evolve and work. The following three main points support this attitude.

1

UV cut off

The first point concerns the ultraviolet cut-off. The string metric is the one in which the ultraviolet cut-off of string theory is kept fixed, i.e. does not depend on the dilaton (at least in perturbation theory). This can be easily seen from the form of the tree-level action in the S-frame, in $d = 3$, in which we include a typical higher-derivative correction, $\alpha' R^2$:

$$\Gamma^{\rm SF} = -\frac{1}{2\lambda_{\rm s}^2}\int d^4x\,\sqrt{|g|}\,e^{-\phi}\left(R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \alpha' R^2\right). \tag{2.119}$$

Obviously, higher-derivative corrections become relevant, in this frame, as soon as $R \sim \lambda_{\rm s}^{-2}$ and independently of ϕ. By contrast, the equivalent E-frame action would read:

$$\Gamma^{\rm EF} = -\frac{1}{2\lambda_{\rm s}^2}\int d^4x\,\sqrt{|\widetilde{g}|}\,\left(\widetilde{R} - \frac{1}{2}\widetilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \alpha' e^{-\phi}\widetilde{R}^2\right). \tag{2.120}$$

indicating that higher-derivative corrections become relevant at $\widetilde{R} \sim \lambda_{\rm s}^{-2}e^{\phi}$, which is again the (dilaton-dependent) string scale expressed in Planck units.

2

The second point concerns masses. Tree-level string masses are dilaton-independent in the S-frame, but depend on ϕ in the E-frame since the S-frame relation $g^{\mu\nu}p_\mu p_\nu = m^2$ becomes $\widetilde{g}^{\mu\nu}p_\mu p_\nu = m^2 e^{\phi}$ in the E-frame.

3

The third point concerns couplings. It is only an illusion that, by going to the E-frame, we make the Newton constant G_N dilaton-independent. The physical quantity is not so much Newton's constant but the gravitational force, say between two heavy strings. The force will be proportional to $G_N m_1 m_2$. In the S-frame, masses are dilaton-independent and $G_N \sim e^{\phi}$, while, in the E-frame, G_N is fixed but all masses scale like $e^{\phi/2}$. The same

dilaton (in)dependent parameters

physical result is obtained for the gravitational force but, clearly, it is much more economical to attribute the dilaton dependence of Newton's force to Newton's constant rather than to a universal dilaton dependence of all masses. We think that this is a good example of how physics is frame-independent, while our own physical intuition may prefer one frame to the other.

For all these reasons we shall interpret our results mainly in the string frame. Nonetheless, we shall often use other frames at intermediate stages of the calculation, to make it as simple as possible.

3 Initial conditions

We have already mentioned that, in standard non-inflationary cosmology, initial conditions have to be fine-tuned to an incredible accuracy in the far past (i.e. at $t \sim t_{\rm P} \sim 10^{-43}$ s), or else it is impossible to explain some of the most striking properties of today's Universe, such as its homogeneity and flatness.

In the pre-big bang scenario, having extended time to the past of the big bang event can certainly help solving the fine-tuning problems of standard cosmology: after all, most of these come from a "shortage of time" for things to happen early on after the big bang. Yet it is important to discuss the conditions under which a pre-big bang phase can naturally generate a patch that can evolve into the Universe we live in. In other words: What does the fine-tuning problem look like if, accepting hints from scale-factor duality, we assume perturbative, yet generic, initial conditions?

APT

In this section we shall address this kind of questions by considering, for simplicity, a class of space-times endowed with a regular (time-like and null) asymptotic past, and by imposing on these geometries a condition of asymptotic-past-triviality (APT) (see Subsection 3.2). We are aware of the fact that this is already a schematization of a more general set-up where one would simply assume that the Universe, almost everywhere and almost all the time, is in a chaotic, highly entropic perturbative state in which the arrow of time itself is ill-defined. Within such a more general set-up a mechanism for generating a low-entropy subsystem has to be found, so that, within this subsystem, a time arrow (the direction of increasing entropy) can possibly be identified, and the organized Universe we happen to be living in can develop.

time

We shall argue in Subsection 3.3 that such a mechanism can be identified in the well-known phenomenon of gravitational collapse, and that the low-entropy region corresponding to our own Universe lies in the interior of the closed trapped surface that emerged from the collapse. We shall illustrate in Subsection 3.4 the particular cases of spherical and planar symmetry, comment on the problem of chaotic oscillations in Subsection 3.5, and finally present our conclusion on the fine-tuning problem in Subsection 3.6.

In our simplified approach we will treat one of these regions as representing, by itself, the full space-time manifold. Such a simplifying procedure is by no means new in general relativity: asymptotic flatness (asymptotic spatial triviality, in our terminology) is often assumed in order to prove rigorous theorems. Yet, causality guarantees that the collapse of a star within our galaxy can only depend on local conditions, and this allows for a simplified treatment of the problem.

In a similar spirit we shall assume that the dynamics that gave rise to our Universe can be approximately described by a space-time with an asymptotically trivial past. Thanks to gravitational instability, this evolved towards the formation of a closed trapped surface and of a space-like singularity of the big bang type. Assuming that the actual singularity is avoided, thanks to string corrections to general relativity, this process may give rise, eventually, to a (post-big bang) phase resembling the one of standard hot big bang cosmology.

In the next subsection, to approach the question of whether our present Universe may arise from generic initial conditions, we will relax the homogeneity assumption adopted in Section 2 and consider more general solutions of the string cosmology equations.

3.1 Quasi-homogeneous and inhomogeneous solutions

If we want to explain the large-scale homogeneity and isotropy of our Universe out of a cosmological phase preceding the big bang, we have to consider the case of non-homogeneous cosmologies with the hope to find homogeneity, isotropy, and flatness as asymptotic features of late-time attractors. The question, in particular, is how to obtain space-time manifolds where spatial gradients are subdominant with respect to time derivatives, starting from initial conditions in which they are of the same order. In a rather commonly used terminology, we would like to see "velocity-dominated" cosmologies emerge, at late times, from non-velocity-dominated initial conditions. This is what ordinary, slow-roll inflation supposedly does.

If we start from very generic inhomogeneous conditions, it is conceivable that, as in the case of chaotic inflation [439], only some spatial patches will end up exhibiting the required homogeneity and flatness to be consistent with our Universe, while inhomogeneity and anisotropy will prevail almost everywhere else. In the context of such a type of background evolution it is clear that, as a necessary preparatory exercise, we need to understand the nature of solutions that are not too inhomogeneous, leaving momentarily aside the problem of showing how patches of this kind may emerge at late times.

ok

Such a program was started in [603] and further extended in [134]. The analysis performed in [603] was based on the general equations for the gravidilaton system in $D = 4$, together with the assumption of regularity at past infinity. It was shown that, on a constant-time hypersurface (defined in the synchronous gauge), sufficiently isotropic initial patches with $\nabla g \sim \dot{g}$ and with a growing dilaton would inevitably expand, in the S-frame, and evolve

towards homogeneity and strong coupling. Within those patches, a reliable expansion in spatial gradients [75] can be justified, the evolution quickly becomes velocity-dominated, and a Kasner-like singularity is approached according to the following generalization of the solution (2.88), (2.89):

$$ds^2 = dt^2 - \sum_a e_i^a(x)\ e_j^a(x)\ (-t)^{2\alpha_a(x)} dx^i dx^j\ ,$$
$$\phi = \left[\sum_i \alpha_i(x) - 1\right] \log(-t), \qquad 1 = \sum_i \alpha_i^2(x)\ , \qquad t < 0\ . \tag{3.1}$$

If the above equations represented a general quasi-homogeneous solution, the Z_2^d generalization of scale-factor duality would be explicitly respected in the form $\alpha_i(x) \to -\alpha_i(x)$ (for any subset of indices i), while keeping all $e_i^a(x)$ and $\bar{\phi}$ fixed. However, the momentum constraints still have to be imposed on Eq. (3.1): they are trivial in the homogeneous case, but quite non-trivial in the presence of spatial gradients. These constraints preserve the simplest scale-factor duality [$\alpha_i(x) \to -\alpha_i(x)$ for all i], but explicitly break its Z_2^d generalization [134]. The full symmetry is expected to be recovered as the solution approaches the singularity.

The (S-frame) local expansion rate, for the above metric, is given by the trace Θ of the volume expansion tensor $\Theta_{\mu\nu}$ [615],

$$\Theta_{\mu\nu} = h_\mu^\alpha h_\nu^\beta \nabla_{(\alpha} n_{\beta)}, \qquad h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \qquad \Theta \equiv h^{\mu\nu}\Theta_{\mu\nu} = \frac{\sum_i \alpha_i(x)}{t}, \quad t < 0, \tag{3.2}$$

where n_μ is the unit time-like vector normal to the (space-like) homogeneity hypersurfaces. Under the Kasner constraint of Eq. (3.1), it is clear that the maximal rate of expansion is reached where all α_i are equal and negative [603]. Hence the most isotropic patches are inflated most during this Kasner-like phase, suggesting that most of the space becomes fairly homogeneous and isotropic. This is true in spite of the fact that, unlike in ordinary inflation, the modulus σ^2 of the shear tensor [615],

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \qquad \sigma_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{3}\Theta h_{\mu\nu} \tag{3.3}$$

is not washed away as a Kasner-like singularity is approached [424] (unless higher-order corrections play an important role in this respect [333]).

ok

In a subsequent work [134] these arguments were generalized in several directions. First, the equations were extended to an arbitrary number of dimensions, showing that in $D \neq 4$ no qualitatively new results emerge: in particular there is no critical value of D below which the phenomenon of BKL oscillations (to be described in Subsection 3.5) takes place. Secondly, quasi-homogeneous solutions were constructed with the inclusion of the NS-NS two-form $B_{\mu\nu}$: again, they look very similar to the known homogeneous solutions, provided the exponents are allowed to depend on space. The only non-trivial point consists, again, in imposing the momentum constraints that explicitly break $O(d,d)$ symmetry.

?????

The case of a quasi-homogeneous axion field in $D = 4$ was also considered in [134]. Since the axion and $B_{\mu\nu}$ are related by a (Poincaré) duality transformation that mixes space and time, this case is genuinely different from the one discussed above. Actually, the case of a homogeneous axion shows interesting new features, such as the blowing up of the dilaton both at very late and at very early times. These results, however, have to be taken with much precaution since some of the approximations may break down before such strong-coupling regimes are entered.

In [134] a conjecture was made that, as one approaches past infinity, there is a wide basin of attraction towards a trivial Milne geometry, which is nothing but a wedge of empty, flat Minkowski space in convenient coordinates. If this conjecture were true, we would conclude that the asymptotic past of pre-big bang cosmology is indeed trivial. However, the conjecture was proved not to be quite correct [167, 423], through the construction of exact inhomogeneous solutions of various kinds. Only some of them exhibit a Milne-like behaviour at very early times, while, more often than not, the asymptotic past consists of weakly interacting waves, endowed with the symmetries of the solutions, travelling on top of a trivial background [167, 423, 629]. Examples of this kind were constructed by various techniques.

In the case of cylindrical symmetry, exact, non-homogeneous and anisotropic solutions, containing both a dilaton and an axion fields, were already discussed in [61]. Up to a trivial time-reversal transformation, those solutions approach, near the late-time, a special case of the background (3.1), while, at very early times, an oscillatory behaviour prevails. The authors of [61] have also discussed symmetry properties of the solutions under $O(d, d)$ duality transformations.

Inhomogeneous, cylindrical backgrounds for the bosonic massless modes of heterotic and type-IIB superstrings have also been obtained [247] by applying T- and S-duality transformations (as well as the usual Geroch transformations) to vacuum solutions of the Einstein equations exhibiting a two-dimensional group (G_2) of Abelian isometries. The inhomogeneous, string theory version of the Mixmaster (Bianchi-IX-type) cosmology, in particular, has been presented in [247]. Other inhomogeneous Einstein–Rosen string cosmologies were studied with a similar technique in [167], by applying the global duality symmetries of the superstring effective action to homogeneous models of Bianchi-VI_h-type. Again, in these solutions, the conjecture of an asymptotic-past Milne metric is not usually satisfied. New inhomogeneous string cosmologies were also obtained in [436] by applying a discrete (mirror-like) symmetry of the string effective action to $D = 4$ to vacuum solutions of general relativity.

On a more phenomenological ground, an interesting model containing a spherically symmetric inhomogeneity was studied at the non-perturbative level in [62]. Such a treatment allows the investigation of primordial black hole formation in string cosmology, an issue that we shall reconsider in the context of tensor perturbations in Section 5 (see also [437]).

The question of how robust the existence of inflationary solutions is, in the presence of various axion-like backgrounds, was addressed in [248]; it was shown that in some cases the parameter space leading to inflation shrinks down to the empty set. Such a phenomenon may well be related to the generic existence of chaos and BKL oscillations, which will be discussed later, in Subsection 3.5.

The question of the asymptotic-past behaviour of inhomogeneous string cosmology was addressed in general in [423] within G_2 space-times, admitting two Abelian space-like Killing vectors. Once more, the asymptotic past, rather than empty and Milne-like, was found to contain weakly interacting waves. The inclusion of a Maxwell field was finally considered in [629]: even in this case, while the behaviour near the singularity is Kasner-like, the asymptotic past is filled with weakly interacting gravitational, dilatonic, and electromagnetic waves.

As a general remark all these special, exact, non-homogeneous solutions (and general, quasi-homogeneous solutions) can shed important light on the evolution of perturbations discussed in Sections 4–7, in particular when perturbation theory becomes inadequate. We shall occasionally make use of this connection later in this report. The conclusion that we wish to draw, for the time being, is that a simple structure in the asymptotic past (although not necessarily as trivial as Milne's) can be consistently assumed for the generic solution. This suggests to replace the assumption of past regularity by a somewhat stronger postulate, that of asymptotic past triviality. Such a concept will be introduced and developed in the rest of this section.

APT

3.2 Asymptotic past triviality and symmetries

We start from the general postulate that the pre-collapse, pre-big bang evolution of the Universe can be described in terms of the low-energy, tree-level action of string theory. Taking a generic closed superstring theory, this reads (see Section 2):

$$S = -\frac{1}{2}\lambda_s^{1-d}\int d^{d+1}x\,\sqrt{|g|}\,e^{-\phi}\left[R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}(dB)^2 - 2\,\Lambda\right] \quad , \qquad (3.4)$$

where dB is the (three-form) field strength associated with the antisymmetric tensor $B_{\mu\nu}$. A further simplification comes from assuming that we are dealing with so-called critical superstring theory, the case in which the tree-level (and actually the all-order-perturbative) cosmological constant Λ vanishes. This requires a total of $D = 10$ space-time dimensions [346]. If $D \neq 10$ there will indeed be an effective cosmological constant $O(\lambda_s^{-2})$ preventing the existence of any low-curvature solution of the field equations.

Equation (3.4) receives corrections *either* when curvatures become $O(\lambda_s^{-2})$, *or* when the coupling e^{ϕ} becomes $O(1)$. If such corrections are both negligible, it sometimes becomes useful to perform a change of variable by going to the E-frame, already introduced in the

previous sections. In general, this is done by defining [575, 615]:

$$g_{\mu\nu} = \widetilde{g}_{\mu\nu} e^{\frac{2}{d-1}(\phi-\phi_0)} \ . \qquad (3.5)$$

E frame

Using the Einstein metric $\widetilde{g}$ (and dropping the tilde), the result for the action (3.4) is simply:

$$S = -\frac{1}{2}\lambda_{\rm P}^{1-d} \int d^{d+1}x \sqrt{|g^{(E)}|} \ \left[R - \frac{1}{d-1}\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{-\frac{4}{d-1}\phi}(dB)^2 \right] \ , \qquad (3.6)$$

where $\lambda_{\rm P}^{d-1} = e^{\phi_0}\lambda_{\rm s}^{d-1}$ is the present value of the Planck length, if we take ϕ_0 to coincide with the present (constant) value of the dilaton.

Although the use of the E-frame could simplify some calculations, and we shall see examples of this below, it should be kept in mind that the form of the corrections is no longer so simple. For instance, as already stressed at the end of Section 2, higher-derivative corrections become important when the E-frame curvature is $O(\lambda_{\rm P}^{-2}e^{\frac{2}{d-1}\phi} = \lambda_{\rm s}^{-2})$, i.e. when it reaches a dilaton-dependent critical value. Similarly, having a constant Newton "constant" in this frame is a mere illusion, because (even tree-level) string masses now depend upon ϕ (as already illustrated in Subsection 2.5). For these reasons, although physical results are frame-independent, we shall always describe them with reference to the original S-frame metric in which the string length $\lambda_{\rm s}$ is constant.

Let us finally remark that the two frames coincide today if the dilaton is eventually fixed at its present value ϕ_0. Similarly, as we shall see, the assumption of APT leads to a very large and negative value of the initial dilaton which, if finite, would also allow the identification of the two frames in the far past. However, the two E-frames that coincide with the S-frame at $t = \pm\infty$ differ from each other by an enormous conformal factor, i.e. by a huge blowing-up of all physical scales.

The classical equations that follow from varying (3.4) or (3.6), besides being generally covariant, are also invariant under a two-parameter group of (global) transformations acting as follows:

$$\phi \to \phi + c, \qquad\qquad g_{\mu\nu} \to \lambda^2 g_{\mu\nu}, \qquad (3.7)$$

where λ and c are constant parameters (using general covariance, the latter symmetry is also equivalent to an overall rescaling of all the coordinates). Indeed, both actions are simply rescaled by a constant factor under this group. Note that these two symmetries depend crucially on the validity of the tree-level, low-energy approximation, and on the absence of a cosmological constant. Loop corrections clearly spoil invariance under dilaton shifts, while lower-derivatives (a cosmological constant) or higher-derivatives (α') corrections spoil invariance under a rescaling of the metric. The importance of dealing with critical superstring theory now becomes evident: if wewould consider non-supersymmetric string theories, a cosmological constant would almost certainly be generated at some finite order of the loop expansion, which would change completely the large-distance properties and would spoil the symmetries of the field equations.

The immediate consequence of the above two symmetries is that solutions of the field equations depend on two free parameters, a dimensionless one related to the overall value of ϕ, and a dimensionful one corresponding to an overall length scale. As a consequence, neither the value of the fundamental string length nor that of the Planck length are of relevance to the classical solutions. The importance of this point on the issue of fine-tuning will become clear later in this section.

Only if (and when) the solutions are driven into the high-curvature regime will higher-derivative corrections (involving the string length) will become relevant and break scale invariance. The analysis presented in this section can only be trusted up to this point, or until the strong coupling regime is reached. Which of the two limitations will be reached first depends, in turn, on the two above-mentioned classical moduli. For sufficiently small initial coupling the high-curvature regime is encountered first, while, for sufficiently small initial curvature, the opposite is true. Both possibilities will be discussed in the last part of this report (see Section 8).

As already mentioned, the way we now introduce (and exploit) the concept of APT in string cosmology parallels the familiar use of "asymptotic flatness" in general relativity [93, 545, 513]. In both cases some assumption of asymptotic simplicity is made, the asymptotic past in our case, spatial infinity in the more familiar case. It seems indeed physically (and philosophically) satisfactory to identify the beginning with simplicity. However, simplicity should not be confused with complete triviality: a rigorously empty and flat Universe, besides being uninteresting, is also very special, i.e. non-generic. By contrast, asymptotically trivial Universes, though initially simple, are also generic in a precise mathematical sense that we shall now discuss.

In the weak-coupling, small-curvature, low-energy regime under consideration, all heavy string modes can be integrated out, and one is left with an effective theory of the massless fields. Some of these fields, the graviton, the dilaton, and the NS-NS two-form, are always present. Others (e.g. gauge fields and higher-order forms) may vary, depending on the particular superstring theory under study or, in an M-theory perspective, on the particular M-theory vacuum around which the generic solution is looked for.

For the moment, let us restrict our attention to those three universal massless fields whose low-energy small-coupling dynamics is controlled by the effective action (3.4). The general solution of the field equations, in this regime, will consist of an arbitrary linear superposition of (long-wavelength) gravitational, dilatonic and axionic waves. From the point of view of target space (taken here, for the sake of illustration, to be $(3+1)$-dimensional), the generic solution depends upon four arbitrary functions of three coordinates [425] related to the metric, plus two more each for the dilaton and for the axion field associated to $B_{\mu\nu}$. We will now see how these arbitrary functions appear in the asymptotic expansion of our fields.

Let us assume, in accordance with APT, the existence, in the far past, of Bondi–Sachs-

AXION = KR!!!

like coordinates r, v, θ, φ in which our fields, as $r \to \infty$, take the asymptotic forms [93, 545]

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{f_{\mu\nu}(v, \theta, \varphi)}{r} + \dots , \tag{3.8}$$

and let us count the number of arbitrary functions needed to specify our initial data up to gauge transformations. This problem has been widely discussed in the general relativistic literature and the result (up to functions of less than three coordinates) is the one we have mentioned: a total of $4+2+2 =$ 8 functions of three coordinates are needed (see for instance [425]).

Interestingly, there is an exact correspondence between this "target-space" counting and a "world-sheet" counting. In the latter, those eight arbitrary functions correspond to eight arbitrary functions of three-momentum entering the most general physical (i.e. on shell) vertex operator describing gravitons, dilatons, and the antisymmetric tensor field (which, in four dimensions, is equivalent to a pseudoscalar, also called Kalb–Ramond axion). It is amusing to compare this number with that of marginal operators, able to deform the trivial conformal field theory on a flat target space into another conformal theory. These are associated with physical vertex operators for the graviton, the dilaton and the NS-NS two-form (see for instance [346]). It is easy to verify that such (1,1) vertex operators depend on the same number of arbitrary functions as our APT data.

For instance, a physical graviton is associated with a vertex operator depending on two physical-polarization vectors, each of which depends arbitrarily upon the three-momentum of the on-shell graviton [346]:

$$V^i_{\rm grav} = \epsilon^i(k)_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}, \qquad\qquad i = 1, 2. \tag{3.9}$$

Similar considerations apply to the dilaton and two-form vertex operators. They lead to a total number of eight functions of space, needed to define the most general deformation of the conformal field theory around its flat space-time limit.

From this point of view we see that APT conditions are generic, in a technical sense. Yet APT is not a general property of initial conditions. For instance, it looks as if, by choosing APT, we have eliminated the possibility of "white holes", objects that can emit but not absorb matter and that could possibly violate the second law of thermodynamics. In a sense, by our APT postulate, we are enforcing the second law, at least in that part of space-time that will give rise to our Universe.

3.3 Pre-big bang inflation as gravitational collapse

For simplicity, we will only illustrate here the simplest case of gravidilaton system already compactified to four space-time dimensions. Through the field redefinition (3.5), our problem is reduced to the study of a massless scalar field minimally coupled to gravity. It is well known that such a form of matter cannot give inflation (since it has positive pressure).

fantastic!!!!!!!!

Instead, it can easily lead to gravitational collapse. Thus, in the E-frame, the problem becomes that of finding out under which conditions gravitational collapse occurs, if asymptotically trivial initial data are assigned. Gravitational collapse usually means that the (Einstein) metric (hence the volume of 3-space) shrinks to zero at a space-like singularity. However, typically, the dilaton blows up at that same singularity. Given the relation (3.5) between the Einstein and the (physical) string metric, we can easily imagine that the latter blows up near the singularity, as implied by the phase of dilaton-driven inflation illustrated in Section 2.

How generically does a gravitational collapse take place? Let us recall the singularity theorems of Hawking and Penrose [514, 372], which state that, under some general assumptions, singularities are inescapable in general relativity. Looking at the validity of those assumptions in the case at hand, one finds that all but one are automatically satisfied. The only condition to be imposed is the existence of a closed trapped surface (a closed surface from which future-directed light rays converge). Rigorous results [163] show that this condition cannot be waived: sufficiently weak initial data do not lead to closed trapped surfaces, to collapse, or to singularities. Sufficiently strong initial data do. But where is the borderline? This is not known in general, but precise criteria do exist for particularly symmetric space-times, e.g. for those endowed with spherical or plane symmetry (see Subsection 3.4).

However, no matter what the general collapse/singularity criterion will eventually turn out to be, we do know, from the classical symmetries described in the previous subsection, that such a criterion cannot depend *either* on an overall additive constant in ϕ, *or* on an overall multiplicative factor in $g_{\mu\nu}$.

A characterization of APT initial data can be made [129] following the pioneering work of Bondi, Sachs, Penrose, and others. Since our initial quanta are assumed to consist of massless gravitons and dilatons, their past infinity is null: it is the so-called $\mathcal{I}^-$ boundary of the Penrose diagram (see Fig. 3.1). APT means that dilaton and metric can be expanded near $\mathcal{I}^-$ in inverse powers of $r \to \infty$, while advanced time v and two angular variables, θ and φ, are kept fixed. We shall thus write:

$$\phi(x^\lambda) = \phi_0 + \frac{f(v,\theta,\varphi)}{r} + O\left(\frac{1}{r}\right) , \tag{3.10}$$

$$g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + \frac{f_{\mu\nu}(v,\theta,\varphi)}{r} + O\left(\frac{1}{r}\right) . \tag{3.11}$$

The null wave data on $\mathcal{I}^-$ are: the asymptotic dilatonic waveform $f(v,\theta,\varphi)$, and two polarization components, $f_+(v,\theta,\varphi)$ and $f_\times(v,\theta,\varphi)$, of the asymptotic gravitational waveform $f_{\mu\nu}(v,\theta,\varphi)$, whose other components can be gauged away. The three functions f, f_+, $f_\times$ of v,θ,φ are equivalent to six functions of r,θ,φ with $r \geq 0$, because the advanced time v ranges over the full line $(-\infty,+\infty)$. This is how the six arbitrary functions of the generic solution to the gravidilaton system are recovered.

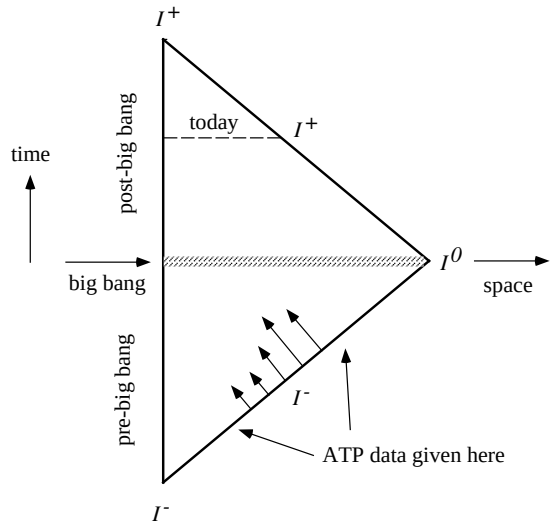

Figure 3.1: *Penrose diagram for a possible model of complete string-cosmology scenario.*

Of particular interest here are the so-called News functions, simply given by

$$N(v,\theta,\varphi) \equiv \partial_v f(v,\theta,\varphi), \qquad N_+ \equiv \partial_v f_+, \qquad N_\times \equiv \partial_v f_\times, \tag{3.12}$$

and the "Bondi mass" given by:

$$M_-(v) = \frac{1}{4\pi}\int d^2\Omega M_-(v,\theta,\varphi), \qquad g_{vv} = -\left(1 - \frac{2M_-(v,\theta,\varphi)}{r}\right) + O\left(\frac{1}{r}\right). \tag{3.13}$$

The Bondi mass and the News are connected by the energy–momentum conservation equation, which tells us that the advanced-time derivative of $M_-(v)$ is positive-semidefinite, and related to incoming energy fluxes controlled by the News:

$$\frac{dM_-(v)}{dv} = \frac{1}{4}\int d^2\Omega \left(N^2 + N_+^2 + N_\times^2\right). \tag{3.14}$$

The physical meaning of $M_-(v)$ is that it represents the energy brought into the system (by massless sources) by advanced time v. In the same spirit, one can define the Bondi mass $M_+(u)$ at future null infinity $\mathcal{I}^+$. It represents the energy still present in the system at retarded time u. If only massless sources are present, the so-called ADM mass is given by

$$M_-(+\infty) = M_+(-\infty) = M_{\rm ADM}\ , \tag{3.15}$$

while $M_-(-\infty) = 0$, and $M_+(+\infty) = M_{\rm C}$ represents the mass that has not been radiated away even after waiting an infinite time, i.e. the mass that underwent gravitational collapse [164]. Collapse (resp. no-collapse) criteria thus aim at establishing under which initial conditions one expects to find $M_{\rm C} > 0$ (resp. $M_{\rm C} = 0$).

Since, as we shall see in the particular case of spherical and planar symmetry, collapse criteria *i*) do not involve any particularly large number, and *ii*) do not contain any intrinsic scale, but just dimensionless ratios of various classical scales, we expect:

- Gravitational collapse to be quite a generic phenomenon.
- Nothing to be there to fix either the size of the horizon or the value of ϕ at the onset of collapse.

Generically, randomly, chaotically, some regions of space at some moment in time will undergo gravitational collapse and will form horizons and singularities therein. When this is translated into the S-frame, regions of space-time within the horizon actually undergo a period of inflation in which the initial values of both the Hubble parameter and ϕ are left arbitrary. In Subsection 3.6 we shall see that such an arbitrariness provides an answer to the fine-tuning allegations that have been moved [588, 394] to the pre-big bang scenario. Before dealing with that issue, however, we shall discuss in the following two subsections cases endowed with special isometries, for which much more is known than for the general case.

3.4 The case of spherical and planar symmetry

In the spherically symmetric case, many authors have studied the problem of gravitational collapse of a minimally coupled scalar field, both numerically and analytically. In the former case we should recall the existence of well-known results [161] pointing at mysterious universalities near critical collapse (i.e. at the borderline situation in which the collapse criteria are just barely met). In this case, a very small black hole forms. This is not the case we are really interested in, for the reasons we just explained. We shall thus turn, instead, to what happens when the collapse criteria are largely fulfilled. For this we make use of the classical results presented in [163, 164, 165, 166].

In the case of spherical symmetry there are no gravitational waves, so that null wave data consist of just an angle-independent, asymptotic dilatonic waveform $f(v)$, with the associated scalar News $N(v) = f'(v)$. A convenient system of coordinates is the double null system (u, v), such that

$$\phi = \phi(u, v)\,, \qquad ds^2 = -\Omega^2(u, v)\, du\, dv + r^2(u, v)\, d\omega^2\,, \tag{3.16}$$

where $d\omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$. The field equations are conveniently re-expressed in terms of the three functions $\phi(u, v)$, $r(u, v)$ and $m(u, v)$, where the local mass function $m(u, v)$ is defined by:

$$1 - \frac{2m}{r} \equiv g^{\mu\nu}\, (\partial_\mu r)\, (\partial_\nu r) = -\frac{4}{\Omega^2}\, \frac{\partial r}{\partial u}\, \frac{\partial r}{\partial v}\,. \tag{3.17}$$

One then gets the following set of evolution equations for m, r and ϕ:

$$2\frac{\partial r}{\partial u}\,\frac{\partial m}{\partial u} = \left(1-\frac{2m}{r}\right)\,\frac{r^2}{4}\,\left(\frac{\partial\phi}{\partial u}\right)^2, \qquad 2\frac{\partial r}{\partial v}\,\frac{\partial m}{\partial v} = \left(1-\frac{2m}{r}\right)\,\frac{r^2}{4}\,\left(\frac{\partial\phi}{\partial v}\right)^2 \tag{3.18}$$

$$r\,\frac{\partial^2 r}{\partial u\partial v} = \frac{2m}{r-2m}\,\frac{\partial r}{\partial u}\,\frac{\partial r}{\partial v}, \qquad r\,\frac{\partial^2\phi}{\partial u\partial v} + \frac{\partial r}{\partial u}\,\frac{\partial\phi}{\partial v} + \frac{\partial r}{\partial v}\,\frac{\partial\phi}{\partial u} = 0. \tag{3.19}$$

The quantity

$$\mu(u,v) \equiv \frac{2m(u,v)}{r} \tag{3.20}$$

plays a crucial rôle in the problem. If μ stays everywhere below 1, the field configuration will not collapse but will finally disperse at infinity as outgoing waves. By contrast, if the mass ratio μ can reach anywhere the value 1, this signals the formation of an apparent horizon $\mathcal{A}$ (a closed trapped surface), whose location is indeed defined by the equation $\mu(u,v)=1$. The above statements are substantiated by some rigorous inequalities [166], stating that:

$$\frac{\partial r}{\partial u} < 0\,, \qquad \frac{\partial m}{\partial v} > 0\,, \qquad \frac{\partial r}{\partial v}\,(1-\mu) > 0\,, \qquad \frac{\partial m}{\partial u}\,(1-\mu) < 0\,. \tag{3.21}$$

Thus, in weak-field regions ($\mu < 1$), we have $\partial_v r > 0$, while, as $\mu > 1$, we have $\partial_v r < 0$, meaning that the outgoing radial null rays ("photons") emitted by the sphere $r = \text{const}$ become convergent, instead of having their usual behaviour. This is nothing but the signature of a closed trapped surface.

The emergence of a closed trapped surface implies the existence of a future singular boundary $\mathcal{B}$ of space-time where a curvature singularity occurs [371, 165]. Furthermore, the behaviour of various fields near the singularity is just that of a quasi-homogeneous phase of dilaton-driven inflation as described by Eqs. (3.1). This highly non-trivial result strongly supports the idea that pre-big bang inflation in the S-frame is the counterpart of gravitational collapse in the E-frame.

Reference [165] gives the following sufficient criterion on the strength of characteristic data, considered at some finite retarded time u

$$\frac{2\Delta m}{\Delta r} \geq \left[\frac{r_1}{r_2}\,\log\left(\frac{r_1}{2\Delta r}\right) + \frac{6r_1}{r_2} - 1\right], \tag{3.22}$$

where r_1 and r_2 define two spheres, such that $r_1 \leq r_2$ and $r_2 \leq 3r_1/2$, , $\Delta r = r_2 - r_1$ is the width of the "annular" region between the two spheres, and $\Delta m = m_2 - m_1 \equiv m(u,r_2) - m(u,r_1)$ is the mass "contained" between the two spheres, i.e. more precisely the energy flux through the outgoing null cone $u = \text{const}$, between r_1 and r_2. Note the absence of any intrinsic scale (in particular of any short-distance cut-off) in the above criterion. The theorem proved in [165] is not exhausted in the above statement. It contains various bounds as well, in particular: *i*) an upper bound on the retarded time at which the closed trapped surgace (i.e. a horizon) is formed, *ii*) a lower bound on the mass, i.e. on the radius of the collapsing region.

The latter quantity is very important for the discussion of the previous subsection since it gives, in the equivalent S-frame problem, an upper limit on the Hubble parameter at the beginning of the phase of inflation. Such an upper limit depends only on the size of the advanced-time interval satisfying the collapse criterion; since the latter is determined by the scale-invariant condition (3.22), the initial scale of inflation will be classically undetermined.

The above criterion is rigorous, but probably too conservative. It also has the shortcoming that it cannot be used directly on $\mathcal{I}^-$, since $u \to -\infty$ on $\mathcal{I}^-$. In Ref. [129] a less rigorous (or less general) but simpler criterion, directly expressible in terms of the News (i.e. on $\mathcal{I}^-$), was proposed on the basis of a perturbative study. It has the following attractive form:

$$\sup_{\substack{v_1,v_2 \\ v_1 \le v_2}} \mathrm{Var}(N(x))_{x\in[v_1,v_2]} > C = O(1/4)\,, \tag{3.23}$$

where:

$$\mathrm{Var}\,(N(x))_{x\in[v_1,v_2]} \equiv \langle (N(x) - \langle N\rangle_{[v_1,v_2]})^2\rangle_{x\in[v_1,v_2]}\,. \tag{3.24}$$

Thus $\mathrm{Var}\,(g)_{[v_1,v_2]}$ denotes the "variance" of the function $g(x)$ over the interval $[v_1, v_2]$, i.e. the average squared deviation from its mean value.

According to this criterion the largest interval satisfying (3.23) determines the size of the collapsing region and thus, through the collapse inflation connection, the initial value of the Hubble parameter. It would be interesting to confirm the validity of the above criterion, and to determine more precisely the value of the constant appearing on its r.h.s., through more analytic or numerical work. This problem has been studied by two perturbative numerical analyses [468, 158] and also, more recently, by a numerical approach that takes into account the full non-linear evolution [460], and stresses the possible role of non-linearities in triggering the collapse of a long train of low-amplitude waves, even though the criterion (3.23) would suggest that they remain perturbative.

Let us discuss now another toy model example of gravitational collapse, that of the collision of plane waves, following closely Refs. [245] and [101]. The advantage of this case is that it can often be solved analytically, thereby providing a useful laboratory for investigating properties that we would like to be shared by more realistic situations (see some comments at the end of this subsection). For this reason we shall treat this example in considerable detail.

In order to include eventually the NS-NS two-form, we will start from the low-energy effective action in a form that already exhibits the $O(d,d)$ non-compact symmetry expected from the $d = D-2$ Abelian isometries of the problem[1]. This is achieved by working with a "reduced action" living in the non-trivial (two-dimensional) subspace spanned by time and by the common direction of propagation of the two waves. Following [469] we write:

$$S = \int dx^0 dx^1 \sqrt{-g} e^{-\overline{\phi}} \left[R + \partial_\alpha \overline{\phi} \partial^\alpha \overline{\phi} + \frac{1}{8} \mathrm{Tr}\left(\partial_\alpha M^{-1} \partial^\alpha M\right)\right]. \tag{3.25}$$

[1] In this subsection, for simplicity, we have denoted by d what in the rest of the report would be $d-1$.

Here the metric $g_{\mu\nu}$ is taken to be block-diagonal, with blocks g_{ij} and $g_{\alpha\beta}$ (the Roman indices $(i, j, \ldots)$ span the transverse directions, while the Greek indices take the values 0 and 1). We arranged the components g_{ij} in a d-dimensional matrix G and the non-vanishing components B_{ij} in another matrix B. M is then the usual $2d$-dimensional matrix containing G and B (see Eq. (2.55)), while $\overline{\phi} = \phi - (1/2)\log\det G + \text{constant}$ is the corresponding shifted dilaton (see Subsection 1.4). The reduced action (3.25) is manifestly invariant under the global $O(d, d)$ transformations (2.62).

The corresponding equations of motion are written most conveniently by going to the conformal gauge for the 2-metric, $g_{\alpha\beta} = e^F \eta_{\alpha\beta}$, and by working with "light-cone" coordinates, $\sqrt{2}u = (x^0 - x^1), \sqrt{2}v = (x^0 + x^1)$. They take the explicitly $O(d, d)$-invariant form

$$\partial_u \partial_v \exp(-\overline{\phi}) = 0, \qquad \partial_u \left(e^{-\overline{\phi}} M^{-1} \partial_v M \right) + \partial_v \left(e^{-\overline{\phi}} M^{-1} \partial_u M \right) = 0; \tag{3.26}$$

$$\partial_u^2 \overline{\phi} - \partial_u F \partial_u \overline{\phi} + \frac{1}{8} \mathrm{Tr} \left(\partial_u M^{-1} \partial_u M \right) = 0, \quad \text{and the same with } u \to v; \tag{3.27}$$

$$\partial_u \partial_v \overline{\phi} - \partial_u \partial_v F + \frac{1}{8} \mathrm{Tr} \left(\partial_u M^{-1} \partial_v M \right) = 0. \tag{3.28}$$

The two colliding waves are defined as having their fronts at $u = 0$ and $v = 0$, and thus to collide at $u = v = 0$ (i.e. at $x^0 = x^1 = 0$). The two waves are *not* assumed to be impulsive, i.e. their energy density can have any (finite) support at positive u and v, respectively.

Space-time is naturally divided in four regions: Region I, defined by $u, v < 0$, is the space-time in front of the waves, before any interaction takes place. It is trivial Minkowski space-time with $B = 0$ and a constant perturbative dilaton ($\exp(\phi_0) \ll 1$). It will be convenient to fix the constant in the definition of $\overline{\phi}$ so that, in region I, $\overline{\phi} = 0$. Region II, defined by $u > 0$, $v < 0$, represents the wave coming from the left before the collision. Metric and dilaton depend only on u:

$$ds^2_{\mathrm{II}} = -2dudv + G^{\mathrm{II}}_{ij}(u) dx^i dx^j, \qquad B_{ij} = B^{\mathrm{II}}_{ij}(u), \qquad \phi = \phi^{\mathrm{II}}(u), \tag{3.29}$$

and we can consistently set $F = 0$. Region III can be similarly described up to interchange of u and v. Note that we have not assumed any special shape for G, so that our results, so far, hold irrespectively of the polarization of the waves. Finally, in region IV ($u > 0$, $v > 0$), i.e. in the interaction region, we write

$$ds^2_{\mathrm{IV}} = -2e^F dudv + G^{\mathrm{IV}}_{ij} dx^i dx^j, \tag{3.30}$$

and F, G^{IV}, ϕ^{IV} and B^{IV} are all functions of both u and v. Of course, the metric must be continuous along with its derivative on the boundary lines between the four regions. The same must be true for the dilaton ϕ and the antisymmetric field B.

Let us begin by solving the equations in region II. The only non-trivial equation is (3.27), which (dropping the subscripts II) reads

$$\ddot{\overline{\phi}} = \frac{1}{8} \mathrm{Tr} \left[\left(M^{-1} \dot{M} \right)^2 \right], \tag{3.31}$$

where the dot indicates the derivative with respect to u. It is now useful to change variables from u to $\tilde{u}$, with

$$\frac{d}{du} = e^{-2\phi/d}\frac{d}{d\tilde{u}}. \tag{3.32}$$

After some algebra we can rewrite (3.31) as:

$$e^{\overline{\phi}/d}\left(e^{-\overline{\phi}/d}\right)'' = -\frac{1}{d^2}\phi'^2 - \frac{1}{4d}\left\{\mathrm{Tr}\left[\left(G^{-1}G'\right)_t^2\right] - \mathrm{Tr}\left[\left(G^{-1}B'\right)^2\right]\right\}, \tag{3.33}$$

where $\left(G^{-1}\dot{G}\right)_t$ is the traceless part of $\left(G^{-1}\dot{G}\right)$, and the prime denotes the derivative with respect to $\tilde{u}$.

It can easily be shown that all terms on the r.h.s. of Eq. (3.33) are negative-definite. Hence, for any non trivial wave, $e^{-\overline{\phi}/d}$, which is constant and identically equal to 1 in region I, must acquire a non-vanishing, negative, and never increasing derivative in region II. Thus, $e^{-\overline{\phi}/d}$ must vanish at some finite $\tilde{u} = \tilde{u}^*$. Returning now to the coordinate u, we see that, if the dilaton is bounded (a necessary assumption if we want to use the tree-level effective action), there exists a finite $u = u^*$ where $e^{-\overline{\phi}/d}$ vanishes. Correspondingly, also $\det G$ vanishes, and the metric of the transverse space will collapse to zero proper volume, thereby producing a (coordinate) singularity. The same arguments can be repeated in region III, where $\det G$ has to vanish at some finite $v = v^*$, with $\overline{\phi} \to +\infty$. These conclusions generalize a well-known result in $D = 4$ general relativity [576, 630] to any D and to non trivial antisymmetric fields, .

Let us finally analyse region IV (where the interaction between the two waves occurs) dropping, for simplicity, the subscript IV from all functions. We begin by using Eq. (3.26), which tells us that $e^{-\overline{\phi}}$ is the sum of a function of u and a function of v. The unique function of this type that matches the boundary conditions with region I is

$$e^{-\overline{\phi}(u,v)} = e^{-\overline{\phi}_{\rm II}(u)} + e^{-\overline{\phi}_{\rm III}(v)} - 1. \tag{3.34}$$

We see that $e^{-\overline{\phi}(u,v)}$ must vanish on a hypersurface that joins the coordinate singularities in regions II and III and is contained in the region $u \le u_0$, $v \le v_0$ within region IV.

Let us now introduce two new sets of coordinates that simplify the analysis in region IV. One set is of the light-cone type:

$$r = r(v) = 2e^{-\overline{\phi}_{\rm III}(v)} - 1, \qquad s = s(u) = 2e^{-\overline{\phi}_{\rm II}(u)} - 1, \tag{3.35}$$

while the second set is of the $t - x$ kind:

$$\xi = \frac{1}{2}(r + s) = e^{-\overline{\phi}(u,v)} \sim -t, \tag{3.36}$$

$$z = \frac{1}{2}(s - r) = e^{-\overline{\phi}_{\rm II}(u)} - e^{-\overline{\phi}_{\rm III}(v)}. \tag{3.37}$$

Note that the coordinates r, s run from $+1$ to -1 in region IV, with their sum always positive except on the singular boundary where $r + s = 0$. Going from the original coordinates to

either of the new sets changes only the conformal factor of the 2-metric. We may thus write:

$$ds^2_{\rm IV} = -e^f d\xi^2 + e^f dz^2 + G_{ij}dx^i dx^j = -2e^f dr ds + G_{ij}dx^i dx^j \quad , \tag{3.38}$$

while the shifted dilaton is simply

$$\overline{\phi} = -\log \xi = -\log(r+s)/2. \tag{3.39}$$

Let us discuss in some detail the case of $B = 0$ and parallel polarized waves following [245]. In each region, we take a diagonal G with $G_{ii} = e^{\lambda+\psi_i}$, where $\sum \psi_i = 0$ and $\lambda = (1/d)\log\det G$ goes to $-\infty$ at $u = u^*$ in region II, and at $v = v^*$ in region III. In the new coordinates, Eq. (3.26) becomes trivial, while the equations for the ψ_i and λ are decoupled and can be solved separately. They are also formally the same, so the solutions have the same structure [576]:

$$\begin{aligned} -(r+s)^{\frac{1}{2}}\psi_i(r,s) = &\int_{\rm s}^{1} ds' \left[(1+s')^{\frac{1}{2}}\psi_i(1,s')\right]_{,s'} P_{-\frac{1}{2}}\left[1+2\frac{(1-r)(s'-s)}{(1+s')(r+s)}\right] \\ &+\int_{r}^{1} dr' \left[(1+r')^{\frac{1}{2}}\psi_i(r',1)\right]_{,r'} P_{-\frac{1}{2}}\left[1+2\frac{(1-s)(r'-r)}{(1+r')(r+s)}\right] , \end{aligned} \tag{3.40}$$

where $P_{-1/2}(x)$ are Legendre functions written in standard notation, and the commas denote partial derivatives. The same expression holds for λ, with the obvious replacements. At this point ϕ can easily be recovered once λ and $\overline{\phi}$ are known.

We see that λ and the ψ_i are singular on the hypersurface $\xi = r + s = 0$. So, in a very general way, we find that the collision of two plane waves leads to a (curvature) singularity in the space-time, whatever the number of dimensions. Although we do not have, at the moment, such an explicit solution for the general case, the discussion given in the previous section makes us believe that a curvature singularity will always emerge along the hypersurface $\xi = 0$.

For the solutions (3.40), the asymptotic behaviour for $\xi \to 0$ is easily found by taking the large-argument limit of the Legendre function [576] (see also [630]):

$$\psi_i(\xi,z) \sim \epsilon_i(z)\log\xi, \qquad \lambda(\xi,z) \sim \kappa(z)\log\xi, \qquad f(\xi,z) \sim a(z)\log\xi. \tag{3.41}$$

The coefficients multiplying the logarithm are functions of z, whose range, on the singular surface, goes from -1 to 1. One easily finds

$$\begin{aligned} \epsilon_i(z) = \; &\frac{1}{\pi\sqrt{1+z}}\int_{z}^{1} ds \left[(1+s)^{\frac{1}{2}}\psi(1,s)\right]_{,s}\left(\frac{s+1}{s-z}\right)^{\frac{1}{2}} + \\ &+\frac{1}{\pi\sqrt{1-z}}\int_{-z}^{1} dr \left[(1+r)^{\frac{1}{2}}\psi(r,1)\right]_{,r}\left(\frac{r+1}{r+z}\right)^{\frac{1}{2}} , \end{aligned} \tag{3.42}$$

$$a(z) = \frac{1}{4}\sum \epsilon_i^2(z) + \frac{d}{4}\kappa^2(z) - 1 \, , \tag{3.43}$$

with $\kappa(z)$ given by the same expression as ϵ_i, but with ψ_i replaced by λ. The sum of the ϵ_i must be zero according to the definition of the ψ_i.

The asymptotic form of the metric is

$$ds^2_{\rm IV} = -\xi^{a(z)} d\xi^2 + \xi^{a(z)} dz^2 + \xi^{\kappa(z)} \sum \xi^{\epsilon_i(z)} \left(dx^i\right)^2 \ , \tag{3.44}$$

while

$$\phi \sim -\left(1 + \frac{d}{2}\kappa(z)\right) \log \xi \ . \tag{3.45}$$

Going over to cosmic time $\xi = t^{\frac{2}{a(z)+2}}$ gives the metric in Kasner form with exponents

$$p_1(z) = \frac{a(z)}{a(z)+2}, \qquad p_i(z) = \frac{\kappa(z) + \epsilon_i(z)}{a(z)+2}. \tag{3.46}$$

The following relations are immediately verified:

$$\phi = \left(\sum_{\alpha=1}^{D-1} p_\alpha(z) - 1\right) \log t, \qquad \sum_{\alpha=1}^{D-1} p_\alpha^2(z) = 1. \tag{3.47}$$

The behaviour of the fields near the singularity is thus of Kasner type, modified, as usual, by the presence of the dilaton. Note that, at the two tips of the singularity, ϵ_i and κ diverge in such a way that the Kasner exponents near the tips are simply $p_1 = 1$, $p_i = 0$. This corresponds to a (contracting) Milne-like metric, which, being non-singular, nicely matches the non-singular behaviour in regions II and III. Away from these two points, κ and ϵ_i can take any value: it is easy to verify that the whole Kasner sphere can be covered by appropriately choosing the initial data.

It may be interesting to note that, in such a context, it is possible to estimate the entropy generated by the non-linear interaction of the incoming waves, before the Universe enters the phase of inflation, during the intermediate phase dominated by both the velocity and spatial gradients [246]. The total produced entropy is of the order of the product of the focal lengths of the two incoming waves (in Planck units), and satisfies the entropy bounds that will be discussed in Section 8.4.

Let us finally perform $O(d,d)$ boosts on our solutions, in order to introduce a non-trivial $B_{\mu\nu}$ field. We will refer to [101] for a full discussion and only summarize here the final result. Cosmologies with non-trivial $B_{\mu\nu}$ that can be obtained, by an $O(d,d)$ transformation, from a metric having Kasner behaviour near the singularity, generally contract in all dimensions. In even dimensions, at least one inflates when the original metric has just one inflating or just one contracting dimension. In odd dimensions, if the $O(d,d)$ transformation has $\det Q = 0$ and we start from a full inflating metric or from a metric with one inflating dimension, then we are left with one inflating dimension. If $\det P = 0$, then one dimension inflates when we start from a full contracting metric or from a metric having just one contracting dimension.

We should also recall that none of the $O(d,d)$ transformations affects the g_{11} component of the metric, which, therefore, can either inflate or contract.

How do these results generalize to more realistic situations such as the head-on collision of two plane-fronted waves, with a transverse profile that extends only over a finite range? Unfortunately such cases cannot be dealt with analytically, and numerical results are not available either, as far as we know. There have been claims [630], however, that black holes (and singularities therein) are inevitably formed, provided certain conditions are met.

Consider, an an example, two identical waves of transverse size L, moving head on against each other, and carrying, in the centre-of-mass frame, energy density (per unit area) ρ. The conjecture [630] is that a black hole of mass $O(\rho L^2)$ will form, provided

$$L > c(G\rho)^{-1}\,, \tag{3.48}$$

where c is a number $O(1)$.

This conjecture appears to be confirmed by applying a recently proposed method [228] for determining when black holes are formed in point-like particle collisions at finite impact parameter. A simple extension of that method [418] to two finite-size massless waves shows that Eq. (3.48) is indeed a good criterion for black-hole formation with $c \leq 1/4\pi$.

3.5 Adding p-forms and BKL chaotic behaviour

We have already mentioned that adding a NS-NS two-form to the background strongly reduces the parameter space available for inflating the Universe (in string units). However, very generally, a smooth Kasner-like behaviour is still generically valid near the singularity. Spatial gradients become small with respect to time derivatives, and solutions become velocity-dominated, thereby solving, at least in principle, the flatness/homogeneity problems.

This result appears to be in contrast with what is known to happen in a standard general relativity context. Indeed, Belinskii, Khalatnikov and Lifshitz (BKL) postulated a long time ago [76] that the approach to a big crunch singularity (the time-reversal of our situation) is typically chaotic, in contrast with the naive expectation that, near the singularity, kinetic terms should dominate over spatial gradients since ${}^{(3)}R \sim a^{-2} \ll R \sim t^{-2}$, for a scale factor going to zero with a small power of t. This would certainly be true for an isotropic situation. However, because of the constraints on the exponents $\alpha_i = (d\ln a_i/d\ln t)$ of the (general relativistic) Kasner solution,

$$\sum \alpha_i = \sum \alpha_i^2 = 1, \tag{3.49}$$

the only non-trivial vacuum cosmologies are anisotropic.

As a consequence, there are gradient terms that are subdominant and others that, instead, grow faster than the extrinsic curvature itself (the expansion contribution to cur-

vature), and start dominating at some point. When this happens the Kasner exponents suddenly change and a new Kasner epoch starts, until another spatial gradient dominates ..., and so on indefinitely. One can visualize this behaviour as the motion of a particle in a billiard, where the Kasner-like phases are the straight motions while the sudden changes in the Kasner exponents correspond to reflections. Unless the "billiard" in question is particularly simple, a chaotic behaviour then follows. This phenomenon goes under the name of BKL oscillations.

It was known for some time that BKL oscillations occur in general relativity for any $D < 11$. However, the phenomenon was known to disappear if a massless, minimally coupled scalar field is added. The dilaton of string cosmology is just such a field, and therefore BKL oscillations do not appear in gravidilaton string cosmology, as explicitly shown in [58]. One way to see this is to realize that, in the presence of a dilaton, an isotropic solution is perfectly allowed, and therefore all spatial gradients can be kept under control. Thus, possibly after a small number of oscillations, the Kasner exponents end up in a region (close to the isotropic point) where no more oscillations occur until the singularity.

Adding the NS-NS two-form does not appear to change the situation with respect to BKL oscillations, as shown in many of the examples of inhomogeneous string cosmology discussed in Subsection 3.1. The same is true in the presence of various kinds of axion fields (see e.g. the case of the Type IIB string cosmology discussed in [248]), or in the context of the gravidilaton–axion–Maxwell system giving rise to inhomogeneous Gowdy-type solutions [500, 629]. On the basis of these examples it was believed for some time that BKL oscillations were the exception, rather than the rule, for string cosmology.

The surprise came when it was pointed out [190, 191] that BKL oscillations reappear generically if all the massless bosonic background fields of superstring theory (or M-theory) are turned on. Note that, even under the assumption of asymptotic past triviality, there is no reason to artificially put to zero any of these massless backgrounds. According to APT, the generic initial state should be an arbitrary superposition of weakly interacting, low-frequency waves of all kinds. Setting arbitrarily to zero some of them would amount to fine-tuning the initial state, and could even be unstable under quantum effects.

The emergence of chaos discovered in [190, 191] could have been anticipated, since we can view the gravidilaton system in $D = 4$ as coming from dimensional reduction of a pure gravity theory in more dimensions. Upon dimensional reduction, the higher dimensional graviton generates both the dilaton and additional gauge fields. The electric and/or magnetic components of these fields, unless artificially turned off, will grow and cause BKL behaviour. Let us now go in some length into this matter, nevertheless referring to the original articles for full details.

The starting point is the low-energy tree-level bosonic action of a generic string theory

(or of M-theory) which, in the Einstein frame, can be written in the form:

$$S = -\int d^D x \sqrt{g} \left[R(g) + ... + \sum_p c_p e^{\lambda_p \varphi} (dA_p)^2 \right], \tag{3.50}$$

where the index p labels the various p-forms that are present in a given superstring model, and c_p, λ_p are numerically coefficients. There are actually possible additional terms, e.g. Chern-Simons terms, but these are not relevant to this discussion. One then assumes the inhomogeneous-Kasner ansatz of Subsection 3.1, rewritten in the Einstein frame as:

$$\begin{aligned} & ds^2 = dt^2 - \sum_a e_i^a(x)\ e_j^a(x)\ (-t)^{2p_a(x)} dx^i dx^j, \qquad \phi = p_\varphi\ \log(-t)\,, \\ & \sum_a p_a = 1\,, \qquad\qquad p_\varphi^2 = 1 - \sum_a p_a^2, \qquad\qquad t < 0\,. \end{aligned} \tag{3.51}$$

In order to check whether such a non-chaotic, non-oscillatory ansatz is an asymptotic solution one has to check whether all neglected terms remain parametrically small as $t \to 0$. Several terms are possibly dangerous. Dilaton gradients are always harmless, but metric gradients are "down" with respect to the time-derivative terms by a factor $|t|^{2g_{ijk}}$, where g_{ijk} is the following combination of Kasner exponents, $g_{ijk} = 1 + p_i - p_j - p_k$. In order for these terms to be negligible, it is necessary that all g_{ijk} be positive. Furthermore, the equations of motion for the various forms turn out to imply the constancy of both the "magnetic" components, $F_{j_1 \ldots j_{p+1}}$, and of the "electric" components, $\sqrt{g} e^{\lambda_p \varphi} F^{0 i_1 \ldots i_p}$, of the field strength $F_{p+1} \equiv dA_p$. When such a result is inserted back in the complete equations of motion one finds further constraints on the Kasner exponents and on the "dilaton couplings" λ_p, which need to be satisfied in order for the solution (3.51) to hold.

The explicit analysis of [190, 191] shows that in any known $D = 10$ superstring theory, as well as in $D = 11$ supergravity (the low-energy limit of M-theory), the couplings λ_p are just what is needed to reduce the parameter space for the validity of (3.51) to a set of zero measure. The generic case is instead chaotic, and can be described in terms of motion on a billiard in a 9-dimensional hyperbolic space [193]. The reflection rules in the billiard form a group that turns out to be the Weyl group of the hyperbolic Kac–Moody algebras of E_{10} or BE_{10}, depending on the particular superstring theory under consideration. Similar groups were also found to emerge in the case of Kaluza–Klein compactification of pure gravity in $D = d + 1 \geq 4$ [194]. The Weyl group is now related to the Kac–Moody algebra AE_d, and the disappearance of chaos for $D \geq 11$ gets related to the fact that these algebras are hyperbolic only for $d < 10$.

Let us finally mention a connection between the BKL chaotic behaviour and the peculiar features of some homogeneous string cosmology models, in particular those with a non-trivial antisymmetric field $B_{\mu\nu}$. We have already seen that the presence of such a backgound tends to reduce drastically (if not completely) the phase space left for inflationary pre-big bang behaviour. In Ref. [192] it was shown that this can be understood again as motion in

a billiard. Whenever the initial Kasner exponents are such as to give inflation, some walls are hit a certain (finite) number of times until the ensuing reflections bring the S-frame Kasner exponents to be positive, thereby stopping the inflationary behaviour and turning it into a contraction.

From the physical viewpoint the implications of all these results are not so obvious, also because the conditions for BKL behaviour are just met at the tree level and to lowest order in α'. It is possible that the inclusion of either α' or loop corrections will bring the oscillations to a stop (after a relatively small number of them [190, 191]). It is not clear, however, which kind of string-scale-curvature hypersurface will be generated by the whole process. It is unlikely that it will be very simple and homogeneous, but it is not obvious that it will not contain isotropic and homogeneous enough patches for our Universe to emerge from.

3.6 Is pre-big bang cosmology fine-tuned?

The two arbitrary parameters discussed in the previous subsection are very important, since they determine the range of validity of our description. In fact, since both curvature and coupling increase during the initial phase of dilaton-driven inflation, the low-energy and/or tree-level description is bound to break down at some point. The smaller the initial Hubble parameter (i.e. the larger the initial horizon size) and the smaller the initial coupling, the longer we can follow the phase of low-energy inflation through the effective action equations, and the larger the number of reliable e-folds we shall gain.

This does answer, in our opinion, the objections raised [588] to the pre-big bang scenario according to which it is fine-tuned. The situation here actually resembles that of chaotic inflation [439]. Given some generic (though APT) initial data, we should ask which is the distribution of sizes of the collapsing regions and of couplings therein. Then, only the "tails" of these distributions, i.e. those corresponding to sufficiently large and sufficiently weakly coupled regions will produce Universes like ours, the rest will not. The question of how likely a "good" big bang is to take place is not very well posed and can be greatly affected by anthropic considerations [129]. Furthermore, asking for a long enough period of low-energy inflation amounts to setting upper limits on two arbitrary moduli of the classical solutions.

Figure 3.2 is a $(2+1)$-dimensional sketch of a possible pre-big bang Universe: an original "sea" of dilatonic and gravity waves leads to collapsing regions of different initial size, possibly to a scale-invariant distribution of them. Each one of these collapses is reinterpreted, in the S-frame, as the process by which a baby Universe is born after a period of pre-big bang inflationary "pregnancy", the size of each baby Universe being determined by the duration of the corresponding pregnancy, i.e. by the initial size of (and coupling in) the corresponding collapsing region. By using, in particular, the result of Eq. (2.110) one

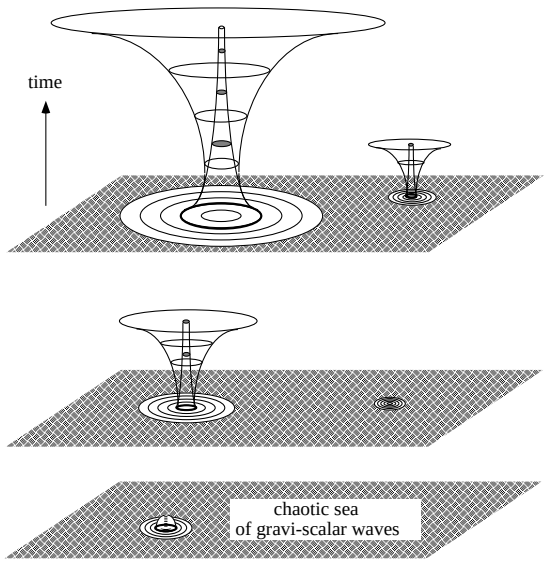

Figure 3.2: *Qualitative space-time illustration of the possible birth of pre-big bang Universes from a chaotic sea of gravidilaton waves. Each baby Universe is simultaneously represented in the S-frame (like an expanding cone) and in the E-frame (like a shrinking hole), and it is followed for an increasing time interval as we move upward in the figure.*

finds that, in $d = 3$, regions initially larger than $10^{19}\lambda_s$ (namely, larger than 10^{-13} cm if $\lambda_s \sim \lambda_P \sim 10^{-32}$ cm) can generate Universes like ours, while smaller ones cannot.

A basic difference between the large numbers needed in (non-inflationary) FRW cosmology and the large numbers needed in pre-big bang cosmology, should be stressed. In the former, the ratio of two classical scales, e.g. of total curvature to its spatial component, which is expected to be $O(1)$, has to be taken as large as 10^{60}. In the latter, the above ratio is initially $O(1)$ in the collapsing/inflating region, and ends up being very large in that same region, thanks to the inflation. However, the (common) order of magnitude of these two classical quantities is a free parameter, and it is taken to be much larger than the classically irrelevant quantum scale. Indeed, the smallness of quantum corrections (which would introduce a scale in the problem) was explicitly checked in [343].

An example of what we have just said is the case of the collision of two finite-front plane waves. It is clear from Eq. (3.48) that all that matters is the ratio of two geometrical classical quantities, the transverse size L and the focal distance $(G\rho)^{-1}$. Neither the Planck nor the string lengths appear in the collapse criterion. Even the appearance of the Newton constant in (3.48) is somewhat misleading: as always in the context of classical general relativity, the Newton constant can be removed by a convenient choice of units of energy, and then everything reduces to geometrical quantities.

We can visualize analogies and differences between standard and pre-big bang inflation

by looking again at the two cases of Figs. 1.2 and 1.3. The common feature in the two pictures is that the fixed comoving scale corresponding to the present horizon was "inside the horizon" for some time during inflation, possibly very deeply inside at its onset. The difference between the two scenarios is just in the behaviour of the Hubble radius during inflation: increasing in standard inflation, decreasing in string cosmology. Thus, while standard inflation is still facing the initial-singularity question, and needs a non-adiabatic phenomenon to reheat the Universe (a kind of small bang), pre-big bang cosmology faces the singularity problem later, combining it with the exit and heating problems (see Section 8).

A much more relevant objection to this approach to initial conditions is the one raised in [190, 192]. If the initial conditions are to be really generic (although satisfying APT), then all massless waves should be allowed. Already with antisymmetric tensor waves the phase of dilaton-driven inflation is becoming non-generic, while, when all massless forms are included, BKL behaviour seems to be the generic situation, at least before strong curvature and/or strong coupling is reached. Understanding the fate of such evolutions calls for a deeper understanding of how string effects may resolve the singularity problem (to be discussed in Section 8).

4 Amplification of quantum fluctuations

fundamental!!!!

If we accept, at least as a working hypothesis, the idea that the pre-big bang scenario illustrated in the previous sections might represent a possible candidate for a self-consistent representation of the primordial cosmological evolution, we are eventually led to the following fundamental questions: Is it possible to find phenomenological effects that can differentiate the pre-big bang scenario from other, more conventional inflationary scenarios, and Are such differences observable today, at least in principle?

In order to answer this question we may note that the accelerated evolution of the Universe is a process accompanied by the production of an enormous amount of radiation. Almost all types of fields are excited, and particles are copiously produced, with an efficiency that reaches its maximum during the explosive "big bang" regime. By studying the properties of the relic cosmic backgrounds we may obtain information about the very early state of the Universe, just as by studying the quantum numbers of the particles produced in a decay process we can get information on the quantum state of the system before the decay.

From the various mechanisms of radiation production, active on a cosmological scale, we will consider here the parametric amplification of the quantum fluctuations of the vacuum, because it is typical of all backgrounds with accelerated kinematics, and it is effective in all inflationary scenarios. It thus represents a good "indicator" of the dynamics of the early Universe [349, 350, 356] and an appropriate starting point to discriminate among different

inflationary models.

Although in this section we will mainly concentrate on the study of the metric fluctuations, our discussion can be easily applied to the evolution of the fluctuations of any background field. The convenient formalism to be adopted for the study of this process is the theory of cosmological perturbations (see [492, 433] for two excellent and comprehensive reviews), which provides a straightforward and (in principle) simple approach to the problem. According to this approach, given a set of background field equations

$$G_{\mu\nu} = T_{\mu\nu}, \tag{4.1}$$

the standard procedure is to perturb (to first order) the metric and the matter sources:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta^{(1)} g_{\mu\nu}, \qquad T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta^{(1)} T_{\mu\nu}, \tag{4.2}$$

expanding around a given cosmological solution. By using the unperturbed equations for the bakground fields, $G^{(0)}_{\mu\nu} = T^{(0)}_{\mu\nu}$, one easily obtains a system of linear equations describing the evolution of the first-order fluctuations of the metric and of the matter fields,

$$\delta^{(1)} G_{\mu\nu} = \delta^{(1)} T_{\mu\nu}, \tag{4.3}$$

and can then discuss, on the basis of these equations, the amount of radiation production.

This perturbative approach is the same for all cosmological scenarios. There are two important differences, however, that characterize pre-big bang models with respect to other inflationary models.

The first difference concerns the higher-dimensional and scalar-tensor (i.e. gravidilaton) nature of the string cosmology backgrounds. As a consequence, there are possible contributions to the amplification of the fluctuations not only from the inflation of the three-dimensional space, but also from the dynamics of the extra-dimensions [264, 25, 207, 296], and from the time evolution of the coupling constants (i.e. of the dilaton), as first pointed out in [299]. The amplification of the perturbations thus becomes possible also for fields conformally coupled to the 3-dimensional metric, unlike in the standard cosmological scenario [90].

The second difference concerns the growing character of the background curvature and of the couplings. In such a background the quantum fluctuations start evolving from an asymptotically flat, zero-coupling state, so that they can be naturally normalized to an initial vacuum fluctuation spectrum. After the amplification their final configuration thus corresponds to a "squeezed vacuum" state [354, 355, 351], and not to a "squeezed-thermal vacuum" [304], or to other, more complicated states. In addition, since the final energy distribution tends to follow (in frequency) the behaviour (in time) of the curvature scale, one typically obtains in such a context a growing (or "blue") spectrum [570, 3, 349, 12, 548, 356]. As a consequence, the peak amplitude of the produced radiation may be high enough to support a picture in which all the radiation filling our present Universe was directly produced

from the quantum fluctuations of the vacuum [601, 275]. This effect may facilitate the detection of a cosmic background of relic gravitational waves [295, 272], as will be discussed in Section 5, but it complicates the generation of a scale-invariant spectrum of curvature perturbations, eventually producing the observed CMB anisotropy (see Section 7).

A further consequence of the growth of the curvature scale, however, is that the comoving amplitude of the perturbations may even grow outside the horizon, instead of being frozen as in standard inflationary models. This effect, implicitly contained in the earlier pioneering studies of cosmological perturbations [570], was first explicitly pointed out in [2], and only later independently rediscovered in a string cosmology context [320] and in the context of scalar-tensor gravity [64]. Such a growth of the amplitude may require an appropriate and non-conventional choice of the gauge [115] in order to restore the validity of the linear approximation, and to allow the application of the standard perturbative formalism.

In the following subsections we will discuss in some detail the above effects, typical of the evolution of perturbations in the context of the pre-big bang scenario.

4.1 "Frame" independence

The computation of the spectrum according to the theory of cosmological perturbations [492] requires a series of formal steps, the first of which is the choice of the "frame", i.e. of the basic set of fields (metric included) used to parametrize the action. Any theory is in general characterized by a preferred set of fields, but one might wonder what happens when there are more than one preferred choices (this is a typical situation of all scalar-tensor models of gravity, as in the case of the gravidilaton string effective action).

In string theory there is a preferred frame (the so-called S-frame), in which the coupling to a constant dilaton is unambiguously fixed (at the tree level) for all fields and to all orders in the higher-derivative α' expansion. Also, the metric of this frame is the one to which fundamental strings are minimally coupled [551, 314] (in other words, the world-paths of strings freely evolving in a curved background are geodesic surfaces of this metric). In the S-frame, and in $d+1$ dimensions, the gravidilaton effective action takes the form

$$S = \frac{1}{2\lambda_{\rm s}^{d-1}} \int d^{d+1}x \sqrt{|g|}\, e^{-\phi} \left[-R + \omega \left(\nabla_\mu \phi\right)^2 \right] \tag{4.4}$$

(for the sake of generality we have included here the Brans–Dicke coefficient ω, which in our context could represent a dilaton self-coupling possibly arising from higher-loop corrections [346]; in particular, $\omega = -1$ for the lowest-order string effective action).

For any ω, however, this action can always be rewritten in the (more conventional) E-frame, in which the graviton and dilaton kinetic terms are diagonalized in the standard canonical form, and the dilaton is minimally coupled to the metric. Consider, in fact, the field redefinition $g \to \tilde{g}$ performed with the help of a new scalar variable ψ:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} e^{\psi}. \tag{4.5}$$

By expressing the scalar curvature R in terms of $\widetilde{g}$ [615, 575], the action (4.4) becomes

$$S = \frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|\widetilde{g}|}\, e^{-\phi+\frac{(d-1)}{2}\psi} \left[-\widetilde{R} + d\widetilde{\nabla}^2\psi + \frac{d(d-1)}{4}\left(\widetilde{\nabla}_\mu\psi\right)^2 + \omega\left(\widetilde{\nabla}_\mu\phi\right)^2 \right], \tag{4.6}$$

where the tilde denotes geometrical quantities computed with respect to $\widetilde{g}$. By setting ($\omega =$ const, $\omega > -d/(d-1)$):

$$(d-1)\psi = 2\phi, \qquad \widetilde{\phi} = \phi\left[\frac{2d + 2\omega(d-1)}{d-2}\right]^{1/2}, \tag{4.7}$$

and neglecting a total derivative, we are led eventually to the E-frame action of general relativity for the new variables $\widetilde{g}, \widetilde{\phi}$:

$$S = \frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|\widetilde{g}|} \left[-\widetilde{R} + \frac{1}{2}\left(\widetilde{\nabla}_\mu\widetilde{\phi}\right)^2 \right]. \tag{4.8}$$

Please note the "right" canonical coefficient in front of the dilaton kinetic term (we remember that we are using the signature $g_{00} > 0$) and the possible change of sign from the S-frame action (4.4) (where $\omega = -1$, to lowest order). The E-frame is thus preferred for performing canonical quantization, for identifying the particle content of the theory, and for defining the effective low-energy masses and couplings (see also Section 6).

However, the field equations for g, ϕ, obtained by varying the action (4.4), are different from the equations for $\widetilde{g}, \widetilde{\phi}$, obtained by varying the action (4.8). As a consequence, also the perturbation equations (4.3), in the S-frame, are different from the E-frame perturbation equations, and one might wonder which is the "right" frame to be used, when computing cosmological effects to be compared with present observations.

This is a false problem, however, because the (final) observable variables, such as the spectral amplitude, the spectral energy density (see Subsection 4.4), are exactly the same in both frames [320] (an obvious consequence of the fact that physical measurable quantities cannot be changed by field redefinitions such as the transformation (4.5); see also [19] for a recent discussion of the equivalence between the String and Einstein frames). The formal reason of this frame-independence is that the perturbation equations are indeed different in the two frames, but the classical solutions around which the expansion is performed are also different, and the two differences exactly compensate each other. This result can be generally proved by perturbing the action, better than the equations of motion (see Section 4.3). It is instructive, however, to give also a particular example, as will be done here by considering tensor metric perturbations propagating in a $(d+1)$-dimensional external background.

We will start the computation in the S-frame, where the background equations obtained from the action (4.4) can be written explicitly as

$$\begin{aligned} &R_\mu{}^\nu + \nabla_\mu\nabla^\nu\phi + (\omega+1)\left[\delta_\mu^\nu\left(\nabla_\alpha\phi\right)^2 - \delta_\mu^\nu\Box\phi - \nabla_\mu\phi\nabla^\nu\phi\right] = 0, \\ &R + \omega\left(\nabla_\alpha\phi\right)^2 - 2\omega\Box\phi = 0. \end{aligned} \tag{4.9}$$

We shall consider the transverse, traceless part of the metric perturbations,

$$\delta^{(1)}\phi = 0, \qquad \delta^{(1)} g_{\mu\nu} = h_{\mu\nu}, \qquad \delta^{(1)} g^{\mu\nu} = -h^{\mu\nu}, \qquad \nabla_\nu h_\mu{}^\nu = 0 = h_\mu{}^\nu, \tag{4.10}$$

propagating in a higher-dimensional, spatially flat, factorizable background with d "external" and n "internal" dimensions, with scale factors $a(t)$ and $b(t)$, respectively. Also, we shall assume that the translations along the internal dimensions are isometries of the full, perturbed metric (see [327] for a more general situation in which also the internal gradients of the fluctuations are non-zero, and see the comment after Eq. (4.25) for backgrounds with non-factorizable geometry). The fluctuations of the external space $h_{ij}(x_i, t)$ are then conveniently described in the synchronous gauge [425], where $(i, j = 1, ..., d;\ m, n = d+1, ..., d+n)$:

$$\begin{aligned} &g_{00} = 1, \qquad g_{0i} = 0, \qquad g_{ij} = -a^2\delta_{ij}, \quad g_{mn} = -b^2\delta_{mn}, \\ &h_{00} = 0, \qquad h_{0i} = 0, \qquad g^{ij}h_{ij} = 0, \qquad \partial_j h_i{}^j = 0. \end{aligned} \tag{4.11}$$

In this gauge

$$\begin{aligned} &\delta^{(1)}\Gamma_{0i}{}^j = \frac{1}{2}\dot{h}_i{}^j, \qquad \delta^{(1)}\Gamma_{ij}{}^0 = -\frac{1}{2}\dot{h}_{ij}, \\ &\delta^{(1)}\Gamma_{ij}{}^k = \frac{1}{2}\left(\partial_i h_j{}^k + \partial_j h_i{}^k - \partial^k h_{ij}\right); \end{aligned} \tag{4.12}$$

to first order in h, the perturbation of the dilaton equation is found to be trivially satisfied, and there are no contributions from the terms inside the square brackets in (4.9). The perturbation equations are thus (remarkably) ω-independent [299]:

$$\delta^{(1)} R_\mu{}^\nu - \left(\delta^{(1)} g^{\nu\alpha}\Gamma_{\mu\alpha}{}^0 + g^{\nu\alpha}\delta^{(1)}\Gamma_{\mu\alpha}{}^0\right)\dot{\phi} = 0. \tag{4.13}$$

For $\mu, \nu \neq i, j$, they are trivially satisfied. The (i, j) components of the Ricci tensor, on the other hand, lead to the higher-dimensional covariant d'Alembert operator [296, 208, 299]

$$\delta^{(1)} R_i{}^j = -\frac{1}{2}\left(\ddot{h}_i{}^j + d\frac{\dot{a}}{a}\dot{h}_i{}^j + n\frac{\dot{b}}{b}\dot{h}_i{}^j - \frac{\nabla^2}{a^2}h_i{}^j\right) \equiv -\frac{1}{2}\Box h_i{}^j = 0, \tag{4.14}$$

where $\nabla^2 = \partial_i^2$ and the indices of h are raised and lowered with the unperturbed metric. Using the identities (see also [277])

$$\begin{aligned} &g^{jk}\dot{h}_{ik} = \dot{h}_i{}^j + 2Hh_i{}^j, \\ &g^{jk}\ddot{h}_{ik} = \ddot{h}_i{}^j + 2\dot{H}h_i{}^j + 4H\dot{h}_i{}^j + 4H^2 h_i{}^j, \end{aligned} \tag{4.15}$$

we are finally led to the S-frame tensor perturbation equation [299]

$$\Box h_i{}^j - \dot{\phi}\dot{h}_i{}^j = 0. \tag{4.16}$$

In terms of the conformal time coordinate η, such that $dt = ad\eta$, this wave equation can also be rewritten (for each polarization mode h) as follows,

$$h'' + (d-1)\frac{a'}{a}h' + n\frac{b'}{b}h' - \phi' h' - \nabla^2 h = 0. \tag{4.17}$$

Let us now repeat the computation in the E-frame, where the background equations following from the action (4.8) can be written as

$$\widetilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\widetilde{\phi}\partial_\nu\widetilde{\phi}, \qquad \widetilde{g}^{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\widetilde{\phi} = 0. \tag{4.18}$$

Perturbing to first order, and using Eqs. (4.10):

$$\delta^{(1)}\widetilde{R}_\mu{}^\nu = 0 \tag{4.19}$$

(the perturbation of the scalar Klein–Gordon equation is trivially satisfied). In the synchronous gauge, we can still apply Eq. (4.14) (for the "tilded" variables), and we are finally led to the E-frame perturbation equation

$$\widetilde{\Box}\widetilde{h}_i{}^j = 0. \tag{4.20}$$

In conformal time, and for each polarization component $\widetilde{h}$,

$$\widetilde{h}'' + \left[(d-1)\frac{\widetilde{a}'}{\widetilde{a}} + n\frac{\widetilde{b}'}{\widetilde{b}}\right]\widetilde{h}' - \nabla^2\widetilde{h} = 0. \tag{4.21}$$

This last equation seems to be different from the S-frame equation (4.17). We have to recall, however, that the conformal time is the same in both frames, $d\eta = dt/a = d\widetilde{t}/\widetilde{a} = d\widetilde{\eta}$ (see Section 2.4), and that, according to Eq. (4.5),

$$\widetilde{a} = ae^{-\phi/(d+n-1)}, \quad \widetilde{b} = be^{-\phi/(d+n-1)}, \quad (d-1)\frac{\widetilde{a}'}{\widetilde{a}} + n\frac{\widetilde{b}'}{\widetilde{b}} = (d-1)\frac{a'}{a} + n\frac{b'}{b} - \phi'. \tag{4.22}$$

We thus have the same equation for h and $\widetilde{h}$, the same solution and, as a consequence, the same spectrum when the solution is expanded in Fourier modes.

In other words, the *observable* results of the perturbative analysis are "frame-independent" and, when computing the spectrum, we are allowed to use the frame that is most convenient for practical purposes.

4.2 Choice of the "gauge" for scalar perturbations

Within each frame, we have a possible additional ambiguity due to the choice of the gauge, i.e. the choice of the parametrization of the given metric manifold. The step corresponding to this choice is more delicate than the choice of the frame: in spite of the fact that the final observable spectrum has to be certainly independent of the given coordinate system, the perturbative analysis *is*, on the contrary, "gauge-dependent". It is possible, for instance, that a linearized description of the perturbations is valid in a given system of coordinates, and yet that the linear approximation is broken in a different system. This problem obviously disappears in the context of a fully covariant and gauge-invariant (to all orders)

description of perturbations [234, 112]. For the perturbative approach usually adopted in a cosmological context [492], however, gauge invariance holds in the linear approximation only, and breaks down at higher orders.

An important example of this effect is provided, in string cosmology, by the time evolution of the scalar (metric and dilaton) perturbations. We will discuss this effect in the E-frame (dropping the "tilde", for simplicity), starting from the unperturbed equations (4.18), and using, as in the previous subsection, a homogeneous, Bianchi-I-type, higher-dimensional background, with $d+n$ factorized structure:

$$g_{\mu\nu} = \mathrm{diag}(a^2, -a^2\delta_{ij}, -b^2\delta_{mn}), \qquad \phi = \phi(\eta). \tag{4.23}$$

We shall consider, in particular, the exact background solution parametrized by

$$\begin{aligned} &a = (|\eta|)^{\alpha}, \qquad b = (|\eta|)^{\beta}, \qquad \phi = \sqrt{\frac{2}{d+n-1}}\,\frac{n-d-\sqrt{d+n}}{1+\sqrt{d+n}}\,\ln(|\eta|), \\ &\alpha = \frac{\sqrt{d+n}+1-2n}{(1+\sqrt{d+n})(d+n-1)}, \qquad \beta = \frac{\sqrt{d+n}-1+2d}{(1+\sqrt{d+n})(d+n-1)}, \end{aligned} \tag{4.24}$$

and corresponding to the conformal-time parametrization of the vacuum solution already introduced in Eq. (2.114). For $\eta < 0$, $\eta \to 0_-$, this unperturbed solution describes a phase of accelerated evolution, growing curvature and growing dilaton, in which the horizon expands more slowly (to be more precise, shrinks faster) than the scale factor. It is thus a good background candidate to amplify metric fluctuations. In the S-frame it describes a phase of superinflation and dynamical dimensional reduction, in which d dimensions are expanding and n dimensions are contracting, with scale factors related by the duality transformation $b = a^{-1}$.

Let us perturb the metric and the dilaton around this solution, by using the isometries of the factorizable geometry (4.23) and assuming that all dynamical variables depend only on the "external" coordinates x_i, $i = 1, ..., d$ (so that the translations along the internal dimensions are also isometries of the full, perturbed background) [300]. In this case, modes with different rotational transformation properties are decoupled, and the scalar component of the background perturbations can be written in general as [492, 300]:

$$\begin{aligned} &\delta^{(1)}\phi = \chi, \qquad ds^2 = \left(g_{\mu\nu} + \delta^{(1)}g_{\mu\nu}\right)dx^\mu dx^\nu = \\ &a^2\,(1+2\varphi)\,d\eta^2 - a^2\left[(1-2\psi)\,dx_i^2 + 2\partial_i\partial_j E dx^i dx^j + 2\partial_i B dx^i d\eta\right] - b^2(1-2\xi)dx_m^2, \end{aligned} \tag{4.25}$$

where all variables depend only on η and x^i. It should be noted that this approach is appropriate to the standard Kaluza–Klein scenario, but it cannot be applied to non-factorized geometrical structures such as those appearing in the context of the Randall–Sundrum scenario [530]. In this last case, the study of scalar metric and dilaton fluctuations (see for

instance [99, 336]) requires a more general metric decomposition [596]. In this report, however, we will restrict our discussion to the perturbations of higher-dimensional Kaluza–Klein backgrounds of the standard, factorizable type.

In our case, it is important to stress that the six functions $\varphi,\ \psi,\ E,\ B,\ \xi,\ \chi$ are *not* invariant under local infinitesimal transformations of coordinates,

$$x^\mu \to x^\mu + \epsilon^\mu(x), \qquad g_{\mu\nu} \to g_{\mu\nu} - \nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu. \tag{4.26}$$

However, if we consider infinitesimal coordinate transformations depending on two scalar parameters ϵ^0 and ϵ,

$$\epsilon^\mu = (\epsilon^0, \partial^i \epsilon, 0), \qquad \eta \to \eta + \epsilon^0, \qquad x^i \to x^i + \partial^i \epsilon, \tag{4.27}$$

and thus preserving the scalar character of the perturbations (in other words, coordinate transformations that do not add vector or tensor components to the perturbed metric), then the scalar variables transform as

$$\begin{aligned} &\varphi \to \varphi - \frac{a'}{a}\epsilon^0 - \epsilon^{0\prime}, \qquad \psi \to \psi + \frac{a'}{a}\epsilon^0, \qquad \xi \to \xi + \frac{b'}{b}\epsilon^0 \\ &E \to E - \epsilon, \qquad B \to B + \epsilon^0 - \epsilon', \qquad \chi \to \chi - \phi'\epsilon^0, \end{aligned} \tag{4.28}$$

and it is always possible to define a set of variables that are gauge-invariant, in the linear approximation. For instance [300]:

$$\begin{aligned} &\Phi = \varphi + \frac{1}{a}[(B - E')a]', \qquad \Psi = \psi - \frac{a'}{a}(B - E'), \\ &\Xi = \xi - \frac{b'}{b}(B - E'), \qquad X = \chi + \phi'(B - E'). \end{aligned} \tag{4.29}$$

The first two variables (also called Bardeen potentials [53, 553]) are phenomenologically important, since their spectral amplitudes Φ_k, Ψ_k, at the recombination era are directly related (via the Sachs–Wolfe effect) to the large-scale CMB anisotropy $(\Delta T/T)_k$, currently observed by astrophysical experiments (see Section 7).

A natural and convenient choice of the gauge is thus the so-called longitudinal (or conformally Newtonian) gauge, in which $E = 0 = B$, and the perturbed metric depends directly on the Bardeen potentials, according to Eqs. (4.29). In this gauge the coordinates are totally fixed because, according to the variation of the metric under the transformation (4.28), the choice $E = 0 = B$ completely fixes ϵ^0 and ϵ^i, and leaves no residual degrees of freedom.

In this gauge we can now write explicitly the set of perturbed field equations

$$\begin{aligned} &\delta^{(1)} R_{\mu\nu} = \partial_\mu \phi \partial_\nu \chi, \\ &\delta^{(1)} g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + \nabla^2 \chi - g^{\mu\nu} \delta^{(1)} \Gamma_{\mu\nu}{}^\alpha \partial_\alpha \phi = 0, \end{aligned} \tag{4.30}$$

obtained from the unperturbed Einstein equations (4.18). From the $(i, j \neq i)$ component of the perturbation equations we obtain a useful relation between three perturbation variables,

$$\varphi = (d-2)\psi + n\xi. \tag{4.31}$$

The $(0, i)$ components give a constraint, while the (i, i) and (m, m) components of the perturbed Einstein equations, combined with the $(0, 0)$ component, provide the following interesting system of coupled equations for the "external" and "internal" perturbations ψ and ξ [300, 326]:

$$\begin{aligned}
&(d-1)\left\{\Box\psi + \psi'\left[3(d-1)\frac{a'}{a} + 3n\frac{b'}{b}\right]\right\} = -n\left\{\Box\xi + \xi'\left[3(d-1)\frac{a'}{a} + 3n\frac{b'}{b}\right]\right\}, \\
&d\left\{\Box\psi + \psi'\left[3(d-1)\frac{a'}{a} + \frac{b'}{bd}\left(2(d-1)(n-1) + nd\right)\right]\right\} = \\
&= -(n-1)\left\{\Box\xi + \xi'\left[\frac{a'}{a(n-1)}\left(3dn - d - n + 1\right) + 3n\frac{b'}{b}\right]\right\},
\end{aligned} \tag{4.32}$$

where $\Box \equiv (\partial^2/\partial\eta^2) - \nabla^2$ denotes here the usual (flat-space) d'Alembert operator.

This system can easily be diagonalized to find the (time-dependent) linear combination of ψ and ξ that represents the true "propagation eigenstates". For our purpose, however, the asymptotic behaviour of the modes ψ_k, ξ_k can be simply obtained (modulo logarithmic corrections) by inserting into the previous system the power-law ansatz

$$\psi_k = A(-\eta)^x, \qquad\qquad \xi_k = B(-\eta)^x. \tag{4.33}$$

For the background solution (4.24) one then finds that there are, in the $|k\eta| \to 0$ limit, non-trivial solutions for the coefficients A and B only if $x = 0$ or $x = -2$. This means that, asymptotically,

$$\psi_k = A_1 + \frac{A_2}{\eta^2}, \qquad\qquad \xi_k = B_1 + \frac{B_2}{\eta^2}, \tag{4.34}$$

and the amplitude is not frozen, unlike in the standard inflationary scenario. Because of the growing mode, $\psi_k \sim \eta^{-2}$, the scalar perturbation amplitude blows up in the limit $\eta \to 0_-$. Otherwise stated, at any given time η there is always a low enough frequency band for which the typical fluctuations have a dimensionless amplitude much larger than 1 [115] (see Eq. (4.75) in the next subsection). This breaks the validity of the linear approximation and is not consistent, in general, with the perturbative expansion around a homogeneous background.

This conclusion cannot be avoided, be it by moving to a different frame (as discussed in the previous subsection), or by changing the number of dimensions, since the previous result is dimensionality-independent. For a $d = 3$ isotropic background, in particular, one finds in the longitudinal gauge $\varphi = \psi$, and the Fourier modes $\nabla^2\psi_k = -k^2\psi_k$ satisfy the Bessel equation

$$\boxed{\psi_k'' + \frac{3}{\eta}\psi_k' + k^2\psi_k = 0} \tag{4.35}$$

(from Eq. (4.32)). The exact solution, in the limit $|k\eta| \to 0$, has the asymptotic expansion [5]

$$\varphi_k = c_1 \ln|k\eta| + \frac{c_2}{\eta^2} \tag{4.36}$$

(c_1, c_2 are integration constants), with the same growing mode $\sim \eta^{-2}$ as in higher dimensions.

It is important to point out, however, that such a growing mode is absent for tensor metric perturbations. If we consider the tensor perturbation equation (4.21), in the same background (4.24), we obtain indeed for each Fourier mode the Bessel equation

$$h_k'' + \frac{1}{\eta} h_k' + k^2 h_k = 0, \tag{4.37}$$

with the asymptotic solution

$$h_k = c_1 + c_2 \ln|k\eta|, \qquad |k\eta| \to 0_-. \tag{4.38}$$

All modes tend to stay constant, asymptotically, modulo a logarithmic growth that can be easily kept under control if the phase of accelerated evolution is not infinitely extended in the high-curvature regime but is bounded, for instance, by the Planck (or string) scale (as expected).

This is a first signal that the growing mode of scalar perturbations, appearing in the longitudinal gauge, might be a pure gauge effect, i.e. an artefact of the particular choice of coordinate system. This suspicion is confirmed by the application of the "fluid flow" approach [368, 433] to the perturbations of a scalar-tensor background. In this approach, the evolution of density and curvature inhomogeneities can be described in terms of covariant scalar variables, which are gauge-invariant to all orders [112].

Let us consider, for simplicity, the isotropic case $d = 3$, $n = 0$. There are two covariant variables, Δ and C, defined in terms of the momentum density of the scalar field $\nabla\phi$, of the spatial curvature ${}^{(3)}R$, and of their derivatives. By expanding around the dilaton-driven background (4.24), which in our case reduces to

$$a = (-\eta)^{1/2}, \qquad \phi = -\sqrt{3}\ln(-\eta), \qquad -\infty < \eta < 0, \tag{4.39}$$

one finds for such variables, in the linear approximation, the asymptotic solution ($|k\eta| \ll 1$) [115]:

$$\Delta_k = \text{const}, \qquad c_k = \text{const} + A_k \ln|k\eta|. \tag{4.40}$$

Such variables tend to stay constant outside the horizon, with at most a logarithmic variation (as in the tensor case), which is not dangerous. Indeed, in terms of Δ and C, the amplitude of density and curvature fluctuations can be consistently computed using the linear approximation (for all modes), and their spectral distribution (normalized to an initial vacuum spectrum, see the next subsection) turns out to be exactly the same as the tensor spectral distribution (4.38), which is bounded.

The same result, on the other hand, can also be directly obtained in the longitudinal gauge simply by neglecting the growing mode of Eq. (4.36), as noted in [321]. This suggests that it should be possible to move to a more appropriate gauge, in which the growing mode is suppressed, to restore the validity of the linear approximation.

This is exactly what happens in the off-diagonal gauge (also called "uniform-curvature" gauge [381]), defined by $\psi = 0 = E$,

$$ds^2 = a^2 \left[(1+2\varphi)\, d\eta^2 - dx_i^2 - 2\partial_i B dx^i d\eta \right], \tag{4.41}$$

which represents another complete choice of coordinates, with no residual degrees of freedom (just as the longitudinal gauge). In this gauge there are two variables for scalar perturbations, φ and B, and the perturbation equations are solved, asymptotically, by

$$\varphi_k = c_1 + c_2 \ln|k\eta|, \qquad B_k = c_1 \eta \ln|k\eta| + \frac{c_2}{\eta}. \tag{4.42}$$

The growing mode is thus completely absent for homogeneous perturbations (since it is $\partial_i B$ that contributes to the perturbed metric). The growing mode is still present in the off-diagonal part of the metric for non-homogeneous perturbations, but it is suppressed with respect to the longitudinal gauge, since $\delta^{(1)} g \sim k B_k \sim k\eta\psi_k \sim \eta^{-1}$, instead of $\psi_k \sim \eta^{-2}$. The lower is k the stronger is the suppression, and this is enough for the linear approximation to be valid, as explicitly checked with a computation up to second-order corrections [115]. This off-diagonal gauge is in general useful for the study of scalar perturbations also in the context of generalized scalar-tensor models of gravity and of the higher-loop string effective action, as discussed for instance in [383].

Even better, one may choose a gauge in which $\psi = 0 = B$. In this gauge, E is related to the Bardeen potential Ψ by $E \sim \eta^2 \Psi$, according to Eq. (4.29). On the other hand, the gauge-invariant Bardeen potential evolves asymptotically (in all gauges) like the longitudinal variable ψ, i.e. like η^{-2}. Since E enters the metric perturbations with two spatial derivatives, it follows that, in this gauge, $\delta^{(1)} g_{ij} \sim (k\eta)^2 \Psi_k$, which is asymptotically frozen, and sufficiently small (at small $|k\eta|$) for the linear approximation to be valid.

To conclude this subsection, let us stress that the asymptotic growth of perturbations is a typical problem of string cosmology, with no counterpart in the standard inflationary scenario. Consider, for instance, a $d = 3$, conformally flat and accelerated background, which can be conveniently parametrized in the negative range of the conformal time coordinate by a generic power-law scale factor, as follows:

$$a = (-\eta)^\alpha, \qquad \eta < 0. \tag{4.43}$$

For α ranging over the whole real axis we can represent in this way all classes of inflationary backgrounds introduced in Section 1.3, and reported in Table 1. Consider then the E-frame perturbation equation (4.21), which in this case reduces to the Bessel equation

$$h_k'' + \frac{2\alpha}{\eta} h_k' + k^2 h_k = 0, \tag{4.44}$$

$\alpha = -1$	de Sitter	$\dot{a} > 0, \ddot{a} > 0, \dot{H} = 0$
$\alpha < -1$	power-inflation	$\dot{a} > 0, \ddot{a} > 0, \dot{H} < 0$
$-1 < \alpha < 0$	superinflation	$\dot{a} > 0, \ddot{a} > 0, \dot{H} > 0$
$\alpha > 0$	accelerated contraction	$\dot{a} < 0, \ddot{a} < 0, \dot{H} < 0$

Table 1: Four classes of accelerated backgrounds, parametrized by $a = (-\eta)^{\alpha}$.

with asymptotic solution (for $|k\eta| \to 0$):

$$h_k = A + B \int^{\eta} \frac{d\eta'}{a^2(\eta')} = A + B\,|\eta|^{1-2\alpha}\,. \tag{4.45}$$

The amplitude tends to stay constant for $\alpha < 1/2$, while it tends to grow for $\alpha > 1/2$ (for $\alpha = 1/2$ the growth is simply logarithmic). In the E-frame the growing mode thus appears only for a phase of accelerated contraction (see Table 1), and it is absent in the standard inflationary context where the background is always expanding (even in the E-frame).

In the pre-big bang scenario, on the contrary, the comoving amplitude of metric fluctuations may be growing, in the standard gauges, even in the case of tensor perturbations. In some cases the validity of the linear approximation may be restored in an appropriate gauge, as for the scalar case discussed in this subsection. If this is impossible, the growth of perturbations is a physical effect, signalling a (quantum) instability of the given background [406, 408]. In any case, however, the energy spectrum of the perturbations can always be consistently estimated by truncating the comoving amplitude to the frozen part of the asymptotic solution [118]. This is a consequence of a duality property of the perturbation equations that will be discussed in Subsection 4.5.

4.3 Canonical variables and normalization

The linearized equations for the classical evolution of the perturbations have been obtained, in the previous subsection, by perturbing directly (to first order) the equations of motion of the background fields. The same equations, however, can also be obtained with a different approach, by expanding the action up to terms quadratic in the first-order fluctuations:

$$g \to g + \delta^{(1)} g, \qquad \delta^{(2)} S \equiv S\left[\left(\delta^{(1)} g\right)^2\right]. \tag{4.46}$$

normal modes, defined

The variation of the perturbed action with respect to $\delta^{(1)}g$ gives in fact a set of linearized equations, identical to Eqs. (4.3).

This second method is longer, in general, but has an advantage: the diagonalization of the quadratic perturbed action $\delta^{(2)}S$ defines the "normal modes" of oscillation for the total system formed by the metric plus the matter sources. The so-called normal modes are the variables that diagonalize the kinetic part of the action and that, once quantized, satisfy canonical commutation relations [492]. They are required to normalize the initial amplitude of the perturbations to a quantum spectrum of zero-point (vacuum) fluctuations, and to study their amplification in the course of the cosmological evolution.

Let us illustrate this point starting with the case of tensor metric perturbations. The second-order action for tensor perturbations in a cosmological background was first written in [260], in the E-frame and in $d = 3$ dimensions (see also [492, 16]). Such a result can be easily generalized to the S-frame [277] and to the $(d+1)$-dimensional, Bianchi-I-type metric background used in the previous subsections, starting from the unperturbed, low-energy gravidilaton string effective action (in units $2\lambda_s^{d-1} = 1$):

FLUCT. of metric

$$S = -\int d^{d+n+1}x\sqrt{-g}e^{-\phi}\left[R + (\partial_\mu\phi)^2\right], \tag{4.47}$$

and considering the transverse, trace-free variable $h_{\mu\nu}$, defined by the first-order perturbation (4.10). As already discussed, the fluctuations of the d-dimensional external space are conveniently described in the synchronous gauge of Eq. (4.11). In this gauge, we expand to order h^2 the controvariant components of the metric,

$$\delta^{(1)}g^{\mu\nu} = -h^{\mu\nu}, \qquad \delta^{(2)}g^{\mu\nu} = h^{\mu\alpha}h_\alpha{}^\nu, \tag{4.48}$$

FLUCTUATIONS of metric

the volume density,

$$\delta^{(1)}\sqrt{-g} = 0, \qquad \delta^{(2)}\sqrt{-g} = -\frac{1}{4}\sqrt{-g}h_{\mu\nu}h^{\mu\nu}, \tag{4.49}$$

and so on for $\delta^{(1)}R_{\mu\nu}$, $\delta^{(2)}R_{\mu\nu}$ (see [277] for the details of the explicit computation). The second-order perturbed action has then contributions from $\sqrt{-g}$, R and $g^{\mu\nu}$:

$$\delta^{(2)}S = -\int d^{d+n+1}xe^{-\phi}\left[\delta^{(2)}\left(\sqrt{-g}R\right) + \delta^{(2)}\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)\right]. \tag{4.50}$$

Integrating by parts, using the unperturbed equations of motion, and neglecting total derivative terms, we finally arrive at the quadratic action:

2nd order fluctuation action

$$\delta^{(2)}S = \frac{1}{4}\int d^{d+n+1}xa^d b^n e^{-\phi}\left(\dot{h}_i^j\dot{h}_j^i + h_i^j\frac{\nabla}{a^2}h_j^i\right). \tag{4.51}$$

Decomposing the fluctuations of the d-dimensional spatial metric into the two physical polarization modes h_+, $h_\times$,

$$h_i^j h_j^i = 2\left(h_+^2 + h_\times^2\right), \tag{4.52}$$

decomposition of FLUCTUATIONS

We are at the heart of question !!!!!!!!!!!!!!!

4.53 IS the FUND. description of "VAC FLUCTUATIONS" pump field = dilaton +moduli

scalar h

and introducing the conformal time coordinate $d\eta = adt$, we eventually obtain,for each mode, the effective scalar action

$$\delta^{(2)} S_h = \frac{1}{2}\int d\eta a^{d-1} b^n e^{-\phi}\left(h'^2 + h\nabla^2 h\right), \tag{4.53}$$

dilaton

where h is now a scalar variable standing for either of the polarization amplitudes h_+, $h_\times$. The variation with respect to h now gives exactly Eqs. (4.17), i.e. the S-frame tensor perturbation equation (the sameas was obtained by direct perturbation of the equations of motion).

moduli

The above action describes a scalar field non-minimally coupled to a time-dependent external field (also called "pump field"), represented in this case by the dilaton and by the internal and external scale factors (the so-called "moduli"). In order to impose the correct normalization to a quantum spectrum of vacuum fluctuations we need, however, the canonical variable that diagonalizes the kinetic part of the action and describes asymptotically a freely oscillating field [212, 352, 492]. In our case such a variable u is defined by [299]

Canon. variable for diag. of Kin.

$$u = zh, \qquad z = a^{(d-1)/2} b^{n/2} e^{-\phi/2}. \tag{4.54}$$

In fact, for each Fourier mode u_k, we get from Eq. (4.53), after integration by parts, an effective action in which the kinetic term appears in the standard canonical form:

4.53

$$\delta^{(2)} S_u(k) = \frac{1}{2}\int d\eta\left(|u_k'|^2 - k^2|u_k|^2 + \frac{z''}{z}|u_k|^2\right), \tag{4.55}$$

and which leads to the canonical evolution equation:

effective potential/ pump field

$$u_k'' + \left[k^2 - V(\eta)\right] u_k = 0, \qquad V(\eta) = \frac{z''}{z}, \tag{4.56}$$

with an effective potential $V(\eta)$ depending only on the external pump field. An accelerated background, in particular, has an effective potential that goes to zero as $\eta \to -\infty$. We thus obtain, asymptotically, the free-field equation

$$u_k'' + k^2 u_k = 0, \qquad \eta \to -\infty, \tag{4.57}$$

and the variable u_k can be normalized to an initial-vacuum fluctuation spectrum,

$$u_k = \frac{1}{\sqrt{2k}} e^{-ik\eta}, \qquad \eta \to -\infty, \tag{4.58}$$

so as to satisfy canonical commutation relations $[u_{k_1}, u_{k_2}^{*\prime}] = i\delta_{k_1 k_2}$. The normalization of the canonical variable u_k automatically fixes the normalization of the metric fluctuations, $h_k = u_k/z$.

It is important to stress that the procedure used to introduce the diagonalized action (4.55) can be exactly repeated in the E-frame. The only difference in this new frame is that

VAC FLUCTUATIONS

there is no dilaton coupling $e^{-\phi}$ in front of the Lagrangian density, and the pump field is simply determined by the E-frame scale factors, $\widetilde{z} = \widetilde{\psi}/\widetilde{h} = \widetilde{a}^{(d-1)/2}\widetilde{b}^{n/2}$. But, according to the transformation rule (4.22), $\widetilde{z} = z$. The effective potential z''/z is thus the same in the two frames, the time evolution of ψ and $\widetilde{\psi}$ is the same and, as a consequence, we find the same spectrum in both frames, as anticipated in Subsection 4.1.

In the case of scalar metric perturbations, the computations are more complicated, in practice, but the procedure is exactly the same in principle. The scalar canonical variable has been computed for a $d = 3$ isotropic background coupled to various types of sources: a perfect fluid [455, 159], one scalar field [554, 490, 571], two scalar fields [213], N scalar fields [28]. Working in the E-frame (and omitting the "tilde", for simplicity) the scalar perturbation of the gravidilaton action (4.8) can be easily performed in the longitudinal gauge, where $\delta^{(1)}\phi = \chi$, $\delta^{(1)}g_{00} = 2a^2\varphi$, $\delta^{(1)}g_{ij} = 2a^2\psi\delta_{ij}$. One then obtains the diagonalized second-order action [492]:

$$\delta^{(2)}S_v(k) = \frac{1}{2}\int d\eta\left(|v_k'|^2 - k^2|v_k|^2 + \frac{z''}{z}|v_k|^2\right), \tag{4.59}$$

where the canonical variable v is defined by

$$v = a\chi + z\psi, \qquad\qquad z = \frac{a^2\phi'}{a'}, \tag{4.60}$$

and is gauge-invariant, as can be explicitly checked by using the infinitesimal transformations (4.28).

This result can be easily generalized to the scalar fluctuations of the higher-dimensional, anisotropic background considered in the previous subsections, and parametrized as in Eq. (4.25). After the diagonalization, the higher-dimensional action for the Fourier modes of scalar perturbations can be written as [300]

$$\delta^{(2)}S_{v,w}(k) = \frac{1}{2}\int d\eta\left(|v_k'|^2 - k^2|v_k|^2 + \frac{z''}{z}|v_k|^2 + |w_k'|^2 - k^2|w_k|^2 + \frac{z''}{z}|w_k|^2\right) \tag{4.61}$$

(modulo a total derivative that does not contribute to the equations of motion). There are two gauge-invariant canonical variables, v and w, which in the longitudinal gauge ($E = 0 = B$) are defined by

$$\begin{aligned} v &= a^{(d-1)/2}b^{n/2}\chi + z\left(\psi + \frac{n}{d-1}\xi\right) \\ w &= z\left[\frac{n(n+d-1)}{(d-1)}\right]^{1/2}\left(\frac{a'}{a\phi'}\xi - \frac{b'}{b\phi'}\psi\right), \end{aligned} \tag{4.62}$$

with the generalized pump field

$$z = \frac{a^{(d-1)/2}b^{n/2}\phi'}{\frac{a'}{a} + \frac{nb'}{(d-1)b}}. \tag{4.63}$$

The definition of the canonical variable and the correct normalization of the perturbations are necessary ingredients to study the growth of the quantum fluctuations, and to determine whether they backreaction may become eventually so large as to destroy the initial homogeneity of a pre-big bang solution (see also Section 3). It is the canonical normalization, in particular, that leads to the standard relation between the amplitude of quantum fluctuations and the curvature scale of the background geometry.

CASE: tensor perturb.

In order to illustate this important point let us consider tensor perturbations in a $d = 3$, isotropic, E-frame metric, evolving in time aymptotically according to Eq. (4.45). The classical perturbation equation (4.44) only determines the time dependence of the fluctuations. The canonical action, however, determines not only the k-dependence through the normalization of u_k, but also the correct canonical dimensions of the fluctuations, $[u] = M$. By inserting in fact the required dimensional factors in the (E-frame) perturbed action,

$$\delta^{(2)} S_h = \frac{1}{32\pi G} \int d\eta a^2 \left(h'^2 + h \nabla^2 h \right), \tag{4.64}$$

we obtain the scalar field action (4.55) with canonical dimensions by setting, for each Fourier mode,

$$h_k(\eta) = \frac{\sqrt{2}}{M_{\rm P}} \frac{u_k(\eta)}{a(\eta)} \tag{4.65}$$

(recall that $8\pi G = M_{\rm P}^{-2}$ with our conventions).

The typical fluctuation amplitude over a comoving length scale r, on the other hand, is determined by the two-point correlation function

$$\xi(\vec{r}) = \langle h(\vec{x}) h(\vec{x} + \vec{r}) \rangle, \tag{4.66}$$

where the brackets denote either a quantum expectation value, if we work in the formalism in which the perturbations are quantized and expanded into annihilation and creation operators (see Subsection 4.4), or a macroscopic ensemble averaged over a spatial volume, if we work in the classical limit. In any case we consider real perturbations ($h_k = h^*_{-k}$), satisfying the isotropy condition (i.e. $|h_k|$ is a function of $k = |\vec{k}|$ only) and the stochastic condition

$$\langle h(\vec{k}) h(-\vec{k}') \rangle = (2\pi)^3 \delta^3(k - k') |h(k)|^2 \tag{4.67}$$

(which is a consequence of their quantum origin). An explicit computation then gives [492, 278]

$$\xi(r) \sim \int \frac{dk}{k} \frac{\sin kr}{kr} |\delta_h(k)|^2, \tag{4.68}$$

where $|\delta_h(k)|$ represents the typical, dimensionless amplitude of tensor fluctuations over a comoving length scale $r = k^{-1}$:

$$|\delta_h(k)| \equiv k^{3/2} |h_k| \sim \left[\xi^{1/2}(r) \right]_{r = k^{-1}}. \tag{4.69}$$

The amplitude of the quantum fluctuations of the metric is now fixed by Eq. (4.65) and by the normalization of the canonical variable, Eq. (4.58). Initially, for $\eta \to -\infty$, all fluctuations oscillate inside the horizon, and

$$|\delta_h(k)| = \frac{k}{aM_{\rm P}} = \frac{\omega}{M_{\rm P}}, \tag{4.70}$$

i.e. the amplitude of a mode is proportional to its proper frequency $\omega = k/a$ (and then it is adiabatically decreasing, in an expanding background). In the opposite regime $\eta \to 0_-$, where the fluctuations are well outside the horizon of an inflationary background with $a \sim |\eta|^\alpha$, we have to use the asymptotic solution (4.45), and we may distinguish three possibilities.

- If $\alpha < 1/2$, then h is frozen, while the canonical variable grows adiabatically, following the growth of the pump field $z = a$ (see Eq. (4.45)): $u_k \sim a(u_k/a)_{\rm hc}$, where "hc" denotes the time of horizon crossing, $|k\eta| \simeq 1$). The final amplitude can thus be expressed in terms of the background curvature scale at the crossing time $|\eta| \simeq 1/k$:

$$|\delta_h(k)| \simeq \frac{k}{M_{\rm P} a_{\rm hc}} \simeq \frac{1}{M_{\rm P}(a\eta)_{\rm hc}} \simeq \left(\frac{H}{M_{\rm P}}\right)_{\rm hc} = \left(\frac{H_1}{M_{\rm P}}\right) |k\eta_1|^{1+\alpha}, \tag{4.71}$$

  where H_1 is some reference scale (a convenient choice is the final curvature scale at the end of the inflationary phase), and α is the conformal power of the scale factor. The spectral amplitude δ_h is frozen in time, and it is then clear from the above equation why pre-big bang models, characterized by a growing (in time) curvature scale ($\alpha + 1 > 0$, see Table 1), are associated to a growing (in frequency) tensor perturbation spectrum.

- If $\alpha > 1/2$, then h is growing according to Eq. (4.45) and, as a consequence, the spectral amplitude δ_h grows not only in frequency but also in time:

$$|\delta_h(k)| \simeq \left(\frac{H}{M_{\rm P}}\right)_{\rm hc} |k\eta|^{1-2\alpha}. \tag{4.72}$$

case: alpha = 1/2

- The low-energy, dilaton-driven pre-big bang solution of Eq. (4.39) corresponds to the limiting case $\alpha = 1/2$. In that case the amplitude of tensor perturbations only acquires a mild, logarithmic dependence [115, 116]:

$$|\delta_h(k)| \simeq \left(\frac{H}{M_{\rm P}}\right)_{\rm hc} |\ln|k\eta|| = \left(\frac{H_1}{M_{\rm P}}\right) |k\eta_1|^{3/2} |\ln|k\eta||, \tag{4.73}$$

  which is under control for inflation scales smaller than Planckian, $H_1 < M_{\rm P}$.

It is important to stress that the behaviour of scalar fluctuations, in the background with $\alpha = 1/2$, is apparently very different from the tensor one, as already discussed in

the previous subsection. By eliminating the dilaton fluctuation χ, and using the scalar perturbation equations, one finds indeed that the scalar longitudinal variable ψ is related to the canonical variable v of Eq. (4.60) by [492]:

$$\psi_k = -\frac{\phi'}{4M_{\rm P}k^2}\left(\frac{v_k}{a}\right)'. \tag{4.74}$$

The normalized amplitude of scalar quantum fluctuations,

$$|\delta_\psi(k)| \equiv k^{3/2}|\psi_k| \simeq \left(\frac{H}{M_{\rm P}}\right)_{\rm hc}|k\eta|^{-2} = \left(\frac{H_1}{M_{\rm P}}\right)|k\eta_1|^{3/2}||k\eta|^{-2}$$
$$= \left(\frac{H_1}{M_{\rm P}}\right)\left(\frac{\eta_1}{\eta}\right)^2|k\eta_1|^{-1/2}, \tag{4.75}$$

is thus unbounded since, for any $|\eta| > |\eta_1|$, we can always find a frequency band $|k\eta_1| \ll 1$ for which $|\delta_\psi|$ is arbitrarily large.

By using the correct quantum normalization, however, it is now easy to check that this difficulty disappears in the off-diagonal gauge [115], where the off-diagonal perturbation variable B satisfies $B_k \sim k\eta\psi_k$ and the amplitude of the corresponding quantum fluctuations is controlled by

$$|\delta_B(k)| \sim |k\eta||\delta_\psi(k)| \sim \left(\frac{H_1}{M_{\rm P}}\right)\left|\frac{\eta_1}{\eta}\right||k\eta_1|^{1/2}. \tag{4.76}$$

For the typical inflation scale of string cosmology, $H_1 \simeq M_{\rm s} < M_{\rm P}$, this amplitude stays smaller than 1 for the whole duration of the pre-big bang phase, $|\eta| > |\eta_1|$, and for all modes with $k < k_1 = |1/\eta_1|$ (higher-frequency modes are not amplified, see the next subsection). It should be mentioned, however, that this conclusion does not apply to *classical* fluctuations, whose initial amplitude is not constrained by the quantum normalization (4.58), and can be fixed to much higher values (see a recent discussion in [128]).

case: Higher derivatives

To conclude this subsection we note that the above procedure used to define the canonical variables can also be applied to more complicated models, and in particular to the actions including the higher-derivative corrections typical of the "stringy", high-curvature regime [482, 140], appearing towards the end of the phase of pre-big bang evolution. To first-order in α', and in the S-frame, such an action can be parametrized as [307]

$$S = \int \frac{d^4x}{2\lambda_{\rm s}^2}\sqrt{-g}e^{-\phi}\left\{-R - \partial_\mu\phi\partial^\mu\phi + \frac{\alpha'}{4}\left[R^2_{GB} - (\partial_\mu\phi\partial^\mu\phi)^2\right]\right\}, \tag{4.77}$$

where $R^2_{GB} \equiv R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the Gauss–Bonnet invariant (see Section 8.2 for further details). Starting with Eq. (4.77), the second-order perturbed action for each polarization mode of tensor fluctuations becomes [277], in conformal time,

$$\delta^{(2)}S_h = \frac{1}{4\lambda_{\rm s}^2}\int d^3x d\eta\left[z^2(\eta)h'^2 + y^2(\eta)h\nabla^2 h\right], \tag{4.78}$$

where

$$z^2(\eta) = e^{-\phi}\left(a^2 - \alpha'\frac{a'}{a}\phi'\right),$$
$$y^2(\eta) = e^{-\phi}\left[a^2 + \alpha'\left(\phi'^2 - \phi'' + \frac{a'}{a}\phi'\right)\right] \tag{4.79}$$

(the inequality of the two pumping fields z and y, which prevents factorization, is a consequence of the breaking of the scale-factor duality symmetry, to this order in α'; see [476] for an action with α' corrections, which preserves scale-factor duality). By setting $u = zh/\sqrt{2}\lambda_s$ we get the action in canonical form,

$$\delta^{(2)} S_h = \frac{1}{2}\int d\eta \left(u'^2 + \frac{z''}{z}u^2 + \frac{y^2}{z^2}u\nabla^2 u\right), \tag{4.80}$$

and the corresponding perturbation equation

$$u_k'' + \left[k^2 - V_k(\eta)\right] u_k = 0, \qquad V_k(\eta) = \frac{z''}{z} - \frac{k^2}{z^2}(y^2 - z^2), \tag{4.81}$$

looks the same as Eq. (4.56), with the only difference that the effective potential is now, in general, k-dependent. The above equation takes into account (to first order) higher-curvatures effects on the perturbation spectrum [277], and provides the starting point for studying the evolution of tensor perturbations in the high-curvature regime typical of the pre-big bang scenario [148]. A similar approach has been developed for the study of scalar and vector fluctuations including α' corrections [150].

modes of metric fluctuations!

4.4 Spectral distribution of the energy density

For all types of perturbations the evolution in time of the canonical modes u_k is described by the Schrödinger-like equation (4.56), with an effective potential that depends on the pump field, and thus on the peculiar kinematics of the given background. For all inflationary backgrounds, however, such a potential tends to vanish, asymptotically, at large positive and negative values of the conformal time.

pump field

Consider, for instance, the evolution of tensor perturbations in the E-frame: the pump field is the scale factor, $z = a$, and $V = a''/a$. For a typical inflationary scenario the background is initially accelerated, parametrized by a power-law scale factor, $a \sim |\eta|^\alpha$, so that $|V| \sim \eta^{-2}$ for $\eta \to -\infty$; at large times the background eventually evolves into the standard radiation-dominated era, $a \sim \eta$, so that $V = 0$ for $\eta \to +\infty$. The analysis of perturbations thus reduces to a problem of scattering of the "wave function" u_k, induced by a potential $V(\eta)$, which is asymptotically vanishing (see Fig. 4.1).

The important difference between the present problem and an ordinary scattering problem, however, is that the differential variable of Eq. (4.56) is *time*, not space. As a con-

VERY IMPORTANT THEOREM!!!

Bogo coeff See Aitchison & Hey on BOGO

sequence, the oscillation frequencies represent *energies*, not momenta. With an initial normalization to a state of positive energy,

$$u_{\rm in} \simeq \frac{e^{-ik\eta}}{\sqrt{2}}, \qquad \eta \to -\infty, \tag{4.82}$$

the final state is thus in general a superposition of positive and negative energy modes,

$$u_{\rm out} \simeq c_+ e^{-ik\eta} + c_- e^{+ik\eta}, \qquad \eta \to +\infty. \tag{4.83}$$

In the context of quantum field theory, it is well known that such a mixing represents a process of pair creation from the vacuum [90]. The so-called Bogoliubov coefficients $c_\pm$ parametrize the unitary transformation connecting the set of $|{\rm in}\rangle$ and $|{\rm out}\rangle$ annihilation and creation operators:

$$\begin{aligned} &\{u_{\rm in}, b_k, b_k^\dagger\} \implies \{u_{\rm out}, a_k, a_k^\dagger\}, \\ &a_k = c_+ b_k + c_-^* b_{-k}^\dagger, \qquad a_{-k}^\dagger = c_- b_k + c_+^* b_{-k}^\dagger, \end{aligned} \tag{4.84}$$

and define the expectation number of pairs produced from the vacuum in the mode k:

$$\overline{n}_{\rm in}(k) = \langle 0|b_k^\dagger b_k|0\rangle = 0, \qquad \overline{n}_{\rm out}(k) = \langle 0|a_k^\dagger a_k|0\rangle = |c_-(k)|^2 \neq 0. \tag{4.85}$$

In a second quantization language, the amplification of perturbations can thus be described as a process of particle production or, equivalently, as the evolution of the initial vacuum into a final "squeezed" vacuum state [351, 352, 354, 355],

????? "squeezed" vac??

$$|0\rangle \implies |S_k\rangle = \Sigma_k |0\rangle, \tag{4.86}$$

I can't find out those outstanding papers!

generated by the two-mode squeezing operator [565]

$$\Sigma_k = \exp\left(s_k^* b_k b_{-k} - s_k b_k^\dagger b_{-k}^\dagger\right).$$

Here $s_k = r_k e^{2i\theta_k}$ is the so-called squeezing parameter, whose modulus and phase are related to the Bogoliubov coefficients $c_\pm(k)$ by [565]

$$c_+(k) = \cosh r_k, \qquad c_-(k) = e^{2i\theta_k} \sinh r_k. \tag{4.88}$$

Itshould be noted that the squeezed-state formalism is particularly convenient for the analysis of the statistical properties of the produced radiation, and for the study of the entropy growth associated to particle production. Representing the effective decoherence of the squeezed density matrix through a suitable "coarse-graining" procedure, which accounts for the loss of information associated to (averaged) macroscopic observations, one finds indeed that the entropy growth associated to particle production, for each mode k, is given by $\Delta S_k = 2r_k$ [297, 298]. In the large squeezing limit, summing over all modes, the total entropy inside a proper volume $V = (aL)^3$ can then be estimated as

?????

$$S \simeq V \int d\omega \omega^2 \ln|c_-(\omega)|^2 = V \int d\omega \omega^2 \ln \overline{n}(\omega), \tag{4.89}$$

PARAMETRIC AMPLIFICATION AS "ANTITUNNELLING"

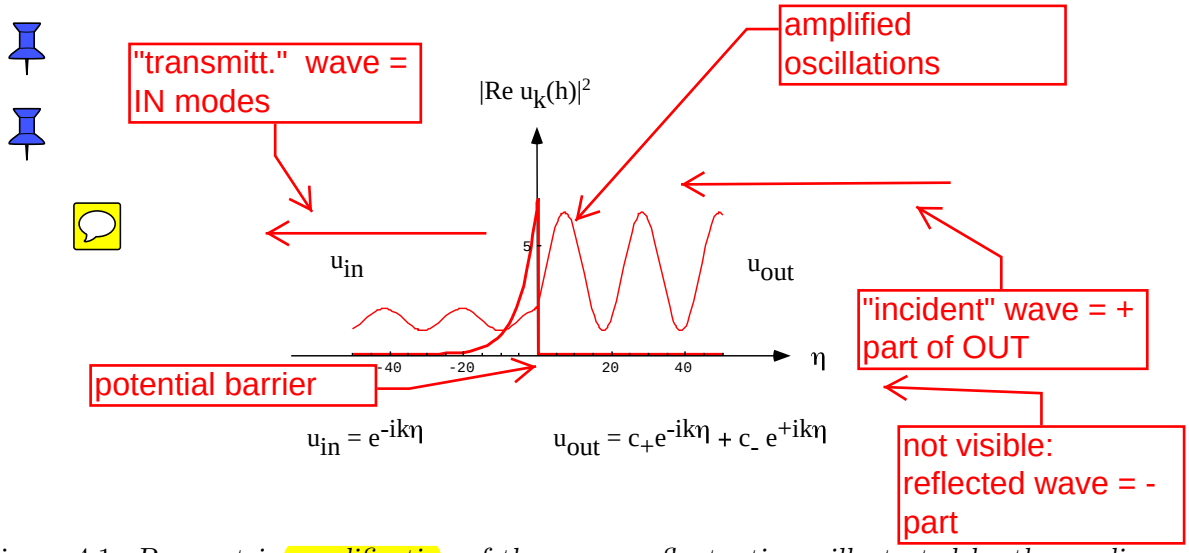

Figure 4.1: *Parametric amplification of the vacuum fluctuations, illustrated by the qualitative evolution of the canonical variable. The plot of the real part of u_k (thin curve) has been obtained through a numerical integration of the canonical perturbation equation for a step-like effective potential (bold curve), exponentially damped in the inflationary region.*

in agreement with different approaches to the same problem [106, 107], and with different definitions of quantum (von Neumann) and informatic (Shannon–Wehrl) entropy [301]. For a growing, power-law spectral distribution $n(\omega)$, typical of pre-big bang models, one obtains a constant entropy per comoving volume, $S \sim (a\omega_1)^3$, where ω_1 is the maximal amplified frequency inside the given volume. This result has useful applications to the problem of the graceful exit, as will be discussed in Section 8.4.

In a semiclassical language, the amplification of perturbations can be represented instead as a parametric amplification [348, 570] of the wave function u_k , i.e. as an "antitunnelling" process [278] in which the positive frequency part of the $|\mathrm{out}\rangle$ mode (4.83) plays the role of the wave incident (from the right) on the barrier and its negative part the role of the reflected wave, while the $|\mathrm{in}\rangle$ mode (4.82) plays the role of the transmitted wave (see Fig. 4.1). The Bogoliubov coefficient c_-, which controls the number of produced particles, can then be written as the ratio of the reflection and transmission coefficients, $|c_-|^2 = R/T$. In the parametric amplification regime, characterized by a large number of produced particles ($\overline{n}_k \gg 1$), one finds $|c_-| \simeq |c_+|$, so that $R \simeq 1$, and then

$$|c_-|^2 \simeq T^{-1}. \tag{4.90}$$

Hence the term antitunnelling.

Quite independently of the adopted language, the energy density of the produced particles, for each mode k (summing over two polarization states), is given by

$$d\rho_k = 2k\overline{n}_k \frac{d^3k}{(2\pi)^3}, \qquad \overline{n}_k = |c_-(k)|^2. \tag{4.91}$$

c-: see 4.88

computation of BOGO c-

The computation of the spectral energy distribution per logarithmic interval of frequency,

$$\frac{d\rho_k}{d\ln k} \equiv k\frac{d\rho_k}{dk} = \frac{k^4}{\pi^2}|c_-(k)|^2, \tag{4.92}$$

thus requires the computation of the Bogoliubov coefficient $c_-(k)$. By recalling Eqs. (4.82), (4.83), this amounts to solving the perturbation equation in the limit of large positive times, with the initial normalization imposed at $\eta = -\infty$.

case:

We will consider, as a simple example, a model of background characterized by two phases: an initial accelerated evolution up to the time η_1, and a subsequent radiation-dominated evolution for $\eta > \eta_1$:

$$\begin{aligned} a &\sim (-\eta)^\alpha, & \eta &< \eta_1, \\ a &\sim \eta, & \eta &> \eta_1. \end{aligned} \tag{4.93}$$

In the first phase the canonical equation (4.56) for tensor perturbations reduces to a Bessel equation,

$$u_k'' + \left[k^2 - \frac{\alpha(\alpha-1)}{\eta^2}\right] u_k = 0, \tag{4.94}$$

with general solution [5]

$$u_k = |\eta|^{1/2}\left[AH_\nu^{(2)}(|k\eta|) + BH_\nu^{(1)}(|k\eta|)\right], \qquad \nu = |\alpha - 1/2|, \tag{4.95}$$

where $H_\nu^{(1,2)}$ are the first- and second-kind Hankel functions, of index $\nu = |\alpha - 1/2|$ determined by the kinematics of the background (in this case, by the time evolution of the scale factor; more generally, however, α parametrizes the evolution of the pump field in conformal time). By using the large argument limit [5] for $\eta \to -\infty$,

$$H_\nu^{(2)}(k\eta) \sim \sqrt{\frac{2}{\pi|k\eta|}}e^{-ik\eta}, \qquad H_\nu^{(1)}(k\eta) \sim \sqrt{\frac{2}{\pi|k\eta|}}e^{+ik\eta}, \tag{4.96}$$

we normalize the solution to a vacuum fluctuation spectrum,

$$A = \sqrt{\pi/4}, \qquad B = 0. \tag{4.97}$$

In the second phase $V = 0$, and we have the simple oscillating solution,

$$u_k = \frac{1}{\sqrt{2k}}\left(c_+e^{-ik\eta} + c_-e^{+ik\eta}\right). \tag{4.98}$$

==> BOGO:

The coefficients $c_\pm$ are now determined by the continuity of u and u' at $\eta = \eta_1$. More precisely, assuming a non-singular and continuous background, the matching would require the continuity of the perturbed metric projected on a space-like hypersurface containing η_1, and the continuity of the extrinsic curvature of that hypersurface [385, 214]; but in many cases these conditions are equivalent to the continuity of the canonical variable u,

and of its first time derivative (see however [135]). Such a matching prescription might require modifications, however, in the case of bouncing backgrounds [518] evolving from contraction to expansion).

When matching the solutions (4.95), (4.98), it is convenient to distinguish the two regimes in which the comoving frequency k is much higher or much lower than the frequency associated to the top of the effective potential barrier, $|V(\eta_1)|^{1/2} \simeq |\eta_1|^{-1}$. In the first case, $k \gg 1/|\eta_1| \equiv k_1$, we can use the large-argument limit of the Hankel functions to find that there is no significant particle production, i.e.

$$|c_+| \simeq 1, \qquad |c_-| \simeq 0. \tag{4.99}$$

Actually, c_- is not exactly zero, but it is exponentially suppressed [12, 264] as a function of frequency, just like the quantum-mechanical reflection probability for a wave function with a total energy well above the top of a potential barrier. Such an effect is, however, negligible for the purpose of a first-approximation estimate in a cosmological context, and in any case it only affects the shape of the very high-frequency tail of the spectrum. A sharp cut-off, in that region, is a good enough approximation.

In the second case, $k \ll 1/|\eta_1| \equiv k_1$, we can use the small-argument limit of the Hankel functions [5],

$$H_\nu^{(2)} \sim a(k\eta_1)^\nu - ib(k\eta_1)^{-\nu}, \qquad H_\nu^{(1)} \sim a^*(k\eta_1)^\nu + ib(k\eta_1)^{-\nu}, \tag{4.100}$$

where a and b are dimensionless numbers of order 1, and we find

$$|c_+| \simeq |c_-| \simeq |k\eta_1|^{-\nu-1/2}, \tag{4.101}$$

corresponding to a spectral distribution:

$$\frac{d\rho_k}{d\ln k} = \frac{k^4}{\pi^2}|c_-(k)|^2 \simeq \frac{k_1^4}{\pi^2}\left(\frac{k}{k_1}\right)^{3-2\nu}, \qquad k < k_1. \tag{4.102}$$

For a comparison with present observations it is convenient to use proper frequencies, $\omega(t) = k/a(t)$, and to express the spectrum in units of critical energy density, $\rho_c(t) = 3M_{\rm P}^2 H^2(t)$. We thus obtain the dimensionless spectral distribution

$$\Omega(\omega,t) = \frac{\omega}{\rho_c(t)}\frac{d\rho(\omega)}{d\omega} \simeq \frac{\omega_1^4}{3\pi^2 M_{\rm P}^2 H^2}\left(\frac{\omega}{\omega_1}\right)^{3-2\nu}, \qquad \omega < \omega_1, \tag{4.103}$$

where

$$\omega_1 = \frac{k_1}{a} \simeq \frac{1}{a\eta_1} \simeq \frac{H_1 a_1}{a} \tag{4.104}$$

is the maximal amplified frequency (approximately, the frequency "hitting" the top of the potential barrier $|V(\eta)|$). Such a distribution can also be rewritten (modulo model-dependent numerical factors) in the final useful form

$$\Omega(\omega,t) \simeq g_1^2 \Omega_\gamma(t)\left(\frac{\omega}{\omega_1}\right)^{3-2\nu}, \tag{4.105}$$

4.103

Curvature scale H^2	**Metric** $a = \|\eta\|^\alpha$	**Bessel index** $\nu = \|\alpha - 1/2\|$	**Spectrum** $\Omega \sim \omega^{3-2\nu}$
de Sitter, constant	$\alpha = -1$	$3 - 2\nu = 0$	flat
power-inflation, decreasing	$\alpha < -1$	$3 - 2\nu < 0$	decreasing
pre-big bang inflation, growing	$\alpha > -1$	$3 - 2\nu > 0$	increasing

Table 2: Accelerated backgrounds and the slope of the spectrum.

where $g_1 = H_1/M_{\rm P}$ is the curvature scale at the transition epoch t_1 (a fundamental parameter of the given cosmological model), and

$$\Omega_\gamma(t) = \frac{\rho_\gamma}{\rho_c} = \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \tag{4.106}$$

is the energy density (in critical units) of the radiation which becomes dominant at $t = t_1$, rescaled down to a generic time t.

It is important to stress that the spectral slope $3-2\nu$ is directly related to the kinematic behaviour of the background during the phase of accelerated evolution, as illustrated in Table 2. In particular, the spectrum tends to follow the behaviour of the curvature scale. In the standard inflationary scenario, represented by a phase of de Sitter-like evolution, Ω does not depend on ω, and one obtains the so-called scale-invariant Harrison–Zeldovich spectrum [364, 631]. For the low-energy, dilaton-driven phase typical of string cosmology (the solution (4.39)) one has $\alpha = 1/2$, and the corresponding spectrum $\Omega \sim \omega^3$ simulates the low-energy (Rayleigh–Jeans) tail of a thermal black-body distribution (see Subsection 5.2). Note, however, that a flat, scale-invariant spectrum with $\nu = 3/2$ can also obtained for $\alpha = 2$, corresponding to a phase of contraction dominated by a cold "dust" source with effective equation of state $p = 0$. This possibility, first pointed out in [273], was recently discussed in [251] for the pre-big bang and ekpyrotic [412, 413] scenarios.

It is also to be remarked that the above results for the spectrum have been obtained by matching, as usual, the (perturbed) induced metric and the extrinsic curvature across the hypersurface of constant total energy density, $\rho + \delta\rho = $ const. In the case of scalar metric perturbations, this implies the continuity of the Bardeen potential Φ and of the variable $\zeta = v/z$ (representing curvature perturbations), even for bouncing backgrounds [384]. Performing the matching across different hypersurfaces may lead, however, to a dif-

ferent spectrum of scalar perturbations (without changing the tensor spectrum), especially in the case of transitions from a collapsing to an expanding phase, and in the case of non-continuous backgrounds, as shown for a special case in [518] and discussed in general in [227] (see Section 7 for possible applications of this effect to the scalar perturbation spectrum of the pre-big bang scenario).

It should be noted, finally, that the above procedure for an approximate determination of the spectrum can be applied to any type of perturbation described by the canonical equation (4.56). Also, it can be easily generalized to the case of n transition scales η_1, η_2, $...\eta_n$, for a background characterized by n phases, each of them approximated by a power-law evolution with parameters $\alpha_1, \alpha_2, \ ... \ , \alpha_n$. In each of these phases we have a general solution with Bessel index $\nu_i = |\alpha_i - 1/2|$, $i = 1, 2, ..., n$. By matching u and u', using the large-argument limit of the Hankel functions for modes above the potential barrier, and the small-argument limit for modes below the barrier, we obtain a spectrum with n different frequency bands. In each band, the slope of the spectrum is only determined by the background kinematics of the two phases in which a given mode "hits" the barrier and "re-enters" the horizon, respectively. The amplitude of the spectrum, on the contrary, keeps track of the relative duration and kinematics of all phases during which the mode stays "under the barrier" [278].

By expanding the coefficients of the exact Bessel solutions (determined by the matching conditions) at each transition scale $|\eta_i| = k_i^{-1}$, and including the next-to-leading terms when the leading ones are zero [135], we can also formulate a set of synthetic prescriptions for a diagrammatic computation of the spectrum [210]. The method requires the drawing of a simple plot in which we insert a vertical line corresponding to each transition, the height of the line being proportional to the associated transition frequency, $k_i = |\eta_i|^{-1}$. It is thus possible to identify at a glance the various frequency bands of the spectrum, and the knowledge of the various Bessel indices ν_i turns out to be sufficient to write down a quick estimate of the various amplitudes and spectral distributions [210]. Such a computation can also be made exact by including the full numerical coefficients in the asymptotic expansion of the Hankel functions.

4.5 Duality of the perturbation equations

In the previous subsections we have applied the usual Lagrangian formalism to describe the evolution in time of the perturbations, and their cosmological amplification. The Hamiltonian formalism, however, is more appropriate for the study of an interesting duality symmetry which connects the amplitude of quantum fluctuations to their conjugate momentum, and which may be useful for estimating their energy–density spectrum [118] (see also [616]).

We shall assume, as discussed in Subsection 4.3, that the perturbations of the effective

action leads to a quadratic Lagrangian density which, in the conformal-time gauge, can be parametrized as

$$\mathcal{L} = \frac{1}{2} z^2 \left(\psi'^2 + \psi \nabla \psi \right), \tag{4.107}$$

where $z(\eta)$ is the pump field and ψ the comoving amplitude of the given fluctuations (see for instance Eq. (4.53)). This Lagrangian may describe not only the (scalar and tensor) metric perturbations, but also the quantum fluctuations of gauge fields and antisymmetric tensors (see Section 7). By introducing the conjugate momentum

$$\pi = \frac{\delta \mathcal{L}}{\delta \psi'} = z^2 \psi', \tag{4.108}$$

we can define the corresponding Hamiltonian density, written in Fourier space as

$$\mathcal{H} = \frac{1}{2} \sum_k \left(z^{-2} |\pi_k|^2 + z^2 k^2 |\psi_k|^2 \right), \tag{4.109}$$

where $\psi_{-k} = \psi_k^*$ and $\pi_{-k} = \pi_k^*$. The first-order perturbation equations then read, for each mode,

$$\psi_k' = z^{-2} \pi_{-k}, \qquad \pi_k' = -z^2 k^2 \psi_{-k}, \tag{4.110}$$

and lead to the decoupled equations

$$\psi_k'' + 2 \frac{z'}{z} \psi_k' + k^2 \psi_k = 0\,, \qquad \pi_k'' - 2 \frac{z'}{z} \pi_k' + k^2 \pi_k = 0. \tag{4.111}$$

It is now easy to check that the transformation

$$\pi_k \to \widetilde{\pi}_k = k \psi_k\,, \qquad \psi_k \to \widetilde{\psi}_k = -k^{-1} \pi_k\,, \qquad z \to \widetilde{z} = z^{-1} \tag{4.112}$$

leaves the Hamiltonian, Poisson brackets, and equations of motion unchanged. It is a duality transformation that reduces to the usual strong–weak coupling duality [561, 523, 556] in the special case in which $z = e^{-\phi/2} \equiv g_{\rm s}^{-1}$ (as for the perturbations of heterotic, four-dimensional gauge bosons [305, 306, 430], see Section 7), and to scale-factor duality [599, 580, 560] in the special case in which $z = a$ (as for tensor metric perturbations). In addition, in the case of electromagnetic perturbations, the contribution of $k\psi_k$ to the Hamiltonian can be identified with that of the magnetic field, whereas π_k is proportional to the electric field: hence, in that case, the transformation (4.112) exactly represents the well-known electric–magnetic duality transformation.

As a consequence of this duality property, the energy-density spectrum of the quantum fluctuations can be consistently estimated by truncating the solution (outside the horizon) to the frozen part of the fluctuations, even in backgrounds where the comoving amplitude ψ grows with time and the growing modes dominate the spectrum. Only in a duality-invariant context is such a truncation consistent, as the contribution to the Hamiltonian of the growing mode of ψ is simply replaced by the contribution of the frozen part of its duality-related conjugate momentum [118].

In order to illustrate this important effect, it is convenient to introduce the new variables

$$\widehat{\psi} = z\psi, \qquad \widehat{\pi} = z^{-1}\pi, \tag{4.113}$$

which are canonically normalized and which transform as ψ, π under the duality transformation (4.112). Our Hamiltonian density (4.109) can then be written, formally[1], in a free canonical form,

$$\mathcal{H} = \frac{1}{2}\sum_{\vec{k}} \left(|\widehat{\pi}_k|^2 + k^2 |\widehat{\psi}_k|^2 \right), \tag{4.114}$$

where the variables $\widehat{\psi}, \widehat{\pi}$ satisfy the Schrödinger-like equations of motion:

$$\widehat{\psi}_k{}'' + \left[k^2 - (z)''z^{-1} \right] \widehat{\psi}_k = 0\,, \qquad \widehat{\pi}_k{}'' + \left[k^2 - (z^{-1})''z \right] \widehat{\pi}_k = 0, \tag{4.115}$$

characterized by two duality-related "effective potentials", $V_\psi(z) = z''/z$ and $V_\pi(z) = z(z^{-1})'' = V_\psi(\widetilde{z})$.

The initial evolution of perturbations, for all modes with $k^2 > \{|V_\Psi|, |V_\Pi|\}$, can now be described by the WKB-like approximate solutions of Eqs. (4.115):

WKB: Wiki

$$\widehat{\psi}_k(\eta) = \left(k^2 - V_\psi \right)^{-1/4} \exp\left[-i \int_{\eta_0}^{\eta} d\eta' \left(k^2 - V_\psi \right)^{1/2} \right],$$
$$\widehat{\pi}_k(\eta) = k \left(k^2 - V_\pi \right)^{-1/4} \exp\left[-i \int_{\eta_0}^{\eta} d\eta' \left(k^2 - V_\pi \right)^{1/2} \right], \tag{4.116}$$

which we have normalized to zero-point vacuum fluctuations, and where the extra factor of k in the solution for $\widehat{\pi}_k$ is inserted for consistency with the first-order Hamiltonian equations. We have ignored a possible relative phase in the solutions. Asymptotically, at $\eta \to -\infty$, we have V_ψ, $V_\pi \to 0$, and the solutions (4.116) reduce to the canonically normalized vacuum fluctuations

$$\widehat{\psi}_k(\eta) = \frac{e^{\,-ik\eta + i\varphi_k}}{\sqrt{k}}\,, \qquad \widehat{\pi}_k(\eta) = \sqrt{k} e^{\,-ik\eta + i\varphi'_k}, \tag{4.117}$$

where φ_k, φ'_k are random phases, originating from the random initial conditions. The initial state thus satisfies

$$\langle z^{-2}\pi^2 \rangle = \langle z^2 (\nabla\psi)^2 \rangle, \tag{4.118}$$

where $\langle \cdots \rangle$ denotes ensemble average, or expectation value if perturbations are quantized. Note that, because of the random phases, the initial state is duality-invariant only on the average, in the sense of the above equation.

[1] Equivalently, we can introduce $\widehat{\psi}, \widehat{\pi}$ via a time-dependent canonical transformation and work with a corrected Hamiltonian $\widehat{\mathcal{H}}$.

In the opposite regime, $k^2 < \{|V_\psi|, |V_\pi|\}$, it is possible to write "exact" solutions to Eq. (4.115) as follows [118],

$$\widehat{\psi}_k(\eta) = z\left[\widehat{A}_k\ \mathrm{Tcos}(z^{-1}, z) + \widehat{B}_k\ \mathrm{Tsin}(z^{-1}, z)\right],$$
$$\widehat{\pi}_k(\eta) = kz^{-1}\left[\widehat{B}_k\ \mathrm{Tcos}(z, z^{-1}) - \widehat{A}_k\ \mathrm{Tsin}(z, z^{-1})\right], \tag{4.119}$$

where $\widehat{A}_k, \widehat{B}_k$ are arbitrary integration constants, and the functions $\mathrm{Tcos}(k,\eta)$, $\mathrm{Tsin}(k,\eta)$ satisfy

$$\left[\mathrm{Tcos}(z^{-1}, z)\right]' = -\frac{k}{z^2}\ \mathrm{Tsin}(z, z^{-1})\,, \qquad \left[\mathrm{Tsin}(z^{-1}, z)\right]' = \frac{k}{z^2}\ \mathrm{Tcos}(z, z^{-1}). \tag{4.120}$$

They are defined through the following "ordered" expansions (see also [492]):

$$\mathrm{Tcos}(z^{-1}, z; k, \eta) = 1 - k\int_{\eta_{\rm ex}}^{\eta} d\eta_1 z^{-2}(\eta_1)\ k\int_{\eta_{\rm ex}}^{\eta_1} d\eta_2 z^2(\eta_2) +$$
$$+\ k\int_{\eta_{\rm ex}}^{\eta} d\eta_1 z^{-2}(\eta_1)\ k\int_{\eta_{\rm ex}}^{\eta_1} d\eta_2 z^2(\eta_2)\ k\int_{\eta_{\rm ex}}^{\eta_2} d\eta_3 z^{-2}(\eta_3)\ k\int_{\eta_{\rm ex}}^{\eta_3} d\eta_4 z^2(\eta_4) + \cdots\ ;$$
$$\mathrm{Tsin}(z^{-1}, z; k, \eta) = k\int_{\eta_{\rm ex}}^{\eta} d\eta_1 z^{-2}(\eta_1) -$$
$$-\ k\int_{\eta_{\rm ex}}^{\eta} d\eta_1 z^{-2}(\eta_1)\ k\int_{\eta_{\rm ex}}^{\eta_1} d\eta_2 z^2(\eta_2)\ k\int_{\eta_{\rm ex}}^{\eta_2} d\eta_3 z^{-2}(\eta_3) + \cdots, \tag{4.121}$$

where, for matching purposes, a convenient choice of the time $\eta_{\rm ex}$, appearing as an arbitrary lower limit of integration, is the horizon-exit time, such that $|k\eta_{\rm ex}| \simeq 1$.

Ansatz:

Suppose now that the background, during the accelerated period, can be parametrized by a power-law evolution of the pump field, i.e. $z \sim |\eta|^\alpha$ for $\eta \to 0_-$. From the general solution (4.121) we obtain, in the limit $|k\eta| \ll 1$,

$$\widehat{\psi}_k = \widehat{A}_k|\eta|^\alpha + \frac{k|\eta|}{1-2\alpha}\widehat{B}_k|\eta|^{-\alpha}, \qquad \widehat{\pi}_k = k\left(\widehat{B}_k|\eta|^{-\alpha} - \frac{k|\eta|}{1+2\alpha}\widehat{A}_k|\eta|^\alpha\right), \tag{4.122}$$

with logarithmic corrections for $\alpha = \pm 1/2$. The corresponding expansion for the comoving amplitudes ψ_k, π_k, is then

$$\psi_k = A_k + B_k|\eta|^{1-2\alpha}, \qquad \pi_k = k\left(B_k - A_k|\eta|^{1+2\alpha}\right). \tag{4.123}$$

They may contain a growing mode, which leads in general to a growing Hamiltonian, and then to a growing energy–density spectrum. The Hamiltonian, however, is always dominated by a joint contribution of the *frozen* modes of ψ_k and π_k. We may consider, in fact, three possibilities.

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- $\alpha > 1/2$. The growing mode dominates ψ_k, while π_k is frozen, as the constant mode is dominant (see Eq. (4.123)). Asymptotically, however, the Hamiltonian (4.114) is momentum-dominated, $\widehat{\pi}^2 \gg k^2\widehat{\psi}^2$, according to Eq. (4.122).

- $\alpha < -1/2$. The comoving amplitude ψ_k is frozen, while the growing mode dominates π_k (see Eq. (4.123)). Asymptotically, however, the Hamiltonian (4.114) is dominated by ψ_k, i.e. $\widehat{\pi}^2 \ll k^2\widehat{\psi}^2$, according to Eq. (4.122).

- $-1/2 < \alpha < 1/2$. The comoving amplitude and the conjugate momentum are both frozen, so that

$$k\widehat{\psi}_k = kz\psi_k = k\left(\frac{\eta}{\eta_{\rm hc}}\right)^{\alpha}\psi_k \sim \sqrt{k}|k\eta|^{\alpha},$$
$$\widehat{\pi}_k = kz^{-1}\pi_k = \left(\frac{\eta}{\eta_{\rm hc}}\right)^{-\alpha}\pi_k \sim \sqrt{k}|k\eta|^{-\alpha}, \tag{4.124}$$

and

$$\mathcal{H}_k = \frac{1}{2}{\rm Max}\left\{\widehat{\pi}_k^2, k^2\widehat{\psi}_k^2\right\}. \tag{4.125}$$

Thus, for super-horizon wavelengths, the Hamiltonian density is always growing but, irrespectively of the value and sign of α, the joint contribution of the constant modes $\psi_k \sim \widehat{A}_k$ and $\pi_k \sim k\widehat{B}_k$ always provides an accurate estimate of the leading contribution to the canonical energy density, which can be written as

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$$\mathcal{H} \simeq \frac{1}{2}\sum_k k^2\left[|\widehat{A}_k|^2|\eta|^{2\alpha} + |\widehat{B}_k|^2|\eta|^{-2\alpha}\right]. \tag{4.126}$$

This estimate, based on the low-energy effective Lagrangian (4.107), could no longer be appropriate when including α' and loop corrections, typical of the high-curvature string phase of pre-big bang models. Such corrections could indeed break the low-energy duality symmetry. In that case, we cannot exclude that other modes in ψ and π will dominate the Hamiltonian over the constant modes. In that case, however, the above estimate can be applied as a *lower bound* on the total energy spectrum, provided the perturbation equations still contain a frozen part in their asymptotic solutions, as shown, for instance, in the example discussed in detail in [277].

5 Relic gravitons

One of the most firm predictions of all inflationary models is the amplification of the traceless, transverse part of the quantum fluctuations of the metric tensor, and the formation of a stochastic background of relic gravitational waves [348, 570, 13, 585], distributed over quite a large range of frequencies (see [13, 17] for a detailed discussion of the stochastic

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properties of such a background, and [353, 15] for a possible detection of the associated "squeezing").

In this section we will apply the formalism and the results established in the previous section to compute the possible spectral distributions of relic gravitons in the context of the pre-big bang scenario. We will then estimate the maximal expected intensity (allowed by phenomenological bounds and by model-dependent constraints) of such backgrounds and discuss their possible detection by present (or near-future) gravitational antennas. For additional studies of this aspect of the pre-big bang scenario we also refer the reader to recent review papers, especially devoted to the primordial graviton background in string cosmology [602, 278, 280, 289, 113, 463].

5.1 Phenomenological bounds on the graviton spectrum

As discussed in the previous section, the spectrum of the tensor perturbations of the metric, amplified by inflation, contains various important parameters: the inflation scale H_1, the slope $3-2\nu$, the end-point frequency ω_1. Such parameters are (partially) constrained by the direct (or indirect) observational bounds existing at present on the energy density of a primordial stochastic background of gravitational waves. To discuss the effects of such bounds, let us recall, first of all, the gravitational spectrum expected in the context of the standard inflationary scenario.

In that context, the overall cosmological evolution can be approximated by a three-phase model of background, in which the accelerated de Sitter (or quasi-de Sitter) epoch is followed by the standard radiation-dominated and matter-dominated eras:

3 inflationary phases

$$\begin{array}{lll} \text{de Sitter inflation,} & a \sim (-\eta)^{-1}, & \eta < \eta_1, \\ \text{radiation domination,} & a \sim \eta, & \eta_1 < \eta < \eta_{\rm eq}, \\ \text{matter domination,} & a \sim \eta^2, & \eta_{\rm eq} < \eta < \eta_0. \end{array} \tag{5.1}$$

The effective potential appearing in the perturbation equation (4.56) is then characterized by two transition scales, inflation $\to$ radiation at $\eta \sim \eta_1$, and radiation $\to$ matter at $\eta \sim \eta_{\rm eq}$. As a consequence, there are two different branches of the spectrum, corresponding to high-frequency modes, $|k\eta_{\rm eq}| > 1$, that are affected by the first background transition only, and low-frequency modes, $|k\eta_{\rm eq}| < 1$, that are affected by both transitions. Following the procedure described in Section 4.4 we can easily compute the corresponding Bogoliubov coefficients, and the resulting spectral energy distribution (4.105), evaluated at the present time t_0, is [543, 243, 3]:

$$\begin{aligned} \Omega_G(\omega, t_o) &\simeq \left(\frac{H_1}{M_{\rm P}}\right)^2 \Omega_\gamma(t_0), && \omega_{\rm eq} < \omega < \omega_1, \\ &\simeq \left(\frac{H_1}{M_{\rm P}}\right)^2 \Omega_\gamma(t_0) \left(\frac{\omega}{\omega_{\rm eq}}\right)^{-2}, && \omega_0 < \omega < \omega_{\rm eq}. \end{aligned} \tag{5.2}$$

Here H_1 is the curvature scale corresponding to the inflation $\to$ radiation transition, ω_1 is the maximal amplified frequency, corresponding today to $\omega_1 \simeq \sqrt{H_1/M_{\rm P}}\ 10^{11}$ Hz; ω_0 is the minimal amplified frequency, corresponding to a mode crossing today the Hubble horizon, $\omega_0 \simeq 10^{-18}$ Hz; $\omega_{\rm eq} \simeq 10^2\omega_0$ is the frequency of a mode crossing the horizon at the epoch of the radiation $\to$ matter transition (the maximal frequency affected by the potential due to the matter era); finally, $\Omega_\gamma(t_0) \sim 10^{-4}$ is the present fraction of critical energy density in radiation.

The expected distribution decreases like ω^{-2} between ω_0 and $\omega_{\rm eq}$; it is flat (i.e. scale-invariant) between $\omega_{\rm eq}$ and ω_1 and then dies off exponentially for $\omega > \omega_1$. The position of the spectrum in the (Ω, ω) plane is completely controlled by H_1, which is the only unknown parameter in this simple example of inflationary model. The existing phenomenological bounds able to constrain the position of the spectrum in the (Ω, ω) plane can thus constrain, indirectly, the inflation scale H_1. At present, there are three main experimental bounds.

A first bound follows from the analysis of the pulsar-timing data: there is at present no detectable distortion of pulsar timing due to the influence of a cosmic graviton background, at the frequency scale $\omega_p \sim 10^{-8}$ Hz. The present level of experimental sensitivity provides the bound [404]

$$\Omega_G(\omega_p) \lesssim 10^{-8}, \qquad\qquad \omega_p \sim 10^{-8}\ {\rm Hz}. \tag{5.3}$$

A second bound comes from the standard nucleosynthesis analysis [557]: the total integrated energy density of the graviton background, at the epoch of nucleosynthesis, cannot exceed, roughly, the energy density of one massless degree of freedom in thermal equilibrium at that epoch. This gives a bound for the integrated spectrum [117],

$$h_{100}^2 \int d\ln\omega \Omega_G(\omega, t_0) \ \lesssim\ 0.5 \times 10^{-5}, \qquad h_{100}^2 = H_0/(100\ {\rm km\ s^{-1} Mpc^{-1}}), \tag{5.4}$$

which also applies to a possible peak of the spectrum, at all scales. Note, however, that the above bound can be relaxed by about an order of magnitude if significant matter–antimatter domains are present at the onset of nucleosynthesis, as recently pointed out in [337].

A third bound (the most stringent one for flat or decreasing spectra) comes from the COBE measurements [567, 77] of the large-scale CMB anisotropy. Those observations imply that the contribution of a graviton background to the CMB anisotropy, at the horizon scale ω_0, cannot exceed the upper bound

$$\left(\frac{\Delta T}{T}\right)_{\omega_0} \sim |\delta_h(\omega_0)| \lesssim 10^{-5}, \tag{5.5}$$

where $|\delta_h(\omega)|$ is the typical fluctuation amplitude (4.69) of tensor perturbations over a proper length scale ω^{-1}. The spectral energy distribution, on the other hand, can be expressed in terms of the amplitude δ_h by noting that, from the E-frame action (4.64) of tensor perturbations, the average proper energy density of the background (summing over

evaporating BH produces gravitons

polarizations) is given by

$$\rho_G = \frac{d\langle E_g\rangle}{a^3 d^3x} = M_{\rm P}^2 \langle |\dot h|^2\rangle. \tag{5.6}$$

Using the Fourier expansion, the isotropy and the stochastic condition (4.67) for h_k, we finally obtain the relation [278]

$$\Omega_G(\omega,t) = \frac{d(\langle E_g\rangle/\rho_c)}{a^3 d^3x\ d\ln\omega} = \frac{1}{3\pi^2}\,|\delta_h(\omega)|^2 \left(\frac{\omega}{\omega_0}\right)^2. \tag{5.7}$$

The COBE bound can thus be rewritten as a condition on the graviton spectrum,

$$\Omega_G(\omega_0) \lesssim 10^{-10}, \qquad\qquad \omega_0 \sim 10^{-18}\ {\rm Hz}. \tag{5.8}$$

A further, model-dependent bound can be obtained by considering the production of primordial black holes [180, 181], that could be in principle closely related to the process of graviton production. The process of black-hole evaporation, at the present epoch, is constrained in fact by a number of astrophysical observations: the absence of evaporation may thus impose an indirect upper limit on the graviton background. In a string cosmology context the best upper limit obtained from primordial black holes turns out to be roughly of the same order as the nucleosynthesis bound (5.4), namely $\Omega_G(\omega_{\rm s}) \lesssim 0.5\times 10^{-5}$ [180], where $\omega_{\rm s}$ is the maximal frequency amplified at the end of the low-energy dilaton-driven phase (see the next subsection). It should be mentioned, finally, that another bound comparable with the nucleosynthesis one, $\Omega_G \lesssim 10^{-5}$, and only valid however in the frequency range 10^{-8} – 10^{-16} Hz, is also obtained from present gravitational lensing experiments [54].

Taking into account all significant bounds, the maximal allowed spectrum for the standard inflationary scenario is represented in Fig. 5.1 by the line labelled "de Sitter". The CMB isotropy bound (5.11) is the most significant one and, when imposed on the spectrum (5.2), it implies [422, 522]

$$H_1 \lesssim 10^{-5} M_{\rm P}. \tag{5.9}$$

Also shown in Fig. 5.1 are two possible spectra for the case in which the accelerated de Sitter phase of Eq. (5.1) is replaced by a phase of power-law inflation, with $a \sim (-\eta)^\alpha$, $\alpha < -1$. In that case the spectrum is decreasing even for $\omega > \omega_{\rm eq}$ (see Table 2), and the isotropy bound on H_1 becomes more and more stringent as the negative slope becomes steeper and steeper [295, 299].

The bound (5.9) on the final inflation scale thus applies to all models characterized by a flat or decreasing energy spectrum. It may be evaded, however, if the spectrum is growing, as in the context of the pre-big bang models that will be considered in the next subsection.

5.2 The graviton spectrum in minimal pre-big bang models

According to the pre-big bang scenario, the very early cosmological evolution starts from a state approaching the string perturbative vacuum, goes through an initial low-energy phase

the poet's corner

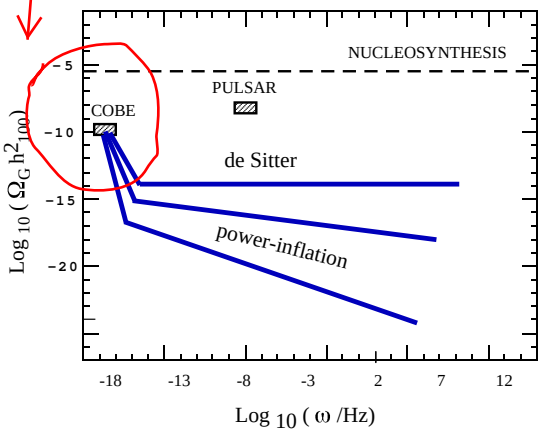

Figure 5.1: *Expected graviton spectra in the standard inflationary scenario.*

driven by the dilaton kinetic energy, and then necessarily reaches a purely "stringy" and quantum regime of high curvatures and large couplings. In that regime, the perturbative expansion of the string effective action becomes inappropriate (see Section 8), and the details of the background kinematics become (with present understanding) uncertain and strongly model-dependent; for the illustrative purposes of this subsection we shall assume that, in spite of these uncertainties, it is still possible to parametrize the kinematics of the string phase by a power-law evolution of the metric and of the dilaton field.

Neglecting the final matter-dominated era, which only affects the low-energy tail of the spectrum (and which is not very relevant to spectra that are growing very rapidly with frequency), we are thus led to consider a simple class of pre-big bang models in which the phase of accelerated evolution preceding the standard radiation era is divided into two physically distinct regimes: an initial, low-energy, dilaton-dominated phase, and a late, high-energy, string phase where α' and loop corrections become important (see Section 8). We shall denote by η_s the time scale marking the transition to the high-curvature string phase and by η_1 the time scale marking the transition to the radiation-dominated cosmology (see Fig. 5.2), associated to a final curvature of order 1 in string units. In the E-frame we may thus parametrize the following class of backgrounds:

PRE BB MODEL 1

$$\begin{array}{llll} \text{dilaton phase,} & a \sim (-\eta)^{1/2}, & \phi \sim -\sqrt{3}\ln(-\eta), & \eta < \eta_s, \\ \text{string phase,} & a \sim (-\eta)^{\alpha}, & \phi \sim -2\beta \ln(-\eta), & \eta_s < \eta < \eta_1, \\ \text{radiation era,} & a \sim \eta, & \phi = \text{const}, & \eta > \eta_1. \end{array} \tag{5.10}$$

The model seems to contain four parameters, α, β, η_1, η_s or, equivalently, the curvature scales and coupling constants at the two transition epochs:

$$H_1, \qquad H_s, \qquad g_1 = e^{\phi_1/2}, \qquad g_s = e^{\phi_s/2}. \tag{5.11}$$

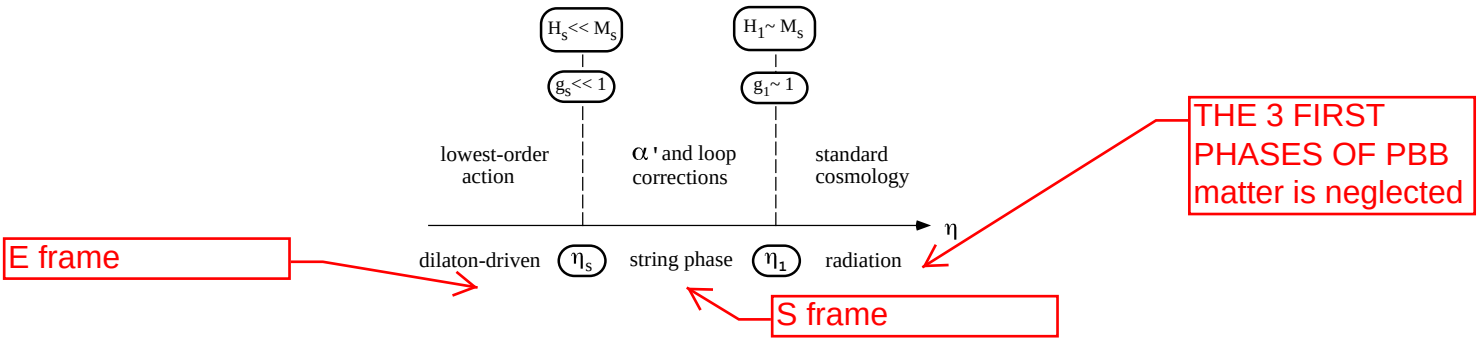

Figure 5.2: *A "minimal" model of pre-big bang background.*

We may expect, however, that during the string phase the curvature scale stays controlled by the fundamental string mass M_s,

$$H_{\text{string phase}} \simeq M_s = e^{\phi/2} M_P, \tag{5.12}$$

so that H keeps constant in the S-frame, while it grows (driven by the dilaton) in the E-frame, where M_P is a constant. This condition has two important consequences.

The first is that the final curvature scale, $H_1 = g_1 M_P$, turns out to be fixed by the present value of the fundamental ratio between the string and the Planck mass [403]:

$$M_s/M_P \simeq 0.3 - 0.03. \tag{5.13}$$

The second is that, in the E-frame, according to the parametrization (5.10), we have $|H_s/H_1| = g_s/g_1$, so that

$$|\eta_1/\eta_s|^\beta = |\eta_1/\eta_s|^{1+\alpha}, \tag{5.14}$$

i.e. $\beta = 1 + \alpha$.

Since H_1 is fixed, and α and β are related, we are thus left with a "minimal" class of pre-big bang models [116, 305], characterized by a two-dimensional parameter space. As a convenient set of physical parameters we may choose, for instance, the duration of the string phase in conformal time, $z_s = \eta_s/\eta_1$, and the value of the string coupling $g_s(\eta_s)$ at the beginning of the high-curvature string phase.

For such a background, the effective potential $V(\eta)$ of Eq. (4.56) is characterized by two transition scales, and the corresponding spectrum by two frequency bands, separated by the limiting frequency ω_s: we have in fact high-frequency modes with $\omega > \omega_s$ hitting the barrier (or crossing the horizon) during the string phase, and low-frequency modes with $\omega < \omega_s$ hitting the barrier during the dilaton phase. By applying the general procedure illustrated in the previous section one then finds the following spectral distribution [116]:

$$\begin{aligned} \Omega_G(\omega, t_o) &\simeq g_1^2 \Omega_\gamma(t_0) \left(\frac{\omega}{\omega_1}\right)^{3-2\nu}, && \omega_s < \omega < \omega_1, \\ &\simeq g_1^2 \Omega_\gamma(t_0) \left(\frac{\omega_1}{\omega_s}\right)^{2\nu} \left(\frac{\omega}{\omega_1}\right)^3, && \omega < \omega_s \end{aligned} \tag{5.15}$$

(modulo logarithmic corrections, see Section 4.2). Such a spectrum has three important properties.

- The low-frequency band is characterized by a nearly thermal (i.e., Rayleigh–Jeans) behaviour, $\Omega \sim \omega^3$, that simulates the low-energy part of a black-body spectrum.
- At high frequencies the slope is at most cubic, but in general flatter, as $\Omega \sim \omega^{3-2\nu}$, with $\nu > 0$.
- The end-point values of the spectrum are approximately known, and fixed in terms of the fundamental ratio $g_1 = H_1/M_{\rm P} \simeq M_{\rm s}/M_{\rm P}$:

$$\omega_1 \simeq g_1^{1/2} 10^{11} \text{ Hz}, \qquad \Omega(\omega_1) \simeq 10^{-4} g_1^2. \tag{5.16}$$

The spectral behaviour of this class of backgrounds, computed by perturbing the lowest-order string effective action, is expected to remain valid even when the α' corrections are included (to first order) in the perturbed action [277]: the spectrum remains the same, only the amplitude is slightly modified, with no appreciable consequence for an order-of-magnitude estimate (as confirmed also by recent numerical simulations, [148]). What remains model-dependent, however, is the position of the break in the spectrum, determined by the coordinates $\omega_{\rm s}$ and $\Omega(\omega_{\rm s})$. For comparison with the standard predictions of the de Sitter inflationary scenario, we have thus plotted in Fig. 5.3 three possible spectra, for a graviton background of pre-big bang origin, corresponding to different durations of the string phase, and to different slopes of the string branch of the spectrum.

It should be mentioned that the full spectrum of tensor metric perturbations has recently been computed for a regular class of minimal pre-big bang models, including in the linearized perturbation equations also the contributions from the higher-derivative and loop corrections needed for a smooth transition to the post-big bang regime [148]. It has been checked, in such a context, that the spectrum of the low-frequency modes $\omega < \omega_{\rm s}$, crossing the horizon in the low-curvature regime, is unaffected by the higher-order corrections. Also, it has been explicitly confirmed that the "string branch" of the spectrum ($\omega > \omega_{\rm s}$) is flatter than at low frequency, and that its amplitude can be reliably estimated also by means of the low-energy perturbation equation. It has been found, however, that if the transition to the post-big bang regime goes through a phase described by the fixed-point solutions of the (first order in α') truncated string effective action (see Subsection 8.3), then the slope of the string branch of the spectrum is at least quadratic, i.e. still too steep for a possible detection by present gravitational antennas (see Subsection 5.4).

In any case, the graviton spectrum of the minimal pre-big bang scenario is monotonically growing and, as a consequence, the most constraining bound is provided by nucleosynthesis, as is evident from Fig. 5.3. The string branch of the spectrum can be extended (at least in principle) to arbitrarily low frequencies, provided that the slope is not too flat, to avoid

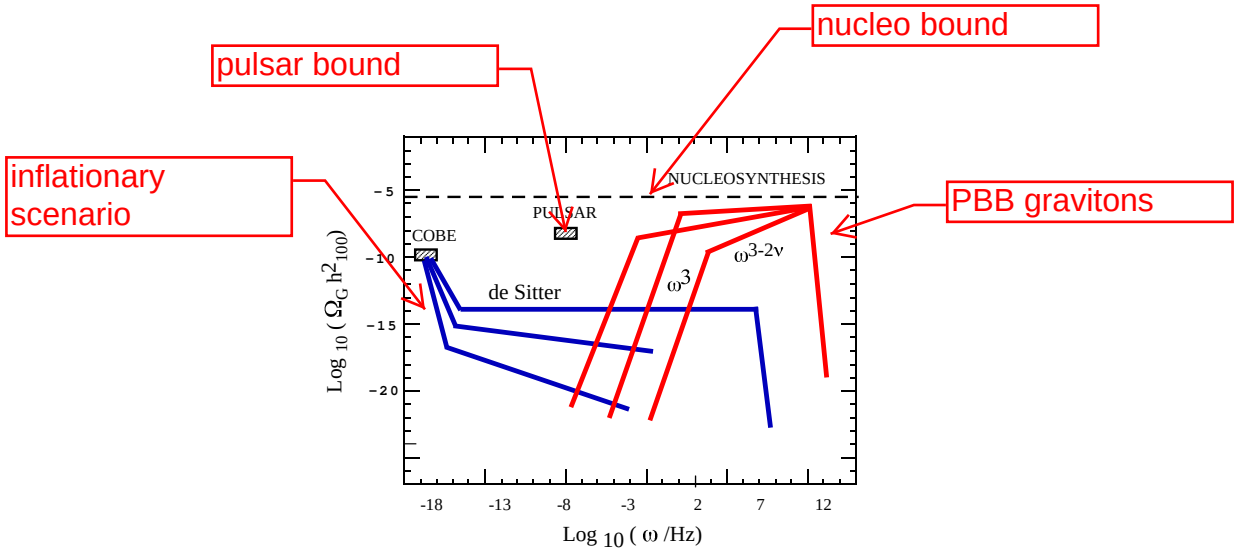

Figure 5.3: *Expected graviton spectra in minimal pre-big bang models, compared with the flat (or decreasing) spectra of the standard inflationary scenario.*

a conflict with the pulsar bound. It should be stressed, however, that the nucleosynthesis bound applies to the total integrated energy density, and thus becomes more and more stringent as the spectrum is flatter and the string phase is longer.

Given our present ignorance of the dynamical details of the string phase, where high-curvature and quantum-loop effects are expected to play a significant role, the precise shape of the spectrum turns out to be (at present) strongly model-dependent (as also stressed by the examples analysed in [148]). It is remarkable that, in spite of this uncertainty, it may be possible to provide an accurate (and rather model-independent) estimate of the theoretically allowed region for the expected spectrum, in the plane of Fig. 5.3.

5.3 Allowed region in the $\omega - \Omega$ plane

The determination of the allowed region requires a precise estimate of the coordinates of the end-point of the spectrum [117], ω_1 and $\Omega(\omega_1)$. The first one is the maximal amplified frequency, corresponding approximately to the production of one graviton per polarization mode and per unit of phase-space volume. The present value of this frequency is:

$$\omega_1(t_0) = \frac{H_1 a_1}{a(t_0)}. \tag{5.17}$$

The second one is the associated peak value,

$$\Omega(\omega_1) = \frac{\omega_1^4}{\pi^2 \rho_c} \tag{5.18}$$

(from Eq. (4.92)). The present value of ω_1 contains two elements of intrinsic uncertainty.

A first uncertainty corresponds to the identification $H_1 = M_s$. It is true that the final transition scale should be controlled by the string mass scale, but the exact value of H_1 might be slightly different from M_s. This uncertainty is expected to be small, however, and can be neglected in a first approximation.

A second (and just moreimportant) uncertainty is associated to the rescaling of ω_1 from the transition time t_1 down to t_0. There are in fact two possibilities. If the CMB radiation that we observe today was all produced by the transition at $t = t_1$, together with the gravitons, then the ratio Ω_G/Ω_γ remains fixed throughout the subsequent cosmological evolution from t_1 down to t_0, the redshift of ω_1 is known and can be computed exactly. If, on the contrary, there was some process of radiation production due to some reheating phase occurring at time scales much later than t_1, then the present value of the ratio Ω_G/Ω_γ can be arbitrarily diluted with respect to its original value at t_1, corresponding to a redshift of ω_1 which is in principle unknown and arbitrarily large.

To take this last possibility into account, we may introduce the phenomenological parameter δs, representing the fraction of the present CMB entropy density due to all reheating processes following the end of the string phase at $t = t_1$. This parameter is defined by

$$\delta s = \frac{s_0 - s_1}{s_0}, \tag{5.19}$$

where s_0 and s_1 are the thermal entropy density of the CMB background at t_0 and t_1, defined in terms of the temperatures (T_0, T_1) and of the numbers (n_0, n_1) of particle species contributing (with their own statistical weight) to the entropy:

$$s_0 = \frac{2\pi^2}{45} n_0 (a_0 T_0)^3, \qquad s_1 = \frac{2\pi^2}{45} n_1 (a_1 T_1)^3. \tag{5.20}$$

By using $n_1/n_0 \sim 10^3$, and assuming thermal equilibrium at $t = t_1$, we then obtain [117]:

$$\begin{aligned} \omega_1(t_0) &\simeq T_0 \left(\frac{M_s}{M_P}\right)^{1/2} (1-\delta s)^{1/3}, \\ \Omega_G(\omega_1, t_0) &\simeq 7 \times 10^{-5} h_{100}^{-2} \left(\frac{M_s}{M_P}\right)^2 (1-\delta s)^{4/3}, \end{aligned} \tag{5.21}$$

where $T_0 = 2.7$ K $\simeq 3.6 \times 10^{11}$ Hz is the present CMB temperature.

At fixed δs, the main uncertainty on the end-point values is thus controlled by the fundamental ratio M_s/M_P, as illustrated in Fig. 5.4, where the boxes around the end point display an uncertainty corresponding to the allowed range $0.01 < (M_s/M_P) < 0.1$. By varying δs, the end point moves along the thin line with slope ω^4. The present peak intensity is thus depressed for large values of δs; however, as we can see from the picture, even if 99% of the present entropy had been produced during the latest stage of evolution, the intensity of the relic graviton background would stay well above the full line labelled "de Sitter", which represents the most optimistic prediction of the standard inflationary scenario in this frequency range.

coupling constants depend on dilaton!!!!!

g = e^2phi

We note, finally, that the same numerical value of the ratio $M_{\rm s}/M_{\rm P}$ also controls the amplitude of the bulk graviton spectrum possibly produced in the context of a pre-big bang scenario with large extra-dimensions [35, 31]. Indeed, even if the D-dimensional Planck scale is smaller (for larger volumes of the internal space), the string scale is correspondingly lowered, so as to keep the grand-unified gauge coupling in the above allowed range.

The above discussion refers to the minimal models illustrated in the previous subsection, where the end point and the peak of the spectrum are strongly correlated, according to Eq. (5.21). Actually, what is really fixed in the pre-big bang scenario is the maximal height of the peak, but not necessarily the position in frequency, and it is not impossible, in more complicated, *non-minimal* models, to shift the peak to lower frequencies.

In the minimal models, in fact, the end of the string phase coincides with the freezing of the coupling parameter $g = e^{\phi/2}$, and with the beginning of the standard radiation era. In non-minimal models [278] (see also [45]) the coupling is still very small at the end of the string phase, $g(t_1) \ll 1$, the dilaton keeps growing in a decelerated way while the curvature starts decreasing, and the produced radiation becomes dominant only much later (such models are also important in the context of reheating, see [132]).

The main difference between the two classes of models is that in the non-minimal case the effective potential $V(\eta)$ appearing in the canonical perturbation equation is non-monotonic, so that the highest frequency modes re-enter the horizon during the intermediate dilaton-driven phase that follows the string phase and precedes the beginning of the radiation era. This modifies the slope of the spectrum in the high-frequency sector, with the possible appearance of a negative power [278] (see also [262]). The gravity-wave spectrum may thus become non-monotonic, and the peak may no longer coincide with the end point, as illustrated in Fig. 5.4. A similar behaviour for the spectrum is also obtained in the context of non-minimal pre-big bang models that include a phase of "thermal inflation", or a period of early matter domination [480]. A phase of matter-dominated pre-big bang evolution can also produce a flat graviton spectrum [251], but in that case the intensity of the relic background is much smaller than the peak values reported in Fig. 5.4.

These are good news from an experimental point of view, because they suggest the possibility of a large detectable signal, even at frequencies much lower than the gigahertz band. However, they also provide a warning against too naive an interpretation and extrapolation of possible future experimental data, because of the complexity of the parameter space of the string cosmology models. It should be noted, for this purpose, that the spectrum associated to the string phase could be monotonic only on the average, and could locally oscillate even in minimal models [133], as illustrated by the wavy line appearing in the high-frequency branch of the spectrum of Fig. 5.4.

Given the present theoretical uncertainties, it seems appropriate to define the maximal allowed region for the expected graviton background, i.e. the region spanned by the spectrum in the $\{\omega, \Omega\}$ plane when all its parameters are varied. Such a region is illustrated

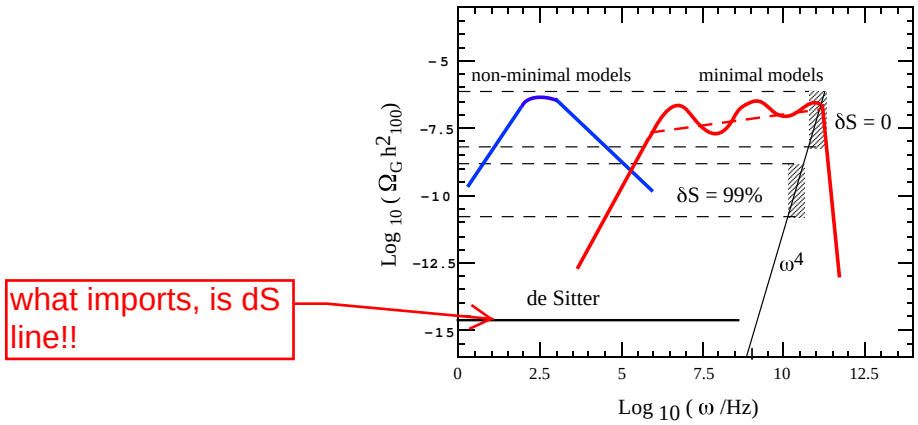

Figure 5.4: *Peak and end point of the graviton spectrum in minimal and non-minimal models, as a function of δs. The boxes at the end of each strip represent the uncertainty due to the unknown value of the ratio $M_{\rm s}/M_{\rm P}$.*

in Fig. 5.5, for the (phenomenologically interesting) high-frequency range $\omega > 1$ Hz. The figure emphasizes the possible, large enhancement (of about eight orders of magnitude) of the expected graviton background in pre-big bang models, with respect to the standard inflationary scenario. The maximal allowed intensity of the graviton background coincides with the upper border of the strip with $\delta s = 0$ in Fig. 5.4, i.e. from Eq. (5.21):

$$\Omega_G^{\rm max} \simeq 10^{-6} h_{100}^{-2}. \tag{5.22}$$

Note that this upper limit is automatically compatible with the nucleosynthesis bound, Eq. (5.4), even if the spectrum is nearly flat from the end point down to the hertz scale.

This maximal amplitude can be conveniently expressed in terms of a useful experimental parameter, the strain density S_h, defined by [255]

$$\langle h(\nu) h^*(\nu')\rangle = \frac{1}{2}\delta(\nu-\nu') S_h(\nu), \tag{5.23}$$

and related to the energy density (5.7) (using the stochastic condition (4.67)) by [278, 463]:

$$S_h(\nu) = \frac{3H_0^2}{4\pi^2\nu^3}\,\Omega_G(\nu), \qquad \nu = \omega/2\pi. \tag{5.24}$$

The minimal experimental sensitivity required for a direct detection (by a single gravitational antenna) of the maximal background (5.22) of pre-big bang gravitons is thus given by [117]

$$\left(S_h^{\rm min}\right)^{1/2} \simeq 3\times 10^{-26}\left(\frac{\rm kHz}{\nu}\right)^{3/2}\ {\rm Hz}^{-1/2}. \tag{5.25}$$

This shows that the minimal sensitivity needed to reach the border of the allowed region grows with the frequency band to which the detector is tuned, even if such a border is the

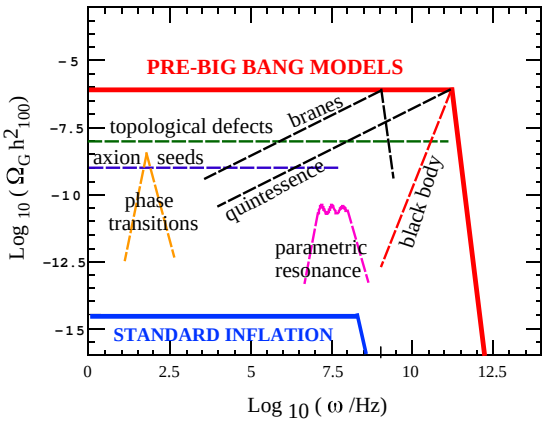

Figure 5.5: *Allowed region for the vacuum-fluctuation spectrum in string cosmology and in the standard inflationary scenario, compared with other possible tensor perturbation spectra of primordial origin.*

same at all frequency scales (see Eq. (5.22) and Fig. 5.5). This effect may favour –from an experimental point of view– detectors working at lower frequencies. Unfortunately –from a theoretical point of view– the probability of a large background intensity seems to be larger at higher frequencies, since the typical spectrum of pre-big bang models is growing. It is important to stress, however, that the minimal required sensitivity (5.25) may be somewhat relaxed, for realistic detection strategies in which the search for a stochastic background proceeds through the cross-correlation of the outputs of two independent detectors (see the next subsection).

The relic background associated with the parametric amplification of the vacuum fluctuations is compared, in Fig. 5.5, with other possible backgrounds of primordial origin, obtained with different mechanisms: graviton radiation from cosmic strings [137, 68] and other topological defects [473], from axion seeds [608], from bubble collision at the end of a first-order phase transition [590], and from parametric resonance of the inflaton oscillations [411, 66]. The spectrum obtained from a phase transition, plotted in the figure, refers to a typical reheating temperature $T_{\rm r} \sim 10^8 - 10^9$ GeV, but higher peak values are possible for higher $T_{\rm r}$ [590].

Also shown in the figure is the linearly growing spectrum obtained in recently proposed models of "quintessential" inflation [510, 334] and of "brane-world" inflation [550], and a thermal black-body spectrum corresponding to a present temperature $T_0 \sim 1$ K. In the standard inflationary scenario, a thermal background might originate at the Planck scale, when the temperature is high enough to maintain gravitons in thermal equilibrium. Such a background, however, should be strongly diluted (with respect to the present CMB radiation) by the action of the subsequent inflationary phase, occurring at curvature scales

lower than Planckian. As a consequence, the surviving spectrum should correspond today to an effective temperature so depressed as to be practically invisible. In the context of the pre-big bang models, on the contrary, a thermal spectrum like that of Fig. 5.5 could be simulated by the sudden transition of the low-energy dilaton-driven phase to the standard radiation-dominated era – corresponding, for instance, to a minimal model of Section 5.2 with $\eta_s \sim \eta_1$.

It may be noted that all the primordial backgrounds of Fig. 5.5 are higher (in intensity) than the background obtained from the amplification of the vacuum fluctuations in the standard inflationary scenario, but not higher than the analogous background expected in the context of the pre-big bang scenario. In any case, given the rich phenomenology of possible spectra, it seems appropriate to report at this point the experimental sensitivities of present gravitational antennas, referred in particular to the frequency range of Fig. 5.5. This will be the subject of the next subsection.

5.4 Experimental sensitivities

The limiting sensitivity (5.25) (corresponding to the upper border of the allowed region in Fig. 5.5) is unfortunately too high for the detectors that are at present in operation: the cryogenic, resonant-mass, gravitational antennas (see [520] for a review). In fact, if we take for instance the resonant bar NAUTILUS [173], the resonance frequencies are at 907 and 922 Hz and, at these frequencies, the strain sensitivity is approximately

$$\sqrt{S_h} \simeq 5 \times 10^{-22}\ \mathrm{Hz}^{-1/2}. \tag{5.26}$$

With some improvements, the sensitivity is expected to reach in a few years the target value $8 \times 10^{-23}\ \mathrm{Hz}^{-1/2}$. Similar resonance frequencies (around 900 Hz) and similar sensitivities are also typical of the other resonant detectors such as ALLEGRO [361], AURIGA [528], and EXPLORER [506]. The resonant bar NIOBE [374], however, is working at a smaller frequency (~ 700 Hz) and could reach, in principle, a slightly better sensitivity.

Interferometric detectors, such as LIGO [4], VIRGO [145], GEO600 [448] and TAMA300 [405], are already basically built and will soon become operative. The best planned sensitivity, for the first-generation instruments, is around

$$\sqrt{S_h} \simeq 10^{-22}\ \mathrm{Hz}^{-1/2}, \tag{5.27}$$

very similar to the present bar sensitivities. This sensitivity, however, is available at a smaller frequency band, $\nu = 100$ Hz, and thus corresponds to a smaller limit in terms of the energy density of the graviton background (see Eq. (5.24)). In practice, the interferometers of first generation will be directly sensitive to $\Omega_G h_{100}^2 \sim 10^{-1} - 10^{-2}$, better than resonant bars, which can at most reach the critical density, but still well above the border of the allowed region.

In order to approach the border sensitivity $\Omega_G h_{100}^2 \sim 10^{-6}$, we must wait for the second (and higher) generation of interferometers. The Advanced LIGO project [444], for instance, might lead to a two orders of magnitude improvement of the sensitivity (5.27), at $\nu = 100$ Hz, thus reaching the borderline (5.22). An even better result is expected for the space interferometer LISA [379], whose goal is to reach the strain sensitivity [362] $\sqrt{S_h} \simeq 4 \times 10^{-21}$ $\mathrm{Hz}^{-1/2}$ in the frequency band from 3 to 10 mHz, corresponding (in this band) to the remarkable limit $\Omega_G h_{100}^2 \sim 10^{-11}$.

Another interesting possibility concerns resonant detectors with spherical (or truncated icosahedron) geometry, such as the TIGA [387, 481] or SFERA [39] projects. Hollow spheres [172] are particularly promising: a large copper–aluminium detector (4 – 5 m in diameter), pushed to its extreme quantum limit may reach a strain sensitivity[2] $\sqrt{S_h} \simeq 3 \times 10^{-24}$ $\mathrm{Hz}^{-1/2}$, with resonant frequencies of 600 and 2500 Hz (the resonance frequency can be tuned, in principle, by varying the geometrical radius of the sphere). Such a limiting sensitivity can be approached by the recently proposed wide-band "dual-sphere" detector [156], which consists of a massive solid sphere suspended inside a larger hollow one. In that case, a sapphire detector of an overall size of 2.6 m could reach a strain sensitivity $\sqrt{S_h} \simeq 10^{-23} \mathrm{Hz}^{-1/2}$, for frequencies between 1 and 4 kHz.

Concerning higher frequencies (well above the kilohertz), the only possibility at present seems to be the use of resonant electromagnetic cavities [512, 534] or waveguides [187]. There are also recent experimental studies for detecting small displacements based on two coupled microwave cavities [80]: such a detector reaches the maximum sensitivity when the frequency of the incident gravitational wave is equal to the frequency-difference of the two cavity-modes. In this way, it seems possible at present to reach the strain sensitivity $\sqrt{S_h} \simeq 10^{-20}$ $\mathrm{Hz}^{-1/2}$ for frequencies around the kilohertz range [160]. Unfortunately, however, this value is well above the minimal required sensitivity (5.25).

As shown explicitly by the above examples, the limiting sensitivity (5.25) seems not to be within easy reach of present technology, for the case of a single detector. In addition, an unambiguous detection of a stochastic gravity-wave background with a single experimental apparatus would require a complete and exact knowledge of all the intrinsic instrumental noises, and of all the backgrounds of different origins that could interact with the gravitational antenna. Fortunately, an efficient answer to both difficulties is known: it is provided by the cross-correlation of the outputs of two (or more) detectors [162, 255, 13, 17].

Suppose that the outputs of two detectors, $s_i(t)$, $i = 1, 2$, are correlated over an integration time T, to define a signal:

$$S = \int_{-T/2}^{T/2} dt \; dt' s_1(t) s_2(t') Q(t - t'). \tag{5.28}$$

Here $Q(t)$ is a real "filter" function, determined so as to optimize the signal-to-noise ratio

[2] M. Cerdonio and L. Conti, private communication.

(SNR), defined by an ensemble average as:

$$\mathrm{SNR} = \langle S\rangle/\Delta S \equiv \langle S\rangle \left(\langle S^2\rangle - \langle S\rangle^2\right)^{-1/2}. \tag{5.29}$$

The outputs $s_i(t) = h_i(t) + n_i(t)$ contain the physical strain induced by the cosmic background, h_i, and the intrinsic instrumental noise, n_i. The two noises are supposed to be uncorrelated (i.e. statistically independent), $\langle n_1(t)n_2(t')\rangle = 0$, and much larger in magnitude than the physical strains, $|n_i| \gg |h_i|$. Also, the cosmic background is assumed to be isotropic, stationary and Gaussian, with $\langle h_i\rangle = 0$. It follows that [17]:

$$\langle S\rangle = \int_{-T/2}^{T/2} dt\ dt' \langle h_1(t)h_2(t')\rangle Q(t-t'). \tag{5.30}$$

The physical strain is given by the projection of the metric fluctuation $h_{ab}(x_i,t)$ (computed at the detector position $x = x_i$) over the detector response tensor D_i^{ab}, which characterizes the spatial orientation of the arms of the i-th detector. If $\hat{n}$ is a unit vector specifying a direction on the two-sphere, and $e_{ab}(\hat{n})$ is the gravity-wave polarization along $\hat{n}$, we can expands the strain in momentum space as

$$\begin{aligned} h_i(t) &= \int dp \int d^2\hat{n}\ h_A(p,\hat{n}) F_i^A(\hat{n}) e^{2\pi i p(\hat{n}\cdot\vec{x}_i - t)}, \\ F_i^A(\hat{n}) &= e_{ab}^A(\hat{n}) D_i^{ab}, \end{aligned} \tag{5.31}$$

where $d^2\hat{n}$ denotes the angular integral over the unit two-sphere, and the index A labels the two polarization modes of the wave; $F_i^A(\hat{n})$ is the so-called "pattern function", depending on the geometry of the detector. Note that in the above equation we have used the proper momentum $\vec{p} = \vec{k}/a$, even if the cosmological variation induced by the expansion of the scale factor is negligible over a typical experimental time scale T. Note also that, for a better comparison with experimental variables, we have used units in which $h = 1$, so that, for a gravity wave, $p = \nu = \omega/2\pi$ (but not in general for massive waves, see Section 6.4).

For an isotropic, unpolarized and stationary background we can then use the stochastic condition to define [17, 463]

$$\langle h_A^*(p,\hat{n}), h_{A'}(p',\hat{n}')\rangle = \frac{1}{4\pi}\delta_{AA'}\delta(p-p')\delta^2(\hat{n},\hat{n}')\frac{1}{2}S_h \tag{5.32}$$

(the integration over the angular variables $d^2\hat{n}, d^2\hat{n}'$ then reproduces Eq. (5.23)). By inserting the momentum expansion into Eq. (5.30), and assuming, as usual, that the observation time T is much larger than the typical time intervals $t - t'$ for which $Q \neq 0$, we finally obtain, using Eqs. (5.23) and (5.24):

$$\langle S\rangle = NT\frac{3H_0^2}{8\pi^2}\int \frac{dp}{p^3}\gamma(p)Q(p)\Omega(p), \tag{5.33}$$

where $Q(p)$ is the filter function in momentum space, and $\gamma(p)$ is the so-called "overlap-reduction function":

$$\gamma(p) = \frac{1}{N}\int \frac{d^2\hat{n}}{4\pi} F_1(\hat{n})F_2(\hat{n})e^{2\pi i p\hat{n}\cdot(\vec{x}_2-\vec{x}_1)}. \tag{5.34}$$

Here N is an overall normalization factor, depending on the type of the detectors (for two interferometers, a convenient choice is $N = 2/5$, see [17]; see [463] for the normalization of different antennas).

We now need to compute the variance ΔS^2, which, for uncorrelated noises much larger than the physical strains, can be expressed as [17]:

$$\Delta S^2 \simeq \langle S^2\rangle = \int_{-T/2}^{T/2} dtdt'd\tau d\tau' \langle n_1(t)n_1(\tau)\rangle\langle n_2(t')n_2(\tau')\rangle Q(t-t')Q(\tau-\tau'). \tag{5.35}$$

Introducing the (one-sided) noise power spectrum in momentum space, $P_i(p)$, defined by

$$\langle n_i(t)n_i(\tau)\rangle = \frac{1}{2}\int dp P_i(p)e^{-2\pi i p(t-\tau)}, \tag{5.36}$$

and assuming as before that T is much larger than the typical correlation intervals $t - t'$, $\tau - \tau'$, we obtain

$$\Delta S^2 = \frac{T}{4}\int dp P_1(p)P_2(p)Q^2(p). \tag{5.37}$$

The optimal filtering is now determined by the choice (see [17] for details)

$$Q(p) = \lambda\frac{\gamma(p)\Omega(p)}{p^3 P_1(p)P_2(p)}, \tag{5.38}$$

where λ is an arbitrary normalization constant. With such a choice one finally arrives, from Eq. (5.33) and (5.38), to the optimized signal-to-noise ratio:

$$\mathrm{SNR} = \frac{\langle S\rangle}{\Delta S} = N\frac{3H_0^2}{4\pi^2}\left[T\int \frac{dp}{p^6}\frac{\gamma^2(p)\Omega^2(p)}{P_1(p)P_2(p)}\right]^{1/2}. \tag{5.39}$$

The background can then be detected, with a detection rate γ, and a false-alarm rate α, if [17]:

$$\mathrm{SNR} \geq \sqrt{2}\left(\mathrm{erfc}^{-1}2\alpha - \mathrm{erfc}^{-1}2\gamma\right). \tag{5.40}$$

Armed with the above results, it is now possible to estimate the minimum detectable background at a given confidence level, for any pair of gravitational antennas of given noise power spectrum P_i and overlap γ. The correlation of all existing pairs of interferometers has been studied in [44] and [17]. With the sensitivities of the first-generation interferometers, for an observation time $T = 4$ months, a detection rate of 0.95, a false-alarme rate of 0.05, and a flat graviton spectrum $\Omega_G = \mathrm{const}$, the minimum detectable value is

$$\Omega_G \mathrm{h}_{100}^2 \simeq 5\times 10^{-6}. \tag{5.41}$$

This value can be improved only slightly by increasing the number of correlated detectors. With the planned sensitivity of Advanced LIGO, however, one obtains the much better limit [13, 17]:

$$\Omega_G h^2_{100} \simeq 5 \times 10^{-11}. \tag{5.42}$$

The correlation between two resonant bars, and between a bar and an interferometer, has been studied in [40, 41, 614, 44]. The minimum detectable values for the VIRGO-AURIGA or VIRGO-NAUTILUS or AURIGA-NAUTILUS pairs is at present [463] $\Omega_G h^2_{100} \sim$ a few $\times 10^{-4}$ (depending on the bar orientation) for a flat spectrum, one year of observation, at the 90% confidence level. The correlation of two spherical detectors seems to be more promising, however. Two spheres (of 3 m diameter) located at the same site could reach a sensitivity [614] $\Omega_G h^2_{100} \sim 4 \times 10^{-7}$. Also, studies reported in [172] suggest that the correlation of hollow spheres could reach $\Omega_G h^2_{100} \sim 10^{-9}$.

All the above results refer to a flat graviton spectrum. The signal-to-noise ratio, and therefore the sensitivity, depend, however, on $\Omega_G(p)$, and then on the slope of the spectrum (see Eq. (5.39)), as also stressed in [463, 46]. The sensitivity to a growing spectral distribution such as that predicted by pre-big bang models has been computed in particular in [14, 593] and [45, 47], for a pair of LIGO and a (virtual) pair of VIRGO interferometers, respectively. The minimum detectable intensity of the background, in that case, does not show any improvement with respect to a flat spectrum; on the contrary, the sensitivity could even suffer from a slight decrease, if the maximum of the signal is near the border of the accessible frequency band, where the noise of the interferometer is higher.

Finally, the cross-correlation in the case of space interferometers such as LISA has recently been discussed in detail [594] by taking into account that, at lower frequencies, the sensitivity to a stochastic graviton background of primordial inflationary origin is fundamentally limited by the possible presence of other astrophysical backgrounds, generated at much later times, due for instance to binary systems of compact objects that cannot be resolved as individual sources. A detailed analysis, performed for the cross-correlation of two identical LISAs, leads to a minimum detectable signal $\Omega_G h^2_{100} \sim 10^{-12}$, in the mHz range, for a flat spectrum. The three arms of LISA could also be combined in such a way as to simulate two inteferometers located in the same place; unfortunately, they would be rotated by an angle of $\pi/4$ with respect to one another, and the corrresponding overlap function would be identically zero over the whole frequency range. This is the reason why two space interferometers are needed, even in that case.

To conclude this subsection, let us recall the best (direct) experimental upper bound existing at present on the energy density of a stochastic graviton background, which has been obtained from the cross-correlation of the two resonant bars NAUTILUS and EXPLORER [37, 38]:

$$\Omega_G h^2_{100} \lesssim 60, \qquad\qquad \nu \simeq 907 \text{ Hz}. \tag{5.43}$$

This results corresponds to a very small observation time ($T \sim 12$ hours), and it is thus far

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VERY IMPORTANT!!!

from the theoretical upper limit (5.25), but the future perspectives seem to be promising, as reported in the above discussion. The cross-correlations of advanced (ground-based) interferometers, space interforemeters, spheres, hollow spheres, are all expected to cross the borderline sensitivity (5.25), and then to explore, in a very near future, the allowed region of the string cosmology spectrum. Thus, future data from gravitational antennas will directly constrain the parameter space of the minimal and non-minimal classes of pre-big bang models considered in this section.

We should add, finally, that useful information and constraints on the relic graviton background will also be obtained from future measurements of the CMB polarization [559, 400, 432]. The presence of a gravitational-wave background should in fact induce a characteristic "curl component" in the large scale CMB polarization [401], which might be detectable by dedicated experiments (already at the sensitivity level of the Planck satellite [521]) for the gravitational spectra predicted by the standard inflationary models. No polarization signal is expected to be detectable, on the contrary, for the graviton spectra of pre-big bang models, whose amplitude is extremely suppressed at the large scales accessible to the polarization experiments (see, however, Ref. [251] for possible exceptions).

6 Relic dilatons

The effective action of string cosmology always contains at least two fundamental fields, the metric and the dilaton. The amplification of the dilaton fluctuations leads to dilaton production, just like the amplification of the transverse, traceless components of the metric fluctuations leads to graviton production. The possible formation of a stochastic background of relic dilatons [271, 321, 276], with statistical and squeezing properties similar to those of the graviton background, is one of the most peculiar phenomenological aspects of pre-big bang models.

Banks/Kaplan/ Nelson massive dilaton

The main physical difference with the graviton background, discussed in the previous section, concerns the possible contribution of the mass to the dilaton spectrum (see Subsection 6.1). The dilaton should indeed be massive, according to conventional models of supersymmetry breaking [52]. Also, the dilaton *must* be massive if it is coupled non-universally to macroscopic matter with gravitational strength [577, 237], in order to avoid unacceptable violations of the equivalence principle. The value of the dilaton mass, however, is largely unknown from a theoretical point of view. There are phenomenological constraints on the mass, but they depend on the effective coupling of dilatons to macroscopic matter, which we shall discuss in Subsection 6.2. The value of the coupling, in turn, is strongly model-dependent [197, 198], and phenomenologically constrained, at present, only by tests of the equivalence principle and of Newtonian gravity available on a macroscopic scale [253, 380].

Taylor /vene dilaton coupling

As a consequence, the present amplitude of the relic dilaton background, controlled by the string and dilaton mass, is more uncertain than in the graviton case. This is disap-

pointing, on the one hand, as it prevents the existence of an unambiguous link between the spectral properties of the dilaton background and the parameters of pre-big bang models. On the other hand, however, it leaves open two interesting phenomenological possibilities: a relic background of non-relativistic dilatons could represent a significant fraction of the present dark matter density [271] (see Subsection 6.3) and, if the mass is small enough, it could be detectable by the cross-correlation of gravitational antennas [287] already at the sensitivity level of enhanced interferometric detectors [317], as will be discussed in Subsection 6.4.

6.1 Dilaton production

In this subsection we will estimate the spectral distribution of the dilatons obtained from the amplification of the vacuum fluctuations, in the context of pre-big bang models. There are, of course, also other cosmological mechanisms of production, such as particle collisions at high (Planckian) temperature, coherent oscillations around the minimum of the scalar potential (and more exotic mechanisms, such as dilaton radiation from cosmic strings; see for instance [200]). We will concentrate here on the parametric amplification because, even if the temperature remains too low to allow thermal production, and oscillations are avoided through a symmetry that ensures the coincidence of the minima of the potential at early and late times [199], quantum fluctuations cannot be eliminated; also, they may be expected to represent the dominant source [457] when the inflation scale is not smaller than about 10^{16} GeV, as in the context of the pre-big bang scenario.

The production of dilatons through the amplification of the quantum fluctuations of the scalar background is very similar to the amplification of tensor perturbations, discussed in Section 5. Unlike the graviton case, however, the analysis of dilaton perturbations is complicated by their coupling to the scalar component of metric perturbations, and to the scalar perturbations of the matter sources. The matter sources may even be absent from the pre-big bang phase, but they are certainly not absent when dilatons re-enter the horizon, in the radiation- and matter-dominated epoch.

In order to discuss this effect, we shall work in the E-frame, in the longitudinal gauge, and we shall assume that the d-dimensional, homogeneous and isotropic, unperturbed background is driven by the dilaton and by a perfect-fluid model of matter sources. Our independent variables are the scalar metric, dilaton and fluid perturbation, respectively φ, ψ, χ, $\delta\rho$, δp, δu_i, defined by:

$$ds^2 = a^2\left[d\eta^2\,(1+2\varphi) - (1-2\psi)\,dx_i^2\right], \qquad \delta\phi = \chi,$$
$$\delta T_0^0 = \delta\rho, \qquad \delta T_i^j = -\delta p\,\delta_i^j, \qquad \delta T_i^0 = \frac{p+\rho}{a}\delta u_i. \tag{6.1}$$

In the E-frame the fluid sources are non-trivially coupled to the dilaton, and the unper-

turbed equations take the form (in units $16\pi G = 1$):

$$2R_\mu{}^\nu - \delta_\mu^\nu R = \partial_\mu\phi\partial^\nu\phi + \delta_\mu^\nu\left[V - \frac{1}{2}(\nabla\phi)^2\right] + T_\mu{}^\nu$$
$$\nabla_\mu\nabla^\mu\phi + \frac{\partial V}{\partial\phi} + cT = 0, \tag{6.2}$$

where the numerical factor $c = \sqrt{2/(d-1)}$ arises from the conformal transformation of the S-frame string cosmology equations (see for instance Eq. (2.113)). The perturbation of these equations, for the $i \neq j$ components, gives $\varphi = \psi$, while the combination of the perturbed $(0,0), (i,i)$ and dilaton equation leads to the vector-like equation for the doublet Z_k, representing the Fourier components of the metric and of the dilaton fluctuations [321]:

$$Z_k'' + 2\frac{a'}{a}\mathcal{A}Z_k' + \left(k^2\mathcal{B} + \mathcal{C}\right)Z_k = 0, \qquad Z_k^\dagger = (\psi_k, \chi_k). \tag{6.3}$$

Here $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are 2×2, time-dependent mixing matrices, whose explicit form depends on the unperturbed solutions, on the equation of state $\gamma(t) = p/\rho$, on the model of fluid perturbations $\epsilon(t) = \delta p/\delta\rho$ and, finally, on the possible presence of a dilaton potential $V(\phi)$. In particular, for an unperturbed background in which the dilaton can be parametrized by $\phi = \beta \ln a$, with $\beta = \text{const}$, the mixing matrices are given by

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2}(2d - 3 + d\epsilon), & -\frac{1-\epsilon}{4(d-1)}\beta \\ -(d-1)[\frac{cd}{2}(1-d\epsilon) + \beta], & \frac{1}{2}(d-1) - \frac{c}{4}(1-d\epsilon)\beta \end{pmatrix}, \tag{6.4}$$

$$\mathcal{B} = \begin{pmatrix} \epsilon, & 0 \\ -c(1-d\epsilon)(d-1), & 1 \end{pmatrix}, \tag{6.5}$$

$$\mathcal{C} = \begin{pmatrix} 2(d-2)\frac{a''}{a} + (d-2)[d-4+d\epsilon + \frac{1-\epsilon}{2(d-1)}\beta^2]\left(\frac{a'}{a}\right)^2, & \frac{\epsilon+1}{2(d-1)}V'a^2 \\ cd(d-2)(\epsilon-\gamma)[d(d-1) - \frac{\beta^2}{2}]\left(\frac{a'}{a}\right)^2 + 2(d-2)V'a^2, & [V'' - \frac{c}{2}(1-d\epsilon)V']a^2 \end{pmatrix}, \tag{6.6}$$

where the primes denote differentiation with respect to conformal time, except for the potential, where $V' = \partial V/\partial\phi$. Once Z is known, the density and velocity contrast are fixed in terms of ψ and χ by two additional equations $\delta\rho = \delta\rho(\psi, \chi)$, $\delta u = \delta u(\psi, \chi)$, following, respectively, from the perturbation of the $(0i)$ and (00) background equations:

$$\partial_i\left[2(d-1)\left(\frac{a'}{a}(d-2)\psi + \psi'\right) - \chi\phi'\right] = (\rho + p)a\delta u_i, \tag{6.7}$$

$$\nabla^2\psi - d\frac{a'}{a}\psi' - \left[d(d-2)\left(\frac{a'}{a}\right)^2 - \frac{d-2}{2(d-1)}\phi'^2\right]\psi =$$
$$= \frac{1}{2(d-1)}\left(\phi'\chi' + \frac{\partial V}{\partial\phi}a^2\chi + a^2\delta\rho\right). \tag{6.8}$$

An exact computation of the dilaton fluctuations thus requires an exact solution of the coupled equations (6.3). Also, the correct normalization of the dilaton spectrum to the

quantum fluctuations of the vacuum requires in general the diagonalization of the system *metric + dilaton + fluid sources*, to find the normal modes of oscillation satisfying canonical commutation relations, as discussed in Subsection 4.3.

This program can be easily performed if we limit ourselves to a dilaton-dominated pre-big bang phase with negligible matter sources ($T_{\mu\nu} = 0 = \delta T_{\mu\nu}$), and we consider the high-frequency part of the spectrum, i.e. those modes re-entering the horizon during the radiation-dominated phase, characterized by adiabatic fluid perturbations ($\gamma = \epsilon = 1/d$) and by the dilaton frozen ($\beta = 0$) at the minimum of the non-perturbative potential ($\partial V/\partial\phi = 0$). In that case the canonical variables are known, and one finds that the perturbation equations are decoupled in the radiation era, where the canonical variable for the dilaton fluctuation is simply (in $d = 3$) $\overline{\chi} = a\chi$, and satifies the (covariant) Klein–Gordon equation (from the system (6.3)):

$$\overline{\chi}_k'' + (k^2 + m^2a^2)\overline{\chi}_k = 0. \tag{6.9}$$

For relativistic modes, of proper momentum $p = k/a \gg m$, the solution describes (in conformal time) free plane-wave oscillations, and we thus recover for the dilaton the same spectrum as in the graviton case [321]. Working in the framewok of minimal pre-big bang models (see Subsection 5.2) we find, in particular, that the dilaton modes amplified during the low-energy phase are characterized by the distribution $\Omega_\chi(p) \sim p^3 \ln^2 p$. In order to include the possible corrections due to the modified kinematics of the high-curvature string phase, we may thus parametrize the high-frequency, relativistic branch of the dilaton spectrum as

$$\Omega_\chi(p,t) = g_1^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \left(\frac{p}{p_1}\right)^\delta, \qquad m < p < p_1. \tag{6.10}$$

Here $0 \le \delta \le 3$ is the slope parameter corresponding to a growing, but unknown, spectral index; $g_1 = H_1/M_{\rm P}$ and $p_1 = k_1/a \simeq H_1 a_1/a$ are the parameters associated to the transition scale $H_1 \simeq M_{\rm s}$ of minimal pre-big bang models. It should be noted that, unlike the graviton case, we no longer identify proper momentum p and proper frequency ω, as we are dealing, in principle, with massive particles for which $\omega^2 = p^2 + m^2$.

The above spectral distribution cannot be extrapolated down to momentum scales re-entering the horizon after equilibrium, since in the matter era the perturbation equations (6.3) are no longer decoupled, even if the dilaton background is frozen at the minimum of the potential (they remain coupled not only in the longitudinal gauge, but also in the uniform-curvature gauge, defined in Eq. (4.41)). In addition, for modes of low enough momentum $p \lesssim m$, we must take into account the non-relativistic correction to the above spectrum (indeed, even if the mass term is negligible at the inflation-radiation transition scale, i.e. $k < ma_1$, the proper momentum is redshifted with respect to the mass in the subsequent radiation era, and all modes tend to become non-relativistic).

When including the mass term, the general solution of Eq. (6.9) in the radiation era can be written in terms of the parabolic cylinder functions [5], and its matching to the

(massless) pre-big bang solution provides the correct inclusion of the mass contributions to the spectrum. The details of the computation are the same as in the case of massive axions [224], and will be reported in Section 7. Here we simply note that if a mode becomes non-relativistic well inside the horizon, i.e. when $p(t) > H$, then the number of produced dilatons is fixed after re-entry, when the mode is still relativistic, and the effect of the mass is a simple rescaling of the energy density: $\Omega_\chi \to (m/p)\Omega_\chi$. If, on the contrary, a mode becomes non-relativistic outside the horizon, when $p(t) < H$, then the final energy distribution turns out to be determined by the background kinematics at the exit time, and the effective number of non-relativistic dilatons has to be adjusted by continuity, in such a way that Ω_χ has the same spectral distribution as in the absence of mass. The two regimes are separated by the limiting momentum scale p_m of a mode that becomes non-relativistic just at the time it re-enters the horizon [271, 321]:

$$p_m = p_1 \left(\frac{m}{H_1}\right)^{1/2}. \tag{6.11}$$

Taking into account such mass corrections, we may thus add to the relativistic dilaton spectrum (6.10) two lower-frequency, non-relativistic bands (see also [292]):

$$\Omega_\chi(p,t) = g_1^2 \frac{m}{H_1}\left(\frac{H_1}{H}\right)^2\left(\frac{a_1}{a}\right)^3\left(\frac{p}{p_1}\right)^{\delta-1}, \qquad p_m < p < m, \tag{6.12}$$

$$\Omega_\chi(p,t) = g_1^2 \left(\frac{m}{H_1}\right)^{1/2}\left(\frac{H_1}{H}\right)^2\left(\frac{a_1}{a}\right)^3\left(\frac{p}{p_1}\right)^{\delta}, \qquad p < p_m. \tag{6.13}$$

Note that these spectral distributions evolve in time like a^{-3}, and thus keep constant during the matter-dominated era, for $t > t_{\rm eq}$. Note also that, for simplicity, the non-relativistic corrections have been referred to a single relativistic branch, with $\delta = 3$ for the low-energy pre-big bang phase, and with $\delta \leq 3$ for the string phase. However, the mass corrections can be extended without difficulty to a more general situation in which there are two (or more) branches in the relativistic spectrum, as in minimal pre-big bang models characterized by the two frequency bands $p < p_{\rm s}$ and $p > p_{\rm s}$ (separated by the string scale $p_{\rm s} = H_{\rm s}$). In such a case, the non-relativistic corrections will affect only the lower, or also the higher, frequency band of the spectrum, depending on whether $m < p_{\rm s}$ or $m > p_{\rm s}$, respectively.

If the dilaton mass is large enough, the present amplitude of a possible relic dilaton background is thus controlled by the (unknown) value of its mass. The cosmological bounds on the spectrum can thus be translated into phenomelogical bounds on the mass, depending (but only weakly) on the spectral index δ. Such bounds are to be combined with other, model-dependent bounds on the mass related to the strength of the effective dilaton coupling to macroscopic matter, which will be discussed in the next subsection.

6.2 Effective coupling to macroscopic matter

??????? The motion of macroscopic test bodies in an external, non-trivial, gravidilaton background is in general non-geodesic, if the body posseses a non-negligible dilatonic charge. Suppose in fact that the interaction of macroscopic matter with the external scalar-tensor background is described by the effective action of Subsection 4.1:

$$S = \int d^4x\sqrt{-g}e^{-\phi}\left[-R + \omega\left(\nabla\phi\right)^2\right] + S_M(g,\phi) \tag{6.14}$$

(in units $2\lambda_s^2 = 1$), where the Brans–Dicke parameter ω is equal to -1 for the lowest-order action of string theory, and we have added the action S_M for the matter fields describing a test body non-trivially coupled to the metric and to the dilaton. The variation with respect to $g_{\mu\nu}$ and ϕ provides respectively the field equations (with $16\pi G = 1$):

$$\begin{aligned} &G_{\mu\nu} + \nabla_\mu\nabla_\nu\phi - (\omega+1)\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi + \left(\frac{\omega}{2}+1\right)g_{\mu\nu}\left(\nabla\phi\right)^2 = \frac{1}{2}e^\phi T_{\mu\nu}, \\ &R + \omega\left(\nabla\phi\right)^2 - 2\omega\nabla^2\phi + e^\phi\sigma = 0, \end{aligned} \tag{6.15}$$

containing two source terms: the "gravitational charge density" $T_{\mu\nu}$ (the dynamical stress tensor), and the dilatonic charge density σ:

$$\sqrt{-g}\;T_{\mu\nu} = 2\frac{\delta S_M}{\delta g^{\mu\nu}}, \qquad\qquad \sqrt{-g}\;\sigma = \frac{\delta S_M}{\delta\phi}. \tag{6.16}$$

The combination of the above equations, and the use of the contracted Bianchi identity $\nabla_\nu G^{\mu\nu} = 0$, then implies the covariant conservation of the stress tensor, $\nabla_\nu T^{\mu\nu} = 0$, *for any* ω, *provided* $\sigma = 0$. If the dilatonic charge is non-zero, however, the gravitational charge density is not separately conserved; for $\omega = -1$, in particular, we obtain from Eqs. (6.15):

$$\nabla^\nu T_{\mu\nu} + \sigma\nabla_\mu\phi = 0. \tag{6.17}$$

To obtain the equation of motion of the test body we now integrate this equation over a space-like hypersurface, and perform a multipole expansion of the external metric and dilaton field around the world line $x^\mu(\tau)$ of the centre of mass of the body [507]. In the point-particle approximation, the gravitational and dilatonic charge densities are defined by

$$\begin{aligned} T^{\mu\nu}(x') &= \frac{p^\mu p^\nu}{\sqrt{-g}p^0}\delta^{(3)}\left(x' - x(\tau)\right), \\ \sigma(x') &= q\frac{m^2}{\sqrt{-g}p^0}\delta^{(3)}\left(x' - x(\tau)\right) \end{aligned} \tag{6.18}$$

(where $p^\mu = mu^\mu = mdx^\mu/d\tau$), and we obtain the general equation of motion [285]

$$\frac{du^\mu}{d\tau} + \Gamma_{\alpha\nu}{}^\mu u^\alpha u^\nu + q\nabla^\mu\phi = 0, \tag{6.19}$$

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where q is the dilatonic charge per unit of gravitational mass (representing the relative strength of scalar to tensor forces).

It is now evident that a non-zero q induces deviations from pure geodesic motion, and that the experimental tests on the gravitational forces provide limits on the allowed values of the dilaton charge. Large values of q can only be allowed if the dilatonic interactions, on a macroscopical scale, are suppressed by a sufficiently small range. We may also note, for future reference, that the relative acceleration between two world lines satisfying Eq. (6.19), with infinitesimal separation η_μ, is given by a generalized equation of "geodesic deviation" [285]:

$$\frac{D^2\eta^\mu}{D\tau^2} + R_{\beta\alpha\nu}{}^\mu u^\alpha u^\nu \eta^\beta + q\eta^\beta \nabla_\beta \nabla^\mu \phi = 0. \tag{6.20}$$

We shall come back to this point in Subsection 6.4.

A precise estimate of q, in a string theory context, should start from the effective action describing the coupling of the dilaton to the fundamental quark and lepton fields building up ordinary macroscopic matter. Including all possible loop corrections, such an action can be written in the general form

$$S = \int d^{d+1}x\sqrt{-g}\left[-Z_R(\phi)R - Z_\phi(\phi)\left(\nabla\phi\right)^2 - V(\phi) + \frac{1}{2}Z_k^i(\phi)\left(\nabla\psi_i\right)^2 + Z_m^i(\phi)\psi_i^2\right], \tag{6.21}$$

where Z^i are the dilatonic "form factors", to all orders in the loop expansion, and where we have used, for simplicity, a scalar model of fundamental matter fields ψ_i. The effective masses (m_i) and dilaton couplings (g_i) of the fields ψ_i can then be defined in terms of the canonical rescaled variables $\widehat{\psi}_i$, which diagonalize the kinetic term and have canonical dimensions. In the low-energy limit we may expand the $\phi-\psi$ interaction Lagrangian around the value ϕ_0 of the dilaton, which extremizes the effective potential [577, 321]:

$$L(\phi,\psi_i) \equiv Z_m^i\psi_i^2 = \frac{1}{2}m_i^2\widehat{\psi}_i^2 + \frac{1}{2}g_i\phi\widehat{\psi}_i^2 + ... \tag{6.22}$$

Performing the computation in both the String and Einstein frames [285], we find that the dilatonic charges per unit of gravitational mass, respectively q_i and $\widetilde{q}_i$ in the two frames, are related by (in units $16\pi G = 1$):

$$q_i \simeq \frac{g_i}{m_i^2} = \left[\frac{\partial}{\partial\phi}\ln\left(\frac{Z_m^i}{Z_k^i}\right)\right]_{\phi=0} = \widetilde{q}_i - 1. \tag{6.23}$$

Unless $\widetilde{q}_i$ is fine-tuned to 1 (corresponding to an exact Brans–Dicke theory for the field ψ_i), the dilatonic charge is non-vanishing in both frames and, in general, is non-universal, as the form factors $Z_i(\phi)$ are different for different fields.

As originally stressed in [577], this suggests a large dilatonic charge ($q_i \sim 40 - 50$) for the confinement-generated components of hadronic masses, and a smaller charge, but of gravitational intensity ($q_i \sim 1$), for leptonic masses (see also [237, 30]). If this is the case,

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the total charge of a macroscopic body is large (in gravitational units), and composition-dependent.

Consider in fact a macroscopic body of mass M, composed of B baryons with mass and charge m_b, q_b, and Z electrons with mass and charge m_e, q_e. The total dilatonic charge per unit mass, $q = \sum_i m_i q_i / \sum_i m_i$, for $Z \sim B$, $m_e \ll m_b$, $q_e \ll q_b$ is then

$$q \simeq \frac{B m_b q_b}{M} = \left(\frac{B}{\mu}\right) q_b, \tag{6.24}$$

where $\mu = M/m_b$ is the mass of the body in units of baryonic masses. Since $B/\mu \sim 1$, the total dilaton charge of a macroscopic body is controlled by the dilaton coupling to baryons, $q \sim q_b \gg 1$, it is of the same order of magnitude in the String and Einstein frames, $q \sim \tilde{q}$, and it is composition-dependent (as B/μ depends on the internal nuclear structure), with variations, across different types of ordinary matter, which are typically of order [285]

$$\frac{\Delta q}{q} \simeq \Delta\left(\frac{B}{\mu}\right) \sim 10^{-3}. \tag{6.25}$$

The only way to escape the conclusion that the dilaton charge of macroscopic bodies is composition-dependent and large is to assume that the coupling of the dilaton to the fundamental matter fields is universal (the same for all fields, to all orders in the loop expansion), and to choose appropriate values of the parameters in the resulting scalar-tensor effective action. This is not impossible, as illustrated by the model discussed in [197, 198]. In such a context, if the effective coupling is appropriately suppressed ($q \ll 1$), the dilaton mass can be very small, or even vanishing. The precise measurements of the gravitational force over quite a wide range of distances provide in fact exclusion plots in the plane (q^2, m^2), which constrain the dilaton charge as a function of its mass (see for instance [253] for a comprehensive compilation of experimental bounds on possible "Yukawa" deviations from Newtonian gravity, both for universal and composition-dependent interactions).

For later use, let us recall here the phenomenological bound on the dilaton mass for a large, composition-dependent dilaton charge [253]:

$$\boxed{m \gtrsim 10^{-4}\ \text{eV}.} \tag{6.26}$$

In the opposite, small-coupling regime, and in the mass range appropriate to gravitational antennas, i.e. from 10^{-14} to 10^{-12} eV, corresponding to the frequency range from 10 to 1000 Hz, the phenomenological bounds [253] can be parametrized as

$$\begin{aligned} &\log q^2 \lesssim -7, && 1\ \text{Hz} \lesssim m \lesssim 10\ \text{Hz}, \\ &\log q^2 \lesssim -7 + \log(m/10\ \text{Hz}), && 10\ \text{Hz} \lesssim m \lesssim 1\ \text{kHz}, \end{aligned} \tag{6.27}$$

for universal dilaton interactions, and

$$\begin{aligned} &\log q^2 \lesssim -8, && 1\ \text{Hz} \lesssim m \lesssim 10\ \text{Hz}, \\ &\log q^2 \lesssim -8 + \log(m/10\ \text{Hz}), && 10\ \text{Hz} \lesssim m \lesssim 1\ \text{kHz}, \end{aligned} \tag{6.28}$$

for composition-dependent interactions.

We can now discuss the intensity of the relic-dilaton background, taking into account two distinct phenomenological possibilities: 1) (heavy) massive dilatons, gravitationally (or more strongly) coupled to macroscopic matter, and 2) massless (or very light) dilatons, universally (and weakly enough) coupled to matter. In the last case, the conclusions about a possible detection of the dilaton background produced by pre-big bang models are not completely negative, as we shall illustrate in Subsection 6.4.

6.3 Bounds and allowed windows for heavy (strongly coupled) dilatons

Let us first discuss the large-coupling case $q \gtrsim 1$ (the most unfavourable, probably, for a direct detection). In such a case $m \gtrsim 10^{-4}$ eV, so that today even the highest mode present in the spectrum, $p_1(t_0)$, is non-relativistic (see Eq. (5.16)):

$$p_1(t_0) = \frac{k_1}{a(t_0)} \simeq \left(\frac{H_1}{M_{\rm P}}\right)^{1/2} 10^{11}\ {\rm Hz} \simeq \left(\frac{M_{\rm s}}{M_p}\right)^{1/2} 10^{-4}\ {\rm eV} < m \tag{6.29}$$

(since $M_{\rm s} < M_{\rm P}$). In other words, the branch (6.12) of the dilaton spectrum extends from p_m to p_1. We shall start assuming that the slope is steep enough, $\delta > 1$, for the high-frequency branch from p_m to p_1 to represent the dominant part of the spectrum. In that case all the cosmological bounds that apply to the total integrated dilaton energy density will become δ-independent [271, 321] (the case $\delta < 1$ will be considered below).

When the spectrum becomes non-relativistic, the dilaton energy density $\rho_\chi \sim a^{-3}$ starts to grow with respect to the radiation energy density, $\rho_\gamma \sim a^{-4}$. In particular, when the Universe enters the matter-dominated era, the non-relativistic part of the dilaton spectrum remains frozen (in critical units) at the value reached at the time of the matter-radiation equilibrium, $\Omega_\chi(t_{\rm eq})$. The present value of the dominant (non-relativistic) branch of the spectrum can thus be expressed in terms of the equilibrium scale as:

$$\Omega_\chi(t_0, p) \equiv \Omega_\chi(t_{\rm eq}, p) \simeq \left(\frac{H_1}{M_{\rm P}}\right)^2 \left(\frac{m^2}{H_1 H_{\rm eq}}\right)^{1/2} \left(\frac{p}{p_1}\right)^{\delta-1}, \qquad p_m < p < p_1, \tag{6.30}$$

where we have used Eq. (6.12), and the fact that $(H_1/H_{\rm eq})^2 (a_1/a_{\rm eq})^3 = (H_1/H_{\rm eq})^{1/2}$, assuming radiation-dominance from t_1 to $t_{\rm eq}$. We recall that, for the equilibrium curvature scale, $H_{\rm eq} \sim 10^6 H_0 \sim 10^{-55} M_{\rm P}$.

A first important constraint [238, 240, 205] is now obtained by imposing that the energy density stored in the coherent oscillations of non-relativistic dilatons, integrated over all modes, remains smaller than critical, at all times:

$$\int^{p_1} d\ln p\ \Omega_\chi(t_0, p) \lesssim 1. \tag{6.31}$$

For $\delta > 1$ this gives the stringent constraint

$$m \lesssim \left(\frac{H_{\rm eq} M_{\rm P}^4}{H_1^3}\right)^{1/2}, \tag{6.32}$$

which represents, in this context, the most restrictive upper bound on the mass of (not yet decayed) dilatons. For 100 keV $\lesssim m \lesssim$ 100 MeV a more restrictive constraint on Ω_χ is provided in principle by the observations of the astrophysical background of diffuse γ-rays [200]; such a range of masses, however, tends to be excluded, as will be shown below, in the context of pre-big bang models where the inflation scale is controlled by the string mass, $H_1 \simeq M_{\rm s}$.

The critical density bound (6.32) can be evaded if the dilatons are heavy enough to decay, so that the energy stored in their coherent oscillations was dissipated into radiation before the present epoch. The decay scale H_d is fixed by the decay rate (for instance into two photons) as

$$H_d \simeq \Gamma_d \simeq m^3/M_{\rm P}^2 \tag{6.33}$$

(which implies, in particular, that the produced dilatons have not yet decayed, today, only if $m \lesssim 100$ MeV). The decay, however, generates radiation, reheats the Universe, and is associated in general to a possible entropy increase

$$\Delta S \simeq (T_r/T_d)^3, \tag{6.34}$$

where T_r and T_d are the final reheating temperature and the radiation temperature immediately before dilaton decay, respectively. The decay process is thus the source of additional phenomenological constraints on the dilaton spectrum, which are complementary to the critical bound (6.32), as they imply in general a lower bound on the dilaton mass.

In order to discuss the consequences of dilaton decay let us note, first of all, that the induced reheating is significant ($T_r > T_d$, $\Delta s > 1$) provided dilatons decay when they are dominant with respect to radiation, i.e. for $H_d < H_i$, where H_i is the curvature scale marking the beginning of dilaton dominance. Such a scale is defined by the condition $\Omega_\chi(t_i) = \Omega_\gamma(t_i)$, where $\Omega_\chi(t_i)$ is obtained by integrating over all modes the dominant branch (6.12) of the non-relativistic dilaton spectrum. For $\delta > 1$ this leads to

$$m H_1/M_{\rm P}^2 \simeq a_1/a_i, \tag{6.35}$$

from which

$$H_i \simeq m^2 H_1^3/M_{\rm P}^4 \tag{6.36}$$

(we have assumed that such a scale belongs to the radiation era, $H_i > H_{\rm eq}$; otherwise, the bound we obtain is less stringent).

At the given scale H_i, the radiation temperature is fixed by the Einstein equations (and by the Sefan law $\rho \sim T^4$) as

$$T_i = k(M_{\rm P} H_i)^{1/2}, \tag{6.37}$$

where k is a dimensionless numerical factor of order unity, which disappears from the final estimate. The radiation temperature at the epoch of dilaton decay, T_d, is then obtained by the adiabatic rescaling

$$T_d = \left(\frac{a_i}{a_d}\right) T_i = \left(\frac{H_d}{H_i}\right)^{2/3} T_i \simeq k \left(\frac{m^{10}}{M_{\rm P} H_1^3}\right)^{1/6}, \tag{6.38}$$

since the scale factor evolves like $t^{2/3}$ during the phase dominated by the non-relativistic dilatons. After the decay, on the other hand, the Universe becomes radiation-dominated, with a reheating temperature fixed by the decay scale as

$$T_r = k(M_{\rm P} H_d)^{1/2} = k \left(\frac{m^3}{M_{\rm P}}\right)^{1/2}, \tag{6.39}$$

and with a final, total entropy increase [271, 321]:

$$\Delta S \simeq \left(\frac{H_1^3}{m M_{\rm P}^2}\right)^{1/2}. \tag{6.40}$$

This entropy injection can in principle disturb nucleosynthesis and/or baryogenesis [238, 240, 205], and we should consider two possibilities.

If the reheating temperature T_r is too low to allow nucleosynthesis, $T_r < 1$ MeV, i.e. $m \lesssim 10$ TeV, we must assume that nucleosynthesis occurred before, and we must impose

$$\Delta S \lesssim 10, \qquad\qquad m \lesssim 10 \text{ TeV}, \tag{6.41}$$

to avoid destroying the light nuclei already formed (more precise bounds can also be determined through a detailed analysis of photodissociation [239] and hadroproduction [217]).

If the reheating temperature is large enough to allow nucleosynthesis, $T_r > 1$ MeV, i.e. $m \gtrsim 10$ TeV, the only possible constraint comes from primordial baryogenesis. The bound is model-dependent, but the constraint

$$\Delta S \lesssim 10^5, \qquad\qquad m \gtrsim 10 \text{ TeV}, \tag{6.42}$$

seems to be sufficient [238, 240, 205] not to wash out any pre-existing baryon–antibaryon asymmetry (this bound could be evaded in the case of low-energy baryogenesis occurring at a scale $H < H_d$).

The above constraints can be easily extended to the case $\delta \leq 1$, see [271, 321]. Consider, for instance, the critical density bound. For $0 \leq \delta \leq 1$ the non-relativistic spectrum (6.30) has a peak at $p = p_m = p_1(m/H_1)^{1/2}$, which dominates the integral (6.31), and which leads to a total integrated energy density:

$$\Omega_\chi(t_0) \simeq \left(\frac{H_1}{M_{\rm P}}\right)^2 \left(\frac{m^2}{H_{\rm eq} H_1}\right)^{1/2} \left(\frac{m}{H_1}\right)^{\frac{\delta-1}{2}}. \tag{6.43}$$

The bound $\Omega_\chi < 1$ now imposes [271, 321]

$$m \lesssim M_{\rm P}^{\frac{4}{\delta+1}} H_{\rm eq}^{\frac{1}{\delta+1}} H_1^{\frac{\delta-4}{\delta+1}}, \qquad \delta \leq 1, \tag{6.44}$$

which for $\delta = 1$ exactly reduces to the condition (6.32).

For the dilaton decay the situation is similar: if $\delta \leq 1$ the total energy density is dominated by the contribution of $p = p_m$, and all bounds become δ-dependent. The scale H_i of dilaton dominance, obtained by equating $\Omega_\chi(p_m)$ to Ω_γ in the radiation era, becomes $H_i \simeq m^{\delta+1} H_1^{4-\delta} M_{\rm P}^{-4}$, and the corresponding increase of entropy is

$$\Delta S = \left(\frac{T_r}{T_d}\right)^3 = \left(\frac{H_i}{H_d}\right)^{1/2} \simeq \left(m^{\delta-2} M_{\rm P}^{-2} H_1^{4-\delta}\right)^{1/2}. \tag{6.45}$$

Taking into account all bounds, we obtain the allowed region in the plane (m, H_1) illustrated in Fig. 6.1. The allowed values are below the upper (bold) solid lines for a growing ($\delta \geq 1$) spectrum [271], and below the lower (thin) solid lines for a flat ($\delta = 0$) spectrum [342, 238, 240]. For $0 < \delta < 1$ the allowed region interpolates between the two limiting cases illustrated in the figure.

To the left, the region is bounded by the tests of the equivalence principle, Eq. (6.26), as we are considering dilatons with large couplings to macroscopic matter, $q \gtrsim 1$. To the right, the mass is bounded by the condition $m < M_{\rm P}$ (not shown in the picture), which has to be imposed to avoid overcritical density in case $m > H_1$, when the produced dilatons are already non-relativistic from the beginning [321] (in that case there are no additional bounds, as dilatons decay before becoming dominant). Finally, the upper limit on the inflation scale, $H_1 < M_{\rm P}$, is required to avoid overcritical production of massless particles since, for a growing spectrum, $\int d \ln p \Omega(p) \sim g_1^2 = (H_1/M_{\rm P})^2$ (see for instance Eq. (6.10)).

In the standard inflationary scenario the spectrum tends to be flat, and the "natural" value for the dilaton mass required by models of supersymmetry breaking [52], $m \sim 1$ TeV, is not consistent with the preferred value of the inflation scale, $H_1 \sim 10^{-5} M_{\rm P}$, typically determined by the grand-unification phase transition (this is the so-called problem of the dilaton mass). In string cosmology the spectrum tends to be growing, the allowed region "inflates" in parameter space, and the TeV mass scale becomes marginally compatible with the wanted inflation scale. Unfortunately, however, in a string cosmology context, H_1 is expected to be determined by the string scale, and thus fixed around $(0.1 - 0.01) M_{\rm P}$, so that the problem remains.

A possible solution of this problem can be obtained in the presence of a late, post-inflationary reheating phase, preceding nucleosynthesis and dilaton decay, and producing a suitable dilution of the original dilaton density by the factor [276]

$$\Omega_\chi \to \Omega_\chi (1 - \delta s)^{4/3} \left(\frac{n_f}{n_b}\right)^{4/3}, \qquad \delta s = \frac{(s_f - s_b)}{s_f}. \tag{6.46}$$

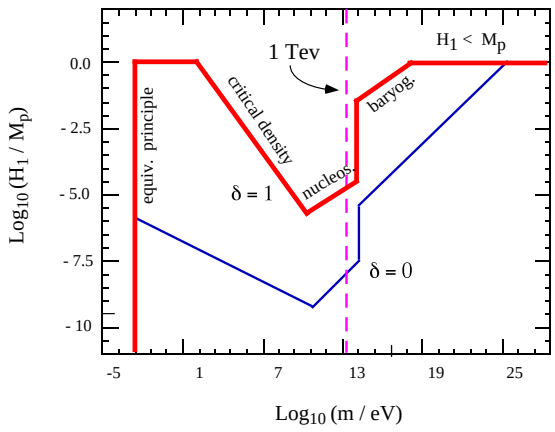

Figure 6.1: *Allowed region for a growing (below the bold lines) and flat (below the thin lines) dilaton spectrum, for strongly coupled dilatons.*

Here s_b, s_f, n_b, n_f are, respectively, the thermal entropy density of the CMB radiation and the number of particles species in thermal equilibrium, at the beginning (t_b) and at the end (t_f) of this reheating process. An efficient reheating, $s_f \gg s_b$, $\delta s \to 1$, reduces in a significant way the dilaton fraction of critical density Ω_χ: as a consequence, the scale (6.36) of dilaton dominance is lowered, the decay temperature (6.38) is raised, and the bounds on m following from the entropy constraint ($\Delta S < 10$) and the critical bound ($\Omega_\chi < 1$) are relaxed. Such a reheating is possible, for instance, in the context of models of "intermediate scale" inflation [531], or "thermal" inflation [458].

Even without such a relaxation, the minimal pre-big bang scenario leaves open two interesting mass windows, obtained by intersecting the allowed region of Fig. 6.1, for $\delta \geq 1$, with the allowed value of the string mass scale [403],

$$0.01 \lesssim M_{\rm s}/M_{\rm P} \lesssim 0.1 \tag{6.47}$$

(under the assumption $H_1 \simeq M_{\rm s}$). It follows that there are two possible ranges for the dilaton mass [276]:

$$10^{-4}\ {\rm eV} \lesssim m \lesssim 10\ {\rm keV}, \qquad\qquad m \gtrsim 10\ {\rm TeV} \tag{6.48}$$

illustrated in Fig. 6.2. The vertical solid line at the 100 MeV mass scale, shown in Fig. 6.2, corresponds to a dilaton decay rate of the same order as the present Hubble scale, $H_d \simeq m^3/M_{\rm P}^2 \simeq H_0$. To the right of this line all produced dilatons have already decayed, while to the left, and in particular inside the window of Eq. (6.48), dilatons are still around us, and could in principle be observed.

If, in addition, the mass is within the restricted range

$$100\ {\rm eV} \lesssim m \lesssim 10\ {\rm keV}, \tag{6.49}$$

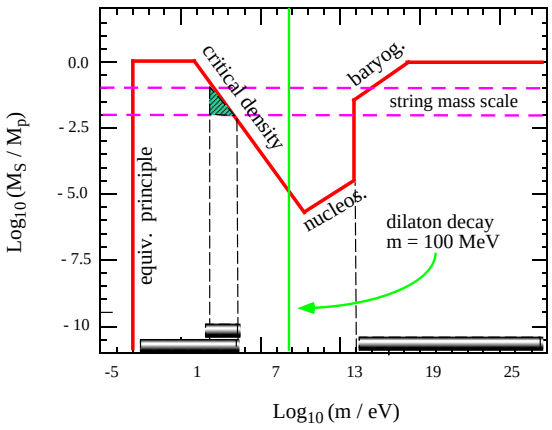

Figure 6.2: *Allowed mass windows for heavy (strongly coupled) dilatons in minimal pre-big bang models with $\delta \geq 1$. The region within the shaded triangle corresponds to a significant dilaton contribution to the present cold dark matter density.*

obtained by intersecting the string mass scale with the critical density bound (6.32), then the produced dilatons could even saturate the critical density value, or at least could provide a significant fraction of the present cold dark-matter density [271], with

$$0.01 \lesssim \Omega_\chi \lesssim 1, \tag{6.50}$$

according to Eq. (6.30). It is important to recall, in such a context, that the possibility of a small dilaton mass is not theoretically excluded (in spite of the "standard" TeV mass range), as shown for instance by models of supersymmetry breaking with light dilatons [250]. Also, it is important to stress that the range of masses saturating the critical density bound moves to lower and lower values as the spectrum becomes flatter and flatter (see Eq. (6.44) with $H_1 = M_s$). For $\delta \to 0$, in particular, the critical bound is saturated by

$$m = H_{eq} \, (M_P/M_s)^4 \, , \tag{6.51}$$

corresponding to masses as small as 10^{-19} eV (or lower).

Such values of mass are excluded neither theoretically nor experimentally, as discussed in Subsection 6.2, provided the dilaton coupling to macroscopic matter is sufficiently small, $q \ll 1$. It is then possible to imagine a cosmological scenario in which a relic background of non-relativistic dilatons, produced as a consequence of the pre-big bang evolution, could provide today a significant contribution to the critical energy density and, simultaneously, have a mass in the appropriate range to stimulate a resonant response of present gravitational antennas [287] (i.e. $m \sim 10^{-12} - 10^{-14}$ eV). In that case, the high intensity of the background (not constrained by nucleosynthesis, as in the graviton case), could possibly compensate the weakness of the dilatonic charge of the antennas, and could determine

in principle a detectable signal. This interesting possibility will be discussed in the next subsection.

6.4 Detection of ultra-light dilatons

A stochastic background of relic dilatons, produced by a phase of pre-big bang evolution, and light enough to have a lifetime larger than the present Hubble scale, could interact today in two ways with the existing gravity wave detectors: either indirectly, through a geodesic coupling of the detector to the scalar part of the induced metric fluctuations contained in the Riemann tensor [83, 82, 111, 564, 464, 493], or even directly through the non-geodesic coupling of the dilaton to the scalar charge of the detector [285, 287] (see Eqs. (6.19), (6.20)).

In the first case the coupling has a gravitational strength, but the amplitude of scalar metric perturbations, at the typical resonant scales of present detectors, could be much lower than the amplitude of the original dilaton fluctuations. In the second case the detector could be coupled to a background of very high intensity (of order 1 in critical units), but with a coupling constant strongly suppressed for phenomenological reasons, as discussed in Subsection 6.2.

In order to take into account both possibilities, we will estimate the scalar-to-noise ratio associated to a stochastic dilaton background by assuming that the physical strains $h_i(t)$ $(i = 1, 2)$, induced on a pair of gravitational antennas, vary in time like the scalar fluctuations $\phi(x_i, t)$ that perturb the detectors, and are proportional to the scalar pattern functions $F_i(\hat{n})$ through the scalar charges q_i of the detectors [287, 317]:

$$h_i(t) = \phi(x_i, t) F_i(\hat{n}), \qquad F_i(\hat{n}) = q_i e_{ab}(\hat{n}) D_i^{ab}. \tag{6.52}$$

Here $e_{ab}(\hat{n})$ is the polarization tensor of the scalar wave propagating along the $\hat{n}$ direction, and D_i^{ab} is the detector response tensor (see Subsection 5.4); finally, $q_i = 1$ for the coupling to scalar metric fluctuations, while $q_i \ll 1$ for the direct coupling of long-range scalar fields.

We expand the strain in momentum space, taking into account the mass of the scalar fluctuations ($\vec{p} = p\hat{n}$),

$$\begin{aligned} h_i(t) &= \int_0^\infty dp \int d^2\hat{n}\ F_i(\hat{n}) \left[\phi(p, \hat{n}) e^{2\pi i [p\hat{n}\cdot\vec{x}_i - E(p)t]} + \text{h.c.}\right], \\ E(p) &= (\widetilde{m}^2 + p^2)^{1/2}, \qquad \widetilde{m} = (m/2\pi), \end{aligned} \tag{6.53}$$

and we use the stochastic condition which, for a scalar spectrum $\Omega(p)$,

$$\rho = \rho_c \int d\ln p\ \Omega(p) = \frac{M_{\rm P}^2}{16\pi} \langle |\dot{\phi}|^2 \rangle, \tag{6.54}$$

can be written as

$$\langle \phi^\star(p, \hat{n}), \phi(p', \hat{n}') \rangle = \frac{3H_0^2 \Omega(p)}{8\pi^3 p E^2(p)} \delta(p - p') \delta^2(\hat{n} - \hat{n}'). \tag{6.55}$$

By imposing optimal filtering [17], and following the same steps as in Subsection 5.4 for the graviton case, we finally arrive at the signal-to-noise ratio [317]:

$$\mathrm{SNR} = \left(\frac{H_0^2}{5\pi^2}\right)\left[2\,T\int_0^\infty \frac{dp}{p^3\,(p^2+\tilde{m}^2)^{3/2}}\,\frac{\Omega^2(p)\,\gamma^2(p)}{P_1(\sqrt{p^2+\tilde{m}^2})\,P_2(\sqrt{p^2+\tilde{m}^2})}\right]^{1/2}, \qquad (6.56)$$

where T is the total integration time, $P_1(|\nu|), P_2(|\nu|)$ are the one-sided noise spectral density of the detectors, defined as Fourier transforms of the frequency $\nu = E(p) = (\widetilde{m}^2 + p^2)^{1/2}$, and $\gamma(p)$ is the overlap reduction function, defined in momentum space by:

$$\gamma(p) = \frac{15}{4\pi}\int d^2\hat{n} F_1(\hat{n}) F_2(\hat{n}) e^{2\pi i p\hat{n}\cdot(\vec{x}_2-\vec{x}_1)} \qquad (6.57)$$

(normalized so that $\gamma = 1$ for coincident and coaligned interferometers). For $m \to 0$ and $q \to 1$ we recover the signal-to-noise ratio (5.39), modulo a different normalization of γ.

The result (6.56) determines the cross-correlation and the detectability of a signal induced by a massive scalar spectrum $\Omega(p)$. We may distinguish three phenomenological possibilities.

If m is much larger than the typical sensitivity band ν_0 of the antenna, then the noises P_i, estimated at the frequency $\nu = (\widetilde{m}^2+p^2)^{1/2} \gg \nu_0$, tend to infinity, and we cannot expect to detect a signal. If, on the contrary, $m \ll \nu_0$ (which includes the massless case, discussed in [43]), then the detectors in their sensitivity band will respond to the relativistic branch of the spectrum, whose integrated amplitude is constrained by the nucleosynthesis bound, $\Omega \lesssim 10^{-6}$ (see Subsection 5.1). In addition, the response is possibly suppressed by the factor $q^2 \ll 1$, because the dilaton field has a long range. As a consequence, we may expect in such a case a signal not larger (and possibly much smaller) than the signal produced by a relic graviton background.

The interesting possibility corresponds to the dilaton mass in the sensitivity band of the detectors ($m \sim \nu_0$), where the overlap is large and the noises as small as possible. In that case we have a resonant response also to the non-relativistic branch ($p < m$) of the spectrum, which is not constrained by nucleosynthesis and can approach the critical density bound, as emphasized in the previous subsection. In order to discuss the possibility of detection let us then consider the most favourable case, in which the non-relativistic branch dominates the spectrum and saturates the critical density, $h_{100}^2 \int^m d\ln p\Omega \sim 1$ (more precisely, we should impose the saturation of Ω at $\simeq 0.35$, to be consistent with present dark-matter estimates [586]). Let us suppose, also, that the spectrum is peaked around $p \sim m$, and that the contribution of the relativistic branch $p > m$, if present, is negligible.

The integral (6.56), in such a case, can be estimated by integrating over the non-relativistic modes only, and approximating P_i with their constant values at $\nu = m$. We consider the ideal case of two identical detectors ($P_1 = P_2 = P$, $q_1 = q_2 = q$), with maximal overlap ($\gamma = 1$) at $\nu = m$ and integration time $T = 10^8$ s. The dilaton background is then

detectable, SNR > 1, provided [287]:

$$m^{5/2} P(m) \lesssim q^2 \times 10^{-32}\ \mathrm{Hz}^{3/2}. \tag{6.58}$$

The intersection of this condition with the noise power spectrum $P(\nu)$ of present gravitational antennas provides a rough estimate of the allowed values of the dilaton mass, possibly compatible with a *direct* detection of a stochastic dilaton background. The mass windows depend, of course, on the values of the dilaton coupling. By using a simple, power-law parametrization [188] of the VIRGO noise curve, for instance, we find [287] that the detectable mass windows extends, for $q^2 = 1$, over the full sensitivity band of the detector (1 Hz to 10 kHz). However, the window decreases as q decreases, and closes completely for $q^2 \lesssim 10^{-8}$.

As already stressed at the beginning of this section (and in Subsection 6.2), a large value of q^2 can only refer, in this context, to the indirect coupling of the gravitational antennas to the (scalar) metric perturbation spectrum generated by the dilatons. It seems unlikely, however, that such a spectrum, even including the non-relativistic corrections due to the dilaton mass, can be high enough (at the kHz scale) to saturate the critical density bound, as assumed to obtain the condition (6.58).

For the direct coupling to the dilaton background, on the other hand, we have to recall that the maximum allowed value of q^2, for composition-dependent and -independent interactions, are summarized in Eqs. (6.27), (6.28)). Taking into account such bounds, we are led to the situation illustrated in Fig. 6.3, where the noise power spectrum of VIRGO [188] (bold curve) is compared with the condition (6.58) at fixed q (thin dashed lines) and with the maximal allowed values of q^2, for universal (bold upper lines) and composition-dependent (thin lower lines) scalar interactions. A scalar background of nearly critical density, non-universally coupled to macroscopic matter, turns out to be only marginally compatible with detection (at least, at the level of sensitivity plotted in the picture). If the coupling is instead universal (for instance, as in the dilaton model discussed in [197]) but the scalar is not exactly massless, then there is a mass window open to detection:

$$10^{-14}\ \mathrm{eV} \lesssim m \lesssim 10^{-12}\ \mathrm{eV}. \tag{6.59}$$

The above results, to be interpreted only as a first qualitative indication, are confirmed by a more realistic analysis of the cross correlation of the two LIGO interferometers [317], performed by applying Eq. (6.56) to the phenomenological power spectrum:

$$\Omega(p) = \delta\ \Omega_0\, (p/\widetilde{m})^{\delta}\,, \qquad\qquad p \lesssim \widetilde{m}, \qquad\qquad 0 < \delta \leq 3. \tag{6.60}$$

In spite of the reduced value of the overlap factor we find (for any δ, but with $\Omega_{\mathrm{s}} \sim 1$) a significant SNR in the relevant mass range, already at the level of the Enhanced LIGO configuration. No possibility of detection, even with the Advanced LIGO configuration, is found instead for a non-relativistic spectrum peaked outside the detector sensitivity band, such

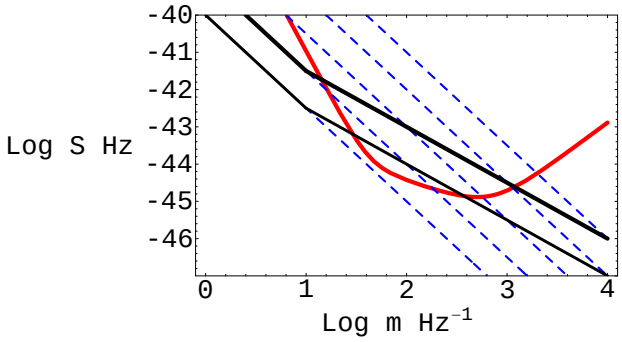

Figure 6.3: *The noise spectrum of VIRGO (bold curve) is compared with the maximal values of q^2 allowed by gravitational phenomenology, in two cases: universal (bold upper lines) and composition-dependent (thin lower lines) scalar interactions. The thin dashed lines correspond, from left to right, to $q^2 = 10^{-8}$, 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}. The region compatible with a detectable signal is above the noise spectrum and below the upper bounds on q^2.*

as the "minimal" spectral distribution (6.12), (6.13). It should be stressed, however, that interferometric detectors are particularly unfavoured for the detection of a non-relativistic scalar background, because their response tensor D_{ab} is traceless: spherical resonant detectors represent more promising devices, and it seems appropriate to conclude this section by reporting the possible enhanced level of SNR when a massive scalar background interacts with the monopole mode of a resonant sphere [174].

For a massive scalar wave there are in fact two possible pattern functions, one associated to the geodesic coupling of the detector to the scalar component of metric fluctuations [464] (the Riemann term in Eq. (6.20)), the other related to the direct, non-geodesic coupling of the charge q_i of the detector to the spatial gradients of the scalar background [285]. In terms of the transverse and longitudinal decomposition of the polarization tensor of the wave, with respect to the propagation direction $\hat{n}$,

$$T_{ab} = (\delta_{ab} - \hat{n}_a\,\hat{n}_b), \qquad\qquad L_{ab} = \hat{n}_a\,\hat{n}_b\,, \tag{6.61}$$

the two pattern functions can be written, respectively, as [317]

$$F(\hat{n}) = D^{ab}\left(T_{ab} + \frac{\widetilde{m}^2}{E^2}\,L_{ab}\right), \qquad\qquad F_q(\hat{n}) = q\frac{p^2}{E^2}D^{ab}\,L_{ab}. \tag{6.62}$$

For the differential mode of an interferometer [255] $D^{ab}\delta_{ab} = 0$, so that, in both cases, the pattern function of massive waves turns out to be proportional to the massless pattern function and, for non-relativistic modes, it is highly suppressed by the factor $(p/E)^2$ [317]. The monopole mode of a resonant sphere, on the contrary, is characterized by the response

tensor $D^{ab} = \delta^{ab}$ [82]. The non-geodesic pattern function $F_q(\hat{n})$ is still suppressed, but the geodesic part is not, and the overlap function (6.57) takes the simple form

$$\gamma(p) = \frac{15}{2\pi} \left(\frac{3\widetilde{m}^2 + 2p^2}{\widetilde{m}^2 + p^2} \right)^2 \frac{\sin(2\pi p d)}{pd}, \tag{6.63}$$

where d is the spatial separation of the detectors. The response to scalar spectra peaked at $p \ll m$ is thus strongly enhaced with respect to interferometric detectors (at least in the case of geodesic coupling).

Beside this effect there is a second possible enhancement, for flat enough spectra, associated to a faster growth of SNR with the observation time T. Consider, for instance, the cross-correlation of two spherical detectors according to Eq. (6.56), computed for the spectrum (6.60) with the overlap (6.63). Assuming that the instrumental noises are constant (and finite) for p ranging from 0 to $\widetilde{m}$, it follows that for $\delta < 1$ the momentum integral is dominated by its lower limit, i.e. by the infrared cut-off scale $p_{\rm min}$ determined, for $p \to 0$, by the "minimum" observable frequency interval $\Delta\nu$ associated to the "maximal" observation time T [174]:

$$\Delta\nu = (p^2 + \widetilde{m}^2)^{1/2} - \widetilde{m} > T^{-1}, \qquad\qquad p_{\rm min} = (2\widetilde{m}/T)^{1/2}\,. \tag{6.64}$$

This effect modifies, for $\delta < 1$, the final T-dependence of SNR, and enhances the response of the detector with the growth of the observation time. Performing the integral for $\delta < 1$ one finds in fact (modulo numerical factors)

$$\mathrm{SNR} \simeq \frac{\Omega_{\rm s}}{\widetilde{m}\ P(\widetilde{m})} \left(\frac{H_0}{\widetilde{m}} \right)^2 (\widetilde{m}T)^{1-\delta/2}\,, \qquad\qquad \delta < 1, \tag{6.65}$$

to be compared with the more standard result

$$\mathrm{SNR} \simeq \frac{\Omega_{\rm s}}{\widetilde{m}\ P(\widetilde{m})} \left(\frac{H_0}{\widetilde{m}} \right)^2 (\widetilde{m}T)^{1/2}\,, \qquad\qquad \delta \geq 1. \tag{6.66}$$

For $T = 10^7$ s, and a mass in the kilohertz range, the sensitivity of spheres to a flat enough and massive spectra is thus increased by a factor $10^{10(1-\delta)/2}$ with respect to interferometric detectors (see [174] for a more detailed discussion).

We may thus conclude, summarizing the results of this subsection, that both interferometric and resonant gravitational antennas seem to be able to explore the possible presence of a light (but non-relativistic) dilatonic component of dark matter, and thus to constrain the parameters of the pre-big bang models, in a mass range that overlaps with their sensitivity band. This is possible in spite of the fact that the relic dilaton background could be directly coupled to the total mass of the detector, with a charge that is much weaker than gravitational. Spherical detectors are more efficiently coupled to massive scalar waves, but enhanced and advanced interferometers are already sufficient to provide significant constraints on a possible background of relic dilatons.

7 Relic photons, axions and CMB anisotropy

As discussed in the previous sections, the primordial spectrum of (scalar and tensor) metric fluctuations, amplified by a phase of pre-big bang inflation, is characterized in general by a positive and rather steep slope. Because of the high-frequency normalization of the peak of the spectrum, controlled by the string scale, the amplitude of metric fluctuations is strongly suppressed at the large-distance scales relevant to the anisotropy of the CMB radiation observed by COBE [567, 77], or to the density fluctuations spectrum (see for instance [509]) required by the standard mechanism of structure formation. It thus seems difficult, in such a context, to generate a flat spectrum of scalar curvature perturbations directly responsible for the above effects, unless we accept rather drastic modifications of the pre-big bang kinematics, as recently suggested in [251, 227] (and briefly discussed in Subsection 7.2).

The slope of the spectrum, however, depends on the external "pump" field, which amplifies the quantum fluctuations (see Section 4) and which contains, in general, the contribution of all the components of the background (metric, dilaton, moduli fields, ...). Different fluctuations are coupled to different pump fields and may have very different spectra, even in the same background.

It turns out, in particular, that the primordial spectrum of electromagnetic [305, 306] and of "axionic" [175, 182] fluctuations (i.e. the fluctuations of the four-dimensional NS-NS two-form and of other, string- and M-theory-motivated, antisymmetric tensors) may be characterized by a slope much flatter than the slope of the metric perturbation spectrum (in some cases, even decreasing [135, 184, 119, 331, 226], depending on the coupling to the moduli fields, on the anisotropy of space-time, and on the number of extra dimensions). A flat and primordial distribution of (isocurvature) axionic fluctuations, in turn, can directly generate a scale-invariant spectrum of CMB anisotropies through the "seed" mechanism [223, 224, 324], which will be reported in Subsections 7.3, 7.4, or even generate an adiabatic, scale-invariant spectrum of density and curvature perturbations through the "curvaton" mechanism recently discussed in [241, 459, 487, 65, 502, 488, 100], and illustrated in Subsection 7.5.

NS-NS

In addition, the amplification of electromagnetic fluctuations, with a flat enough spectrum, seems to naturally provide the seed fields required to explain the origin of the magnetic fields present on a galactic and intergalactic scale (see [345] for a recent review). We will first illustrate this interesting possibility in the next subsection.

7.1 Large-scale magnetic fields and photon production

In the context of the pre-big bang scenario, the background evolution that amplifies metric perturbations also amplifies the quantum fluctuation of the electromagnetic field, because

of their direct coupling to the dilaton and to the moduli fields. For the heterotic string model, such a coupling is described by the effective action [305]

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}e^{-\phi}F_{\mu\nu}F^{\mu\nu}, \tag{7.1}$$

where $\phi = \Phi_{10} - \ln V_6 \equiv \ln(g^2)$ controls the tree-level four-dimensional gauge coupling (Φ_{10} being the ten-dimensional dilaton field, and V_6 the volume of the six-dimensional compact internal space). The effective coupling to the external geometry may be different, however, if the electromagnetic $U(1)$ symmetry is a component of the Kaluza–Klein gauge group produced in the compactification down to $D = 4$ dimensions [135] (see also [335]).

The electromagnetic field is also coupled to the four-dimensional geometry, of course. If such a coupling is minimal, however, it is also conformally invariant, and therefore does not contributes to photon production in a conformally flat metric such as that of a typical inflationary background, unless conformal invariance is broken at the classical [589] or quantum [218] level. Even in that case, however, photon production turns out to be rather suppressed (with respect to graviton production, for instance).

In order to discuss the amplification induced by the dilaton, in a conformally flat metric $g_{\mu\nu} = a^2\eta_{\mu\nu}$, it is convenient to use the radiation gauge $A_0 = 0$, $\partial_i A^i = 0$ explicitly. Because of conformal invariance, the coupling to the metric disappears from the action (7.1); after partial integration the action becomes, in this gauge,

$$S = \frac{1}{2}\int d\eta e^{-\phi}\left(A_i' A^{i\prime} + A_i\nabla^2 A^i\right), \tag{7.2}$$

and can be easily diagonalized by setting

$$\psi_i = zA_i, \qquad\qquad z = e^{-\phi/2}. \tag{7.3}$$

For each polarization mode we then obtain the effective action

$$S_\psi = \frac{1}{2}\int d\eta\left(\psi'^2 + \psi\nabla^2\psi + \frac{z''}{z}\psi^2\right), \tag{7.4}$$

and the canonical evolution equation

$$\psi_k'' + \left[k^2 - V(\eta)\right]\psi_k = 0, \qquad\qquad V(\eta) = \frac{z''}{z} = e^{\phi/2}\left(e^{-\phi/2}\right)'', \tag{7.5}$$

which is exactly the same as the graviton equation (4.56), with the only difference that now the pump field is fully determined by the dilaton. If we use, in particular, the minimal model of background introduced in Subsection 5.2, the spectrum of the amplified electromagnetic fluctuations will be characterized by the same parameters as the graviton spectrum. In such a case, the constraints on the electromagnetic spectrum will provide constraints on the parameters of the model and then, indirectly, on the associated graviton spectrum.

An important application of the "dilatonic" amplification of the electromagnetic fluctuations is the possible production of "seed" for the galactic magnetic fields [305, 430]. The origin of the large-scale cosmic magnetic fields (with coherence scale $\gtrsim 10$ kpc, and typical amplitude $\sim 10^{-6}$ G) is in fact, to a large extent, still an open problem [345]. Almost all mechanisms invoked to generate the large-scale fields, e.g. the galactic "dynamo" [508, 632], require the presence of small, primordial seeds to trigger the subsequent magnetic amplification. Many mechanisms of seed production have been suggested [365, 589, 595, 532, 263, 218, 157, 414] [81, 124, 338, 185] [67, 252, 201, 471, 291], trying to bypass the difficulty –due to conformal invariance– that forbids a standard inflationary amplification of the electromagnetic fluctuations. In the context of the pre-big bang models, on the contrary, the seeds can be produced directly from the electromagnetic fluctuations of the vacuum, thanks to the accelerated growth of the dilaton during the pre-big bang phase (with possible additional contributions from dilaton oscillations, during preheating [328]).

Consider in fact the two-parameter model of Eqs. (5.10):

$$\begin{aligned} &\text{dilaton phase,} && e^{-\phi/2} \sim (-\eta)^{\sqrt{3}/2}, && \eta < \eta_{\rm s}, \\ &\text{string phase,} && e^{-\phi/2} \sim (-\eta)^{\beta}, && \eta_{\rm s} < \eta < \eta_1, \\ &\text{radiation era,} && e^{-\phi/2} \sim \text{const}, && \eta > \eta_1. \end{aligned} \tag{7.6}$$

The effective dilatonic potential of Eq. (7.5) grows during the dilaton phase, keeps growing during the string phase, and goes to zero in the radiation era. Because of the two background transitions, there are two bands of the spectrum, corresponding to modes hitting the barrier in the dilaton phase, $\omega < \omega_{\rm s}$, and in the string phase, $\omega > \omega_{\rm s}$. By applying the standard procedure of Subsection 4.4 we obtain:

$$\begin{aligned} \Omega_{\rm em}(\omega, t_o) &\simeq g_1^2 \Omega_\gamma(t_0) \left(\frac{\omega}{\omega_1}\right)^{3-2\mu}, && \omega_s < \omega < \omega_1, \\ &\simeq g_1^2 \Omega_\gamma(t_0) \left(\frac{\omega_1}{\omega_s}\right)^{1+2\mu-\sqrt{3}} \left(\frac{\omega}{\omega_1}\right)^{4-\sqrt{3}}, && \omega < \omega_s \end{aligned} \tag{7.7}$$

where $\mu = |\beta - 1/2|$. By recalling that $\beta = 1 + \alpha$ (see Eq. (5.14)), we find that the electromagnetic spectrum thus has the same parameters as the graviton spectrum (5.15), but the slope is in general flatter. Even flatter spectra are possible for vector fields of Kaluza–Klein origin [135].

The generation of fluctuations of a strength large enough to seed the galactic dynamo constrain the electromagnetic spectrum at the megaparsec scale, and imposes in particular a bound that, in the most conservative form, can be written as follows [589]:

$$\Omega_{\rm em}(\omega_G, t_{\rm gal}) \gtrsim 10^{-34} \Omega_\gamma(t_{\rm gal}), \qquad \omega_G \sim (1\ \text{Mpc})^{-1} \sim 10^{-14}\ \text{Hz}, \tag{7.8}$$

where $t_{\rm gal}$ is the epoch of galaxy formation, i.e. $a_{\rm gal} \sim 10^{-2} a_0$. Remarkably, this condition is compatible with the condition of negligible backreaction,

$$\Omega_{\rm em}(\omega) < \Omega_\gamma \tag{7.9}$$

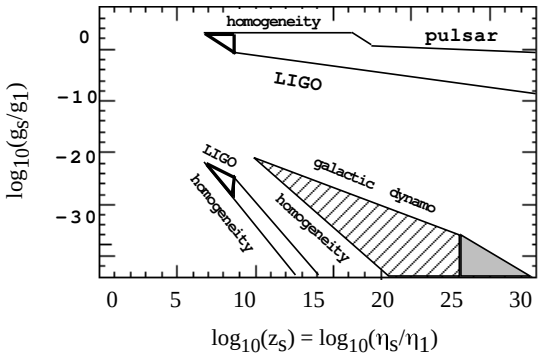

Figure 7.1: *Allowed region in the parameter space of minimal pre-big bang models for the production of seed fields for the galactic dynamo (shaded and hatched area). The other two regions correspond to the production of a graviton background detectable by advanced interferometric antennas.*

(at all frequency scales), required by the consistency of the perturbative approach around a homogeneous unperturbed background.

There is indeed a wide region of parameter space in which the two conditions are consistent without fine-tuning [305], as illustrated by the shaded and hatched areas of Fig. 7.1 (the hatched area refers to the dilaton branch of the spectrum, $\omega < \omega_s$; the shaded area to the string branch, $\omega > \omega_s$). As shown in the picture, the production of seeds is strong enough to satisfy the needs of the dynamo mechanism (Eq. (7.8)), provided the coupling is sufficiently small at the beginning of the string phase:

$$\log_{10} \exp(\phi_s/2) \lesssim -20, \tag{7.10}$$

and the duration in time of the string phase is sufficiently long:

$$|\eta_s/\eta_1| \gtrsim 10^{10} \sim e^{23}, \tag{7.11}$$

(in cosmic time, $\Delta t \gtrsim 23\lambda_s$). For illustrative purposes, the plots of Fig. 7.1 have been constructed by taking $g_1 = 1$ as a reference value. We also note that, besides the condition (7.9), there are additional bounds on the energy spectrum of the magnetic fields because of a possible (classical) production of gravitational waves induced by the magnetic stress tensor [144]. Flat enough spectra are not strongly constrained, however, and we are assuming that this is indeed the case for the electromagnetic distribution $\Omega_{em}(\omega)$ that we are considering. Possible additional bounds on the primodial magnetic amplitude, due to Faraday rotation effects, may follow in principle from future measurements of the CMB polarization, as discussed in [329].

Also shown in Fig. 7.1 are the allowed regions of parameter space compatible with the detection of the relic graviton background produced in the context of minimal pre-big

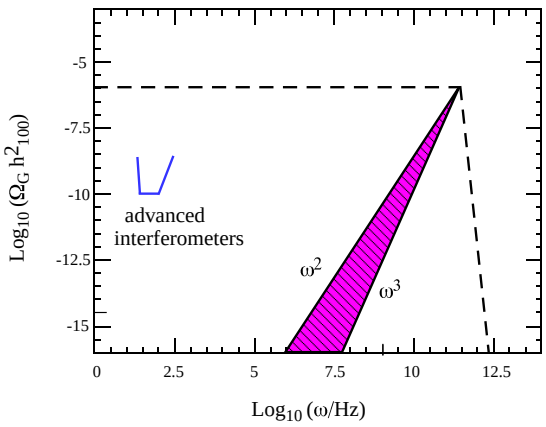

Figure 7.2: *Allowed region for the graviton spectrum in minimal pre-big bang models (shaded area), assuming that the amplification of the electromagnetic fluctuations is strong enough to seed the galactic dynamo.*

bang models. We have used the typical sensitivity expected for the cross-correlation of two second-generation interferometers, with the characteristic of Advanced LIGO (see Eq. (5.42)). The allowed region is then defined by the condition $\Omega_G(10^2 \text{ Hz}) \geq 10^{-10}$, imposed on the spectrum (5.15), together with the pulsar bound (5.3), and the condition of negligible backreaction ($\Omega_G < 1$ at all ω and t), required for the validity of our perturbative computation of the spectrum around a homogeneous background (for a detailed implementation of all bounds see [274]). It turns out that there are two allowed regions, corresponding to tensor perturbations whose comoving amplitude stays constant outside the horizon during the string phase (upper region) or grows (lower region). At the left-end of each region, the marked triangular area corresponds to the possible detection of low-energy modes, crossing the horizon during the dilaton-dominated phase; the rest of the allowed region refers to modes crossing the horizon during the string phase.

It should be noted that the gravitationally allowed regions have no possible overlap with the electromagnetic ones, implying that the interesting phenomenological possibilities of seed production and of graviton detection tend to exclude each other [274]. Indeed, if we restrict to the class of backgrounds consistent with the production of magnetic seeds (i.e. inside the shaded and hatched regions of Fig. 7.1), then the allowed slope of the high-frequency (string) branch of the graviton spectrum is constrained to vary between 2 and 3 [278], with a consequent drastic reduction of the allowed region for Ω_G, as illustrated in Fig. 7.2. In that case, the graviton background is certainly outside the range of currently available detectors (resonant bars and ground-based interferometers), which reach their maximal sensitivity in the frequency band $10^2 - 10^3$ Hz.

According to the above analysis, a future detection of cosmic gravitons in the hertz to

kilohertz range would seem to exclude a dilatonic origin of the seed magnetic fields, directly amplified from the fluctuations of the vacuum. Conversely, an absence of signals from the relic graviton background would seem to support, indirectly, a primordial dilatonic origin of the cosmic magnetic fields. It must be stressed, however, that such conclusions, based on the plots of Fig. 7.1, cannot be applied in general to the pre-big bang scenario: they are only pertinent to the class of minimal models introduced in Subsection 5.2, and could be evaded in the context of more complicated models of background.

A final remark concerns the fact that the photon spectrum (7.7) is in general flatter than the graviton spectrum (5.15), suggesting the possibility that the electromagnetic fluctuations may also act as seeds for the (approximately) scale-invariant spectrum of temperature anisotropies of the CMB radiation [306, 275] and for the density perturbations $\delta\rho/\rho$ required for the structure formation. According to a detailed analysis [224], however, such a possibility has to be discarded in favour of an analogous possibility based on the fluctuations of the fundamental antisymmetric tensors appearing in the bosonic sector of string theory, which can be amplified with a perfectly flat spectrum [175, 182], as will be illustrated in the next subsection.

7.2 Large-scale CMB anisotropy and axion production

In the context of the standard inflationary scenario, the observed CMB anisotropy, at very large angular scales, can be consistently generated by the primordial spectrum of scalar metric fluctuations Φ_k through the Sachs–Wolfe (SW) effect [546], $(\Delta T/T)_k \sim \Phi_k$. Indeed, when the metric fluctuations are directly amplified by the accelerated evolution of the background, their spectral distribution is controlled by the value of the Hubble scale at the time of horizon crossing, $\Phi_k \sim (H/M_{\rm P})_k$. For a standard de Sitter (or quasi-de Sitter) inflationary solution, H is constant in time, so that the spectrum is scale-invariant. A typical normalization of the spectrum, corresponding to inflation occurring roughly at the GUT scale [422],

$$\frac{H}{M_{\rm P}} \sim \frac{\text{GUT curvature scale}}{\text{PLANCK scale}} \sim 10^{-5}, \tag{7.12}$$

is thus perfectly consistent with the anisotropy observed at the present horizon scale [567, 77], $\Delta T/T \sim 10^{-5}$, and with the fact that the spectrum is scale-invariant.

In the context of the pre-big bang scenario, on the contrary, the curvature scale grows with time, so that the spectrum of the metric fluctuations tends to grow with frequency, as illustrated in Section 4. In addition, the natural inflation scale corresponds to the string scale, so that the normalization of the spectrum, at the end-point frequency k_1, is controlled by the ratio

$$\left(\frac{H}{M_{\rm P}}\right)_{k_1} \sim \frac{\text{STRING curvature scale}}{\text{PLANCK scale}} \sim 10^{-1} - 10^{-2}. \tag{7.13}$$

As a consequence, for a typical pre-big bang model, the slope of the spectrum is too steep,

and the normalization too high, to be compatible with the COBE observations. The slope is so steep, however, that the contribution of the metric fluctuations to $\Delta T/T$ is certainly negligible at the COBE scale. So, on the one hand there is no contradiction with observations, namely the COBE data cannot be used to rule out pre-big bang models [283]. On the other hand, the problem remains: How to explain the observed CMB anisotropy?

There are two possible answers to this question. The first one relies on somewhat drastic modifications of the kinematics of the pre-big bang phase illustrated in the previous sections, in such a way as to generate a primordial scale-invariant spectrum *directly* from the amplification of the vacuum fluctuations of the metric, as in the standard inflationary scenario. The second possibility relies on the *indirect* generation of scale-invariant metric perturbations through the isocurvature fluctuations of another field, which may have a flat spectrum even in the context of the pre-big bang solutions illustrated up to now in this paper. We will discuss in detail the second possibility in the following subsections. Here we will briefly report on the first possibility.

There are at present two possible examples of pre-big bang models characterized by a flat primordial spectrum of scalar perturbations [251, 227]. They are both based on the introduction of a negative, exponential dilaton potential $V(\phi)$, which can be parametrized in the E-frame as follows (in units $M_{\rm P}^2 = 2$):

$$V(\phi) = -V_0 e^{-\phi/\sqrt{\epsilon}}, \qquad\qquad 0 < \epsilon < 1, \tag{7.14}$$

and which is similar to the attractive potential existing between the colliding branes in the context of the ekpyrotic scenario [412, 413].

With such a potential, the corresponding (E-frame) pre-big bang solution describes an accelerated contraction, controlled by the dimensionless parameter ϵ,

$$a(t) \sim (-t)^{\epsilon} \sim (-\eta)^{\epsilon/(1-\epsilon)}, \tag{7.15}$$

which has to match the subsequent phase of decelerated, radiation-dominated expansion. By assuming the existence of a non-singular bounce for the background solution, and assuming the continuity of the Bardeen potential Φ and of the curvature perturbation $\zeta = v/z$ across the bounce (their continuity is guaranteed by performing the matching on the constant density hypersurface), one then finds that after the bounce the spectrum of Φ is determined by the pump field (7.15), according to the rules derived in Section 4, namely

$$k^3 \left|\Phi_k\right|^2 \sim k^{3-2\nu}, \qquad\qquad \nu = \left|\frac{\epsilon}{1-\epsilon} - \frac{1}{2}\right|. \tag{7.16}$$

A flat spectrum of scalar perturbations can then be obtained for $\epsilon = 2/3$, as noted in [251], corresponding to a dilaton potential that simulates a "dust" matter source with vanishing effective pressure $p = 0$.

Alternatively, as recently discussed in [227], a flat spectrum of scalar metric perturbations can be obtained from the background solution (7.15) by assuming a very slow

contraction, i.e. $0 < \epsilon \ll 1$ as in the ekpyrotic scenario [412, 413], and assuming that the matching to the expanding phase is performed across a hypersurface different from that of constant energy density (for instance, across the zero-shear hypersurface [227]). In that case the scalar spectrum after the bounce turns out to be different from Eq. (7.16), and is given by

$$k^3 \left|\Phi_k\right|^2 \sim k^{-2\epsilon/(1-\epsilon)}. \tag{7.17}$$

The case $\epsilon \ll 1$ thus reproduces a (nearly) flat spectrum, for both the pre-big bang and ekpyrotic scenarios.

This result can be easily explained by noting that, in the transition from a contracting to an expanding phase, the asymptotic solutions of the scalar perturbation equations are a linear combination of a constant and of a growing mode before the transition, and of a constant and a decaying mode after it. Consider, for instance, a bouncing background evolving from the contracting solution (7.15) valid for $\eta < -\eta_1$, to the standard, radiation-dominated expansion $a \sim \eta$, valid for $\eta > \eta_1$. The asymptotic solutions for the super-horizon modes of the Bardeen potential, in the longitudinal gauge, can be written as

$$\begin{aligned} \Phi_k^1(\eta) &= \frac{A_1(k)}{\eta a_1^2(\eta)} + B_1(k), \qquad\qquad \eta < -\eta_1, \\ \Phi_k^2(\eta) &= \frac{A_2(k)}{\eta a_2^2(\eta)} + B_2(k), \qquad\qquad \eta > \eta_1, \end{aligned} \tag{7.18}$$

where $a_1 \sim (-\eta)^{\epsilon/(1-\epsilon)}$, and $a_2 \sim \eta$. Here A_1, B_1 are determined by the initial normalization to a vacuum fluctuation spectrum, while A_2, B_2 are to be determined by the matching conditions. It is then clear that different matching prescriptions may lead to different solutions for $B_2(k)$, the dominant mode of the post-big bang expanding phase, and then to different final spectra.

Given a space-like hypersurface, with normal vector $n_\mu(\eta, \vec{x})$, the matching of two background manifolds [386] has to be performed on the induced metric $q_{\mu\nu}$ and on the extrinsic curvature $K_{\mu\nu}$ of the given hypersurface, where

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \qquad\qquad K_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu\right). \tag{7.19}$$

For the given example of bouncing background the metric will be continuous, i.e. $[g_{\mu\nu}]_\pm = 0$, where

$$[g]_\pm = \lim_{\eta \to \eta_1^+} g(\eta) - \lim_{\eta \to -\eta_1^-} g(-\eta); \tag{7.20}$$

the curvature, however, will be characterized by a "jump" from negative to positive values, $[K_{\mu\nu}]_\pm = S_{\mu\nu}$ (if we neglect the high-energy corrections, which are expected to smooth out the singularity, see Section 8). This introduces in general a surface stress tensor, which in our case corresponds to a tension (i.e. to a negative pressure), needed to avoid the singularity and to implement the transition from contraction to expansion.

The matching of the metric fluctuations can now be performed by perturbing the Israel junction conditions on the chosen hypersurface, and leads to

$$[\delta g_{\mu\nu}]_{\pm} = 0, \qquad \left[\delta K^{\nu}_{\mu}\right]_{\pm} = \delta S^{\nu}_{\mu}. \tag{7.21}$$

On a general hypersurface one then obtains that the dominant coefficient B_2, determining the post-big bang asymptotic expansion of the Bardeen potential, is in general related to the pre-big bang coefficients A_1, B_1 by [227]

$$B_2(k) = f_A A_1(k) + f_B B_1(k) + f_{\rm s}\delta p_{\rm s}, \tag{7.22}$$

where $f_A, f_b, f_{\rm s}$ are (possibly k-dependent) coefficients, and $\delta p_{\rm s}$ represents the perturbation of the surface tension. The final spectrum $k^3|B_2(k)|^2$ thus depends, in general, not only on the choice of the matching hypersurface, but also on its perturbed tension (i.e. on the details of the regularizing mechanism that is responsible for the bouncing transition).

The matching across the hypersurface of total constant energy density ($\rho + \delta\rho = {\rm const}$) gives in particular $f_A = 0$ [227] so that, assuming a negligible $\delta p_{\rm s}$, B_2 simply inherits the spectrum of the constant mode B_1, and one is led to the "standard" result (7.16) for the scalar perturbation spectrum. The matching across a different hypersurface leaves, however, $f_A \neq 0$, so that the final spectrum is dominated by the growing mode of the pre-big bang solution and (again assuming a negligible $\delta p_{\rm s}$) one is led to the "new" result reported in Eq. (7.17), which leads to a flat spectrum for all backgrounds with $\epsilon \ll 1$.

In the rest of this section we will adhere, however, to the standard matching prescription at constant energy density, and we will concentrate on the alternative possibility of an indirect generation of scale-invariant metric perturbations through the fluctuations of another background field. Such an auxiliary field, in a string cosmology context, can be naturally identified with the axion, whose quantum fluctuations, unlike the metric fluctuations, can be directly amplified with a flat spectrum, even in the context of the minimal pre-big bang models (without the contribution of a dilaton potential).

To explain the reason of such a difference, first pointed out in [175, 182], let us consider the four-dimensional reduced action for the antisymmetric tensor $H_{\mu\nu\alpha}$,

$$S = \frac{1}{12}\int d^4x\sqrt{-g}e^{-\phi}H_{\mu\nu\alpha}H^{\mu\nu\alpha}, \tag{7.23}$$

where

$$H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\alpha\mu} + \partial_\alpha B_{\mu\nu}, \qquad B_{\mu\nu} = -B_{\nu\mu}. \tag{7.24}$$

The equation of motion,

$$\partial_\nu\left(\sqrt{-g}e^{-\phi}H^{\nu\mu\alpha}\right) = 0, \tag{7.25}$$

can be automatically satisfied by introducing the "dual" axion field σ:

$$H^{\mu\nu\alpha} = \frac{e^{\phi}}{\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}\partial_\beta\sigma \equiv e^{\phi}\eta^{\mu\nu\alpha\beta}\partial_\beta\sigma \tag{7.26}$$

($\eta^{\mu\nu\alpha\beta}$ is the totally antisymmetric, covariant tensor), and the effective action (7.23) may be rewritten

$$S = \frac{1}{2}\int d^4x\sqrt{-g}e^{\phi}\partial_\mu\sigma\partial^\mu\sigma \tag{7.27}$$

(note the coupling of σ to e^{ϕ} instead of $e^{-\phi}$). In conformal time, and for a conformally flat metric

$$S = \frac{1}{2}\int d\eta a^2 e^{\phi}\left(\sigma'^2 + \sigma\nabla^2\sigma\right), \tag{7.28}$$

the action can be diagonalized by setting

$$\psi = z\sigma, \qquad\qquad z = ae^{\phi/2}, \tag{7.29}$$

so that the evolution equation for the axion canonical variable ψ is formally the same,

$$\psi_k'' + \left[k^2 - V(\eta)\right]\psi_k = 0, \qquad\qquad V(\eta) = z''/z, \tag{7.30}$$

but is characterized by a pump field z different from that of gravitons or photons. Assuming a smooth evolution from a power-law pre-big bang background, with $z \sim |\eta|^\alpha$, to the standard radiation era, the axion spectrum Ω_σ is then determined by the background kinematics according to the general prescriptions of Subsection 4.4, i.e.

$$\Omega_\sigma(\omega) \sim \omega^{3-2\nu}, \qquad\qquad \nu = |\alpha - 1/2|, \qquad\qquad z \sim |\eta|^\alpha. \tag{7.31}$$

Consider now the low-energy branch of the spectrum, for modes hitting the effective potential barrier during a higher-dimensional dilaton-driven phase, which we assume to be characterized by three isotropically expanding dimensions, with scale factor $a(\eta)$, and n "internal" dimensions, with scale factor $b_i(\eta)$. The background evolution is known, determined by the $d = 3 + n$, anisotropic, S-frame solutions of the string effective action, Eqs. (2.88) and (2.89). The scale factors, and the dimensionally reduced dilaton ϕ, can then be expressed in conformal time as

$$\begin{aligned} &a \sim |\eta|^{\beta_0/(1-\beta_0)}, \qquad b_i \sim |\eta|^{\beta_i/(1-\beta_0)}, \qquad \Phi_d = \frac{\sum_i \beta_i + 3\beta_0 - 1}{1-\beta_0}\ln|\eta|, \\ &\phi = \Phi_d - \ln V_n = \Phi_d - \sum_i \ln b_i \sim \frac{3\beta_0 - 1}{1-\beta_0}\ln|\eta|, \end{aligned} \tag{7.32}$$

where β_0, β_i are $n+1$ parameters related by the "Kasner-like" condition

$$3\beta_0^2 + \sum_i \beta_i^2 = 1 \tag{7.33}$$

($\beta_0 < 0$ for inflationary expansion), and Φ_d is the dilaton field appearing in the d-dimensional, non-reduced action,

$$\int d^{d+1}x\sqrt{-g_{d+1}}e^{-\Phi_d} = V_n\int d^4x\sqrt{-g_4}e^{-\Phi_d} = \int d^4x\sqrt{-g_4}e^{-\phi}. \tag{7.34}$$

Combining the behaviour of a and ϕ we obtain the power α of the pump field, and the corresponding spectral index, which are in general given by :

$$\alpha = -\frac{5\beta_0 - 1}{2(1-\beta_0)}, \qquad 3 - 2\nu = 3 - |2\alpha - 1| = \frac{3\beta_0 + 1}{1 - \beta_0}. \tag{7.35}$$

In the particular case of an isotropic n-dimensional subspace, $\beta_i = \beta$, $3\beta_0^2 + n\beta^2 = 1$, one can also re-express the spectral index (7.35) in terms of the conformal power $s = \beta/(1-\beta_0)$ of the modulus, as originally done in [175], or even in terms of a parameter $r = (\dot{V}_n/2V_n)(\dot{V}_3/V_3)^{-1}$, measuring the relative evolution in time of the external and internal volumes [479]. What is important, in our context, is that a flat spectrum (i.e. $\Omega_\sigma =$ const, $3 - 2\nu = 0$) is now perfectly allowed for $\beta_0 = -1/3$. For an isotropic d-dimensional background, in particular, $\beta_0 = -1/\sqrt{d}$, so that a flat spectrum exactly corresponds to $d = 9$, i.e. just the number of dimensions of critical superstring theory! (see also [135]). Flat (or nearly flat) spectra are also possible for different exact solutions of the low-energy string effective action [177, 178, 324], even in four dimensions if the background is anisotropic [331], or if the dilaton has an appropriate exponential potential and it is non-minimally coupled to the axion [251].

It should be stressed that for gravitons, on the contrary, a flat spectrum cannot be obtained in the above background, quite irrespective of the number and dynamics of the internal dimensions. By using Eq. (7.32) we find indeed that, independently of β_0, β_i, the graviton pump field $ae^{-\phi}$ always has the same power, $\alpha = 1/2$,

$$z = ae^{-\phi/2} \sim |\eta|^{1/2}, \tag{7.36}$$

so that the spectrum (modulo logarithmic corrections) is always characterized by the known cubic slope $3 - 2\nu = 3 - |2\alpha - 1| = 3$. A flat spectrum, in the background (7.32), is also forbidden for heterotic photons (coupled to the dilaton as in the example discussed in the previous subsection). The situation is different, however, for Kaluza–Klein photons, whose spectrum is more sensitive to the dynamics of the internal dimensions [135].

It is important to note, finally, that in the different backgrounds obtained through $SL(2,R)$ transformations mixing the axion and dilaton fields, different evolution equations are generally obtained for the metric, the dilaton, and the axion fluctuations. After a diagonalization, and a redefinition of new canonical variables, one finds however that the evolution equations for the two fundamental scalar degrees of freedom, i.e. the gauge-invariant (Bardeen) potential and the pseudoscalar axion field, are unaltered by the $SL(2,R)$ transformation. Thus, the blue spectral tilt of metric perturbations and a possibly flat slope of the axion spectrum remain the same as in the standard, dilaton-dominated background solutions [175, 182] (see also [437] for more details on the axion spectra).

A flat, primordial spectrum of axion fluctuations can be used for an indirect generation of the observed CMB anisotropies in two ways. A first possibility is the "seed" mechanism

[220, 221]: indeed, even if the axion fluctuations are negligible as sources of the metric background, $\rho_\sigma \ll \rho_c$, their inhomogeneous stress tensor generates metric fluctuations according to the standard gravitational equations, and the metric, in its turn, is source of the temperature anisotropies through the usual Sachs–Wolfe effect:

$$\rho_\sigma/\rho_c \sim \Phi \sim \Delta T/T. \tag{7.37}$$

Such an indirect production of anisotropies could work, in principle, because the contribution (7.37) to $\Delta T/T$ is quadratic in the axion field, and not linear as in the case of metric perturbations. So, even if the amplitude of the axion fluctuations is still normalized at the string curvature scale, the square of the amplitude is not very far from the expected value 10^{-5}:

$$\frac{\Delta T}{T} \sim \Phi \sim \sigma^2 \sim \left(\frac{\text{STRING curvature scale}}{\text{PLANCK scale}}\right)^2. \tag{7.38}$$

The string normalization (7.13), on the other hand, is imposed at the end point of the spectrum (typically, at the gigahertz scale), while COBE observations constrain the spectrum at the present horizon scale ($\sim 10^{-18}$ Hz): it is then evident that a very small, "blue" tilt in the axion spectrum is sufficient to make compatible the string and COBE normalizations, Eqs. (7.38) and (7.12).

The seed mechanism, however, seems to have difficulties in reproducing the peak structure of the observed anisotropies at smaller angular scales, as will be discussed in Subsection 7.3. A more realistic scenario is probably offered by the "curvaton" mechanism [459]: if the axion is heavy enough, the initial and flat primordial spectrum of isocurvature axion fluctuations can be eventually converted, after the axion decay, into a final scale-invariant spectrum of curvature fluctuations. The standard results for the temperature anisotropies then apply, with the only difference that the metric perturbations are not primordial, but indirectly generated by the axion decay. This possibility will be discussed in Subsection 7.5, after a brief report on the seed mechanism, which will be presented in the next two subsections.

7.3 Massless axions as seeds

The quantum fluctuations of the so-called "universal" (or Kalb-Ramond) axion σ (i.e. the dual of the field strength of the NS-NS two-form appearing in the string effective action) can be amplified with a scale-invariant distribution of the spectral energy density, $\Omega_\sigma(k,\eta) = \rho_c^{-1}(d\rho_\sigma(k,\eta)/\rho_c d\ln k)$ (as shown in the previous subsection); they are thus a possible candidate for seeding the large-scale anisotropy in the context of the pre-big bang scenario. To illustrate this possibility we will proceed in two steps.

We will first show that the scalar metric fluctuation on a given scale k, at the time the scale re-enters the horizon, is precisely determined by the axion energy distribution

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evaluated at the conformal time of re-entry, $\eta_{\rm re} \simeq k^{-1}$:

$$\Phi_k(\eta_{\rm re}) \sim \Omega_\sigma(k, \eta_{\rm re}). \tag{7.39}$$

Secondly, we will show that the dominant contribution of a given scale to the Sachs–Wolfe effect comes from the time that scale re-enters the horizon, in such a way that the final temperature spectrum exactly reproduces the primordial seed spectrum:

$$(\Delta T/T)_k \sim \Phi_k(\eta_{\rm re}) \sim \Omega_\sigma(k, \eta_{\rm re}). \tag{7.40}$$

These two results are far from being trivial, as they are consequences of the particular time dependence of the Bardeen spectrum induced by axion fluctuations (the same results do not apply, for instance, to the case of electromagnetic fluctuations). We will give here only a sketch of the arguments leading to the above results, which are interesting in themselves, apart from their possible string cosmology applications. For a detailed derivation we refer the reader to the original papers [223, 224].

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Let us first compute the spectrum of metric perturbations seeded by a flat, primordial distribution of axion fluctuations. We define, as usual, the power spectrum of the Bardeen potential, $P_\Phi(k)$, in terms of the Fourier transform of the two-point correlation function,

$$\int \frac{d^3k}{(2\pi k)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} P_\Phi(k) = \langle \Phi(x)\Phi(x')\rangle \tag{7.41}$$

(the brackets denote spatial average, or expectation value if perturbations are quantized), and we recall that the square root of this function, evaluated at a comoving distance k^{-1}, represents the typical amplitude of fluctuations on a scale k:

$$\left(\langle \Phi(x)\Phi(x')\rangle\right)^{1/2}_{|x-x'|=k^{-1}} \sim k^{3/2}|\Phi_k| \tag{7.42}$$

(see also Eq. (4.69)). We also define the power spectrum of the seed stress tensor, in the same way (no sum over μ, ν):

$$\int \frac{d^3k}{(2\pi k)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} P^\nu_\mu(k) = \langle T^\nu_\mu(x) T^\nu_\mu(x')\rangle - \langle T^\nu_\mu(x)\rangle^2. \tag{7.43}$$

Metric fluctuations and seed fluctuations, on the other hand, are related by the cosmological perturbation equations [221]. By taking into account the important contribution of the off-diagonal components of the axion stress tensor one finds, typically, that the Bardeen spectrum P_Φ and axion energy–density spectrum P_ρ are related by:

$$P^{1/2}_\Phi(k) \sim G \left(\frac{a}{k}\right)^2 P^{1/2}_\rho(k), \tag{7.44}$$

where $\rho = T^0_0$, and

$$\begin{aligned} \int \frac{d^3k}{(2\pi k)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} P_\rho(k) &= \langle \rho_\sigma(x)\rho_\sigma(x')\rangle - \langle \rho_\sigma(x)\rangle^2 \\ &\sim \langle \sigma'^2(x)\sigma'^2(x')\rangle - \langle \sigma'^2(x)\rangle^2 + \ldots \end{aligned} \tag{7.45}$$

Note that the two-point correlation function of the energy density becomes a four-point function of the seed field, since the energy is quadratic in the axion field.

Using now the stochastic average condition for the axion field,

$$\langle \sigma'(\mathbf{k},\eta)\sigma'^{*}(\mathbf{k}',\eta)\rangle = (2\pi)^3\delta^3(k-k')\Sigma(\mathbf{k},\eta), \tag{7.46}$$

it turns out that the energy–density spectrum reduces to a convolution of Fourier transforms,

$$P_\rho(k) \sim \frac{k^3}{a^4}\int d^3p\ \Sigma(p)\Sigma(|k-p|) + ...\ \ . \tag{7.47}$$

Also, for a flat enough axion spectrum, one finds that the above integral is dominated by the region $p\eta \sim 1$ [224]. By expressing the convolution through the spectral energy density Ω_σ, evaluating the Bardeen potential at the time of re-entry $\eta_{\rm re} \sim k^{-1}$, and using Eq. (7.44), one is finally led to relating the Bardeen spectrum and the axion spectrum as follows:

$$P_\Phi^{1/2}(k,\eta_{\rm re}) \sim k^{3/2}\,|\Phi_k(\eta_{\rm re})| \sim \Omega_\sigma(k,\eta_{\rm re}). \tag{7.48}$$

We should now explain why we are interested in the metric fluctuations evaluated at the time of re-entry.

To this purpose, let us compute the seed contribution to $\Delta T/T$. In the multipole expansion of the temperature anisotropies (in terms of the Legendre polynomials P_ℓ),

$$\left\langle \frac{\delta T}{T}(\mathbf{n})\frac{\delta T}{T}(\mathbf{n}')\right\rangle_{(\mathbf{n}\cdot\mathbf{n}'\equiv\cos\vartheta)} = \frac{1}{4\pi}\sum_\ell (2\ell+1)C_\ell P_\ell(\cos\vartheta)\ , \tag{7.49}$$

the coefficients C_ℓ, at very large angular scales ($\ell \ll 100$), are determined by the Sachs–Wolfe effect as [221]:

$$C_\ell^{\rm SW} = \frac{2}{\pi}\int d\ (\ln k)\left\langle\left[\int_{k\eta_{\rm dec}}^{k\eta_0} d(k\eta)\ k^{3/2}(\Psi+\Phi)(\mathbf{k},\eta)j_\ell'\left(k\eta_0-k\eta\right)\right]^2\right\rangle. \tag{7.50}$$

Here Φ and Ψ are the two independent components of the gauge-invariant Bardeen potential (see Subsection 4.2), j_ℓ are the spherical Bessel functions, and a prime denotes the derivative with respect to their argument. Equation (7.50) takes into account both the "ordinary" and the "integrated" Sachs–Wolfe contribution, namely the complete distortion of the geodesics of the CMB photons (due to shifts in the gravitational potential), from the time of decoupling $\eta_{\rm dec}$ down to the present time η_0. By inserting the Bardeen potential determined by the axion field, we find that the time-integral is dominated by the region $k\eta \sim 1$. Using Eq. (7.48) we finally obtain

$$C_\ell^{\rm SW} \sim \int d(\ln k)\ k^3|\Phi_k(\eta_{\rm re})|^2|j_\ell(k\eta_0)|^2, \tag{7.51}$$

which gives the multipole coefficients in terms of the axion distribution $\Omega_\sigma(k,\eta_{\rm re})$, through Eq. (7.48). This explains why it was important to evaluate the Bardeen spectrum at the time of re-entry, $\eta_{\rm re} \sim k^{-1}$.

From this final expression we can extract, in particular, the value of the quadrupole coefficient C_2 [224]:

$$C_2 \simeq \Omega_\sigma^2(k_0, \eta_0) \simeq \left(\frac{M_{\rm s}}{M_{\rm P}}\right)^4 \left(\frac{k_0}{k_1}\right)^{n-1}, \tag{7.52}$$

where n is the spectral index (sufficiently near to 1) characterizing the primordial axion distribution, k_0 is the comoving scale of the present horizon, and k_1 the end point of the spectrum, namely the maximal amplified comoving frequency. We have assumed, in the context of a minimal model of pre-big bang, that the peak amplitude of the axion spectrum, at the end-point frequency, is controlled by the fundamental ratio between string and Planck masses, as it was for the graviton and dilaton spectra computed in the previous sections. Note, however, that in the low-frequency band corresponding to the large-scale observations the axion spectrum contains the enhancement factor $(k/k_{\rm eq})^{-2}$ (see Eq. (5.2)), which for $k = k_0$ almost exactly cancels the present radiation energy density, $\Omega_\gamma(t_0) \sim 10^{-4}$.

The quadrupole coefficient, on the other hand, is at present determined by COBE as [50]

$$C_2 = (1.9 \pm 0.23) \times 10^{-10}. \tag{7.53}$$

This experimental value, inserted into Eq. (7.52), provides an interesting, direct relation between the string mass and the spectral index of the temperature anisotropy. Using $k_1/k_0 \simeq (M_{\rm s}/M_{\rm P})^{1/2} 10^{29}$ (see Eq. (5.16)), and using (as a reference value) the following bounds on the spectral index [78]:

$$1 \leq n \leq 1.4 \tag{7.54}$$

(we have excluded here the allowed values $0.9 \leq n \leq 1$, to avoid introducing an infrared cut-off in the massless axion spectrum), we obtain in fact an allowed range for the string-mass scale, $0.0037 \lesssim M_{\rm s}/M_{\rm P} \lesssim 3.2$, which is compatible with the theoretical expectations [403], $0.01 \lesssim M_{\rm s}/M_{\rm P} \lesssim 0.1$. Conversely, the theoretically expected range for $M_{\rm s}$ implies a spectral index $1.06 \lesssim n \lesssim 1.2$, which is also in agreement with the observational results.

This promising result, however, is only a first step towards a more complete comparison of the seed model with observations, comparison that must include also smaller angular scales, where precise experimental data have recently been obtained [483, 475, 203, 363], and will become available in the next few years [470, 521].

A numerical computation of the anisotropy power spectrum (7.50), at intermediate angular scales (up to $\ell \sim 10^3$), has already been performed as reported in [479, 608]. The resulting spectrum shows the familiar peak structure, typical of all inflationary models. However, as was already shown very clearly by a first analysis [479], the height and the position of the first acoustic peak can be reproduced (at least approximately) only if the axion spectrum grows fast enough with frequency. In particular, using the notation of the previous subsection, one must require at least $3 - 2\nu > 0.1$ (but $3 - 2\nu \gtrsim 0.3$ seems to be favoured by the data).

This requirement, however, is in general incompatible with the high-frequency normalization typical of the pre-big bang scenario (if the amplitude is large enough to match observations at the COBE scale and the slope is too steep, then the energy density blows up and becomes overcritical at the string scale).

A possible solution, suggested in [479], exists in the context of "non-minimal" models of background, leading to a slope that is steep enough at low frequency, and flatter at high frequency, so as it limits the peak value. Since the axion spectrum depends both on the number and on the kinematics of the internal dimensions (see Subsection 7.2), such a spectrum could be produced during a phase of dynamical dimensional reduction, in which the low-frequency modes are frozen outside the horizon while the Universe is dimensionally different from the subsequent phase of freezing of the high-frequency modes.

A more detailed analysis [608], based on the recent BOOMERanG-98 [203] and MAXIMA-1 [363] data, shows indeed that it is not impossible to fit the height of the first acoustic peak, observed around $\ell \sim 200$, by using a (pre-big bang-motivated) axion spectrum, which grows with a steep slope up to a limiting scale k_b, and which is pretty flat above k_b:

$$3 - 2\nu \simeq 0.33, \qquad k < k_b, \qquad 3 - 2\nu \simeq 0, \qquad k > k_b. \tag{7.55}$$

In order to be efficient, the break point k_b has to be around the equilibrium scale, $k_b \sim k_{\rm eq}$. However, the precise position of the break is not strongly constrained, and one may exploit the possibility of lowering the break scale, to lower the height of the second (and subsequent) acoustic peaks, for a better comparison with data (such a shift does not substantially affect the height of the first peak).

A break in the axion spectrum is thus effective to match the *height* of the peaks, but is not sufficient to fit the *position* of the first acoustic peak, determined with small uncertainty by recent data [483, 475, 203, 363] around $\ell \sim 200$. The position depends on the full set of parameters of a given cosmological model and, in the context of the seed mechanism based on massless axions, a peak at $\ell \sim 200$ seems to require a moderately closed Universe , with $\Omega_m + \Omega_\Lambda > 1$. For instance [608],

$$\Omega_m \simeq 0.4, \qquad \Omega_\Lambda \simeq 0.85 \tag{7.56}$$

provides a good fit of the data, also compatible with current supernovae results [517]. It should be noted, to confirm the validity of a such model, that if the axion spectrum has a break it is also possible to fit the dark-matter power spectrum, in reasonable agreement with observations.

However, even adjusting the break point k_b and the value of $\Omega_m + \Omega_\Lambda$, the width of the peak is still not in very good agreement with the present data, as the resulting normalized χ^2 is about 1.8, which confirms the model only at the 30% confidence level [608]. Nonetheless, it is important to stress that the position of the second acoustic peak predicted by the axion model is different from the one indicated by the standard inflationary predictions (and by

present data). In addition, the axion model also predicts a small peak, or "hump", around $\ell \sim 100$, due to the "isocurvature" (rather than adiabatic, i.e. isoentropic) nature of the induced metric perturbations. These two peculiar predictions will be extremely important to confirm or disprove definitely the axion model, when the future satellite observations [470, 521] will become available.

Also, the axion fluctuations σ_k are described by a Gaussian, stochastic variable, but the induced metric fluctuations depend on the axion energy–momentum tensor: they are quadratic in σ_k and therefore non-Gaussian, in general. This may change the estimate of the confidence level (i.e. of the probability) that the axion model leads to the measured CMB anisotropy spectrum.

It should be stressed, to conclude this subsection, that the isocurvature character of the primordial axion spectrum considered here is due to the fact that the axions are treated as seeds, i.e. inhomogeneous fluctuations of a background that *is not* axion-dominated. As a consequence, the predicted anisotropies are different from previous models of isocurvature axion perturbations [42, 558, 496]. So, even if this seed mechanism based on isocurvature axion perturbations were definitely ruled out by future observations, this would not preclude the possibility of adiabatic axion perturbations (see for instance [409]), where the axions produced during the phase of pre-big bang inflation represent a significant fraction of the present cold dark-matter density.

7.4 Massive axion spectra

In the previous subsection we have discussed the CMB anisotropies seeded by a stochastic background of massless axion fluctuations, amplified by a phase of pre-big bang inflation. It is possible, or even likely, however, that the axion fluctuations amplified by inflation become massive in the post-inflationary era: it is thus important, also in view of future applications other than the seed mechanism, to discuss whether, even in the massive case, the string normalization of the primordial spectrum may be consistent with the large-scale normalization determined by the CMB anisotropy, in spite of some important differences that appear in the spectrum and in the computation of the Sachs–Wolfe effect.

In this subsection we shall keep assuming that the axion background is negligible, $\langle\sigma\rangle = 0$, and that the whole contribution to the axion energy density ρ_σ comes from the vacuum fluctuations, as in the previous subsection. There are, however, various important differences. A first difference between the massless and massive axion cases concerns the relation between ρ_σ and the Bardeen potential Φ. In the massless case the perturbation equations, taking into account the important contribution of all the off-diagonal terms of the axion stress tensor, lead to

$$\Phi_k \simeq G \left(\frac{a}{k}\right)^2 \rho_\sigma(k) \tag{7.57}$$

(see Eq. (7.44)). In the massive case, on the contrary, the axion stress tensor can be

approximated as a diagonal, perfect-fluid stress tensor, and we obtain [324]

$$\Phi_k \sim G a^2 \eta^2 \rho_\sigma(k). \tag{7.58}$$

Also, the convolution (7.47) for the axion energy density is dominated by the region $p \sim \eta^{-1}$ in the massless case, and by $p \sim k$ in the massive case. In the massless case the integrated Sachs–Wolfe effect is dominant, while it is the ordinary one that dominates in the massive case [324].

In spite of these differences, the final result for the temperature anisotropies is the same as before, with the quadrupole coefficient determined by the axion spectral energy density as [324]

$$C_2 \simeq \Omega_\sigma^2(k_0, \eta_0). \tag{7.59}$$

In the massive case, however, the axion spectrum is affected by non-relativistic corrections, which are to be computed by including the mass contribution to the canonical evolution equation (7.30):

$$\psi_k'' + \left[k^2 - \frac{z''}{z} + m^2 a^2\right] \psi_k = 0, \qquad \psi_k = z\sigma_k. \tag{7.60}$$

To this purpose, let us consider a simple background transition at $\eta = \eta_1$, from an initial pre-big bang phase in which the axion is massless, to a final radiation-dominated phase in which the dilaton freezes to its present value, $\phi = \text{const}$, and the axion acquires a mass, which is small in string units. Therefore, for $\eta < \eta_1$, we then have $m = 0$, and the usual Hankel solution is normalized to a vacuum fluctuation spectrum (see Section 4.4),

$$\psi_k(\eta) = \frac{\sqrt{\pi}}{2} |\eta|^{1/2} H_\nu^{(2)}(|k\eta|), \tag{7.61}$$

where ν depends on the pre-big bang evolution of the pump field $z = a e^{\phi/2}$. In the subsequent radiation era, $\eta > \eta_1$, the effective potential z''/z is vanishing, and the perturbation equation reduces to

$$\psi_k'' + \left(k^2 + \alpha^2 \eta^2\right) \psi_k = 0, \tag{7.62}$$

where we have put

$$m^2 a^2 = \alpha^2 \eta^2, \qquad \alpha = m H_1 a_1^2, \tag{7.63}$$

using the time behaviour of the scale factor $a \sim \eta$.

For the high-frequency modes that are relativistic at the transition scale, $m \ll k/a_1$, the mass term can be neglected, and we can match the solution (7.61) to the plane-wave solution

$$\psi_k = \frac{1}{\sqrt{2}k} \left[c_+(k) e^{-ik\eta} + c_-(k) e^{ik\eta}\right], \tag{7.64}$$

to obtain:

$$c_\pm = \pm c(k) e^{\pm ik\eta}, \qquad |c(k)| \simeq (k/k_1)^{-\nu - 1/2} \tag{7.65}$$

(we neglect, for simplicity, numerical factors of order 1, which will be absorbed into the uncertainty relative to the inflation scale, represented by the overall numerical coefficient g_1 in front of the final spectrum.) In the relativistic regime the amplified axion perturbation then takes the form:

$$\sigma(\mathbf{k},\eta) = \frac{c(\mathbf{k})}{a\sqrt{k}}\sin(k\eta), \tag{7.66}$$

and the corresponding energy density, $\rho_\sigma^{\rm rel}(k) \sim (\sigma'/a)^2$, integrated over all modes, leads to the spectral distribution (4.102), and to the usual relativistic spectrum (normalized to critical units):

$$\Omega_\sigma(p,t) = g_1^2\Omega_\gamma(t)\left(\frac{p}{p_1}\right)^{3-2\nu}, \qquad m < p < p_1, \tag{7.67}$$

where $p = k/a$ is the proper momentum (see Subsection 4.4).

In the radiation era the proper momentum is redshifted with respect to the rest mass, and all axion modes tend to become non-relativistic. When the mass term is no longer negligible, on the other hand, the general solution of Eq. (7.62) can be written in terms of parabolic cylinder functions [5]. For an approximate estimate of the axion field in the non-relativistic regime, however, it is convenient to distinguish two cases, depending on whether a mode k becomes non-relativistic inside or outside the horizon. Defining as k_m the limiting comoving frequency of a mode that becomes non-relativistic ($k_m = ma_m$) at the time it re-enters the horizon ($k_m = H_m a_m$), we find, in the radiation era [324],

$$k_m = \frac{H_m a_m}{H_1 a_1} H_1 a_1 = k_1\left(\frac{m}{H_1}\right)^{1/2}. \tag{7.68}$$

We will consider the two cases $k \gg k_m$ and $k \ll k_m$ separately.

In the first case we can rewrite Eq. (7.62) as

$$\frac{d^2\psi_k}{dx^2} + \left(\frac{x^2}{4} - b\right)\psi_k = 0, \qquad x = \eta(2\alpha)^{1/2}, \qquad -b = k^2/2\alpha, \tag{7.69}$$

and the general solution we give has the form

$$\psi = AW(b,x) + BW(b,-x), \tag{7.70}$$

where $W(b,x)$ are the Weber parabolic cylinder functions (see [5], Chap. 19). In order to fix the integration constants A and B we can match the solutions (7.70) and (7.66) in the relativistic limit where

$$\frac{k^2}{m^2a^2} = \frac{k^2}{\alpha^2\eta^2} = \frac{-4b}{x^2} \gg 1. \tag{7.71}$$

In this limit, since we are considering modes that become non-relativistic when they are already inside the horizon,

$$\left(\frac{k}{k_m}\right)^2 \sim \frac{k^2}{\alpha} \sim (-b) \gg 1, \tag{7.72}$$

we can expand the W functions for b large with x moderate [5]. Matching to the plane-wave solution (7.64), we obtain $A = 0$, and

$$\psi_k \simeq \frac{c(\mathbf{k})}{\alpha^{1/4}} W(b, -x). \tag{7.73}$$

In the opposite, non-relativistic limit $x^2 \gg |4b|$, the expansion of the Weber functions gives [5]

$$\psi_k \simeq \frac{c(\mathbf{k})}{(\alpha\eta)^{1/2}} \sin\left(\frac{m}{H}\right) \tag{7.74}$$

(we have used $x^2/4 = ma\eta/2 \sim m/H$). The corresponding axion field is then (inside the horizon)

$$\sigma(\mathbf{k}, \eta) = \frac{c(\mathbf{k})}{a\sqrt{ma}} \sin\left(\frac{m}{H}\right), \qquad k > k_m. \tag{7.75}$$

The associated energy density, $\rho_\sigma^{\rm non-rel}(k) \sim (\sigma'/a)^2 \sim (ma/k)\rho_\sigma^{\rm rel}(k)$, differs from the previous one by the non-relativistic rescaling m/p, thus leading to the spectrum

$$\Omega_\sigma(p, t) = g_1^2 \frac{m}{H_1} \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3 \left(\frac{p}{p_1}\right)^{2-2\nu}, \qquad p_m < p < m, \tag{7.76}$$

where $p_m = k_m/a$ (see Eq. (7.67)).

Finally, we consider the case of a mode that becomes non-relativistic when it is still outside the horizon, $k \ll k_m$. In this case, we cannot use the large $|b|$ expansion as $|b| < 1$, and it is convenient to express the general solution of Eq. (7.69) in the form

$$\psi = A y_1(b, x) + B y_2(b, x) \ , \tag{7.77}$$

where y_1 and y_2 are the even and odd parts of the parabolic cylinder functions [5]. The matching to (7.66), in the relativistic limit $x \to 0$, gives $A = 0$ and

$$\psi_k \simeq c(\mathbf{k}) \left(\frac{k}{2\alpha}\right)^{1/2} y_2(b, x). \tag{7.78}$$

In the non-relativistic limit $x^2 \gg |b|$ we use the relation [5]

$$y_2 \sim [W(b, x) - W(b, -x)] \sim \frac{1}{\sqrt{x}} \sin\frac{x^2}{4}, \tag{7.79}$$

which leads to

$$\psi_k \simeq \frac{c(\mathbf{k})}{(\alpha\eta)^{1/2}} \left(\frac{k^2}{\alpha}\right)^{1/4} \sin\left(\frac{m}{H}\right), \tag{7.80}$$

from which

$$\sigma(\mathbf{k}, \eta) = \frac{c(\mathbf{k})}{a\sqrt{ma}} \left(\frac{k}{k_1}\right)^{1/2} \left(\frac{H_1}{m}\right)^{1/4} \sin\left(\frac{m}{H}\right), \qquad k < k_m \tag{7.81}$$

with an associated spectrum

$$\Omega_\sigma(p,t) = g_1^2 \sqrt{\frac{m}{H_1}} \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3 \left(\frac{p}{p_1}\right)^{3-2\nu}, \qquad p < p_m, \tag{7.82}$$

reproducing the low frequency band of the non-relativistic spectrum already presented in the case of dilatons (see Eq. (6.13)).

However, the above result is valid only if the axion field becomes massive *before* the limiting scale p_m crosses the horizon. Let us call T_m the temperature scale at which the mass turns on (typically, $T_m \sim 100$ MeV if axions become massive at the epoch of chiral symmetry breaking) and p_T the proper frequency re-entering the horizon precisely at the same epoch. The present value of p_T is then

$$p_T(\eta_0) \simeq p_{\rm eq} \left(\frac{T_m}{\rm eV}\right), \tag{7.83}$$

and the spectrum (7.82) is valid for $p_m < p_T$, namely for $m/H_{\rm eq} < (T_m/{\rm eV})^2$. In the opposite case, $p_m > p_T$, the role of the transition frequency, which separates modes that become non-relativistic inside and outside the horizon, is played by p_T, and the lowest-frequency band of the non-relativistic spectrum (7.82) has to be replaced by [324]:

$$\Omega_\sigma(p,t) = g_1^2 \frac{m}{\sqrt{H_1 H_{\rm eq}}} \left(\frac{\rm eV}{T_m}\right) \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3 \left(\frac{p}{p_1}\right)^{3-2\mu}, \; p < p_T, \quad \left(\frac{m}{H_{\rm eq}}\right)^{1/2} \left(\frac{\rm eV}{T_m}\right) > 1 \tag{7.84}$$

(such a spectrum can be easily obtained by imposing the continuity with (7.76) at $p = p_T$).

In both cases, as clearly illustrated in Fig. 7.3, the effect of non-relativistic corrections is to enhance the amplitude of the spectrum at low frequency. The enhancement is proportional to the square root of the axion mass according to Eq. (7.82), if the scale p_m is still outside the horizon at the time when axions become massive. In the opposite case, the enhancement is linear in the axion mass, according to Eq. (7.84). In both cases the slope is the same as that of the massless spectrum, $3 - 2\nu$.

Since the amplitude of the spectrum now depends on the axion mass, the condition imposed by the COBE normalization provides a constraint on the allowed range of masses. This might represent a problem, in general: since the slope cannot be too steep at low frequency (according to Eq. (7.54)), the allowed mass has to be small enough, typically in the range of $H_{\rm eq}$. An explicit analysis [224] then leads to the window 10^{-27} eV $\lesssim m \lesssim 10^{-17}$ eV.

It should be noted, at this point, that the mass of an axion field coupled to matter with a typical gravitational strength is not significantly constrained (unlike the dilaton mass) by the present gravitational experiments, which are indeed to be performed with polarized bodies [538] to be sensitive to pseudoscalar interactions. Also, the Kalb-Ramond (KR) axion, even if (gravitationally) coupled to the QCD topological current, is not to be

QCD vacuum angle

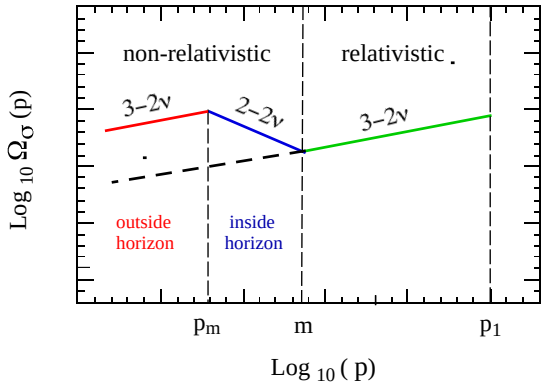

Figure 7.3: *Mass-dependent enhancement of the spectrum at low frequency, due to non-relativistic corrections. If the slope is sufficiently flat, $3 - 2\nu < 1$, then the non-relativistic part of the spectrum may have a peak in correspondence of the frequency mode p_m that becomes non-relativistic at the time it re-enters the horizon.*

necessarily identified with the "invisible" axion [415] responsible for solving the strong CP problem. Other, more strongly coupled pseudoscalars, can play the traditional axion's role.

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In that case, the standard Weinberg–Wilczek formula [619, 623] would give the mass of the appropriate combination of pseudoscalars, which is coupled to the topological charge, while the KR axion would mostly lie along the orthogonal combinations, which remain (almost) massless. In this context it is thus possible that KR axions are produced neither from an initial misalignment of the QCD vacuum angle, nor from the decay of axionic cosmic strings, so that existing cosmological bounds on the axion mass [566] can be evaded. In spite of all this, it is quite likely that the KR axion will be heavier than 10^{-17} eV, in which case, using the above spectra, it would be impossible even in principle to seed the observed CMB anisotropies.

This conclusion, based on the effect illustrated in Fig. 7.3, refers however to a relativistic spectrum characterized by a constant slope. It is quite easy to imagine, however, and to implement in practice, a model of background in which the relativistic axion spectrum is sufficiently flat at low frequency (as required by a fit of the large-scale anisotropy), and much steeper at high frequency. A simple example is illustrated in Fig. 7.4, where we have compared two spectra. The top one is flat everywhere, except for modes becoming non-relativistic outside the horizon. The other one has a relativistic branch, which is flat at low frequency $p < p_s$, and steeper at high frequency $p > p_s$. It is evident that the steeper and the longer the high-frequency branch of the spectrum, the larger is the suppression of the amplitude at low frequency, and the larger is the axion mass allowed by the COBE normalization at the present Hubble scale $\omega = \omega_0$.

This possibility has been analysed through an explicit two-parameter model of back-

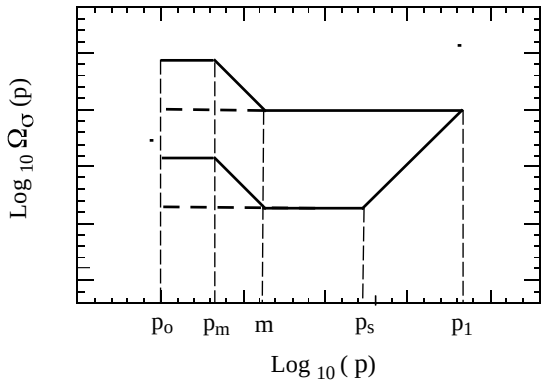

Figure 7.4: *Two examples of axion spectra with non-relativistic corrections. Note the common normalization at the end-point frequency p_1, in spite of the different slopes in the different frequency regimes.*

ground [324], based on exact solutions of the low-energy string-cosmology equations with classical string sources [321, 302, 303]. By fixing the high-frequency normalization at $Ms/M_{\rm P} = 10^{-2}$, the low-frequency slope at $n = 1.2$, and the temperature of mass generation at $T = 100$ MeV, we can use as parameters the slope and the duration of the high-frequency part of the spectrum, measured in particular by $x = \log(p_1/p_{\rm s}) > 0$ and by the corresponding variation of the pump field, $y = \log(z_{\rm s}/z_1) < 0$. By imposing the COBE normalization (7.59), $\Omega_\sigma(p_0) \simeq 10^{-5}$, the seed condition $\Omega_\sigma < 0.1$, and the condition of mass-dominated spectrum, $m > H_{\rm eq}$ (see [324]), the allowed region in parameter space turns out to be consistent with a very wide range of axion masses, from the equilibrium scale $m \sim 10^{-27}$ eV up to $m \sim 100$ MeV (higher masses are not acceptable, because the axion would decay into photons before the present epoch, like dilatons, see Eq. (6.33)). We may thus conclude that there is no fundamental incompatibility between the normalization imposed by the large-scale anisotropy and an axion mass in the expected range of conventional axion models [566]. The difficulties encountered to match the observed spectrum at smaller angular scales, discussed in the previous subsection, remains however .

7.5 Massive axions and adiabatic perturbations

In the previous subsections we have analysed a possible scenario in which no axion background is present in the post-big bang epoch ($\langle\sigma\rangle = 0$), and an *isocurvature, non-Gaussian* spectrum of temperature anisotropies is eventually induced by the quantum fluctuations of the (massless or massive) axion field, amplified by inflation. This scenario, whose typical signatures have been discussed in Subsection 7.3, seems to be strongly disfavoured with respect to more conventional explanations of the CMB anisotropy offered by the standard inflationary models.

In this subsection we will consider a possible alternative picture, based on the following assumptions: *i*) the constant value of the axion background after the pre-big bang phase is displaced from the minimum (conventionally defined as $\sigma = 0$) of the non-perturbative potential $V(\sigma)$ generated in the post-big bang epoch; *ii*) the axion potential is strong enough to induce a phase of axion dominance before its decay into radiation. Under these conditions one finds that the initial isocurvature axion fluctuations, amplified by horizon-exit during inflation, can be converted into *adiabatic* (and *Gaussian*) perturbations of the spatial curvature, associated to super-horizon (scalar) metric perturbations, which maintain the same spectrum as the original axion fluctuations until the time of horizon re-entry: these can then possibly produce the observed CMB anisotropies.

This possibility, based on a mechanism originally pointed out in [486], has recently been discussed in detail (and not exclusively within a string cosmology framework) in [241, 459, 487, 65, 502, 488] for the isocurvature fluctuations of a generic scalar field (called "curvaton" [459]), amplified with a flat spectrum. In the context of the pre-big bang scenario, such a mechanism has been applied, in particular, to the fluctuations of the KR axion [241], as suggested in [437]. Here, after a brief description of the conversion of axion fluctuations into scalar curvature perturbations, we shall discuss the constraints imposed by the CMB data, and its possible consistency with the small-scale normalization and tilts typical of pre-big bang models [100].

The conversion of the axionic isocurvature modes into adiabatic curvature inhomogeneities takes place in the post-big-bang phase, where we assume the dilaton ϕ to be frozen and the axion σ to be displaced from the minimum of its potential. In the E-frame the background evolution is then described by the Einstein equations (6.2), with ϕ simply replaced by σ, and with $c = 0$ since the axion is minimally coupled to the metric. For a conformally flat metric, $g_{\mu\nu} = a^2\eta_{\mu\nu}$, the time and space components of such equations, together with the axion evolution equation, can be written (in conformal time and in $d = 3$) respectively as

$$6\mathcal{H}^2 = a^2\left(\rho_r + \rho_\sigma\right), \qquad 4\mathcal{H}' + 2\mathcal{H}^2 = -a^2\left(p_r + p_\sigma\right), \tag{7.85}$$

$$\sigma'' + 2\mathcal{H}\sigma' + a^2\frac{\partial V}{\partial\sigma} = 0, \tag{7.86}$$

where $\mathcal{H} = a'/a = d(\ln a)/d\eta$, $\rho_r = 3p_r$ is the energy density of the radiation fluid, and

$$\rho_\sigma = \frac{1}{2a^2}\sigma'^2 + V(\sigma), \qquad \rho_\sigma = \frac{1}{2a^2}\sigma'^2 - V(\sigma). \tag{7.87}$$

The combination of Eqs. (7.85) and (7.86) leads to the conservation equation for the radiation fluid, i.e. $\rho_r' + 4\mathcal{H}\rho_r = 0$. In order to perform analytical estimates, we shall also assume here that the axion potential can be approximated by the quadratic form $V(\sigma) = m^2\sigma^2/2$. This is certainly true for $\sigma_i \ll 1$, but it may be expected to be a realistic approximation also for the range of values of σ_i not much larger than 1 (which, as we shall see, is the appropriate range for a normalization of the spectrum compatible with

present data). Actually, for the periodic potential expected for an axion, the value of $|\sigma_i|$ is effectively bounded from above [241].

Let us now consider the scalar perturbations of the Einstein equations (6.2). Working in the longitudinal gauge we set $\delta\sigma = \chi$, and we use the perturbation variables already defined in Eq. (6.1) for the metric and for the fluid sources. We impose, in addition, $\delta p_r = \delta\rho_r/3$ (adiabatic fluid perturbations), and we introduce the convenient velocity potential v_r, such that $\delta u_i = a\partial_i v_r$.

The perturbation of the $i \neq j$ component of the Einstein equations then gives $\varphi = \psi = \Phi$, where Φ is the Bardeen potential, while the perturbation of the $(0,i)$, $(0,0)$, (i,i) components, and of the axion (Klein–Gordon) equation, gives a set of four independent equations for the four scalar variables $\Phi,\ \chi,\ \delta\rho_r,\ v_r$. They can be written, respectively, in the following convenient form:

$$\Phi' + \mathcal{H}\Phi = \frac{1}{4}\chi\sigma' + \frac{1}{3}a^2\rho_r v_r, \tag{7.88}$$

$$\nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) = \frac{1}{4}a^2\left(\rho_\rho\delta_r + \rho_\sigma\delta_\sigma\right), \tag{7.89}$$

$$\Phi'' + 3\mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi = \frac{1}{4}a^2\left(\frac{1}{3}\rho_\rho\delta_r + \delta p_\sigma\right), \tag{7.90}$$

$$\chi'' + 2\mathcal{H}\chi' - \nabla^2\chi + a^2\frac{\partial^2 V}{\partial\sigma^2}\chi = 4\sigma'\Phi' - 2a^2\frac{\partial V}{\partial\sigma}\Phi, \tag{7.91}$$

where we have defined

$$\begin{aligned} &\delta_r = \delta\rho_r/\rho_r, \qquad\qquad \delta_\sigma = \delta\rho_\sigma/\rho_\sigma, \\ &\delta\rho_\sigma = -\Phi\left(\rho_\sigma + p_\sigma\right) + \frac{\sigma'\chi'}{a^2} + \frac{\partial V}{\partial\sigma}\chi, \\ &\delta p_\sigma = -\Phi\left(\rho_\sigma + p_\sigma\right) + \frac{\sigma'\chi'}{a^2} - \frac{\partial V}{\partial\sigma}\chi. \end{aligned} \tag{7.92}$$

The above equations can also be obtained directly from the general system of scalar-perturbation equations (6.3)–(6.8) (used to discuss dilaton production), for the particular case in which $c = 0,\ d = 3,\ \beta = \sigma'/\mathcal{H},\ \gamma = \epsilon = 1/3,\ p = \rho/3$, and with ϕ' replaced by σ'. We note, for later use, that by eliminating δ_r from Eqs. (7.89) and (7.90) we are led to

$$\Phi'' + 4\mathcal{H}\Phi' + \left(2\mathcal{H}' + 2\mathcal{H}^2 + \frac{1}{6}\sigma'^2\right)\Phi - \frac{1}{3}\nabla^2\Phi = \frac{1}{6}\sigma'\chi' - \frac{1}{3}a^2\frac{\partial V}{\partial\sigma}\chi, \tag{7.93}$$

which, together with Eq. (7.91), provides a closed set of coupled differential equations for the two variables Φ and χ. By using the above perturbation equations, together with the background relations (7.85) and (7.86), two useful equations for the evolution of δ_r and v_r can be finally obtained:

$$\delta_r' = 4\Phi' + \frac{4}{3}\nabla^2 v_r, \qquad\qquad v_r' = \frac{1}{4}\delta_r + \Phi. \tag{7.94}$$

We now suppose that we start at $t = t_i$, with a radiation-dominated configuration in which the axion background is initially constant and non-vanishing, $\sigma(t_i) = \sigma_i \neq 0$, $\sigma'(t_i) = 0$, providing a subdominant (potential) energy density, $\rho_\sigma(t_i) = -p_\sigma(t_i) = V_i = m^2\sigma_i^2/2 \ll H_i^2 \sim \rho_r(t_i)$. We also give initial conditions to our system of perturbation equations by assuming a given spectrum of isocurvature axion fluctuations, $\chi_k(t_i) \neq 0$, and a total absence of perturbations for the metric and the radiation fluid, $\Phi(t_i) = \delta_r(t_i) = v_r(t_i) = 0$. The initial values of the first derivatives of the perturbation variables are then dedermined by the momentum and Hamiltonian constraints (7.88), (7.89).

In order to discuss the possible generation of curvature fluctuations we must now specify the details of the background evolution, taking into account that the axion, initially constant and subdominant, starts oscillating at a curvature scale $H_{\rm osc} \sim m$ (see Eq. (7.86)), and eventually decays (with gravitational strength) in radiation, at a scale $H_d \sim m^3/M_{\rm P}^2 < H_{\rm osc}$. During the oscillating phase the kinetic and potential energy densities are equal on an average, so that $\langle p_\sigma \rangle = 0$ and $\langle \rho_\sigma \rangle \sim a^{-3}$ behaves like dust matter. Since $\rho_r \sim a^{-4}$, the radiation energy is always diluted faster, and the axion background tends to become dominant at a scale $H_\sigma(t) \sim m\sigma(t)$.

For an efficient conversion of the initial χ and δ_σ fluctuations into Φ and δ_r fluctuations it is further required [241, 459, 487] that the decay occurs after the beginning of the axion-dominated phase, i.e. $H_\sigma > H_d$. Depending upon the relative values of H_σ and $H_{\rm osc}$ (i.e. depending upon the value of σ_i, in Planck units) we have thus different options which will now be discussed separately.

(1) If $\sigma_i < 1$, and then $H_\sigma < H_{\rm osc}$, the axion starts oscillating (at a scale $H \sim m$) when the Universe is still radiation-dominated. During the oscillations the averaged potential energy density decreases like a^{-3}, i.e. the typical amplitude of the axion background decreases like $a^{-3/2}$, from its initial value σ_i down to the value $\sigma_{\rm dom}$ at which $H = H_\sigma \sim m\sigma_{\rm dom}$. During this period $a \sim H^{-1/2}$ (as the background is radiation-dominated), so that $\sigma_{\rm dom} \sim \sigma_i^4$, and $H_\sigma \sim m\sigma_i^4$. Finally, the background remains axion-dominated until the decay scale $H_d \sim m^3/M_{\rm P}^2$. This model of background is thus consistent for $H_i > H_{\rm osc} > H_\sigma > H_d$, namely for

$$1 > \sigma_i > (m/M_{\rm P})^{1/2}, \tag{7.95}$$

which leaves a wide allowed range for σ_i, if we recall the cosmological bounds on the mass following from the decay of a gravitationally coupled scalar [238, 240] (typically, $m > 10$ TeV to avoid disturbing standard nucleosynthesis).

For this class of backgrounds we can now solve the coupled equations (7.91), (7.93), neglecting the k^2 terms in the (Fourier transformed) perturbation equations, since we are interested in modes well outside the horizon. We find (in cosmic time) that the Bardeen potential Φ grows like t^2 during the initial radiation phase in which the axion is slow rolling, and like $t^{1/2}$ during the period of (radiation-dominated) axion oscillations:

$$\Phi_k(t) \;=\; A\sigma_i\chi_k(t_i)\left[m(t-t_i)\right]^2, \qquad\qquad t_i < t < t_m, \tag{7.96}$$

$$= \Phi_k(t_m) + B\sigma_i\chi_k(t_i)\left[m(t-t_m)\right]^{1/2}, \qquad t_m < t < t_\sigma \tag{7.97}$$

(A and B are dimensionless numbers of order o1), where $t_m \sim H_{\rm osc}^{-1} \sim m^{-1}$ and $t_\sigma \sim H_\sigma^{-1} \sim (m\sigma_i^4)^{-1}$. In the subsequent axion-dominated phase, preceeding the decay, the Bardeen potential finally oscillates around the value $\Phi_k(t_\sigma)$, with a resulting overall generation of curvature perturbations from $\Phi_k = 0$ to

$$\langle\Phi_k(t_\sigma)\rangle \sim \sigma_i\chi_k(t_i) + \frac{\chi_k(t_i)}{\sigma_i} \sim \frac{\chi_k(t_i)}{\sigma_i}, \qquad \sigma_i < 1. \tag{7.98}$$

where $\langle\ldots\rangle$ refers to averages over one oscillation period.

(2) If $\sigma_i \sim 1$, and then $H_\sigma \sim H_{\rm osc} \sim m$, the beginning of the oscillating and axion-dominated phase are nearly simultaneous, and the Bardeen potential only grows during the initial radiation-dominated phase, according to eq. (7.96). The final averaged value of the induced perturbations is now

$$\langle\Phi_k(t_\sigma)\rangle \sim \sigma_i\chi_k(t_i) \sim \chi_k(t_i), \qquad \sigma_i \sim 1. \tag{7.99}$$

(3) Finally, if $\sigma_i >$, and then $H_\sigma > H_{\rm osc}$, the axion starts dominating at the scale $H_\sigma \sim m\sigma_i$, which marks the beginning of a phase of slow-roll inflation, lasting until the curvature drops below the oscillation scale $H_{\rm osc} \sim m$. Such a model of background is consistent for $H_\sigma < H_1$, namely for

$$\frac{H_1}{m} > \sigma_i > 1, \tag{7.100}$$

where H_1 (fixed around the string scale) corresponds to the beginning of the radiation-dominated, post-big bang evolution. During the inflationary phase the slow decrease of the Hubble scale can be approssimated (according to the background equations (7.85) and (7.86)) as $H(t) = \alpha m\sigma_i - \beta m^2(t - t_\sigma)$, where α and β are dimensionless coefficients of order 1. Inflation thus begins at the epoch $t_\sigma \sim 1/m\sigma_i$, and lasts until the epoch $t_m \sim (\sigma_i - 1)/m \sim \sigma_i/m$. Let us estimate, for this background, the evolution of the Bardeen potential generated by the primordial axion fluctuations.

For $t < t_\sigma$ the solution is still given by eq. (7.96), with t_m replaced by $t_\sigma = 1/m\sigma_i$. At the beginning of inflation the averaged amplitude is then $\Phi_k(t_\sigma) \sim \chi_k(t_i)/\sigma_i$. During slow-roll inflation, on the other hand, it is known [492] that, for superhorizon modes, $\Phi_k \sim \dot{H}/H^2$. The final amplitude at the beginning of the oscillating phase is then

$$\langle\Phi_k(t_m)\rangle = \Phi_k(t_\sigma)\left(\frac{\dot{H}}{H^2}\right)_{t_m}\left(\frac{H^2}{\dot{H}}\right)_{t_\sigma} \sim \sigma_i^2\Phi_k(t_\sigma) \sim \sigma_i\chi_k(t_i), \qquad \sigma_i > 1, \tag{7.101}$$

and no further amplification is expected in the course of the subsequent cosmological evolution.

It is amusing to observe that the results (7.98), (7.99), (7.101), which determine the amplitude of the Bardeen potential in the oscillating (axion-dominated) phase preceding the axion decay, can be summarized by an equation that holds in all cases, namely

$$\langle \Phi_k(t_d) \rangle = -\chi_k(t_i) f(\sigma_i), \qquad f(\sigma_i) = \left(c_1 \sigma_i + \frac{c_2}{\sigma_i} + c_3 \right), \tag{7.102}$$

where c_1, c_2, c_3 are numerical coefficients of the order of unity. A preliminary fit [100] based on numerical and analytical integrations of the perturbation equations gives $c_1 = 0.129, c_2 = 0.183, c_3 = 0.019$. The function $f(\sigma_i)$ has the interesting feature that it is approximately invariant under the transformation $\sigma_i \to \sigma_i^{-1}$ and, as a consequence, has a *minimal* value around $\sigma_i = 1$.

Given Φ_k, we can easily compute the final value (before axion decay) of the density constrasts δ_σ, δ_r, and of the variable ζ parametrizing in a gauge-invariant way the strength of curvature perturbations. By using the background equations (7.85) in the axion-dominated phase, $\rho_\sigma \gg \rho_r$, and taking the average of Eq. (7.89), for super-horizon modes, one finds indeed (averaging being understood) $\delta_\sigma(k) \simeq -2\Phi_k$. Similarly, from Eqs. (7.94), and for super-horizon modes, $\delta_r(k) \simeq 4\Phi_k$.

Consider then the variable ζ representing the perturbation of the spatial curvature on uniform density hypersurfaces, which is conserved (outside the horizon) for adiabatic perturbations, and which can be written for a general background as [492]:

$$\zeta = -\Phi - \frac{\mathcal{H}\Phi' + \mathcal{H}^2\Phi}{\mathcal{H}^2 - \mathcal{H}'}. \tag{7.103}$$

Outside the horizon we can use $-\delta_\sigma/2$ for Φ, the sum of the two background equations (7.85) for the denominator $\mathcal{H}^2 - \mathcal{H}'$, and the Hamiltonian constraint (7.89) for the numerator $\mathcal{H}\Phi' + \mathcal{H}^2\Phi$, in order to rewrite ζ in the form

$$\zeta_k = \frac{\rho_\sigma \delta_\sigma(k) - (3/4)(\rho_\sigma + p_\sigma)\delta_r(k)}{4\rho_r + 3(\rho_\sigma + p_\sigma)}. \tag{7.104}$$

In the final phase dominated by an oscillating axion ρ_r is negligible, the (averaged) axion pressure is zero so that, using $\delta_\sigma = -2\Phi = -\delta_r/2$ we finally obtain

$$\zeta_k \simeq \frac{5}{6}\delta_\sigma(k) \simeq -\frac{5}{3}\Phi_k \tag{7.105}$$

(as we have also checked by an explicit numerical integration [100]). This result was derived and presented in [459], using however different notations.

The generated spectrum of super-horizon curvature perturbations is thus directly determined by the primordial spectrum of isocurvature axion fluctuations χ_k, according to Eqs. (7.102) and (7.105). The axion fluctuations, on the other hand, are solutions (with pre-big bang initial conditions) of the perturbation equation (7.91) in the radiation era (no additional amplification is expected, for super-horizon modes, in the axion-dominated

phase), computed for negligible curvature perturbations ($\Phi = 0 = \Phi'$) in the massive, non-relativistic limit (as we are already in the oscillating regime), and outside the horizon.

In order to exploit the results of the previous subsection, we can then set $\chi_k = \psi_k/a$, where ψ_k satisfies Eq. (7.69). The general solution for ψ_k, correctly normalized to a relativistic spectrum of quantum fluctuations, has been given in Eq. (7.78) already, taking also into account the details of the pre-big bang amplification (encoded into the Bogoliubov coefficient $c(k)$). Outside the horizon, $k|\eta| \ll 1$, and for non-relativistic modes, $k \ll ma$, we take (respectively) the limits $-bx^2 \ll 1$ and $-b \ll x^2$, in which $y_2 \sim x = \eta\sqrt{2\alpha}$. By inserting a generic, primordial power-law spectrum with cut-off scale $k_1 = H_1 a_1$, i.e. $c(k) = (k/k_1)^{(n-5)/2}$, and using the identity $\eta/a = \eta_1/a_1 \simeq 1/k_1 a_1$, we finally obtain the generated spectrum of curvature perturbations:

$$k^3 |\Phi_k|^2 = f^2(\sigma_i) k^3 |\chi_k|^2 = f^2(\sigma_i) \left(\frac{H_1}{M_{\rm P}}\right)^2 \left(\frac{k}{k_1}\right)^{n-1}, \qquad k < k_1 \tag{7.106}$$

We have re-inserted the appropriate Planck mass factors, keeping however σ_i dimensionless in Planck units, and we have included into k_1 possible numerical factors of order 1, connecting the cut-off scale to the string mass scale

This result, valid during the axion-dominated phase, has to be transferred to the phase of standard evolution, by matching the (well known [492]) solution of the Bardeen potential in the radiation era (subsequent to axion decay) to the solution prior to decay, which is in general oscillating. The matching of Φ and Φ', conventionally performed at the fixed scale $H = H_d$, shows that the constant asymptotic value (7.102) of super-horizon modes is preserved to leading order by the decay process, modulo a random, mass-dependent correction which typically takes the form $[1 + \epsilon \sin(m/H_d)]$, where ϵ is a numerical coefficient of order 1, and $m \gg H_d$. Such a random factor, however, is a consequence of the sudden approximation adopted to describe the decay process, and disappears in a more realistic treatment in which the evolution equations for ρ_σ and $\delta\rho_\sigma$ are supplemented by the friction term $\Gamma\sigma'/a$ (with a corresponding antifriction term $-\Gamma\sigma'^2/a^2$ in the radiation equation).

In such a way, as we have checked with an numerical integration, the decay process preserves the value of the Bardeen potential prior to decay, damping the residual oscillations; ζ itself follows the same behaviour and is finally exactly a constant. When the axion has completely decayed, and the Universe is again dominated by radiation, we can properly match the standard evolution of Φ in the radiation phase to the constant asymptotic value of Eq. (7.102). The expression we obtain for the Bardeen potential, valid until the epoch of matter–radiation equality, can be written in the form

$$\Phi_k(\eta) = -3\Phi_k(\eta_d)\left[\frac{\cos(kc_s\eta)}{(kc_s\eta)^2} - \frac{\sin(kc_s\eta)}{(kc_s\eta)^3}\right], \qquad \eta_d < \eta < \eta_{\rm eq}, \tag{7.107}$$

where $c_s = 1/\sqrt{3}$ and $\Phi_k(\eta_d)$ is given in Eq. (7.102).

The above expression for the Bardeen potential provides the initial condition for the evolution of the CMB-temperature fluctuations, and the formation of their oscillatory pattern.

It is important to note that, from the above equation, $\Phi_k(\eta_i) = \text{constant}$ and $\Phi_k'(\eta_i) \simeq 0$, where $\eta_d < \eta_i < \eta_{\rm eq}$, and $k\eta_i \ll 1$. This implies [100] that the induced temperature fluctuations $(\Delta T/T)_k$ will oscillate in k just as expected in the case of adiabatic perturbations, producing a peak structure clearly distinguishable (and, at present, observationally favoured) from that produced by isocurvature fluctuations.

Let us now discuss the large scale normalization of the induced anisotropies, starting from the observation that the final amplitude of the super-horizon perturbations (7.106) seems to be unaffected by the non-relativistic corrections to the axion spectrum. The dependence on the axion mass appears, however, when computing the amplitude of the spectrum at the present horizon scale ω_0, in order to impose the corrected normalization to the quadrupole coefficient C_2, namely [222]

$$C_2 = \alpha_n^2 f^2(\sigma_i) \left(\frac{H_1}{M_{\rm P}}\right)^2 \left(\frac{\omega_0}{\omega_1}\right)^{n-1}, \qquad \alpha_n^2 = \frac{4^{\frac{n-3}{2}}}{9} \frac{\Gamma(3-n)\Gamma\left(\frac{3+n}{2}\right)}{\Gamma^2\left(\frac{4-n}{2}\right)\Gamma\left(\frac{9-n}{2}\right)}, \tag{7.108}$$

where the value of C_2 is experimentally determined by COBE according to Eq. (7.53).

The present value of the cut-off frequency, $\omega_1(t_0) = H_1 a_1/a_0$, depends in fact on the kinematics as well as on the duration of the axion-dominated phase (and thus on the axion mass), as follows:

$$\omega_1(t_0) = H_1 \left(\frac{a_1}{a_\sigma}\right)_{\rm rad} \left(\frac{a_\sigma}{a_d}\right)_{\rm mat} \left(\frac{a_d}{a_{\rm eq}}\right)_{\rm rad} \left(\frac{a_{\rm eq}}{a_0}\right)_{\rm mat}, \qquad \sigma_i < 1, \tag{7.109}$$

$$= H_1 \left(\frac{a_1}{a_\sigma}\right)_{\rm rad} \left(\frac{a_\sigma}{a_{\rm osc}}\right)_{\rm inf} \left(\frac{a_{\rm osc}}{a_d}\right)_{\rm mat} \left(\frac{a_d}{a_{\rm eq}}\right)_{\rm rad} \left(\frac{a_{\rm eq}}{a_0}\right)_{\rm mat}, \quad \sigma_i > 1. \tag{7.110}$$

Using $H_0 \simeq 10^{-6} H_{\rm eq} \simeq 10^{-61} M_{\rm P}$, we find that the COBE normalization imposes

$$c_2^2 \alpha_n^2 \sigma_i^{2\frac{(n-4)}{3}} \left(\frac{H_1}{M_{\rm P}}\right)^{\frac{(5-n)}{2}} \left(\frac{m}{M_{\rm P}}\right)^{-\frac{(n-1)}{3}} 10^{-29(n-1)} \simeq 10^{-10}, \qquad \sigma_i < 1, \tag{7.111}$$

$$c_1^2 \alpha_n^2 Z_\sigma^{n-1} \sigma_i^{\frac{(5-n)}{2}} \left(\frac{H_1}{M_{\rm P}}\right)^{\frac{(5-n)}{2}} \left(\frac{m}{M_{\rm P}}\right)^{-\frac{(n-1)}{3}} 10^{-29(n-1)} \simeq 10^{-10}, \quad \sigma_i > 1, \tag{7.112}$$

where $Z_\sigma = (a_{\rm osc}/a_\sigma) > 1$ denotes the amplification of the scale factor during the phase of axion-dominated, slow-roll inflation.

The condition (7.111) is to be combined with the constraint (7.95), the condition (7.112) with the constraint (7.100), which are required for the consistency of the corresponding classes of background evolution. Also, both conditions are to be intersected with the experimentally allowed range of the spectral index. The allowed range of parameters compatible with all the constraints is rather strongly sensitive to the values of the pre-big bang inflation scale H_1. In the context of minimal models of pre-big bang inflation we have $H_1 \sim M_s$, and a flat spectrum ($n = 1, \alpha_n^2 = 1/54\pi$) is inconsistent with the normalization (7.111), (7.112).

A growing ("blue") spectrum is instead allowed (see [100] for a detailed discussion). It turns out, in particular, that there is a wide range of allowed axion masses, but a rather narrow range of allowed values for σ_i, especially in the case $\sigma_i > 1$.

The allowed range of parameters is extended if the inflation scale H_1 is lowered, and a flat ($n = 1$) spectrum may become possible if $H_1 \lesssim 10^{-5}(M_{\rm P}\sigma_i/\alpha_1 c_2)$, for $\sigma_i < 1$, and if $H_1 \lesssim 10^{-5}(M_{\rm P}/\sigma_i\alpha_1 c_1)$, for $\sigma_i > 1$ (see Eqs. (7.111) and (7.112)). A lower value of the string scale could emerge indeed in a recently proposed framework [607] according to which, at strong bare-coupling e^{ϕ}, loop effects renormalize downwards the ratio $M_s/M_{\rm P}$ and allow M_s to approach the unification scale.

In addition, a flat spectrum may be allowed even keeping pre-big bang inflation at a high-curvature scale, provided the relativistic branch of the primordial axion fluctuations is characterized by a frequency-dependent slope, which is flat enough at low frequency (to agree with large-scale observations) and much steeper at high frequencies (to match the string normalization at the end-point of the spectrum). A typical example of such a spectrum, already introduced in the previous subsection, and illustrated in Fig. 7.4, can be parametrized by a Bogoliubov coefficient with a break at the intermediate scale k_s,

$$\begin{aligned} |c_k|^2 &= \left(\frac{k}{k_1}\right)^{n-5+\delta}, & k_s < k < k_1, \\ &= \left(\frac{k_s}{k_1}\right)^{n-5+\delta}\left(\frac{k}{k_s}\right)^{n-5}, & k < k_s, \end{aligned} \tag{7.113}$$

where $\delta > 0$ parametrizes the slope of the break at high frequency. Examples of realistic pre-big bang backgrounds producing such a spectrum of axion fluctuations have been discussed in [324]. It is clear that the steeper (and the longer) the high-frequency branch of the spectrum, the easier the matching of the amplitude to the measured anisotropies (in spite of possible σ_i-dependent enhancements).

Using the generalized input (7.113) for the spectrum of χ_k, the amplitude of the low-frequency ($k < k_s$) Bardeen spectrum (7.106) is to be multiplied by the suppression factor $\Delta = (k_s/k_1)^{\delta} \ll 1$, and the normalization condition at the COBE scale becomes

$$\alpha_n^2 c_2^2 \left(\frac{H_1}{\sigma_i M_{\rm P}}\right)^2 \left(\frac{\omega_0}{\omega_1}\right)^{n-1} \simeq C_2\Delta^{-1}, \qquad \sigma_i < 1, \tag{7.114}$$

$$\alpha_n^2 c_1^2 \left(\frac{\sigma_i H_1}{M_{\rm P}}\right)^2 \left(\frac{\omega_0}{\omega_1}\right)^{n-1} \simeq C_2\Delta^{-1}, \qquad \sigma_i > 1. \tag{7.115}$$

A strictly flat spectrum is now possible, even for $\alpha_1 H_1 = \alpha_1 M_s \simeq 10^{-2} M_{\rm P}$, provided

$$\Delta\left(c_1^2\sigma_i^2 + c_2^2\sigma_i^{-2}\right) \lesssim 10^{-6}. \tag{7.116}$$

It thus becomes possible, in this context, to satisfy the stringent limits imposed by the most recent analyses of the peak and dip structure of the spectrum at small scales [204], which imply $0.87 \leq n \leq 1.06$.

"curvaton"?????????

We may thus conclude that, in the context of the pre-big bang scenario, a "curvaton" model based on the Kalb–Ramond axion is able to produce the adiabatic curvature perturbation needed to explain the observed large-scale anisotropies. The simplest, minimal models of pre-big bang inflation seem to prefer blue spectra. A strictly scale-invariant (or even slightly red, $n < 1$) spectrum is not excluded but requires, for normalization purposes, non-minimal models of pre-big bang evolution leading to axion fluctuations with a sufficiently steep slope at high frequencies.

8 Singularity and "graceful exit"

As already stressed in Section 2, the cosmological solutions of the string effective action, thanks to their symmetries, are characterized by four branches, which are inter-related by time-reversal and scale-factor duality transformations. Two branches are of the pre-big bang type, and evolve *towards* a singularity, while the other two branches are of the post-big bang type, and emerge *from* a singularity. A typical example is illustrated in Fig. 8.1, where we have plotted the asymptotic, d-dimensional vacuum solutions of the lowest-order gravidilaton effective action,

$$a(t) = (\mp t)^{\mp 1/\sqrt{d}}, \qquad \overline{\phi}(t) = -\ln(\mp t), \tag{8.1}$$

which represent the bisecting lines of the "phase space" spanned by the variables $\{\dot{\overline{\phi}}, \sqrt{d}H\}$. The initial conditions of the pre-big bang scenario and the present standard cosmological configuration are thus disconnected by a singularity localized at the future or past end, respectively, of the temporal half-line on which they are defined.

In the case of homogeneous and isotropic metric backgrounds, with or without spatial curvature, there are "no-go theorems" [126, 395, 396, 233] that exclude the possibility of a smooth transition from the pre- to the post-big bang branch of the tree-level solutions, even including a (*local*) dilaton potential $V(\phi)$, and matter sources in the form of perfect fluids and/or an homogeneous NS-NS two-form. In all these cases the singularity cannot be removed. In addition, in the absence of sources and of a dilaton potential, it is known that the singularity also affects anisotropic backgrounds, and can be shown to be generally present in all homogeneous four-dimensional metrics of Bianchi type I, II, III, V, VI_0 and VI_h [312].

We are thus led to the problem of explaining how the Universe may smoothly evolve from the string perturbative vacuum, and from an initial accelerated growth of the curvature and of the string coupling, to a final, decelerated, post-big bang configuration, which is certainly included in the set of string-cosmology solutions, but is possibly disconnected from the initial pre-big bang branch by a curvature singularity. This is the so-called "graceful exit" problem.

The exit from inflation, and a smooth description of the transition from accelerated to decelerated expansion, is a typical problem of all inflationary models (see for instance [55]).

Graceful Exit

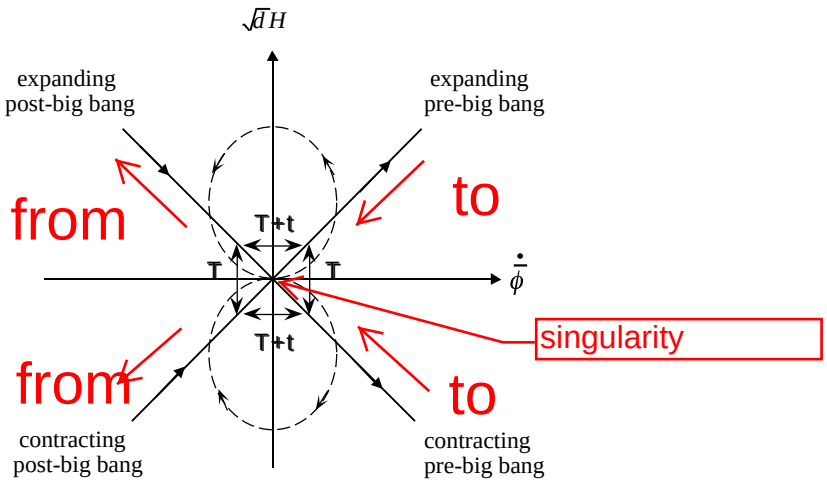

Figure 8.1: *The four branches of the lowest-order string effective action (solid lines), and a possible self-dual solution (dashed curve), characterized by* $a(t) = a^{-1}(-t)$ *and* $\overline{\phi}(t) = \overline{\phi}(-t)$. *The symbols* $\mathcal{T}$ *and* t *represent a T-duality and a time-reversal transformation, respectively.*

In the string cosmology case, however, such a problem is sharpened by the fact that the transition from the pre- to the post-big bang phase is expected to occur at high curvature and strong coupling, possibly in the full quantum-gravity regime. This may suggest that a correct approach to the exit problem is to be developed in a quantum cosmology context, by exploiting the Wheeler-De Witt equation in the string cosmology minisuperspace, and the associated $O(d,d)$ symmetry, which helps avoiding operator ordering ambiguities [310, 410].

In this approach, interesting in itself, the probability of quantum transition from the pre- to the post-big bang branches of the low-energy solutions can be finite and non-vanishing, even if the two branches are classically disconnected by a singularity [310, 322]. This possibility will be presented in detail in Section 9. In this section we shall keep on a classical (i.e. deterministic) description of the exit, which requires the smoothing out of the background singularities.

It should be stressed, in particular, that the phenomenological predictions made in the previous sections were based on the assumption that the singularity can be regularized, and that *i*) a graceful exit does take place; *ii*) sufficiently large scales are only affected by it kinematically, i.e. through an overall redshift of all scales. Of course, one would not only like to know that a graceful exit does take place: one would also like to describe the transition between the two phases in a quantitative way. Achieving this goal would amount to nothing less than a full description of what replaces the big bang of standard cosmology in the pre-big bang scenario. As mentioned in Section 1, this problem of string cosmology is the analogue, in some sense, of the (still not fully solved) confinement problem of QCD.

What is the idea???

The problem of the background regularization is particularly hard because, besides the smoothing out of the curvature, it also requires the smoothing out of the dilaton kinetic energy, and a possible damping of the dilaton (i.e. of the string coupling). We may consider, for a simple illustrative example of this point, the lowest-order gravidilaton action, supplemented by a matter action S_m that describes perfect-fluid sources, with vanishing dilatonic charge, $\delta S_m/\delta\phi = 0$:

$$S = -\frac{1}{2\,\lambda_{\rm s}^{d-1}} \int d^{d+1}x\,\sqrt{|g|}\,e^{-\phi}\,\left[R + (\nabla_\mu\phi)^2 + V - \frac{1}{12}H^2_{\mu\nu\alpha}\right] + S_m. \tag{8.2}$$

By assuming an isotropic, spatially flat metric background, with spatial sections of finite volume, we set

$$\begin{aligned} &\sqrt{-g}\;T_{\mu\nu} = 2\frac{\delta S_m}{\delta g_{\mu\nu}}, && T_\mu{}^\nu = {\rm diag}\left(\rho(t), -p(t)\delta_i^j\right), \\ &g_{\mu\nu} = {\rm diag}\left(1, -a^2(t)\delta_{ij}\right), && \overline{\phi} = \phi - d\ln a, \qquad \phi = \phi(t). \end{aligned} \tag{8.3}$$

For $V = 0$, $H_{\mu\nu\alpha} = 0$, the field equations can be written in the form

$$\dot{\overline{\phi}}^2 - dH^2 = e^{\overline{\phi}}\overline{\rho}, \tag{8.4}$$

$$\dot{H} - H\dot{\overline{\phi}} = \frac{1}{2}e^{\overline{\phi}}\overline{p}, \tag{8.5}$$

$$2\ddot{\overline{\phi}} - \dot{\overline{\phi}}^2 - dH^2 = 0 \tag{8.6}$$

where $\overline{\rho} = \rho a^d$, $\overline{p} = pa^d$ (see Subsection 2.2), and we have used units in which $2\lambda_{\rm s}^{d-1} = 1$. Their combination gives the conservation equation

$$\dot{\overline{\rho}} + dH\overline{p} = 0. \tag{8.7}$$

The above equations can be satisfied by the following particular exact solution

$$\begin{aligned} &H = \frac{a^{\sqrt{d}}}{(1+a^{\sqrt{d}})^2}, && \overline{\rho} = kH^2, && \overline{p} = -\frac{2}{\sqrt{d}}\frac{1-a^{sqrtd}}{1+a^{\sqrt{d}}}\overline{\rho}, \\ &\overline{\phi} = \ln\frac{4d}{k}\frac{a^{\sqrt{d}}}{(1-a^{\sqrt{d}})^2}, && \dot{\overline{\phi}} = \sqrt{d}H\frac{1+a^{\sqrt{d}}}{1-a^{\sqrt{d}}}, \end{aligned} \tag{8.8}$$

where k is an arbitrary integration constant. The integration of $H = H(a)$ then leads to the implicit expression for the scale factor,

$$\frac{t}{t_0} = \frac{1}{\sqrt{d}}\left(a^{\sqrt{d}} - a^{-\sqrt{d}}\right) + 2\ln a. \tag{8.9}$$

This solution is self-dual, in the sense that $a(t) = a^{-1}(-t)$, $\overline{\rho}(a) = \overline{\rho}(a^{-1})$, $\overline{\phi}(a) = \overline{\phi}(a^{-1})$, and describes a smooth evolution of the metric from an initial, accelerated phase of growing curvature and negative pressure, to a final, decelerated phase of decreasing curvature and

positive pressure. The scale factor runs monotonically from zero to infinity, without any curvature singularity.

This example cannot be accepted as a successful model of exit, however, because the dilaton kinetic energy ($\dot{\phi}^2$) blows up at the transition point ($a = 1$), together with the effective string coupling, $\exp\phi$. This implies that both α' and loop corrections to the effective action become non-negligible at the transition, and that the above solution, obtained in the low-energy, tree-level approximation, cannot be trusted. In addition, the curvature is regular in the string frame, but blows up in the Einstein frame defined by

$$\widetilde{a} = ae^{-\phi/(d-1)}, \qquad d\widetilde{t} = dte^{-\phi/(d-1)} \tag{8.10}$$

(see for instance Subsection 1.3). Indeed, by setting $H = d(\ln a)/dt$ and $\widetilde{H} = d(\ln \widetilde{a})/d\widetilde{t}$, we obtain

$$\widetilde{H} = \left(H - \frac{1}{d-1}\dot{\phi}\right) e^{\phi/(d-1)}, \tag{8.11}$$

so that, if the dilaton is unbounded, $\widetilde{H}$ may diverge even if both H and $\dot{\phi}$ are regular.

A successful exit thus requires a smooth dilaton evolution, with the kinetic energy and $g_s^2 = \exp\phi$ bounded everywhere. In addition, the dilaton has to evolve from the initial regime with $\dot{\phi} > 0$ to a final regime with $\dot{\phi} < 0$ (see Fig. 8.1); besides being bounded, at large positive times, its kinetic energy must thus satisfy the condition

$$\dot{\phi} < dH. \tag{8.12}$$

This is not sufficient, however, for a complete exit. An even stronger condition is imposed by requiring that the final, post-big bang configuration may describe (even in the E-frame) a phase of decelerated expansion, after the contraction representing pre-big bang inflation (after all, the String and Einstein frames are doomed to coincide, eventually, in the limit of dilaton stabilization). This implies $\widetilde{H} > 0$ at large positive times or, according to Eq. (8.11),

$$\dot{\phi} < (d-1)H. \tag{8.13}$$

The above constraints can be translated into energy conditions, to be satisfied by the stress tensor of the effective sources driving the exit transition [121, 122].

In spite of such constraints, there are simple examples of regular backgrounds implementing a successful exit by means of appropriate sources, as we shall see in Subsection 8.1. The simplest examples, however, are not very realistic. The most realistic physical mechanism of exit, in the perturbative regime, is probably based on the backreaction of the quantum fluctuations, which leads, however, to including loops (and possibly higher derivative) corrections in the effective action, as we shall discuss in Subsections 8.2 and 8.3. Such corrections suggest possible lines along which a quantitative description of the exit might eventually emerge.

The loop corrections, however, become important in the strong coupling regime, i.e. in the realm of the still largely unknown M-theory [626, 376], as will be discussed in Subsection 8.5. It is thus possible that, in spite of the existing encouraging results, new techniques and/or a deeper understanding of string theory in its non-perturbative regimes need be developed (see [110] for a recent attempt), before a fully satisfactory description of the transition from pre- to post-big bang can be hoped for.

In the next subsection we will start presenting a few simple examples of smooth solutions, without curvature singularities, obtained in the context of the low-energy string effective action.

8.1 Smoothing out the singularity at low energy

It is well known, from the singularity theorems of general relativity (see for instance [372, 371, 615]), that a smooth and complete geometrical background, with all the curvature invariants bounded everywhere, is only allowed if the sources of the gravitational field satisfy appropriate energy conditions.

In the context of the string cosmology equations, the situation is even more constrained, because also the string coupling g_s, parametrized by the dilaton, has to be bounded (and possibly smaller than 1) for a consistent and appropriate use of the string effective action (and of the perturbative expansion). In addition, if the curvature and the dilaton kinetic energy are regular in the S-frame, but the dilaton is free to grow boundlessly, then divergences may appear in the E-frame according to Eq. (8.11). This may occur, for instance, if the regularization of the curvature in the S-frame leads to a phase of constant H and $\dot{\phi}$, as in the context of the higher-order effective action [294, 307], which will be discussed in Subsection 8.3. The importance of a frame-independent regularization is stressed by the fact that some particles, such as the gravitons, follow the E-frame geodesics [399].

In spite of such difficulties, examples of smooth, self-dual solutions with no curvature singularities have been obtained, even in the context of the lowest-order string effective action, and of homogeneous and isotropic backgrounds, by introducing however *ad hoc* a *negative* and *non-local* dilaton potential [477, 319, 310], $V(\overline{\phi}) < 0$. Such a potential, which only makes sense for manifolds with spatial sections of finite volume, $\left(\int d^d x \sqrt{-g}\right)_{t=\text{const}} < \infty$ (for instance, a torus), is non-local because $\overline{\phi}$, to be a generally covariant scalar, must contain the integral of the whole spatial volume (see Subsection 1.4).

Consider, as an example, the action (8.2), with a dilaton potential $V(\overline{\phi})$. The field equations, for an isotropic and spatially flat metric, become ($2\lambda_s^{d-1} = 1$)

$$\dot{\overline{\phi}}^2 - dH^2 - V = \overline{\rho} e^{\overline{\phi}}, \tag{8.14}$$

$$\dot{H} - H\dot{\overline{\phi}} = \frac{1}{2}\overline{p} e^{\overline{\phi}}, \tag{8.15}$$

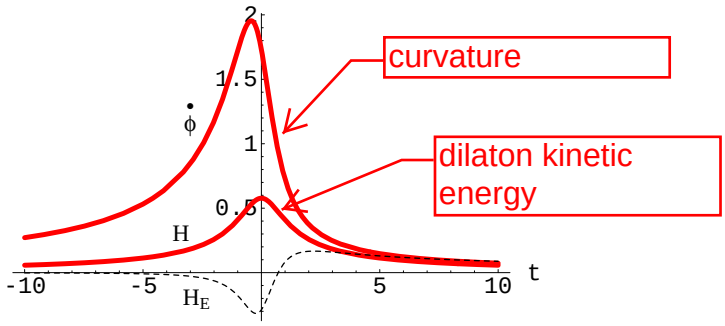

Figure 8.2: *Plot of the solution (8.18) with* $V_0 = 1$, $k = 1$, $d = 3$. *The bold curves show the time evolution of the curvature and of the dilaton kinetic energy in the S-frame. The thin dashed curve shows the Hubble parameter* $H_E = \tilde{H}$ *of the E-frame.*

$$\dot{\overline{\phi}}^2 - 2\ddot{\overline{\phi}} + dH^2 + \frac{\partial V}{\partial \overline{\phi}} - V = 0, \tag{8.16}$$

and the conservation equation (8.7) is still valid. In the vacuum case $\rho = p = 0$, the full set of equations can be integrated exactly [477], and one finds that non-singular solutions are in general allowed. If we consider, for instance, the potential [310]

$$V(\overline{\phi}) = -V_0 e^{4\overline{\phi}}, \tag{8.17}$$

the above equations can be exactly satisfied by the particular solution

$$a = a_0 \left[\frac{k^2 t}{\sqrt{V_0}} + \left(1 + \frac{k^4 t^2}{V_0} \right)^{1/2} \right]^{1/\sqrt{d}}, \qquad \overline{\phi} = \frac{1}{2} \left(\frac{V_0}{k^2} + k^2 t^2 \right), \tag{8.18}$$

where k and a_0 are integration constants.

This is a regular self-dual solution, $a(t)/a_0 = a_0/a(-t)$, characterized by a bounded, "bell-like" shape not only of H but also of the dilaton kinetic energy (unlike the example given in Eq. (8.8)). For $t \to +\infty$ the solution satisfies the conditions (8.12), (8.13) required for a successful exit [121, 122] and, together with its time-reversed partner, it describes indeed a perfect "8-shaped" curve in the plane of Fig. 8.1. Such a solution smoothly interpolates between the asymptotic branches (8.1) of the vacuum solutions, i.e. from the superinflationary expansion $a \sim (-t)^{-1/\sqrt{d}}$, $\overline{\phi} = \sqrt{d} \ln a$ at $t \to -\infty$, to the final state of decelerated expansion and decreasing curvature $a \sim t^{1/\sqrt{d}}$, $\overline{\phi} = -\sqrt{d} \ln a$ at $t \to +\infty$ (see Figs. 8.2 and 8.3).

It is important to note that in this solution the dilaton is monotonically growing (indeed, for $t \to +\infty$, $\phi \sim (\sqrt{d} - 1) \ln t$), but the curvature is regular even in the E-frame, where $\tilde{H}$ smoothly evolves from the pre-big bang accelerated contraction at $\tilde{H} < 0$, to a phase of

Hubble parameter

decelerated expansion at $\widetilde{H} > 0$. For $t \to +\infty$, in fact, $\widetilde{H} \sim t^{(\sqrt{d}-3)/2}$, so that the E-frame curvature is bounded for all $d \leq 9$. The fact that the dilaton keeps growing, however, is not consistent (asymptotically) with the perturbative approach, and with the use of the lowest-order string effective action. In addition, the curvature may blow up in some other frame, different from the String and Einstein ones.

A more satisfactory example of transition, in which the dilaton becomes asymptotically frozen in the radiation-dominated regime, can be implemented by using another form of non-local dilaton potential [319],

$$V(\overline{\phi}) = -V_0 e^{2\overline{\phi}}. \tag{8.19}$$

In that case Eqs. (8.14)–(8.16) can be integrated exactly, for a wide class of (possibly time-dependent) equations of state. Let us report in some detail the integration procedure already outlined in Subsection 2.3, which is also useful for further applications.

We start by defining a new (dimensionless) time coordinate x, such that

$$2dx = L\overline{\rho}dt \tag{8.20}$$

(L is a constant parameter with dimensions of a length). The combination of Eqs. (8.14), (8.16), using the relation $2V = \partial V/\partial\overline{\phi}$, gives

$$\overline{\rho}\left(e^{-\overline{\phi}}\right)' L^2 = 2(x + x_0), \tag{8.21}$$

where a prime denotes the differentiation with respect to x, and x_0 is an integration constant. By assuming a time-dependent equation of state, defined by the function $\Gamma(x)$ such that

$$\Gamma'(x) = \overline{p}/\overline{\rho}, \tag{8.22}$$

Eq. (8.15) can be rewritten as

$$L^2\ \overline{\rho}\,(a'/a)\,e^{-\overline{\phi}} = 2\Gamma(x). \tag{8.23}$$

The matter-conservation equation (8.7), on the other hand, gives

$$L^2\ \overline{\rho}\,'\ e^{-\overline{\phi}} = -d\left(\Gamma^2\right)'. \tag{8.24}$$

By summing Eqs. (8.21), (8.24), and integrating, we are led to defininig the function $D(x)$ such that

$$L^2\ \overline{\rho}\ e^{-\overline{\phi}} = (x + x_0)^2 - d\Gamma^2(x) + \beta \equiv D(x) \tag{8.25}$$

(β is an integration constant), and our system of equations for a and $\overline{\phi}$ can be reduced to quadratures:

$$\frac{a'}{a} = 2\frac{\Gamma(x)}{D(x)}, \qquad \overline{\phi}\,' = -2\frac{x + x_0}{D(x)}. \tag{8.26}$$

We have to satisfy, finally, Eq. (8.14), which determines the constant β through the condition

$$\frac{(x+x_0)^2}{D^2} - \frac{d\Gamma^2}{D^2} - \frac{V}{L^2\overline{\rho}^2} = \frac{1}{D}. \tag{8.27}$$

There are two possibilities. The first one is trivial: $V = 0$ and $\beta = 0$. The second solution, valid for the potential (8.19), is

$$\beta = V_0\ L^2 > 0. \tag{8.28}$$

In such a case the integration of Eqs. (8.26) may provide regular solutions, with bounded curvature, kinetic terms and dilaton, describing a smooth evolution from the pre- to the post-big bang regime [319].

The simplest examples of regular solutions are obtained in the case of perfect fluids, with equation of state $p/\rho = \gamma = \text{const}$. From Eq. (8.22) one finds indeed $\Gamma(x) = \gamma(x+x_1)$, where x_1 is an integration constant and, if $\beta = V_0 L^2 > 0$, it is always possible to choose $x_0,\ x_1$ in such a way that $D(x)$ has no zeros, and the solution is regular. See [319], in particular, for the radiation case with $\gamma = 1/d$. But, in general, we may also describe a smooth evolution, which ends up with radiation, starting from its dual equation of state, $\gamma = -1/d$, corresponding to a gas of stretched strings (typical example of a pre-big bang phase dominated by string sources, see Section 2). If we take, for instance [319],

$$\frac{p}{\rho}(x) = \frac{x}{d(x^2+x_1^2)^{1/2}}, \qquad \Gamma(x) = \frac{1}{d}(x^2+x_1^2)^{1/2}, \tag{8.29}$$

and we choose $x_0 = 0,\ x_1^2 = L^2 V_0$, the integration of Eqs. (8.26) gives in fact

$$a = a_0\left[x + (x^2+x_1^2)^{1/2}\right]^{\frac{2}{d-1}}, \qquad e^{\phi} = a_0^d e^{\phi_0}\left[1 + \frac{x}{(x^2+x_1^2)^{1/2}}\right]^{\frac{2d}{d-1}}, \tag{8.30}$$

$$\rho e^{\phi} = \frac{d-1}{dL^2} e^{2\phi_0}(x^2+x_1^2)^{-\frac{d+1}{d-1}}, \quad pe^{\phi} = \frac{d-1}{dL^2} e^{2\phi_0} x\ (x^2+x_1^2)^{-\frac{3d+1}{2(d-1)}}. \tag{8.31}$$

This solution is self-dual, $a(t)/a_0 = a_0/a(-t)$, and describes a model that is always exapnding, $H > 0$. The Universe, starting at $t \to -\infty$ from the flat space ($H \to 0$) and weak coupling ($e^{\phi} \to 0$) regime, evolves through a superinflationary phase ($a \sim (-t)^{-2/(d+1)}$) dominated by unstable string matter ($p = -\rho/d$), towards the standard radiation-dominated phase ($a \sim t^{2/(d+1)}$, $p = \rho/d$), with frozen dilaton ($\exp\phi \sim \text{const}$). The curvature parameters $H,\ \dot{H}$, and the coupling $\exp\phi$, are everywhere bounded (see Fig. 8.3).

It should be noted that with different, more general choices of x_0 and x_1, the solution contains, in general, an intermediate stage of contraction before the final radiation phase [29], but even in that case the curvature and the dilaton keep bounded. The above solution can also be generalized to anisotropic configurations, describing graceful exit from inflation and, simultaneously, dynamical dimensional reduction [319]. In that case, besides the d expanding dimensions, there are also n shrinking internal dimensions that evolve

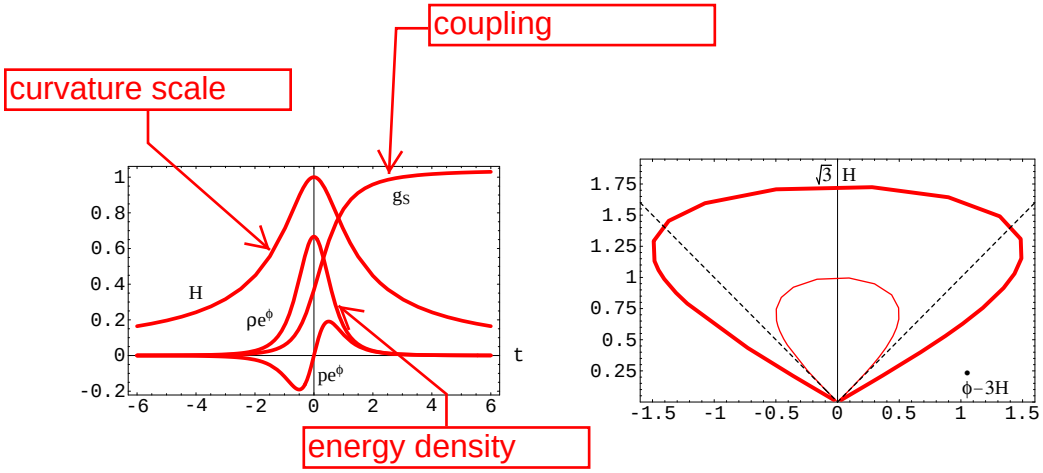

Figure 8.3: *Plot of the solution (8.30), (8.31) with* $\phi_0 = 0$, $L = 1$, $x_1 = 1$, $d = 3$, $a_0 = e^{-2/3}$. *The left panel shows the evolution of the curvature scale, of the coupling, of the energy density and pressure. The right panel shows the trajectory of the above solution (bold curve) in the plane parametrized by the shifted dilaton and by the Hubble parameter. The dashed lines represent the asymptotic vacuum solution (8.1). We have also reported the trajectory of the solution (8.18) (thin curve).*

from accelerated contraction (driven by an internal positive pressure) to a final decelerated contraction (driven by negative pressure).

Without non-local potential and matter sources, smooth solutions can be obtained by relaxing the isotropy and/or homogeneity assumptions. There are indeed particular examples, in the context of the lowest-order string effective action, in which the curvature singularities can be regularized by performing an appropriate $O(d,d)$ transformation, which introduces, however, a non-trivial anisotropic axion background (it is tempting to speculate that this softening of singularities, due to a non-trivial B_{ij} field, could be related to recent developments in the field of non-commutative geometry [219], also induced by a B_{ij} field).

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The simplest example [308] is the $d = 2$ boost of the trivially flat, Milne-type exact solution of the gravidilaton action (8.2) (with $V = 0 = H$);

$$ds^2 = dt^2 - (bt)^2 dx^2 - dy^2 - (dz_i)^2, \qquad \phi = \text{const}, \tag{8.32}$$

or of its (singular) scale-factor duality partner:

$$ds^2 = dt^2 - (bt)^{-2} dx^2 - dy^2 - (dz_i)^2, \qquad \phi = -2\ln|bt|, \tag{8.33}$$

(z_i are "inert" coordinates spanning a n-dimensional Euclidean space). Consider in fact the $O(2,2)$ transformation, in the (x,y) plane, generated by the one-parameter matrix $\Lambda(\gamma)$ such that

$$M \to \widetilde{M} = \Lambda^T M \Lambda\ , \qquad M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix},$$

$$\Lambda(\gamma) = \begin{pmatrix} 1+c & s & c-1 & -s \\ -s & 1-c & -s & 1+c \\ c-1 & s & 1+c & -s \\ s & 1+c & s & 1-c \end{pmatrix}, \qquad c = \cosh\gamma, \quad s = \sinh\gamma \tag{8.34}$$

(here G and B are the 2×2, (x, y) part of the metric and of the antisymmetric tensor). We obtain a new class of exact solutions of the string effective action, with non-trivial $B_{\mu\nu}$, represented by [308]

$$\widetilde{G}(\gamma) = \begin{pmatrix} \frac{(c\mp1)+(c\pm1)b^2t^2}{(c\pm1)+(c\mp1)b^2t^2} & \frac{s(1+b^2t^2)}{(c\pm1)+(c\mp1)b^2t^2} \\ \frac{s(1+b^2t^2)}{(c\pm1)+(c\mp1)b^2t^2} & 1 \end{pmatrix},$$
$$\widetilde{B}(\gamma) = \begin{pmatrix} 0 & \frac{-s(1+b^2t^2)}{(c\pm1)+(c\mp1)b^2t^2} \\ \frac{s(1+b^2t^2)}{(c\pm1)+(c\mp1)b^2t^2} & 0 \end{pmatrix},$$
$$\widetilde{\phi} = -\ln\left[(c \pm 1) + (c \mp 1)b^2t^2\right], \tag{8.35}$$

where the upper and lower signs refer, respectively, to the boost of the solution (8.32) and (8.33).

For such solutions, all the curvature invariants are bounded [308], as well as the dilaton coupling $\exp\phi$, so that the solutions are regular in all frames. They describe a smooth evolution from (anisotropic) contraction to expansion, or vice versa, according to the behaviour of the original metric (the Milne-type solution or its dual). In any case, the boosted backgrounds (8.35) evolve from growing to decreasing curvature, and from growing to decreasing dilaton, as appropriate to a transition from pre- to post-big bang configurations. The new solutions are highly anisotropic, but the associated shear tensor rapidly decays far from the high-curvature regime [308]. Taking into account the backreaction of the produced radiation it is also possible to obtain a more realistic scenario, in which the background is eventually led to a phase of radiation-dominated, isotropic expansion, at constant dilaton [284].

Nappi-Witten:

More complicated, inhomogeneous examples of (non-trivial) regular backgrounds at low energy are provided by the "boosted" version of the Nappi–Witten solution [497]. We start with the following (singular) exact solution of the action (8.2) with $V = 0 = H_{\mu\nu\alpha}$,

$$ds^2 = dt^2 - dx^2 - \tan^{-2} t\, dy^2 - \tan^2 x\, dz^2, \qquad \overline{\phi} = -\ln(\sin 2t) - \ln(\sin 2x), \tag{8.36}$$

representing the direct product of a two-dimensional cosmological metric in the (t, y) plane and a two-dimensional Euclidean black hole in the (x, z) plane. Since the metric is independent of y and z we can apply to it any $O(2, 2)$ transformation in the (y, z) plane, to obtain new classes of inequivalent solutions. In particular, the transformation generated by the 4×4 $O(2, 2)$ matrix

$$\Lambda_{\rm NW}(\alpha) = \begin{pmatrix} \delta & 0 & 0 & \delta \\ 0 & 1 & -1 & 0 \\ 0 & \delta^{-1} & \delta^{-1} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad \delta^2 = \frac{1 - \sin\alpha}{1 + \sin\alpha}, \tag{8.37}$$

generates the one-parameter family of inhomogeneous (and singular) Nappi–Witten cosmologies [497].

As in the previous example, the singularities of the solution (8.36) can be removed by introducing an additional (originally flat) spatial dimension, parametrized by the coordinate w. The model then acquires a larger symmetry, isomorphic to $O(3,3)$, in the subspace (y,z,w). Performing two successive "boost" transformations (with the same parameter γ as in Eq. (8.34)) in the (z,w) plane and in the (y,w) plane, one can obtain a new class of inhomogeneous exact solutions completely free from curvature and dilaton singularities (see [309] for an explicit computation). As an example we report here the curvature scalar and the coupling of the boosted solutions [309]:

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$$e^{\widetilde{\phi}} = \left[(c+1)\sin^2 t + (c-1)\cos^2 t\right]^{-1}\left[(c+1)\cos^2 x + (c-1)\sin^2 x\right]^{-1},$$
$$\widetilde{R} = \frac{16\cos^2 t\ \sin^2 t - 20s^2}{\left[(c+1)\sin^2 t + (c-1)\cos^2 t\right]^2} - \frac{16\cos^2 x\ \sin^2 x - 20s^2}{\left[(c+1)\cos^2 x + (c-1)\sin^2 x\right]^2}. \qquad (8.38)$$

The curvature and the dilaton are bounded, but the background is too inhomogeneous to provide a realistic description of our present Universe. Such solutions could be relevant, however, to descrive the Universe near the transition regime, before the beginning of the standard radiation era.

The same comment applies to another class of regular inhomogeneous solutions of the lowest-order action (8.2), which is obtained even in the absence of torsion ($B_{\mu\nu} = 0$) and dilaton potential, and which can be parametrized in diagonal form as [330]

$$ds^2 = A\left(dt^2 - dx^2\right) - B\left(Cdy^2 + C^{-1}dz^2\right), \qquad \phi = \alpha\ \mathcal{G}_d(\mu t) + \text{const}, \qquad (8.39)$$

where:

$$A(x,t) = e^{\alpha\mathcal{G}_d(\mu t)}\left(\cosh\mu t\right)^{2+\beta}\left(\cosh\frac{\mu x}{2}\right)^{2\beta(\beta+1)},$$
$$B(x,t) = e^{\alpha\mathcal{G}_d(\mu t)}\left(\cosh\mu t\right)\left(\sinh\mu x\right),$$
$$C(x,t) = \left(\cosh\mu t\right)^{1+\beta}\left(\sinh\frac{\mu x}{2}\right)\left(\cosh\frac{\mu x}{2}\right)^{1+2\beta}. \qquad (8.40)$$

Here μ, α and $\beta = \sqrt{\alpha^2+4}$ are constant parameters, and $\mathcal{G}_d(\mu t) = \tan^{-1}(\sinh\mu t)$ is the so-called "Gudermannian amplitude". As shown in [330, 332], there are particular values of α for which the curvature invariants are bounded, the solutions are geodesically complete and describe a phase of growing dilaton coupling, evolving from a minimum up to a maximal value. We note that there is also another family of inhomogeneous solutions of the gravidilaton equations in which all curvature invariants are bounded [519], which is parametrized by a non-diagonal metric and reduces, in the E-frame, to known regular solutions of the Einstein equations with stiff sources [472]. In that case, however, the dilaton grows linearly and boundlessly.

We may thus conclude that, as in general relativity [562, 472], if the metric background is allowed to be inhomogeneous there is no need of a non-local potential and/or antisymmetric

tensor densities to obtain regular cosmological solutions, even in the contex of the lowest order gravidilaton action.

To stress the difficulty in obtaining smooth solutions in the homogeneous and isotropic case, on the contrary, we will conclude this subsection with a simple example, based on a Bianchi-I-type metric, and barotropic (but anisotropic) fluid sources with constant dilaton charge per unit mass:

$$g_{\mu\nu} = \mathrm{diag}\left(1, -a_i^2(t)\delta_{ij}\right), \qquad T_\mu{}^\nu = \mathrm{diag}\left(\rho(t), -p_i(t)\delta_i^j\right),$$
$$p_i = \gamma_i \rho\ , \qquad \sqrt{-g}\sigma = \frac{\delta S_m}{\delta\phi} = \gamma_0\rho\ , \tag{8.41}$$

where S_m is the matter action of Eq. (8.2), and γ_i, γ_0 are $d+1$ constant parameters specifying the "equation of state" of the fluid sources. The corresponding cosmological equations can be written in the form

$$\dot{\overline{\phi}}^2 - \sum_i H_i^2 = \overline{\rho} e^{\overline{\phi}}, \tag{8.42}$$

$$\dot{H}_i - H_i \dot{\overline{\phi}} = \frac{1}{2}\overline{\rho}(\gamma_i + \gamma_0) e^{\overline{\phi}}, \tag{8.43}$$

$$\dot{\overline{\phi}}^2 - 2\ddot{\overline{\phi}} + \sum_i H_i^2 = \gamma_0 \overline{\rho} e^{\overline{\phi}}. \tag{8.44}$$

Following the procedure already applied to the case of the non-local potential (8.19), the above equations can be integrated exactly in terms of the time coordinate (8.20), and we obtain [211]

$$\overline{\phi}\,' = -2(1+\gamma_0)\frac{(x+x_0)}{D(x)}, \qquad (1+\gamma_0) \neq 0, \tag{8.45}$$

$$\frac{a_i'}{a_i} = 2(\gamma_i + \gamma_0)\frac{(x+x_i)}{D(x)}, \qquad (\gamma_i + \gamma_0) \neq 0, \tag{8.46}$$

where

$$L^2 \overline{\rho} e^{-\overline{\phi}} = D(x) \equiv (1+\gamma_0)^2 (x+x_0)^2 - \sum_i (\gamma_i + \gamma_0)^2 (x+x_i)^2, \tag{8.47}$$

and x_i, x_0 are integration constants. In the case $1+\gamma_0 = 0$ and/or $\gamma_i + \gamma_0 = 0$, a slightly different definition of the quadratic form $D(x)$ is obtained, as well as a different expression for $\overline{\phi}\,'$ and a_i', but in that case no smooth solution with bounded curvature and energy density is allowed, for any value of $\gamma_i,\ \gamma_o,\ x_i,\ x_0$. We will thus concentrate on Eqs. (8.45)–(8.47) (see [211] for a discussion of the other possibilities).

By setting $D(x) = \alpha x^2 + bx + c$ it can easily be checked that if the background is isotropic (i.e. γ_i and x_i have the same values along all the d spatial directions), then the discriminant of $D(x)$ is always non-negative, i.e. $\Delta = b^2 - 4\alpha c \geq 0$, so that $D(x)$ necessarily has zeros on the real axis, correponding to singularities both in the curvature and in the dilaton kinetic

energy. Such singularities can be avoided, however, for *anisotropic* backgrounds, provided we accept a model of sources with *negative energy density*, $\rho < 0$.

Consider for instance a model of background in which the spatial geometry is factorizable as the direct product of two conformally flat manifolds with d and n dimensions, respectively, so that we can set:

$$\begin{aligned} &a_i = a_1, \qquad \gamma_i = \gamma_1, \qquad x_i = x_1, \qquad i = 1, \dots d, \\ &a_i = a_2, \qquad \gamma_i = \gamma_2, \qquad x_i = x_2, \qquad i = d+1, \dots d+n, \end{aligned} \tag{8.48}$$

and choose a convenient set of integration constants, such that the linear term in the quadratic form (8.47) disappears. For instance:

$$x_0 = 0, \qquad x_1 = -x_2 \frac{n(\gamma_2+\gamma_0)^2}{d(\gamma_1+\gamma_0)^2}. \tag{8.49}$$

It turns out that $c < 0$, and that the absence of zeros in $D(x)$ can be avoided, $\Delta = -4\alpha c < 0$, provided

$$\alpha = (1+\gamma_0)^2 - d(\gamma_1+\gamma_0)^2 - n(\gamma_2+\gamma_0)^2 < 0. \tag{8.50}$$

With the above assumptions, the integration of Eqs. (8.45), (8.46) leads to the final solution

$$\begin{aligned} &a_i = a_{i0} E_i(x) \, |D(x)|^{\frac{\gamma_i+\gamma_0}{\alpha}}, \qquad E_i(x) = \exp\left[\frac{2x_i(\gamma_i+\gamma_0)}{\sqrt{\alpha c}} \tan^{-1}\left(\frac{\alpha x}{\sqrt{\alpha c}}\right)\right], \qquad i = 1, 2, \\ &e^{\phi} = e^{\phi_0} a_{10}^d a_{20}^n E_1^d(x) E_2^n(x) \, |D(x)|^{-[(1+\gamma_0)-d(\gamma_1+\gamma_0)-n(\gamma_2+\gamma_0)]/\alpha}, \\ &\rho = -\frac{e^{\phi_0}}{L^2} a_{10}^{-d} a_{20}^{-n} E_1^{-d}(x) E_2^{-n}(x) \, |D(x)|^{1-[(1+\gamma_0)+d(\gamma_1+\gamma_0)+n(\gamma_2+\gamma_0)]/\alpha}, \end{aligned} \tag{8.51}$$

where a_{i0}, ϕ_0 are integration constants. As shown in [211], there is a region of non-zero extension in the space of the parameters γ_i, γ_0 where the condition (8.50) is satisfied, together with the other conditions needed to secure that the curvature, the dilaton kinetic energy, the effective string coupling, the energy density are everywhere bounded, that the density goes asymptotically to zero at large times, and that the solution describes in its final configuration d expanding and n contracting dimensions (as appropriate to a smooth transition from a higher-dimensional pre-big bang configuration to a post-big bang regime of dynamical dimensional reduction).

The condition (8.50) implies, however, that $D(x) < 0$ everywhere, and thus $\rho < 0$, according to Eq. (8.47) (the result $D < 0$ in the absence of zeros is independent from the particular choice $b = 0$). Such a result is not surprising, after all, because also the regular solutions obtained in the case of negative non-local potentials correspond to a negative energy density. See also [235] for another class of smooth, self-dual solutions of the low-energy effective action in which the energy density of the matter sources becomes negative, near the transition, because of an "exotic" equation of state.

In the above example the negative energy density goes to zero, asymptotically, far from the transition regime, and can thus be interpreted as the backreaction of the quantum fluctuations outside the horizon [211]. During the initial pre-big bang phase of growing curvature and shrinking horizons, in fact, the quantum fluctuations are stretched outside the horizon, and in such regime they may be characterized by a negative effective gravitational energy density [494, 495], which may favour the transition to the post-big bang branch of the classical solution [344]. Such a negative backreaction is eventually damped to zero when the curvature starts decreasing, the horizon blows up again, and all the fluctuations re-enter inside the horizon and in the regime of positive energy density.

The inclusion of the backreaction of the quantum fluctuation, on the other hand, is also a semiclassical way of taking into account the contribution of the quantum loop corrections to the effective action. Thus, the examples of graceful exit reported in this subsection, based on effective sources (fluid or non-local potential) with $\rho < 0$, which become dominant just around the epoch of transition, suggest the possible importance of the quantum backreaction –and of the loop corrections to the string effective action– for a successful mechanism of curvature and dilaton regularization. This possibility will be illustrated in Subsection 8.3, after a preliminary discussion of the effects of the higher-curvature corrections, which will be presented in the next subsection.

8.2 Growth of the curvature and α' corrections

The effective action of string theory is characterized by two perturbative expansions: the higher-genus expansion in the world-sheet topology, and the order-by-order condition of vanishing conformal anomaly, also called α' expansion [346]. The first one is equivalent to the quantum, field-theoretical loop expansion in powers of the coupling; the second one is equivalent to a higher-curvature expansion in powers of the field gradients (in units of $\lambda_s = (2\pi\alpha')^{-1/2}$). The second one is typical of strings, and disappears in the point-like limit $\lambda_s \to 0$.

These two expansions are indepedent in principle; it is possible, depending on the initial conditions, that the Universe evolving from the string perturbative vacuum reaches the strong-coupling regime when the curvature is still small, and the α' corrections are negligible. Alternatively, the Universe can reach the regime of high curvatures (in string units) when the coupling is small: in that case, only the α' corrections are to be added to the tree-level action.

What is important, in the context of the exit problem, is that such higher-curvature corrections may help the transition to the post-big bang regime, by damping the inflationary growth of the dilaton and of the curvature, and driving the cosmological background to a high-curvature "string phase" of constant curvature and linearly evolving dilaton [307], $H =$ const, $\dot{\phi} =$ const.

Remarkably, such a regularized background can be a solution of the tree-level effective action *to all orders in* α'. Indeed, as shown in [307], for a Bianchi-I-type configuration with $a_i = \exp(H_i t)$, $\phi = ct$, with constant c and H_i, the string cosmology equations reduce (to all orders in α') to a system of $d+1$ algebraic equations for the $d+1$ unknowns $c,\ H_i$. String-phase solutions thus generally exist, provided the algebraic system admits (non-trivial) real solutions.

In addition, the (all orders) equations resemble those of a renormalization group flow (in cosmic time), with the string-phase solution like a "fixed point" of the renormalization equations. It can thus be shown [307] that a string phase solution with $\dot{\phi} < 0$ can play the role of late-time attractor for an isotropic Universe evolving from pre-big bang initial conditions, provided no other fixed point, or singularity, separates the string phase from the string perturbative vacuum (i.e. from the trivial fixed point with $H = 0 = \dot{\phi}$).

A simple example of this possibility, first presented in [307], is based on the following S-frame effective action, truncated to first order in α':

$$S = -\frac{1}{2\lambda_{\rm s}^{d-1}} \int d^{d+1}x \sqrt{|g|} e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{k\alpha'}{4} \left(R^2_{GB} - (\nabla\phi)^4 \right) \right] \qquad (8.52)$$

(k is a number of order 1 depending on the particular string model adopted). Note that we have chosen a convenient field redefinition (preserving the conformal invariance of the σ-model action [482]) that eliminates higher than second derivatives from the equations of motion, but at the price of introducing dilaton-dependent α' corrections.

For a spatially flat anisotropic background, with d external and n internal dimensions,

$$g_{00} = N^2(t), \qquad g_{ij} = -\delta_{ij} e^{\beta(t)}, \qquad g_{ab} = -\delta_{ab} e^{\gamma(t)}, \qquad (8.53)$$

the action becomes

$$\begin{aligned} S = \quad & \int dt e^{d\beta + n\gamma - \phi} \left[\frac{1}{N} \left(-\dot{\phi}^2 - d(d-1)\dot{\beta}^2 - n(n-1)\dot{\gamma}^2 - 2dn\dot{\beta}\dot{\gamma} + 2d\dot{\beta}\dot{\phi} + 2n\dot{\gamma}\dot{\phi} \right) + \right. \\ & \frac{k\alpha'}{4N^3} \left(c_1\dot{\beta}^4 + c_2\dot{\gamma}^4 + c_3\dot{\phi}\dot{\beta}^3 + c_4\dot{\phi}\dot{\gamma}^3 + c_5\dot{\phi}\dot{\beta}\dot{\gamma}^2 + c_6\dot{\phi}\dot{\beta}^2\dot{\gamma} + c_7\dot{\beta}^2\dot{\gamma}^2 + \right. \\ & \left. \left. c_8\dot{\beta}\dot{\gamma}^3 + c_9\dot{\beta}^3\dot{\gamma} - \dot{\phi}^4 \right) \right] , \end{aligned} \qquad (8.54)$$

where $c_1,\ \ldots, c_9$ are dimensionless numerical coefficients depending on d and n (see [307]). By varying the action with respect to ϕ and N, and setting $n = 0$, $\dot{\phi} = x = \text{const}$, $\dot{\beta} = y = \text{const}$ (in the $N = 1$ gauge), we are led to the algebraic equations for an isotropic fixed point:

$$\begin{aligned} & x^2 + d(d-1)y^2 - 2dxy - \frac{k\alpha'}{4} \left(c_1 y^4 + c_3 x y^3 - x^4 \right) - \\ & (dy - x) \left[-2x + 2dy + \frac{k\alpha'}{4} \left(c_3 y^3 - 4x^3 \right) \right] = 0, \end{aligned}$$

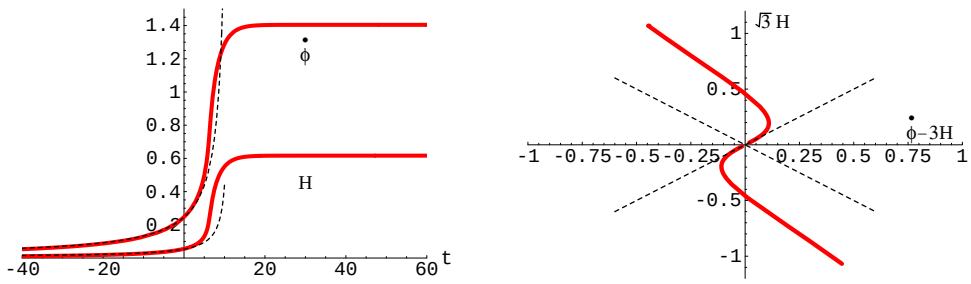

Figure 8.4: *Numerical integration of the isotropic equations for the action (8.54) (solid curves) with $d = 3$, $n = 0$, and with initial conditions on the low-energy pre-big bang branch (dashed curves). The left panel shows the time evolution of $\dot\phi$ and $H = \dot\beta$ for an expanding background. The right panel shows the evolution connecting the vacuum to the non-trivial fixed point, for an expanding solution and its time-reversed counterpart.*

$$x^2 + d(d-1)y^2 - 2dxy - \frac{3}{4}k\alpha'\left(c_1 y^4 + c_3 xy^3 - x^4\right) = 0,$$
$$c_1 = -\frac{d}{3}(d-1)(d-2)(d-3), \qquad c_3 = \frac{4}{3}d(d-1)(d-2), \tag{8.55}$$

which have real solutions for any d from 1 to 9. For instance (in units $k\alpha' = 1$)

$$d = 3, \qquad \dot\phi = \pm 1.40..., \qquad \dot\beta = \pm 0.616..., \tag{8.56}$$

where the same sign has to be taken for $\dot\phi$ and $\dot\beta$. Note that, with the addition of a non-zero $B_{\mu\nu}$ background, anisotropic fixed-point configurations can also be obtained, as shown in [257].

A numerical integration of the isotropic cosmological equations, starting from pre-big bang initial conditions ($H > 0$, $\dot\phi > 0$), shows that the background, when the α' corrections become important, deviates from the tracks of the low-energy, singular solutions, and is eventually attracted to the isotropic fixed point (8.55), as illustrated in Fig. 8.4. In this sense, the curvature singularity is regularized by the α' corrections, at least in the S-frame (but not in the E-frame, as the linear growth of the dilaton is unbounded). Note that a similar curvature regularization is also effective for a static, spherically symmetric solution of the action (8.52) [131]. It should be mentioned that a similar behaviour can also be obtained at low energy (without α' corrections), including the contribution of a supersymmetric gravitino–dilatino condensate, which is able to damp the acceleration of the background and to regularize the S-frame curvature, as shown in [258].

It may be noted, finally, that a solution with $H = \text{const}$, $\dot\phi = \text{const}$ is in general allowed in many higher-derivative models of gravity. Such a fixed point of the cosmological equations, however, is in general disconnected from the trivial fixed point $H = 0 = \dot\phi$ (the perturbative vacuum, in this case) by a singularity, or by an unphysical region in which the curvature

becomes imaginary. For the action (8.52), on the contrary, the constant fixed point is a late-time attractor for all isotropic backgrounds emerging from the string perturbative vacuum, and even for some anisotropic backgrounds, provided sign$\{\dot{\gamma}\}$= sign$\{\dot{\beta}\}$ (the attraction basin of anisotropic configurations covers regions of small but finite size in the space of initial conditions [307]).

This attraction property, unfortunately, is not invariant under field redefinitions, as long as the action is truncated at a given finite order in α'. Also, and most important, the growth of the curvature is stopped, but the transition is not completely performed. Indeed, the final fixed point is in the post-big bang regime $\dot{\overline{\phi}} < 0$, as illustrated in Fig. 8.4, but the transition cannot proceed further towards lower curvatures to complete the transition.

The reason for such an incompleteness, which is manifest as an asymmetry in the plane $\{\dot{\overline{\phi}}, \sqrt{3}H\}$ of Fig. 8.4, is that the action (8.52) is invariant under time reflections, but is not duality-invariant. As a consequence, the regularization concerns the expanding pre-big bang branch of the lowest-order solution, and its time-symmetric counterpart, the *contracting post-big bang* branch. The expanding post-big bang branch remains, however, singular, and thus cannot be smoothly connected to the regularized pre-big bang branch.

It is possible, of course, to modify the action with an additional field redefinition truncated to order α', in such a way as to preserve conformal invariance and to restore the T-duality symmetry (to the same order in α') [476, 397]. The modified action has then self-dual fixed points, but they are not smoothly connected to the perturbative vacuum, as we have checked. See also [123, 149] for a discussion of the existence and position of the fixed points in relation to the scale-factor duality of the higher-order action, and for the difficulty of a higher-order, self-dual model of exit.

The evolution of the background away from the fixed points, towards the low-energy, post-big bang regime, becomes however possible if the action (8.52) is further generalized by the inclusion of quantum loop correction (after all, the dilaton keeps growing while the Universe is trapped in the fixed point, so that higher-order effects in the string coupling are doomed to become eventually important). This point will be illustrated in the next subsection.

It should be noted, finally, that all examples of exit based on a truncated α' expansion are characterized by a certain degree of ambiguity, because the physical properties of such models are field-redefinition-dependent [307], as already stressed. A truly unambiguous example of exit should in principle correspond to the solution of an exact conformal model, which automatically includes all orders in α' (see [462], however, for a different non-perturbative approach to higher-order corrections, based on the inclusion of the exponentially growing density of all massive string-theory states, and on their possible role in regularizing the singularity).

There are in fact higher-curvature, exact conformal-field theory models that smoothly interpolate between two duality-related low-energy solutions [416, 417]. In that cases, how-

ever, it turns out impossible, in general, to describe the high-curvature regime in terms of classical fields such as the metric and the dilaton, and we may lose an intuitive and geometrical understanding of the transition (the conventional space-time interpretation is also lost in other, higher-dimensional, string-theory resolutions of the cosmological singularities [426]).

Exact conformal models, for $D = 2$ and $D = 3$ string-cosmology backgrounds, have been recently studied [215] in the context of gauged Wess–Zumino–Witten models [625], for the $SO(2,1)/SO(1,1)$ and $SO(2,2)/SO(2,1)$ cosets. It turns out that it is possible, in some cases, to avoid the curvature singularities, but the dilaton (and the string coupling) seems to remain unbounded in correspondence of inflationary backgrounds.

To conclude this subsection, devoted to higher-derivative corrections, we recall that a higher-curvature regularization of the cosmological singularity can also be obtained in the context of models implementing the so-called "limiting-curvature hypothesis" [491, 108, 485], applied to string cosmology in both the Einstein [104] and String [230] frames. They are based on a higher-derivative action which, in the S-frame, takes the form

$$S = -\frac{1}{2}\int d^4x\sqrt{|g|}e^{-\phi}\left[R + (\nabla\phi)^2 + c\psi e^{\gamma\psi}I_2 + V(\psi)\right], \tag{8.57}$$

where c and γ are constant parameters, ψ is a scalar Lagrangian multiplier, and

$$I_2 = \left(4R_{\mu\nu}^2 - R^2\right)^{1/2}. \tag{8.58}$$

Finally, $V(\psi)$ is a potential that satisfies appropriate boundary conditions, $V \to \psi^2$ for $|\psi| \to 0$, $V \to$ const for $|\psi| \to \infty$, and has non-trivial zeros.

Numerical and analytical studies then show that this action has regular spatially flat and isotropic solutions, smoothly interpolating from contraction to expansion in the E-frame [104], and from superinflation to decelerated expansion in the S-frame [230], even at constant dilaton. A dynamical dilaton may increase the fraction of phase space for which the curvature is bounded (in the absence of an appropriate potential, however, a dynamical dilaton keeps growing). Unfortunately (and as in other models that will be discussed in the next subsection), the action (8.57) is not derived from a string-theory expansion (the higher-order terms are artificially constructed), so that the obtained results are only valid as (interesting) phenomenological examples of possible approaches to the exit transition.

8.3 Growth of the coupling and loop corrections

According to the pre-big bang scenario, the tree-level effective action (8.2) can provide a correct description of the Universe at low energies, during the initial evolution leading the Universe away from the string perturbative vacuum, until the string coupling remains small, $\exp\phi \ll 1$. However, since ϕ is growing during pre-big bang inflation, in the absence of a

mechanism able to stop the growth of the dilaton, the Universe will tend to evolve towards strong couplings, $\exp\phi \gtrsim 1$. In that regime all the higher-loop contributions will become important, and since the fully corrected action is unknown, in general, we have to restrict our studies to the simplest cases where we know the explicit form of the loop corrections.

In all cases so far analysed in the literature, it was found that the inclusion of appropriate higher-order terms seems to damp the growth of H and $\dot{\phi}$, thus favouring the exit transition. Indeed, in the presence of loop corrections, the "theorems" on the impossibility of branch changing have to be generalized [547]; it becomes possible, in general, to evade the obstructions present at the tree level if the loop functions satisfy appropriate conditions. For a complete and successful transition it seems required, however, that the quantum corrections be included at least to second order [122, 149] (the two examples of smooth homogeneous isotropic solutions presented in Subsection 8.1 correspond indeed to a two-loop (8.19) and to a four-loop (8.17) (non-local) potential).

We wish to recall, at this point, that the quantum loop corrections are generated by the higher-genus expansion of the world-sheet topology of the corresponding sigma-model action [346], and can be represented as an expansion in powers of $g_s^2 = e^{\phi}$. They contain, already to first order, the backreaction of particle production, which becomes important when $\alpha' H^2 e^{\phi} \sim 1$, and thus naturally introduce higher-derivative terms in the effective action, as in general relativity (see for instance [484]), with the only difference that the gravitational coupling is not fixed but is controlled by a dynamical dilaton.

In string theory, as already mentioned, there are additional higher-derivative corrections (typical of the minimal finite size $\lambda_s = (2\pi\alpha')^{1/2}$ of fundamental strings), which are generated by the α' expansion of the sigma model action [346], and which may become important even at small coupling, if the curvature reaches the string scale. The possible effects of such higher-curvature terms, without loop corrections, have been discussed in the previous subsection. Here we shall start considering higher-genus corrections only, by assuming that the strong-coupling regime is reached when the α' corrections are still negligible (this is certainly possible in some appropriate model of cosmological evolution, see [465]). As we have already anticipated, however, both α' and loop corrections seem to be required [279] for a complete transition to the post-big bang regime.

An unrealistic (but instructive) example of curvature regularization operated by the loop corrections is provided by (a generalization of) the so-called CGHS model [139] of dilaton gravity in $D = 2$ dimensions, described by the action

$$S = -\int d^2x\sqrt{-g}e^{-\phi}\left[R + (\nabla\phi)^2\right] + S_{\rm one-loop} \quad ,$$
$$S_{\rm one-loop} = \frac{k}{2}\int d^2x\sqrt{-g}\left(R\nabla^{-2}R + \epsilon\phi R\right). \tag{8.59}$$

Here $k = (N-24)/24$, where N is the number of conformal scalar fields possibly present in the model, and the trace-anomaly term of the loop corrections has been supplemented

by a local counterterm $\epsilon\phi R$, needed to preserve other classical symmetries. The case $\epsilon = 0$ reproduces the original CGHS model, the case $\epsilon = 1$ preserves conformal invariance [202, 544], and has been applied to the string cosmology case in [533].

The cosmological equations for the action (8.59) can be integrated exactly for any constant value of ϵ [323], and it has been shown that for $\epsilon \geq 1$ there are regular solutions without curvature singularities, describing a smooth evolution away from the (two-dimensional version of the) pre-big bang branch of the vacuum solution, $a \sim (-t)^{-1}$, $\phi \sim -2\ln(-t)$. For $\epsilon = 1$ the curvature regularization requires $k < 0$ [533], which is known to correspond to gravitational instabilities [544]. Regular examples that do notspoil the physical requirement $k > 0$ are possible, however, provided $\epsilon > 1$. In that case the smooth solutions describe, for any ϵ, the evolution from a phase of pre-big bang superinflation to a final asymptotic configuration with flat space, $a = \text{const}$, and linearly growing dilaton, $\phi = t$. The solutions can be expressed in parametric form as a function of the monotonic coupling parameter $g_{\rm s}(t) = \exp[\phi(t)/2]$ as follows [323]:

$$a(g_{\rm s}) = e^{\beta} = e^{\beta_0} \left| \frac{g_{\rm s}^2}{\epsilon(r + g_{\rm s}^2) + \epsilon - 2} \right|^{\epsilon/4} \left| \frac{2r + 2 + (\epsilon - 2)g_{\rm s}^2}{g_{\rm s}^2} \right|^{(\epsilon-2)/4} e^{\frac{r-1}{2g_{\rm s}^2}}, \tag{8.60}$$

$$\dot\phi(g_{\rm s}) = \frac{g_{\rm s}^2}{a t_0 r}, \qquad 2\dot\beta = \frac{1}{a t_0 r}\left(1 + \frac{\epsilon}{2} g_{\rm s}^2 - r\right),$$

$$r(g_{\rm s}) = \sqrt{1 + (\epsilon - 2)g_{\rm s}^2 + \frac{\epsilon^2}{4} g_{\rm s}^4}, \tag{8.61}$$

where t_0 and β_0 are integration constants, and $g_{\rm s}(t)$ is given implicitly by

$$\frac{t}{t_0} = \int \frac{dg_{\rm s}^2}{g_{\rm s}^4} r(g_{\rm s})\ a(g_{\rm s}). \tag{8.62}$$

In the above example the curvature and the dilaton kinetic enegy are bounded everywhere. The dilaton, however, keeps growing as $t \to +\infty$. It is true that a monotonic evolution of the dilaton from weak to strong coupling may be equivalent, via S-duality transformations [561, 523], to a smooth interpolation between two different, weak-coupling regimes. In a realistic scenario, however, the final value of the dilaton should go to a finite constant, not to zero. To this aim, a non-perturbative dilaton potential and/or the backreaction of the produced radiation are probably to be included.

Unfortunately, however, the final state of the above regular backgrounds is still characterized by $\dot{\overline{\phi}} > 0$. This means that the conditions of "branch changing" [121] are not satisfied and that the background, in spite of the bounce of the curvature, is still in the pre-big bang regime. As a consequence, it is not ready to be attracted to any stable minimum of the potential, so that dilaton stabilization is impossible [323], even asymptotically, in the context of the above one-loop action.

Similar conclusions apply to the regular solutions of another (four-dimensional) model of one-loop string effective action [34] (which also provided, to the best of our knowledge, the

first example of loop regularization of the cosmological singularity in a superstring-theory context). The action is based on moduli-dependent loop corrections to the gravitational coupling for a heterotic string compactified on a symmetric orbifold [32, 33]. There are no corrections to the Einstein term, only quadratic-curvature corrections, and the action can be written (in the E-frame, and using the notations of [34]) as

$$S = \frac{1}{2}\int d^4x\,\sqrt{|g|}\,\left[R + \frac{1}{2}(\nabla_\mu\phi)^2 + \frac{3}{2}(\nabla_\mu\sigma)^2\right] + S_{\rm one-loop}\quad ,$$
$$S_{\rm one-loop} = \int d^4x\,\sqrt{|g|}\,\left[\lambda e^{\phi} - \delta\xi(\sigma)R^2_{GB}\right]. \tag{8.63}$$

Here σ is the modulus field, $R^2_{GB} = R^2 - 4R^2_{\mu\nu} + R^2_{\mu\nu\alpha\beta}$ is the Gauss–Bonnet invariant, λ and δ are constant parameters (λ is fixed by α', while δ depends on the number of chiral, vector and gravitino massless supermultiplets included in the model). Finally,

$$\xi(\sigma) = \ln\left[2e^{\sigma}\eta^4(ie^{\sigma})\right], \tag{8.64}$$

where η is the Dedekind function [242].

For an isotropic and spatially flat background, the equations have been numerically integrated in [34], and it has been shown that, for $\delta < 0$, the action (8.63) admits regular solutions without singularities, in which the curvature grows up to a maximum and then decreases, with a behaviour that mimics that of the pre-big bang scenario. However, the Hubble parameter turns out to be always positive even in the E-frame, and the dilaton contribution is negligible, so that such solutions do not describe a transition from the pre- to the post-big bang regime.

The same action also admits regular homogeneous and isotropic solutions with positive spatial curvature, $k > 0$ [232]. For such solutions the curvature invariants are bounded, the metric smoothly evolves from contraction to expansion, and there is indeed a branch changing. The dilaton has, however, a very small influence on the background, which is dominated by the modulus and by the spatial curvature. In addition, the dilaton is monotonically decreasing, so that such solutions do not describe the evolution of the Universe away from the string perturbative vacuum, as required by the pre-big bang scenario.

The regular, spatially flat solutions of the action (8.63) can be generalized to the anisotropic case, corresponding to a Bianchi-I-type metric background [407, 628]. The regular, spatially curved ($k > 0$) solutions, however, cannot: for the corresponding anisotropic curved metric, of Bianchi-IX-type, no regular solution of the action (8.63) has been obtained [628]. This seems to be in agreement with an instability of the regular solutions of the action (8.63) against tensor perturbations [406, 408], since the Bianchi-IX-type metric may indeed be regarded as a model of closed Friedmann Universe with a non-linear tensor perturbation, represented by a gravitational wave of given wave-number. In this sense, the curvature regularization operated by the one-loop term (8.63) is generic for spatially flat backgrounds, but non-generic for curved ones [628].

It should be noted, finally, that the action (8.63) has also motivated the study of scalar-tensor models of gravity characterized by the higher derivative, non-minimal coupling $\xi(\phi)R^2_{GB}$, with arbitrary coupling functions $\xi(\phi)$. Even in this case it has been found that there are, for appropriate forms of $\xi(\phi)$, singularity-free, homogeneous and isotropic solutions, without [539] and with [402] spatial curvature, and also (in very special cases) with a scalar potential [11]. But, again, such solutions do not describe the same initial configuration as in the pre-big bang scenario. In addition, the higher-derivative, scalar-tensor coupling is postulated *ad-hoc*, and is not the result of a reliable one-loop computation.

An appropriate loop-corrected action for string cosmology has been discussed instead in [259], considering the heterotic string compactified to four dimensions on a Z_N orbifold, and restricted to the gravidilaton-moduli sector, as in [34], but including in addition all string α' corrections (truncated to first order). The action contains modular-invariant loop corrections to the Kähler potential, known to all orders [209], and one-loop corrections to the gravitational couplings of the higher-derivative terms. The final action, neglecting non-local terms, can be written in the S-frame (using the conventions of [259]) as:

$$\begin{aligned} S &= \frac{1}{2\alpha'}\int d^4x\,\sqrt{|g|}\,e^{-\phi}\,\Big[R+\Big(1+e^{\phi}G(\phi)\Big)\,(\nabla_\mu\phi)^2-\frac{3}{2}(\nabla_\mu\sigma)^2 \\ &\quad +\frac{\alpha'}{4}\Big(1+e^{\phi}\Delta(\sigma)\Big)\Big(R^2_{GB}-(\nabla_\mu\phi)^4\Big)\Big], \end{aligned} \tag{8.65}$$

where $\Delta(\sigma)$ is a modular function depending on the compactification [34], and

$$G(\phi)=\frac{3\alpha}{2}\frac{6+\alpha e^{\phi}}{(3+\alpha e^{\phi})^2}, \tag{8.66}$$

where $\alpha>0$ is a model-dependent constant of order 1.

The cosmological equations of this action, for a homogeneous and spatially flat background, have been numerically integrated [259], starting from an initial pre-big bang regime with $H>0$ and $\dot\phi>0$. The integration shows that, while the α' corrections can drive the Universe to a fixed point with $\dot{\overline{\phi}}<0$ (even in the absence of loop corrections, see the next subsection), the loop corrections to the Kähler potential are essential to drive the Universe to another fixed point with $\dot\phi<2H$, so that both conditions (8.12) and (8.13) (necessary for a graceful exit [121]) can be satisfied. Such conditions are not sufficient, since a regular solution requires that the curvature be bounded also in the E-frame. This result is achieved, however, through the contribution of the (moduli-dependent) higher-derivative terms –at least for the case of small $\dot\sigma$ and/or nearly constant (negative) $\Delta(\sigma)$ discussed in [259].

From the action (8.65), numerical solutions can thus be obtained, which smoothly interpolate from the pre-big bang regime of the S-frame, tree-level effective action, to a final decelerated expansion, with decreasing curvature. The curvature and dilaton kinetic energy are bounded everywhere in the S-frame and E-frame. However, the post-big bang decelerated expansion becomes a phase of the de Sitter inflationary expansion in the E-frame,

because the loop corrections to the dilaton kinetic term do not disappear, asymptotically, but approach a constant value [259]. Also, for all the regular solutions, the dilaton remains growing, logarithmically, as $t \to +\infty$. The dilaton cannot be stabilized being trapped in the minimum of a potential, because it becomes unstable at strong couplings (its kinetic term acquires the "wrong sign" even in the E-frame).

In spite of this property, which means that the solutions do not describe a complete transition to the post-big bang regime, the existence of such solutions represents a remarkable and encouraging result, especially in view of the fact that the higher-order terms of the action (8.65) are not *ad hoc* (except for the truncation of the α' expansion), but are the outcome of a well motivated string theory calculation. A complete example of transition (but with *ad hoc* loop corrections) can be obtained instead, by considering the following higher-order action [122]

$$S = -\frac{1}{2\lambda_s^2}\int d^4x\sqrt{|g|}e^{-\phi}\left[R + (\nabla\phi)^2\right] + S_{\alpha'} + S_q + S_m, \tag{8.67}$$

where $S_{\alpha'}$ contains the α' corrections of Eq. (8.52), S_q provides the appropriate quantum loop corrections, and S_m contains radiation, or a dilaton potential, for the final dilaton stabilization.

The loop corrections are required to give a negative contribution to the energy density, are expected to become strong enough to drive the Universe away from the fixed point, and to disappear fast enough at late times, in order for the dilaton to eventually be captured by a potential or be frozen by radiation production. In the numerical examples presented in [122], the one-loop corrections are suppressed at late times through the introduction of a step function, or with the addition of a two-loop term of opposite sign. A toy model is the following (in units $2\lambda_s^2 = 1$):

$$S_q = \int d^4x\sqrt{|g|}\left(c_1 + c_2 e^{\phi}\right)(\nabla\phi)^4\,, \qquad S_m = c_3\int d^4x\sqrt{|g|}\,(\phi-\phi_0)^2\,e^{\phi-\phi_0}, \tag{8.68}$$

where c_i are dimensionless coefficients. As shown in [122], the numerical integration of the cosmological equations with $c_1 = -c_2 = -10^3$, $c_3 = -10^{-1}$, $\phi_0 = 1$, provides a complete model of graceful exit in which the background smoothly evolves from the vacuum, pre-big bang configuration of the tree-level action, to a final expanding, decelerated configuration, with the dilaton performing damped oscillations around $\phi_0 = 1$. The model can be further improved with the introduction of radiation, which eventually becomes dominant and freezes out the dilaton.

Other examples of a smooth and complete graceful exit, but again with *ad hoc* loop corrections, can be obtained with the following higher-order action [149]:

$$S = -\frac{1}{2\lambda_s^2}\int d^4x\sqrt{|g|}e^{-\phi}\left[R + (\nabla\phi)^2 + \mathcal{L}_{\alpha'} + Ae^{\phi}\mathcal{L}^{(1)}_{\alpha'} + Be^{2\phi}\mathcal{L}^{(2)}_{\alpha'}\right], \tag{8.69}$$

where $\mathcal{L}_{\alpha'}$ is the most general form of the first-order α' corrections to the gravidilaton action [482], on which the condition has been imposed that the equations of motion contain

at most second derivatives:

$$\mathcal{L}_{\alpha'} = \frac{\alpha'}{4}\left[aR^2_{GB} + bG^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + c(\nabla\phi)^2\nabla^2\phi + d(\nabla\phi)^4\right]. \tag{8.70}$$

The coefficients a, b, c, d, in order to reproduce string scattering amplitudes, are constrained by

$$a = -1, \qquad b + 2(c + d) = -2a, \tag{8.71}$$

but are otherwise arbitrary, because of possible further shifts due to field redefinitions, truncated to first order in α' (note that our notation is diferent from [149], where they use a different normalization for the dilaton, i.e. $g_s^2 = e^{2\phi}$; see also [148]). The one-loop and two-loop terms, $\mathcal{L}^{(1)}_{\alpha'}$ and $\mathcal{L}^{(2)}_{\alpha'}$, have the same functional form as $\mathcal{L}_{\alpha'}$, but the coefficients a, b, c, d are in principle replaced by different numbers, a_1, b_1, c_1, d_1 and a_2, b_2, c_2, d_2 , respectively. Finally, the constant parameters A and B control the onset of loop corrections and their late-time suppression for a successful exit.

The study of the fixed points in the weak-coupling regime of the action (8.69), and of the subsequent transition to the decelerated expansion of the post-big bang scenario, has shown that the case $a = -d$, $b = c = 0$ (discussed in [307, 122]) is not the only one that is compatible with a graceful exit. There is indeed a wide region in parameter space that allows fixed points with $\dot{\bar{\phi}} < 0$ and, from the fixed points, the transition can further proceed successfully even if b and c are non-zero [149] and different from b_1, c_1 and b_2, c_2 [148]. In all cases, the exit can be completed by dilaton stabilization, obtained by introducing "by hand" some radiation density ρ_r, coupled to the dilaton through the conservation equation

$$\dot{\rho}_r + 4H\rho_r - \frac{1}{2}\Gamma\dot{\phi}^2 = 0. \tag{8.72}$$

The main problem with such models of exit is that the one-loop and two-loop corrections are not the result of a reliable string theory perturbative expansion. In particular, a successful exit seems to require (as also stressed in [122]) an appropriate sign of the corrections, so that (in our case) $\text{sign}\{A\} = -\text{ sign}\{B\}$. It is unclear, however, whether the required sign may also be the outcome of a correct loop computation (even if the results of [259] seem to provide a positive answer). A successful example of exit, corresponding to a numerical integration of the equations for the action (8.69) with $A = 1$, $B = -2 \times 10^{-3}$, is illustrated in Fig. 8.5.

In addition, the perturbative approach at the two-loop level is self-consistent only for a quite small value of the final coupling, $\exp\phi \ll 1$. A small value of the final dilaton, for the solutions of the action (8.69), seems to require, however, that A and B be both larger than 1 and of the same order (in modulus) [148], in contrast with the usual expectation for the coefficients of a perturbative expansion.

If the string coupling is not kept small enough, on the other hand, the Universe necessarily enters the non-perturbative regime where typical M-theory effects may become

M-theory

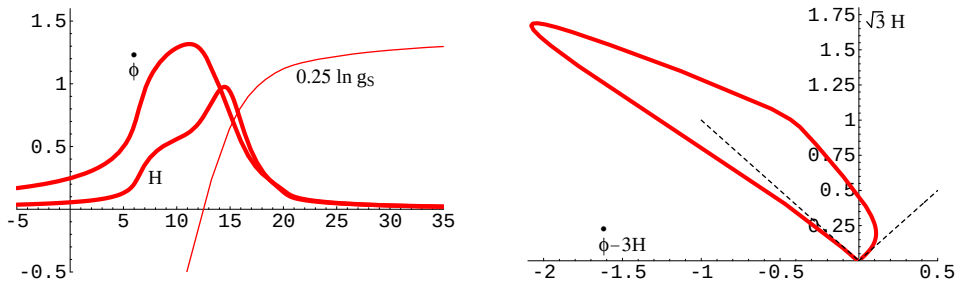

Figure 8.5: *Numerical integration of the cosmological equations for the action (8.69) with* $A = 1$, $B = -2 \times 10^{-3}$, $a = a_1 = a_2 = -d = -d_1 = -d_2 = -1$, $\Gamma = 5.63 \times 10^{-4}$, *and all* $b_i, c_i = 0$. *The left panel shows the time evolution of* $\dot{\phi}$, H *and of the log of the string coupling. The right panel shows the evolution from the vacuum to the fixed point of Fig. 8.4, and the subsequent exit induced by the loop corrections.*

important, as will be illustrated in Subsection 8.5. We shall first present, however, some arguments stressing the importance of the notion of entropy for a model-independent approach to the exit problem.

8.4 Graceful exit and entropy considerations

After the various models of exit reported in the previous subsections, it seems appropriate to recall, at this point, that entropy-related considerations have recently led to *model-independent* arguments in favour of the occurrence of a graceful exit in the pre-big bang scenario. As we shall see, those are physically quite close to the arguments based on backreaction and loop corrections, which we have already discussed.

Let us start by recalling that, as suggested by Bekenstein [72, 74], the entropy of a limited gravity system of energy E and of size R larger than its gravitational radius, $R > R_g \equiv 2G_N E$, is limited by the upper bound

$$S_{\rm Bek} = ER \simeq R_g R\, \lambda_{\rm P}^{-2}. \tag{8.73}$$

According to the holographic principle (see [375, 574] and references therein), on the other hand, the maximal entropy of a system is bounded by $S_{\rm hol}$,

$$S_{\rm hol} = A\lambda_{\rm P}^{-2}, \tag{8.74}$$

where A is the area of the space-like surface enclosing the region of space whose entropy we wish to bound. For systems of limited gravity, since $R > R_g$ and $A = R^2$, the Bekenstein bound (8.73) implies the holography bound (8.74).

How can these bounds be extended to be applied to the whole Universe? At cosmological scales the Universe is *not* a system of limited gravity, since its large-distance behaviour

is determined by the gravitational effect of its matter content through Friedmann's equations. Furthermore, the holography bound obviously fails for sufficiently large regions of space since, for a given temperature, entropy grows like R^3 while area grows like R^2. The generalization of entropy bounds to cosmology turned out to be subtle.

A possible prescription [73] for a cosmological extension is to identify R in Eq. (8.73) with the particle horizon. In this way one can then arrive at the conclusion that the bound is violated sufficiently near the big-bang singularity, implying that the latter is fake (if the bound is always valid). More recently, a similar extension of the holographic bound to cosmology has been proposed [254], arguing that the area of the particle horizon should bound entropy on the backward-looking light cone, according to (8.74). It was soon realized, however, that the such a proposal requires modifications, since violations of it were found to occur in physically reasonable situations. An improvement of the proposal [254], applicable to light-like hypersurfaces, was later made by Bousso [96, 97] (see also [256]).

Of more interest, in our context, are however the attempts made at deriving cosmological entropy bounds on space-like hypersurfaces [231, 604, 48, 91, 392, 127]. These identify the maximal size of a spatial region for which holography works: the Hubble radius [231, 604, 392], the so-called apparent horizon [114], or, finally, a causal-connection (Jeans) scale [127].

For our purpose there is no need here to enter into the relative merits of these various proposals. Rather, we will only outline the physical idea behind them. Consider, inside a quasi-homogeneous Universe, a sphere of radius H^{-1}. We may consider "isolated" bodies, in the sense of Ref. [72], fully contained in the sphere, i.e. with radius $R < H^{-1}$. For such systems, the bound (8.73) holds, and it is saturated by a black hole of size R. We may next consider several black holes inside our Hubble volume, each carrying an entropy proportional to the square of its mass. If two, or more, of these black holes merge, their masses will add up, while the total entropy after the merging, being quadratic in the total mass, will exceed the sum of the initial entropies. In other words, in order to maximize entropy, it pays to form black holes as large as possible.

Is there a limit to this process of entropy increase? The suggestion made in [231, 604, 48, 392, 127], which finds support in old results by several groups [147, 146, 505], is that a critical length of order H^{-1} is the upper limit on how large a classically stable black hole can be. If we accept this hypothesis, the upper bound on the entropy contained in a given region $\mathcal{R}$ of space will be given by the number of Hubble volumes in $\mathcal{R}$, $n_H = VH^3$ times the Bekenstein–Hawking entropy [71, 369] of a black hole of radius H^{-1}, $H^{-2}\lambda_{\rm P}^{-2}$. The two factors can be combined in the suggestive formula:

$$S(\mathcal{R}) < \lambda_{\rm P}^{-2} \int_{\mathcal{R}} d^3x \, \sqrt{h} \, \overline{H} \equiv S_{HB} \, , \qquad (8.75)$$

where $\int_{\mathcal{R}} d^3x \, \sqrt{h}$ is the volume of the space-like hypersurface whose entropy we wish to bound, and $\overline{H}$ differs from one proposal to another, but is, roughly, of the order of the Hubble parameter. Actually, since H is proportional to the trace of the second fundamental

form on the hypersurface, Eq. (8.75) reminds us of the boundary term that has to be added to the gravitational action in order to correctly derive Einstein's equations from the usual variational principle. This shows that the bound (8.75) is generally covariant for $\overline{H} = H$. It can also be written covariantly for the identification of $\overline{H}$ made in [127].

For the qualitative discussion that follows, let us therefore take $\overline{H} = H$, and let us convert the bound to S-frame quantities, taking into account the relation between $\lambda_{\rm P}$ and $\lambda_{\rm s}$, given in Eq. (1.25). We obtain [604]:

$$S(\mathcal{R}) < (VH^3)(H^{-2}\lambda_{\rm s}^{-2}e^{-\phi}) = e^{-\bar{\phi}}H\lambda_{\rm s}^{-2}\ , \tag{8.76}$$

where we have fixed an arbitrary additive constant in the definition of $\bar{\phi}$. Equation (8.76) thus connects very simply the entropy bound of a region of fixed comoving volume to the most important variables occurring in string cosmology (see, e.g., the phase diagram of Fig. 8.1).

An immediate application of the bound (8.76) to the exit problem was pointed out in [604], noting that the bound is initially saturated in the E-frame picture of pre-big bang inflation as a collapse, with corresponding black-hole formation [129]. Since the entropy, and the bound itself, cannot decrease without a violation of the second law, one obtains:

$$\dot{\bar{\phi}} \leq \dot{H}/H\ . \tag{8.77}$$

It is easy to check that this relation holds with the equality sign during the initial, low-energy evolution from the string perturbative vacuum (see Eq. (8.5)). In other words, the Hubble entropy bound is saturated initially *and throughout* the low-energy, dilaton-driven evolution described in [129]. But what happens if the curvature stops its growth, $\dot{H} = 0$ (for instance, during the high-curvature string phase discussed in the previous subsections)? It is quite clear that Eq. (8.77) does not allow H to reach saturation ($\dot{H} = 0$) in the upper-right quadrant of Fig. 8.1, since $\dot{\bar{\phi}} > 0$ there. Instead, saturation of H in the upper-left quadrant (where $\dot{\bar{\phi}} \leq 0$) is perfectly all right. But this implies having attained the sought for branch change!

Let us now look at the loop corrections. Physically, these correspond to taking into account the backreaction from particle production, i.e. from the quantum fluctuations amplified by the cosmological evolution. Let us check when their entropy starts to threaten the bound. Using the results presented in [297, 298, 106, 107], the entropy density carried by quantum fluctuations inside a region of size H^{-1} is given by

$$\sigma_q \sim N_{\rm eff} H^3, \tag{8.78}$$

where $N_{\rm eff}$ is the effective number of species that are amplified. This entropy equals the upper bound (8.76) precisely when

$$N_{\rm eff}\ H^2\ \lambda_{\rm s}^2\ e^{\phi} \sim N_{\rm eff}\ H^2\ \lambda_{\rm P}^2 \sim 1. \tag{8.79}$$

But this is also the line on which the energy density in quantum fluctuations is expected to become critical, and the backreaction of the produced radiation is expected to drive the Universe to the beginning of the standard radiation-dominated phase [275, 135].

Indeed, one can easily estimate the total energy stored in the quantum fluctuations amplified by the pre-big bang backgrounds (for a discussion of generic perturbation spectra, see [135, 119]). The result is, roughly,

$$\rho_q \sim N_{\rm eff}\ H_{\rm max}^4\ , \tag{8.80}$$

where $H_{\rm max}$ is the maximal curvature scale reached around the exit transition. The saturation of the entropy bound is thus equivalent to the condition that the above energy density becomes critical (and induces the exit) just around the transition scale $H_{\rm max}$. On the other hand, we have already argued that $H_{\rm max} \sim M_{\rm s} = \lambda_{\rm s}^{-1}$, and we know that, in heterotic string theory, $N_{\rm eff}$ is in the hundreds. It follows, according to this picture, that the exit may occur when the dilaton satisfies $\exp(\phi_{\rm exit}) \sim N_{\rm eff}^{-1} \sim 10^{-2}$, i.e. at a value of the string coupling very close to its present one. The saturation of the Hubble entropy bound S_{HB} thus supports a picture in which the exit transition occurs to avoid violations of the (generalized) second law of thermodynamics, and it is induced by the backreaction of quantum fluctuations (see also [114] and the numerical examples of [149]).

Remarkably, this also explains why the radiation, which originates from the quantum fluctuations and dominates our final Friedmann phase, has an entropy that roughly corresponds to the number of elementary "Hubble spheres" (i.e. spatial regions of size H^{-1}) contained inside our Universe just after the exit (namely, to the entropy of the quantum fluctuations evaluated at the string scale): $S \sim n_H \sim 10^{90}$. A very large number, on the one hand, but a very small entropy for the total mass and size of our present observable Universe, on the other hand, as often emphasized by Penrose [515]. The pre-big bang scenario may thus neatly explain why the Universe, at the big bang, *looks* so fine-tuned (without being so), and may provide a natural arrow of time in the direction of higher entropy [604].

In conclusion, the picture that finally emerges from all these considerations is best illustrated with reference to the diagram of Fig. 8.6. Two lines are shown, representing boundaries for the possible evolution. The horizontal boundary is forced upon by the large-curvature corrections, while the tilted line in the first quadrant corresponds to the equation $\lambda_{\rm P}^2 H^2 N_{\rm eff} = 1$ that we have just discussed. This line was also suggested as a boundary beyond which copious production of 0-branes would set in [465] (see the next subsection). Thus, depending on initial conditions, the pre-big bang bubble corresponding to our Universe would hit first either the high-curvature or the large-entropy boundary and initiate an exit phase. Hopefully, a universal late-time attractor will emerge guiding the evolution into the FRW phase of standard cosmology.

Needless to say, all these arguments have to be considered, at best, as having heuristic value. If we were to take them seriously, they would suggest that the Universe will never enter the strong-coupling, strong-curvature regime, where the largely unknown M-theory

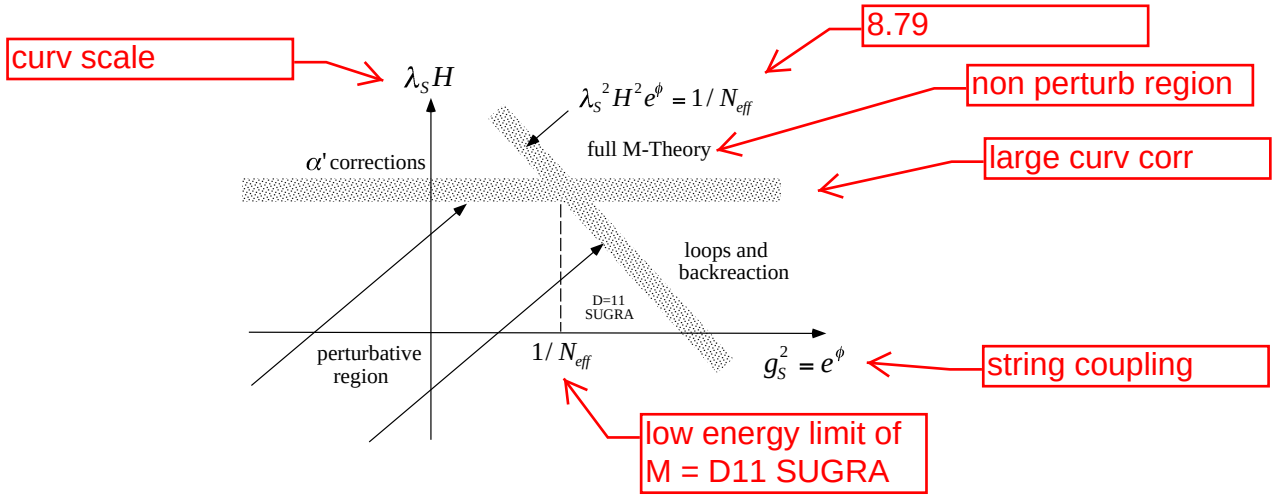

Figure 8.6: *Logarithmic plot of the curvature scale versus the string coupling g_s^2. The two parallel diagonal lines emerging from the bottom-left quadrant represent possible trajectories for the (low-energy) pre-big bang evolution, corresponding to different initial values of the dilaton.*

should be used. The low-energy limit of the latter (the much better understood $D = 11$ supergravity) could suffice to deal with the fundamental exit problem of string cosmology. We refer to the next subsection for a short discussion of this possibility.

8.5 M-theory and brane cosmology

If the growth of the string coupling, which starts during the phase of accelerated pre-big bang evolution, is not stopped at some point by a dilaton potential (or by some other mechanism), the Universe necessarily enters the strong-coupling regime described by the so-called M-theory, which is well approximated, at low-energy, by the known $D = 11$ supergravity theory [626]. In this subsection we will briefly report various ideas on the possibility of smoothing out the big bang singularity, and eventually implement the exit transition, in the context of the low-energy (i.e. small-curvature) limit of M-theory.

We start be recalling the action for the bosonic sector of eleven-dimensional supergravity,

$$S_{11} = \int d^{11}x\sqrt{-g}\left[R - \frac{1}{48}F^2_{ABCD} - \frac{1}{(12)^4\sqrt{-g}}\,\epsilon^{A_1A_2A_3B_1B_2B_3B_4C_1C_2C_3C_4}A_{A_1A_2A_3}F_{B_1B_2B_3B_4}F_{C_1C_2C_3C_4}\right], \quad (8.81)$$

where F is the field strength of the antisymmetric three-form potential A_{BCD}, and the Chern–Simons term arises as a consequence of supersymmetry [186] (upper-case Latin indices run from 0 to 10, Greek indices from 0 to 9, and we follow the conventions of [87]. The ten-dimensional effective action, obtained by compactification of S_{11} on a circle of radius

Chern Simons terms: McMahon

r_{11} (controlled by the dilaton [627]) can be written as

$$S_{11} = \int d^{10}x\sqrt{-g}\left[\, e^{-\phi}\left(R + (\nabla\phi)^2 - \frac{1}{12}H^2_{\mu\nu\alpha}\right) - \frac{1}{48}F^2_{\mu\nu\alpha\beta} - \frac{1}{384\sqrt{-g}}\,\epsilon^{\alpha_1\alpha_2\mu_1\mu_2\mu_3\mu_4\nu_1\nu_2\nu_3\nu_4}B_{\alpha_1\alpha_2}F_{\mu_1\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4}\right], \tag{8.82}$$

where H and F are the field strengths of the antisymmetric potentials $B_{\mu\nu}$ and $A_{\mu\nu\alpha}$, respectively, and we have dropped the one-form potential arising from the dimensional reduction of the metric. After the reduction, we have also rescaled the ten-dimensional metric as $g_{\mu\nu} \to r_{11}\, g_{\mu\nu}$, and we have defined the ten-dimensional dilaton as $\phi = 3\ln r_{11}$.

The above action contains not only the usual string theory action, but also Ramond forms, i.e. higher-rank antisymmetric tensors uncoupled to the dilaton (in the S-frame), and exactly reproducing the Ramond–Ramond sector of the low-energy Type II A superstring action (see for instance [437]). This provides a new, in principle richer context to look for cosmological solutions [452, 391, 446, 183] (see [87] for a complete analysis of spatially flat models), suggesting also the possibility of an M-theory version of the pre-big bang scenario [151].

In such a context, the existence of black p-brane solutions, of the type of those given in [69, 527], provides a first indication of a possible resolution of the problem of curvature singularities [426, 453] (see also [70] for examples including α' corrections). The different branches of a cosmological solution can indeed be related by U-duality transformations [449], but an explicit representation of a smooth exit transition, in this context, is problematic [110]. By U-duality it is possible, however, to transform a singular solution into a non-singular one (as in the case of $O(d,d)$-covariant backgrounds [308, 309]). This possibility has been studied, in particular, in the moduli space of $D = 11$ supergravity compactified on a ten-torus [51].

In addition, the decompactification of the 11-th dimension, produced in the strong-coupling regime as a consequence of the identification of the string coupling with the 11-th dimensional scale factor [627, 376, 377], is possibly associated with a "softening" of curvature singularities, as pointed out in [393]. The study of some explicit examples has shown [249], in particular, that the curvature singularity of FWR models minimally coupled to a set of scalar fields, in four dimensions, can be removed by lifting the solutions to higher dimensions, provided the four-dimensional background M_4 has negative or vanishing spatial curvature ($k \leq 0$). In that case, the regular higher-dimensional backgrounds from which M_4 is obtained by dimensional reduction has only one of the extra dimensions that is dynamical, and for $k = 0$ it coincides with the trivial Minkowski space-time parametrized by Milne coordinates.

A genuinely new effect, in the context of dimensionally reduced supergravity theories, is the presence of p-branes, and the possibility of their production in the strong coupling regime where the brane states become light and (possibly) unstable. This effect might

lead to regularizing the curvature singularities, as first proposed in some pioneering papers on string [427] and solitonic p-brane [529] production in cosmological backgrounds. As suggested in [537], brane production might lead for instance to a phase of constant brane density, and then brane-driven (de Sitter) inflation (reminescent of the old idea of string-driven inflation [591]), thus freezing the growth of the background curvature. We should recall, finally, that brane production in high-energy scattering processes is also allowed for conventional Einstein gravity in models with large extra dimensions [7, 8].

More generally, as suggested in [465], when approaching the singularity one should describe the Universe not in terms of the metric and of the dilaton, but in terms of new low-energy modes, more appropriate to the strong-coupling regime where D(irichelet) p-branes are the fundamental players. Indeed, in the string frame, their tension is controlled by $g_{\rm s}^{-2} = \exp(-\phi)$, so that they are very light at strong coupling: for instance, a Dp-brane of type II A has a mass

$$m \sim g_{\rm s}^{-1} (\alpha')^{-(p+1)/2}. \tag{8.83}$$

In addition, they become unstable at large enough values of the Hubble parameter (an effect very similar to string instability in curved backgrounds [551, 314]), so that they can be copiously produced when approaching the singularity, and in particular around the epoch where [465] $g_{\rm s} H \lambda_{\rm s} \sim 1$. As a consequence of this huge production, the Dp-brane energy density is expected to become critical, inducing a transition towards the standard radiation-dominated phase. If so, the curvature scale $H \sim \lambda_{\rm P}^{-1}$ would be the largest value of curvature probed during the cosmological evolution [465].

More recently, the idea of a brane-dominated Universe has been further developed in the context of the so-called "brane gas cosmology" approach [10, 229] to the M-theory phase of the very early Universe (which generalised previous string-cosmology models [109, 582]). The Universe, in the strong-coupling regime, is assumed to have nine spatial dimensions in a toroidal topological state, and to be filled with a hot gas of p-branes, with all modes in (approximate) thermal equilibrium. This approach aims not only at the resolution of the singularity problem, but also at providing a possible explanation of the number of spatial dimensions of our present Universe, along the lines of [109].

The model contains all the branes that appear in the spectrum of the (ten-dimensional) Type II A string theory. By recalling that M-theory contains the graviton (0-brane), 2-branes and 5-branes as fundamental degress of freedom (see the action (8.81)), the compactification on a circle S^1 leads to 0-branes, strings (1-branes), 2-branes, 4-branes, 5-branes, 6-branes and 8-branes (see Eq. (8.82)) as the fundamental extended objects of the 10-dimensional theory. The total action for the brane gas model is thus the sum of the low-energy effective action for the "bulk" (ten-dimensional) space-time manifold plus the $(p+1)$-dimensional action of all the branes in the gas:

$$S = S_{\rm bulk} + \sum_i S_i({\rm brane}), \tag{8.84}$$

where $S = S_{\rm bulk}$ is the gravidilaton–axion sector of the action (8.82); and the Born–Infeld action of a p-brane, coupled to the bulk via delta function sources with strength fixed by the tension T_p, is given by

$$S_p = T_p \int d^{p+1}\xi \; e^{-\phi} \left[-\det\left(g_{mn} + b_{mn} + 2\pi\alpha' F_{mn}\right)\right]^{1/2} . \tag{8.85}$$

Here g_{mn} is the induced metric on the brane, b_{mn} is the induced antisymmetric tensor, and F_{mn} is a gauge field possibly living on the brane.

All branes contribute to the energy of such a primordial gas with three types of modes [10]. There are winding modes, corresponding to p-branes wrapping around p cycles of the torus, momentum modes, corresponding to the centre-of-mass motion of the brane, and oscillatory modes, corresponding to fluctuations in the directions transverse to the brane. The brane action (8.85) can thus be expanded in terms of these modes, and by taking the average contribution of their stress tensor one obtains the effective "equation of state" for the gas of winding modes;

$$\widetilde{p}/\rho = -p/d \tag{8.86}$$

($\widetilde{p}$ is the pressure, p the rank of the brane). The momentum modes have the "dual" equation of state $\widetilde{p}/\rho = p/d$ (for strings, $p = 1$, one recovers the results of [314, 315]). Transverse oscillations can be viewed as particles living on the brane, and their equation of state is that of "ordinary" matter, with $0 \leq \widetilde{p} \leq 1$. It follows, from the covariant conservation of the stress tensor, that the energy E_p of winding (w) modes increases with the scale factor $a(t)$ as

$$E_p = \rho a^d \sim T_p a^p \tag{8.87}$$

(note, however, that should be included at very high temperature thermal corrections , and that this can modify the effective equation of state, as discussed in [341] for the case of strings near the Hagedorn scale).

The Universe, in the strong-coupling (M-theory) regime, is thus assumed to behave as a hot soup of all modes of all p-branes. The curvature singularity is expected to be avoided because of T-duality exchanging w-modes and momentum modes, and then transforming a phase of increasing temperature into decreasing temperature, with the fixed point of the symmetry acting as a regularizing cut-off for all physical observables [10, 229]. But the most interesting (and probably unique, at present) virtue of this scenario seems to be the possibility of explaining why our present Universe contains just three large spatial dimensions.

Indeed, as the primordial Universe expands, the w-modes tend to become dominant as their energy increases, starting with the largest value of p, according to Eq. (8.87). The effect of the w-modes, on the other hand, is to halt the expansion [582], so that the spatial dimensions can decompactify (i.e. inflate away from the string scale) only if the w-modes disappear.

domain wall problem

They can disappear by annihilation with antiwinding ($\overline{\mathrm{w}}$) modes (the initial number of w and $\overline{\mathrm{w}}$ is assumed to be the same, by symmetry). On the other hand, assuming that the space is periodic, it follows that the world-volume of two p-branes will necessarily intersect in at most $2p+1$ spatial dimensions. In the initial Universe $d = 9$, so that winding p-branes with $p = 8, 6, 5, 4$ have no problem to self-annihilate. We are thus left with 2-branes and strings. The annihilation of the (heavier) winding 2-branes will first allow five dimensions to become large. Within this distinguished torus $\mathcal{T}^5$, the 1-branes (strings) w-modes will then only allow the $d = 3$ subspace $\mathcal{T}^3$ to expand.

In this way, one can understand the origin of the $d = 3$ dimensionality of our Universe and, interestingly enough, can predict a hierarchical structure of the ten-dimensional compact manifold as follows [10]:

$$\mathcal{M} = S^1 \times \left(\mathcal{T}^4 \times \mathcal{T}^2 \times \mathcal{T}^3\right), \tag{8.88}$$

where S^1 comes from the original M-theory compactification on a circle, and the other compact manifolds have been listed in order of growing size, from left to right.

In this context, a possible problem is that, at the end of the decompactification process, at least one w-mode is left per Hubble volume, leading to the well known domain-wall problem. This problem can be solved if the w-modes, before they annihilate, drive the background to a phase of slight contraction, also called "loitering" [549], during which the Hubble horizon becomes larger than the spatial size of the Universe (as in the phase of pre-big bang inflation, when it is seen as a contraction in the E-frame).

By supplementing the string-cosmology equations by terms describing the annihilation of w-modes into string loops [103], it has been shown that during the loitering phase all w-modes are eliminated and that, from this point on, the Universe begins to expand again. The w-modes annihilation leads to breaking T-duality, and it is tempting to speculate that this effect could be related to SUSY breaking, and to the mechanism of dilaton mass generation needed to stabilize the coupling after decompactification [103].

Outside the context of brane gas cosmology, but always in the context of an M-theory approach to the strong coupling regime, a more drastic resolution of the dimensionality problem is represented by the so-called brane-world scenario (inspired by heterotic M-theory [376, 377]), in which our Universe *is assumed* to coincide with a 3-brane embedded in a higher-dimensional bulk manifold [450, 451].

In a realistic picture, of course, our 3-brane is not expected to be the only extended object embedded in the bulk manifold, especially in the regime of large bulk dilaton, and very strong coupling. In that case, the transition to the post-big bang regime could be triggered by a head-on collision against another brane, just as happens in the context of the so-called "ekpyrotic" scenario [412, 413], where the big bang explosion, and the birth of a hot, radiation-dominated Universe, is simulated by the collision of two 3-branes along a (hidden) fifth spatial dimension (see also [325] for a similar but spherically symmetric

Brane-world scenario

process of bubble collision).

In the ekpyrotic scenario the 11-th space-time is assumed to have topology $M^{10} \times S^1/Z_2$, with two branes at the orbifolds fixed points, where space-time has boundaries. Six dimensions are compactified on a Calabi–Yau threefold, leaving the effective theory five-dimensional. The boundary branes are called the visible and the hidden brane, the visible one being identified with our Universe.

The action for this model is the sum of three parts:

$$S = S_{\rm het} + S_I + S_M, \tag{8.89}$$

where S_I represents the brane interactions, S_M describes the matter on the brane created by the collision, and $S_{\rm het}$ is the action of five-dimensional heterotic M-theory. With the conventions of [412]:

$$\begin{aligned} S_{\rm het} &= \frac{M_5^3}{2}\int d^5x\sqrt{-g}\left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{5!}e^{2\phi}F^2_{ABCDE}\right] \\ &- \sum_i 3\alpha_i M_5^3 \int d^4\xi_i \sqrt{-h_i}\left[\, e^{-\phi} - \frac{1}{4!\sqrt{-h_i}}\,\epsilon^{\mu\nu\alpha\beta}A_{ABCD}\partial_\mu X_i^A \partial_\nu X_i^B \partial_\alpha X_i^C \partial_\beta X_i^D\right]. \end{aligned} \tag{8.90}$$

Here M_5 is the five-dimensional Planck mass, F is the field strength of the four-form A_{ABCD}, $h^i_{\mu\nu}$ is the induced metric on the i-th 3-brane, whose embedding in five dimensions is described by the five parametric equations $X_i^A = X_i^A(\xi_i^\mu)$. Finally, the tensions are given by $T_i = \alpha_i M_5^3$.

In the first version of the ekpyrotic scenario [412], besides the two boundary branes, there is a third "bulk" brane, possibly originated from spontaneous bubble nucleation. In that case, the three tensions satisfy the condition $\alpha_2 = -\alpha_1 - \alpha_3$, with $\alpha_3 > 0$ and $\alpha_3 < |\alpha_1|$, and the bulk brane is attracted towards the visible brane, until they collide. The kinetic energy of the bulk brane is then converted into matter and radiation of the visible brane, with a spectrum of density perturbations left impressed by the quantum fluctuations of the bulk brane.

In the second version of the scenario [413], there are instead only the two boundary branes, which collide and then bounce apart, in what is hoped to be a non-singular process. This second possibility also represents an important ingredient in a related, recently proposed "cyclic" model of Universe [573].

In our context, this second possibility is particularly interesting because the collision of the two boundary branes is associated to the collapse of the dimension transverse to the brane: namely, of the 11-th dimension, taking into account the Calabi-Yau compactification. But the shrinking of the eleven-dimensional radius r_{11}, in an M-theory context, is equivalent to a decreasing of the string coupling (recall that $r_{11} = \exp(\phi/3)$). This means that the

collision, and the subsequent beginning of the radiation-dominated, post-big bang evolution, occurs at very weak coupling, in the perturbative regime.

In conclusion, this model suggests that the space-time manifold, after having reached the strong-coupling, M-theory regime as a consequence of the pre-big bang evolution, and after the associated production of branes, might be forced again to the perturbative regime by the attraction (and by the subsequent collapse) of the branes at the boundary of the $D = 11$ space-time. The exit would thus proceed at weak coupling (possibly with a mechanism of radiation production different from the one discussed in the previous subsections). This may perhaps justify the use of the tree-level effective action for the quantum cosmology approach to the exit, presented in the next section.

9 Quantum string cosmology

According to the pre-big bang scenario, the present cosmological state of our Universe should emerge as the result of a long evolution, starting from the string perturbative vacuum. Such an evolution necessarily includes the transition from a phase of growing curvature and strong coupling to a phase of decreasing curvature and (nearly) constant string coupling. The full dynamical description of such a process still contains various problematic aspects, as discussed in the previous section.

Since the transition is expected to occur in the high-curvature (nearly Planckian) regime, where quantum-gravity effects may become important, it is not excluded that a quantum-cosmology approach may be appropriate to describe the "decay" of the string perturbative vacuum into our present, post-big bang Universe. Also in the context of the standard inflationary scenario, in fact, the quantum-cosmology approach is required to describe the "birth" of our classical Universe out of the Planckian regime [613]. With an important difference, however: in the standard scenario the initial state of the Universe is unknown, and has to be fixed through some *ad hoc* prescription. There are various possible choices for the initial boundary conditions [366, 370, 609, 440, 633, 541, 611], leading in general to different quantum pictures of the early cosmological evolution. In the pre-big bang scenario, on the contrary, the initial state is fixed in such a way as to approach, asymptotically, the string perturbative vacuum, and this unambiguously determines the initial wave function.

In a quantum cosmology context the Universe is described by a wave function evolving in superspace, according to the Wheeler–De Witt (WDW) equation [216, 622], and it is always possible, in principle, to compute the transition probability between two different geometrical configurations –in particular, from a pre- to a post-big bang state. We shall report here the results obtained in the context of what may be called a "low-energy" approach to quantum-string cosmology [282], which is based on the lowest-order string effective action, and in which no higher-order (α' and loop) corrections are taken into account into the WDW equation, except those possibly encoded into an effective, non-perturbative dilaton

potential (see [525, 526, 552] for high-curvature contributions to the WDW equation).

Such a low-energy approach, first introduced in string cosmology in [143, 79, 434], is analogous (in the particle case) to low-energy quantum mechanics, where one neglects relativistic and higher-order corrections. It is a first approximation, which is already sufficient to take into account pure quantum-gravity effects, such as the possibility of transitions which are classically forbidden –in particular, transitions between two low-energy geometric configurations which are classically disconnected by a singularity [310, 322, 154]– and thus to motivate a "minisuperspace approach" to the exit problem [279] (see also [474, 511] for a recent quantum-cosmology approach to this problem based on the Bohm–de Broglie ontological interpretation of quantum mechanics).

Quite irrespectively of its applications, quantum-string cosmology may represent a rich and interesting field of research in itself; it thus seems appropriate to recall that it is affected by various conceptual problems already present in the "standard" quantum-cosmology context: the meaning of the probabilistic interpretation [610], the existence and the meaning of a semiclassical limit [79, 434], the unambiguous identification of the time-like coordinate (see however [152]). Other problems affecting the standard scenario, however, disappear. In particular, in the context of the pre-big bang scenario, there is no problem of boundary conditions, which are unambiguously prescribed by the choice of the string perturbative vacuum (see Subsection 9.2); no problem either of operator ordering in the WDW equation (see for instance [36]), as the ordering is fixed by the duality symmetry of the string effective action [310, 410]. This important property of quantum-string cosmology will be illustrated in the next subsection.

9.1 The Wheeler–De Witt equation

The simplest example of quantum-string cosmology model is based on the lowest-order, gravidilaton string effective action

$$S = -\frac{1}{2\,\lambda_{\rm s}^{d-1}} \int d^{d+1}x\,\sqrt{|g|}\,e^{-\phi}\,\left[R + (\nabla_\mu\phi)^2 + V(\phi, g_{\mu\nu})\right], \tag{9.1}$$

where we have included a (possibly non-local and non-perturbative) dilaton potential V (see [153] for more complete models based on the M-theory action). If we are considering, in particular, an isotropic and spatially flat cosmological background,

$$\phi = \phi(t), \qquad\qquad g_{\mu\nu} = {\rm diag}\left(N^2(t), -a^2(t)\delta_{ij}\right), \tag{9.2}$$

the gravidilaton system has only two physical degrees of freedom, the scale factor a and the dilaton (the "lapse" function $N = \sqrt{g_{00}}$ can be arbitrarily fixed by a choice of gauge). The quantum evolution of the system is thus associated to a two-dimensional "minisuperspace", which, assuming spatial sections of finite volume, can be conveniently parametrized by

$$\beta = \sqrt{d}\ln a, \qquad\qquad \overline{\phi} = \phi - \sqrt{d}\,\beta - \ln\int d^dx/\lambda_{\rm s}^d. \tag{9.3}$$

Each "point" $\{\beta(t), \overline{\phi}(t)\}$ of the minisuperspace will then represent a classical solution of the action (9.1).

In terms of the coordinates of the minisuperspace, the action (after integration by parts) can be explicitly written as

$$S = \frac{\lambda_s}{2} \int dt \, \frac{e^{-\overline{\phi}}}{N} \left[\dot{\beta}^2 - \dot{\overline{\phi}}^2 - N^2 \, V(\beta, \overline{\phi}) \right]. \tag{9.4}$$

Its variation with respect to g_{00} defines the total energy density of the gravidilaton system, and leads to the so-called Hamiltonian constraint (in the cosmic time gauge, $N = 1$):

$$\mathcal{H} = \left(\frac{\delta S}{\delta N} \right)_{N=1} = \dot{\overline{\phi}}^2 - \dot{\beta}^2 - V = 0. \tag{9.5}$$

Introducing the canonical momenta

$$\Pi_\beta = \left(\frac{\delta S}{\delta \dot{\beta}} \right)_{N=1} = \lambda_s \, \dot{\beta} \, e^{-\overline{\phi}}, \qquad \Pi_{\overline{\phi}} = \left(\frac{\delta S}{\delta \dot{\overline{\phi}}} \right)_{N=1} = -\lambda_s \, \dot{\overline{\phi}} \, e^{-\overline{\phi}}, \tag{9.6}$$

the Hamiltonian constraint becomes

$$\Pi_\beta^2 - \Pi_{\overline{\phi}}^2 + \lambda_s^2 \, V(\beta, \overline{\phi}) \, e^{-2\,\overline{\phi}} = 0 \ , \tag{9.7}$$

and its differential implementation in the minisuperspace spanned by β and $\overline{\phi}$ finally leads to the WDW equation for the gravidilaton system:

WDW

$$\left[\partial_{\overline{\phi}}^2 - \partial_\beta^2 + \lambda_s^2 \, V(\beta, \overline{\phi}) \, e^{-2\overline{\phi}} \right] \Psi(\beta, \overline{\phi}) = 0. \tag{9.8}$$

It is important to stress that such an equation is manifestly free from operator-ordering problems, as the Hamiltonian (9.7) has a flat metric in momentum space. This is not a special feature of the case we have considered because, thanks to the duality symmetry of the string effective action, the associated WDW minisuperspace is globally flat, and we can always choose a convenient parametrization leading to a flat minisuperspace metric.

For a more general discussion of this point we may add to the effective action (9.1) a non-trivial antisymmetric tensor background, $B_{\mu\nu} \neq 0$. The kinetic part of the action may then be written in compact form as [310, 410] (see Subsection 2.3):

$$S = -\frac{\lambda_s}{2} \int dt e^{-\overline{\phi}} \left[(\dot{\overline{\phi}})^2 + \frac{1}{8} \mathrm{Tr} \, \dot{M} (M^{-1})^{\cdot} \right], \tag{9.9}$$

where $N = 1$, and M is a symmetric $2d \times 2d$ matrix, including the spatial part of the background fields, $G \equiv g_{ij}$, $B \equiv B_{ij}$, and already defined in Eq. (2.55). We recall that this action is invariant under global $O(d, d)$ transformations that leave the shifted dilaton invariant,

$$\overline{\phi} \to \overline{\phi}, \qquad M \to \Omega^T M \Omega, \tag{9.10}$$

where

$$\Omega^T \eta \Omega = \eta, \qquad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{9.11}$$

The Hamiltonian associated to torsion-graviton background,

$$\mathcal{H} = \frac{4}{\lambda_{\rm s}} \mathrm{Tr}\ (M\ \Pi_M M\ \Pi_M), \qquad \Pi_M = \delta S/\delta \dot{M}, \tag{9.12}$$

would seem to have ordering problems, because $[M, \Pi_M] \neq 0$. However, thanks to the $O(d,d)$ properties of M,

$$M \eta M = \eta, \tag{9.13}$$

we can always rewrite the kinetic part of the action in terms of the flat $O(d,d)$ metric η:

$$\mathrm{Tr}\ \dot{M}(M^{-1})\dot{} = \mathrm{Tr}\ (\dot{M}\eta)^2. \tag{9.14}$$

The corresponding Hamiltonian

$$\mathcal{H} = -\frac{4}{\lambda_{\rm s}} \mathrm{Tr}\ (\eta \Pi_M \eta \Pi_M) \tag{9.15}$$

has a flat metric in momentum space, with no ordering problems for the corresponding WDW equation [310, 410]:

$$\left[\frac{\delta^2}{\delta \overline{\phi}^2} + 8\mathrm{Tr}\ \left(\eta \frac{\delta}{\delta M} \eta \frac{\delta}{\delta M} \right) + \lambda_{\rm s}^2 V e^{-2\overline{\phi}} \right] \Psi(M, \overline{\phi}) = 0. \tag{9.16}$$

On the other hand, if we insist on adopting a curvilinear parametrization of the minisuperspace, the ordering fixed by the duality symmetry is exactly the same as the ordering imposed by the requirement of reparametrization invariance, as there are no contributions to the ordered Hamiltonian from the scalar curvature of minisuperspace [36], because minisuperspace is globally flat.

To illustrate this point we may consider the pair of minisuperspace coordinates $\{a, \overline{\phi}\}$, different from the previous pair $\{\beta, \overline{\phi}\}$ used in Eq. (9.4). The kinetic part of the action (9.1) then leads to the (kinetic part of the) Hamiltonian

$$\mathcal{H} = \frac{a^2}{d} \Pi_a^2 - \Pi_{\overline{\phi}}^2 \equiv \gamma^{AB} \Pi_A \Pi_B, \tag{9.17}$$

where

$$\Pi_a = \left(\frac{\delta S}{\delta a} \right)_{N=1} = d\lambda_{\rm s} \frac{\dot{a}}{a^2} e^{-\overline{\phi}}, \tag{9.18}$$

corresponding to the non-trivial 2×2 metric:

$$\gamma_{AB} = \mathrm{diag} \left(\frac{d}{a^2}, -1 \right). \tag{9.19}$$

The quantum operator associated to the Hamiltonian (9.17) has to be ordered, because $[a, \Pi_a] \neq 0$, and its differential representation can be written in general as

$$\mathcal{H} = \frac{\partial^2}{\partial \overline{\phi}^2} - \frac{1}{d}\left(a^2 \frac{\partial^2}{\partial a^2} + \epsilon a \frac{\partial}{\partial a}\right), \tag{9.20}$$

where ϵ is a c-number parameter depending on the ordering. Note that there are no contributions to the ordered Hamiltonian from the minisuperspace scalar curvature [36], which is vanishing for the metric (9.19).

Reparametrization invariance now imposes on the Hamiltonian the covariant d'Alembert form

$$\mathcal{H} = -\nabla_A \nabla^A = -\frac{1}{\sqrt{-\gamma}} \partial_A (\sqrt{-\gamma} \gamma^{AB} \partial_B), \tag{9.21}$$

and consequently fixes $\epsilon = 1$. The kinetic part of the action (9.4), on the other hand, is invariant under the T-duality transformation [599, 580]

$$a \to \widetilde{a} = a^{-1}, \qquad \overline{\phi} \to \overline{\phi}, \tag{9.22}$$

which implies, for the Hamiltonian (9.20),

$$\mathcal{H}(a) = \mathcal{H}(\widetilde{a}) + \frac{2}{d}(\epsilon - 1)\widetilde{a} \frac{\partial}{\partial \widetilde{a}}. \tag{9.23}$$

The invariance of the Hamiltonian requires $\epsilon = 1$, and thus fixes the same quantum ordering as the condition of general covariance in minisuperspace. A similar relation between quantum ordering and duality symmetry can be easily established for more general effective actions, including a larger class of non-minimal gravidilaton couplings [434, 435].

9.2 Wave scattering in minisuperspace

In the absence of dilaton potential, the WDW equation (9.8) reduces to the free d'Alembert equation and provides a plane-wave representation in minisuperspace of the different asymptotic branches of the classical solutions.

Let us recall, in fact, that the solutions of the low-energy effective action (9.4) always contain four branches. For $V = 0$, in particular, we have four (physically different) solutions,

$$a(t) = (\mp t)^{\mp 1/\sqrt{d}}, \qquad \overline{\phi}(t) = -\ln(\mp t), \tag{9.24}$$

related by the duality transformation (9.22) and by time reversal, $t \to -t$. They satisfy the condition

$$\sqrt{d} H \equiv \dot{\beta} = \pm(\mp t)^{-1} = \pm \dot{\overline{\phi}}, \tag{9.25}$$

and may thus be represented as the bisecting lines of the plane $\{\dot{\overline{\phi}}, \dot{\beta}\}$, corresponding to

- expansion $\to \dot{\beta} > 0$,

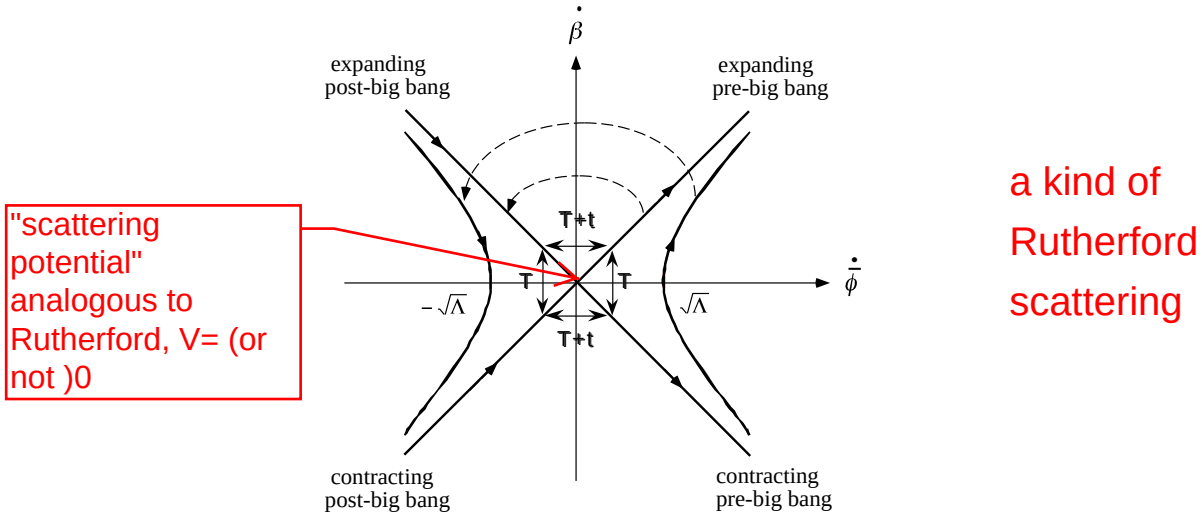

Figure 9.1: *The four branches of the lowest-order string-cosmology solutions, with (hyperbolas) and without (bisecting lines) a positive cosmological constant Λ. The dashed curves represent the quantum transition from a pre- to a post-big bang configuration.*

- contraction $\rightarrow \dot\beta < 0$,
- pre-big bang (growing dilaton) $\rightarrow \dot{\overline\phi} > 0$,
- post-big bang (decreasing dilaton) $\rightarrow \dot{\overline\phi} < 0$,

as illustrated in Fig. 9.1. We may note, for later use, that if we add to the action a positive cosmological constant ($V = \Lambda$), then the classical solutions are represented in the plane $\{\dot{\overline\phi}, \dot\beta\}$ by the hyperbolas

$$\dot{\overline\phi}^{\,2} - (\sqrt{d}H)^2 = \Lambda. \tag{9.26}$$

In that case, the initial configuration is shifted to a state with flat metric and linearly evolving dilaton, but the solution is still characterized by four branches coinciding, asymptotically, with the free configurations of Eq. (9.24).

The $V = 0$ solutions of the WDW equations, on the other hand, can be factorized in the form of plane waves representing free energy and momentum eigenstates:

$$\Psi(\beta, \overline\phi) \sim \psi_\beta^{(\pm)} \psi_{\overline\phi}^{(\pm)} \sim e^{\pm ik\beta \pm ik\overline\phi}, \tag{9.27}$$

where ($k > 0$):

$$\Pi_\beta\, \psi_\beta^{(\pm)} = \pm k\, \psi_\beta^{(\pm)}, \qquad \Pi_{\overline\phi}\, \psi_{\overline\phi}^{(\pm)} = \pm k\, \psi_{\overline\phi}^{(\pm)}. \tag{9.28}$$

By recalling that $\Pi_\beta \sim \dot\beta$, $\Pi_{\overline\phi} \sim -\dot{\overline\phi}$ (see Eq. (9.6)), the above plane waves represent the four branches of Eq. (9.24), defined by $\Pi_\beta = \pm\Pi_{\overline\phi}$, with the following correspondence:

- expansion $\rightarrow \psi_\beta^{(+)}$,
- contraction $\rightarrow \psi_\beta^{(-)}$,
- pre-big bang (growing dilaton) $\rightarrow \psi_{\overline{\phi}}^{(-)}$,
- post-big bang (decreasing dilaton) $\rightarrow \psi_{\overline{\phi}}^{(+)}$.

For an isotropic, low-energy solution [599], the dilaton is growing ($\dot{\phi} > 0$) only if the metric is expanding ($\dot{\beta} > 0$), see Eq. (9.3). If we impose, as our physical boundary condition, that the Universe emerge from the string perturbative vacuum (corresponding, asymptotically, to $\beta \rightarrow -\infty$, $\phi \rightarrow -\infty$), then the initial state $\Psi_{\rm in}$ must represent a configuration with positive eigenvalue of Π_β and opposite eigenvalue of $\Pi_{\overline{\phi}}$, i.e. $\Psi_{\rm in} \sim \psi_\beta^{(+)} \psi_{\overline{\phi}}^{(-)}$ (see also [189] for a detailed discussion of boundary conditions in quantum-string cosmology).

A quantum transition from pre- to post-big bang thus becomes, in this representation, a transition from an initial state

$$\Psi_{\rm in} \sim e^{ik\overline{\phi} - ik\beta}, \qquad \Pi_\beta > 0, \qquad \Pi_{\overline{\phi}} < 0, \tag{9.29}$$

to a final state

$$\Psi_{\rm out} \sim e^{-ik\overline{\phi} - ik\beta}, \qquad \Pi_\beta > 0, \qquad \Pi_{\overline{\phi}} > 0 \tag{9.30}$$

scattering

(see the dashed curves of Fig. 9.1). The associated trajectory describes a monotonic evolution along β, and a reflection along $\overline{\phi}$. The above discussion suggests that we look at the quantum evolution (in minisuperspace) of the initial pre-big bang state as at the scattering, induced by an effective WDW potential, of an incoming wave travelling from $-\infty$ along the positive direction of the axes β and $\overline{\phi}$.

The effective potential, on the other hand, is known to be strongly suppressed as we approach the string perturbative vacuum, $\beta, \overline{\phi} \rightarrow -\infty$. We shall assume that a possible growth of V in the strong-coupling regime is not strong enough to prevent the effective WDW potential from going to zero also at large positive values of β and $\overline{\phi}$, so that $V \exp(-2\overline{\phi}) \rightarrow 0$ for $\beta, \overline{\phi} \rightarrow \pm\infty$. It follows that also the final asymptotic configuration $\Psi_{\rm out}$ can be represented by the free eigenstates $\psi_\beta^{(\pm)}, \psi_{\overline{\phi}}^{(\pm)}$. However, even if the initial state is fixed by the boundary conditions of the pre-big bang scenario, the final state is not, and there are in general four different types of evolution [282], depending on whether the asymptotic outgoing state $\Psi_{\rm out}$ is a superposition of waves with the same Π_β and opposite $\Pi_{\overline{\phi}}$, or with the same $\Pi_{\overline{\phi}}$ and opposite Π_β, and also depending on the identification of the time-like coordinate in minisuperspace [152, 154].

These four possibilities are illustrated in Fig. 9.2, where cases (a) and (b) correspond to $\Psi_{\rm out}^{\pm} \sim \psi_\beta^{(+)} \psi_{\overline{\phi}}^{(\pm)}$, while cases ($c$) and ($d$) correspond to $\Psi_{\rm out}^{\pm} \sim \psi_{\overline{\phi}}^{(-)} \psi_\beta^{(\pm)}$. Also, in cases ($a$) and ($d$) the time-like coordinate is identified with β, in cases (b) and (c) with $\overline{\phi}$. It

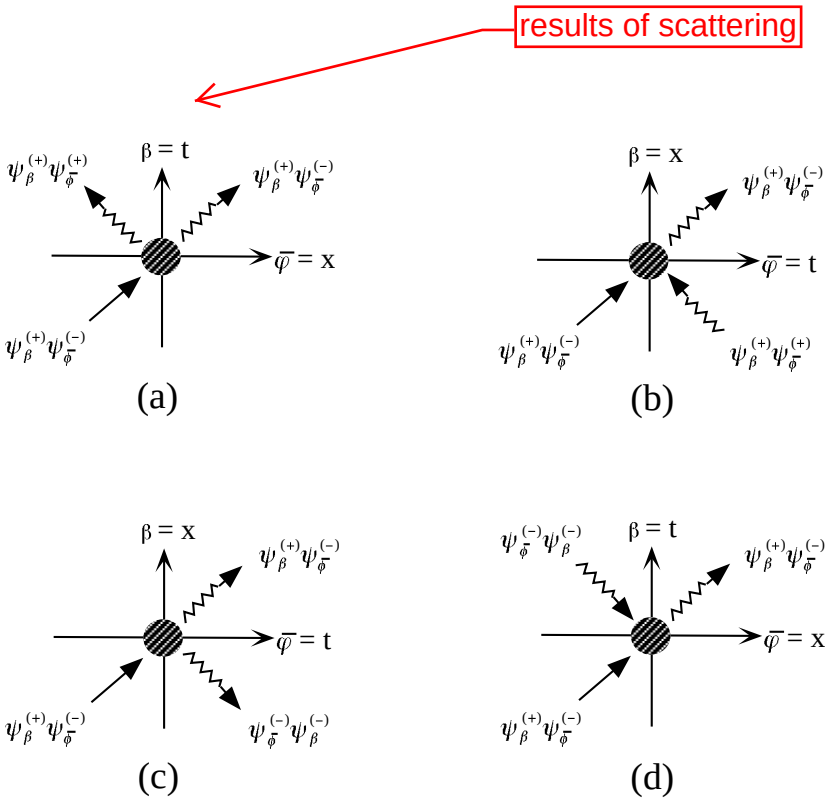

Figure 9.2: *Four different classes of scattering processes for the incoming wave function representing the string perturbative vacuum (straight, solid line). The outgoing state is represented by a mixture of positive and negative eigenfunctions of* Π_β *and* $\Pi_{\bar{\phi}}$.

must be noted that cases (c) and (d) require a duality-breaking dilaton potential, otherwise $[\mathcal{H}, \Pi_\beta] = 0$, and then a reflection along the β axis is impossible, even at the quantum level [310].

The two cases (a) and (c) represent scattering and reflection along the spacelike axes $\bar{\phi}$ and β, respectively. In case (a) the evolution along β is monotonic, so that the Universe always keeps expanding. The incident wave is partially transmitted towards the pre-big bang singularity (unbounded growth of the curvature and of the dilaton, $\beta \to +\infty$, $\bar{\phi} \to +\infty$), and partially reflected back towards the low-energy, expanding, post-big bang regime ($\beta \to +\infty$, $\bar{\phi} \to -\infty$) [310]. In case (c) the evolution is monotonic along the time axis $\bar{\phi}$, but not along β. The incident wave is therefore totally transmitted towards the singularity ($\bar{\phi} \to +\infty$), but in part as an expanding configuration and in part as a contracting one[3].

The other two cases, (b) and (d), are qualitatively different, as the final state is a superposition of positive and negative energy eigenstates, i.e. of modes of positive and negative frequency with respect to the time axes chosen in minisuperspace. In the language of third quantization [542, 421, 357, 358, 359] (i.e. second quantization of the WDW wave function in superspace) they represent a "Bogoliubov mixing", describing the production

[3]R. Ricci, M. Gasperini and G. Veneziano (1996), unpublished.

Bogo mixing

Feynman trick

of pairs of universes from the vacuum. The mode moving backwards in time has to be "re-interpreted", as in quantum field theory, as an "antiuniverse" of positive energy and opposite momentum (in superspace). Since the inversion of momentum, in superspace, corresponds to a reflection of $\dot{\beta}$, the re-interpretation principle in this context changes expansion into contraction, and vice versa.

Case (b), in particular, describes the production of universe–antiuniverse pairs –one expanding, the other contracting– from the string perturbative vacuum [130]. The pairs evolve towards the strong-coupling regime $\overline{\phi} \to +\infty$, so that both members of the pair fall inside the pre-big bang singularity. Case (d) is more interesting, in our context, since there the universe and the antiuniverse of the pair are both expanding: one falls inside the pre-big bang singularity, the other expands towards the low-energy, post-big bang regime, and may expand to infinity, representing the birth of a Universe like ours in a standard Friedmann-like configuration [290].

Possible explicit examples of a quantum representation in minisuperspace of the transition from pre- to post-big bang, based on the above scattering processes, will be given in the next two subsections.

9.3 Birth of the Universe as "quantum reflection"

The simplest way to represent the transition from the string perturbative vacuum to the present, post-big bang regime, is to identify the time-like coordinate in the minisuperspace with β, and to represent the process as a reflection of the incoming wave function along the space-like coordinate $\overline{\phi}$, according to Eqs. (9.29), (9.30) (case (a) of Fig. 9.2). The transition probability is thus controlled by the reflection coefficient

$$R_k = \frac{|\Psi^{(-)}_{-\infty}(\beta, \overline{\phi})|^2}{|\Psi^{(+)}_{-\infty}(\beta, \overline{\phi})|^2}, \tag{9.31}$$

where $\Psi^{(\pm)}_{-\infty}$ are the asymptotic components of the WDW wave function at $\overline{\phi} \to -\infty$, containing the left-moving $(-)$ and right-moving $(+)$ part of the wave along $\overline{\phi}$ (see [152] for a rigorous definition of scalar products in the appropriate Hilbert space, and [154, 155, 339] for a path-integral approach to the computation of the transition amplitudes).

Without potential in the WDW equation there is no transition, of course (in the absence of a dilaton potential, the possible decoherence of the quantum-cosmological system, and its implications for the exit problem, have been discussed in [454]). With an appropriate potential, allowing a smooth classical evolution from pre- to post-big bang, the transition probability R tends to unity; R may be non-zero, however, even if the two branches of the classical solution are causally disconnected by a singularity.

An instructive example, to this purpose, is provided by the duality-symmetric (non-

local) four-loop dilaton potential

$$V_\pm(\beta\overline{\phi}) = \pm V_0 e^{4\overline{\phi}}, \qquad V_0 = \text{const}, \qquad V_0 > 0, \tag{9.32}$$

already introduced in Subsection 8.1 (for generalized exponential potential, with the inclusion of the antisymmetric tensor contribution to the WDW equation, see [467]). The corresponding action (9.4), in $d = 3$, has the following classical solution [310]

$$\overline{\phi} = -\frac{1}{2}\ln\left(\frac{k^2t^2}{\lambda_s^2} \mp \frac{\lambda_s^2 V_0}{k^2}\right), \qquad a = a_0\left|\frac{k^2 t}{\lambda_s^2\sqrt{V_0}} + \left(\frac{k^4t^2}{\lambda_s^4 V_0} \mp 1\right)^{1/2}\right|^{1/\sqrt{3}}, \tag{9.33}$$

where a_0 is an integration constant, and

$$k = \lambda_s \dot{\beta} e^{-\overline{\phi}} = \frac{\sqrt{3}}{\lambda_s^2}\int d^3x\sqrt{-g}e^{-\phi}\dot{\beta} = \text{const} \tag{9.34}$$

is the conserved momentum along the β axis.

For $V < 0$, Eq. (9.33) describes a regular "self-dual" solution –satisfying $a(t)/a_0 = a_0/a(-t)$– characterized by a "bell-like" shape of the curvature scale and of the coupling $e^{\overline{\phi}}$ (see Subsection 8.1). The solutions of the WDW equation (9.8), in the small coupling regime $\overline{\phi} \to -\infty$, can be written as [310]

$$\begin{aligned}\lim_{\overline{\phi}\to-\infty}\Psi_k(\overline{\phi},\beta) &= -\frac{N\pi}{2\sin(ik\pi)}\left[\left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{ik}\frac{e^{-ik(\beta-\overline{\phi})}}{\Gamma(1+ik)} - \left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{-ik}\frac{e^{-ik(\beta+\overline{\phi})}}{\Gamma(1-ik)}\right] \\ &= A_k\psi_\beta^{(+)}\psi_{\overline{\phi}}^{(-)} + B_k\psi_\beta^{(+)}\psi_{\overline{\phi}}^{(+)}\end{aligned} \tag{9.35}$$

(N is a normalization coefficient), and contain the superposition of the initial string perturbative vacuum and of the reflected component, representing the post-big bang regime. Clearly, $R = 1$ for all k, as the two branches of the solution are smoothly connected already at the classical level.

For $V > 0$, on the contrary, the pre- and post-big bang branches are disconnected by an unphysical region, of extension $|t| < \lambda_s^2\sqrt{V_0}/k^2$, where the expansion rate $\dot{\beta}$ and the dilaton coupling $e^{\overline{\phi}}$ become imaginary. Such a region is bounded, on both sides, by a curvature singularity. However, after fixing the boundary conditions with the string perturbative vacuum and in the small coupling regime, the solutions of the WDW equation can be written as [310]:

$$\begin{aligned}\lim_{\overline{\phi}\to-\infty}\Psi_k(\overline{\phi},\beta) &= iN\csc(ik\pi)\left[e^{k\pi}\left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{ik}\frac{e^{-ik(\beta-\overline{\phi})}}{\Gamma(1+ik)} - \left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{-ik}\frac{e^{-ik(\beta+\overline{\phi})}}{\Gamma(1-ik)}\right] \\ &= A_k\psi_\beta^{(+)}\psi_{\overline{\phi}}^{(-)} + B_k\psi_\beta^{(+)}\psi_{\overline{\phi}}^{(+)}.\end{aligned} \tag{9.36}$$

The reflection coefficient

$$R_k = \frac{|\Psi_{-\infty}^{(-)}|^2}{|\Psi_{-\infty}^{(+)}|^2} = e^{-2\pi k} \tag{9.37}$$

quantum effects at the END of PreBB phase

is exponentially suppressed but non-zero, corresponding to a non-vanishing transition probability from the pre- to the post-big bang branches of the above solution.

Exactly the same result (9.37) is obtained for the simplest case of dilaton potential, a positive cosmological constant $V(\beta, \overline{\phi}) = \Lambda$. In that case, by expressing the conserved momentum k in terms of the coupling $g_s = e^{\phi_s/2}$ at the string scale $H_s = \lambda_s^{-1}$, the transition probability between the left and right branches of the hyperbolas of Fig. 9.1, for a three-dimensional portion of space of fixed proper volume Ω_i at $t \to -\infty$, can be written in the form [322]:

$$R(\Lambda, \Omega_i, g_s) = \exp\left\{ -\frac{\sqrt{12}\pi}{g_s^2} \frac{\Omega_i}{\lambda_s^3} \left[\frac{\sqrt{3}}{\lambda_s \sqrt{\Lambda}} + \left(1 + \frac{3}{\lambda_s^2 \Lambda} \right)^{1/2} \right]^{\sqrt{3}} \right\}, \qquad (9.38)$$

Unfortunately, such a transition probability is exponentially suppressed, unless the proper size of the transition volume is small, and the cosmological constant is large, in string units. Note, however, that the probability is peaked in the string coupling regime, with a typical "instantonic" behaviour $R \sim \exp(-1/g_s^2)$: this means that the Universe tends to emerge from the transition in the coupling regime appropriate to the present post-big bang phase.

instanton ??????? ???????

It is important to stress that the quantum-transition probabilities (9.37), (9.38) have a conceptual meaning different from similar results obtained in the context of the standard inflationary scenario. In string cosmology, in fact, the quantum (Planckian) regime is reached *at the end* of the pre-big bang phase, when the Universe is expected *to exit* from (not to enter in) the inflationary regime. Thus, quantum effects are not responsible for inflationary initial conditions [286], whereas they are in the standard inflationary scenario [394]. In spite of this important difference, the result (9.37) is formally very similar to the probability that our Universe may emerge from the Planckian regime obtained in the context of the "tunnelling from nothing" and other similar inflationary scenarios [9, 609, 440, 633, 541], where the probability is also exponentially suppressed, and the coefficient k is inversely proportional to the effective cosmological constant and to the gravitational coupling, $P \sim \exp(-1/\lambda_P^2 \Lambda)$.

The reason for this formal analogy is easy to understand if we recall that, by choosing the string perturbative vacuum as the initial state of the pre-big bang scenario, it follows that in minisuperspace there are only right-moving waves approaching the singularity at $\overline{\phi} \to +\infty$. This is exactly equivalent to imposing tunnelling boundary conditions that select only outgoing modes at the singular space-time boundary [610, 611].

In this sense, the quantum reflection illustrated in this subsection can also be interpreted as a tunnelling process, not *"from nothing"*, however, but *"from the string perturbative vacuum"*. A different minisuperspace representation of the birth of the Universe from the vacuum will be illustrated in the next subsection.

parametric amplification of WDW

9.4 Birth of the Universe as "antitunnelling"

The process of quantum reflection (or tunnelling) is not the only "channel" open to vacuum decay in the string cosmology minisuperspace introduced in the previous subsections. There are also other –in principle more efficient– processes, such as the parametric amplification of the WDW wave function, which could represent the transition to our present cosmological state. We refer, in particular, to the cases (b) and (d) illustrated in Fig. 9.2.

Case (b), discussed in [130], shows indeed that a primordial conversion of expanding into contracting dimensions, corresponding to the production of pairs of Universes from the vacuum, can be efficiently described as the parametric amplification of the wave function. In this subsection we shall report a particular, explicit example of case (d) [290] to show that, with an appropriate model of dilaton potential, the parametric amplification of the wave function can also efficiently represent the transition from the pre- to the post-big bang regime, via pair production from the vacuum. In that case, the birth of the Universe can be described as a process of "antitunnelling from the string perturbative vacuum" (the term "antitunnelling", synonymous of parametric amplification, follows from the fact that the transition probability in that case is controlled by the inverse of the quantum-mechanical transmission coefficient, see Subsection 4.4).

First of all we note that, with a duality-invariant dilaton potential, the string-cosmology Hamiltonian is translationally invariant along the β axis: in that case, the initial expanding pre-big bang configuration keeps expanding, and the *out* state cannot be a mixture of states with positive and negative eigenvalues of Π_β. To implement the process (d) of Fig. 9.2, we thus need a non-local, duality-breaking potential, which contains both the metric and the dilaton, but *not* in the combination $\overline{\phi}$. We shall use, in particular, a two-loop dilaton potential induced by an effective cosmological constant Λ (two-loop potentials are known to favour the transition to the post-big bang regime already at the classical level [319, 122], but only for appropriate repulsive self-interactions, $\Lambda < 0$). We shall assume, in addition, that such a potential is rapidly damped in the large-radius limit $\beta \to +\infty$, and we shall approximate such a damping, for simplicity, by the Heaviside step function θ, by setting

$$V = \Lambda\theta(-\beta)e^{2\phi}, \qquad\qquad \Lambda > 0. \tag{9.39}$$

With such a damping we represent the effective suppression of the cosmological constant, required for the transition to a realistic post-big bang configuration.

Given the above potential, the general solution of the WDW equation can be factorized in terms of the eigenstates of the momentum $\Pi_{\overline{\phi}}$, by setting

$$\Psi(\beta,\overline{\phi}) = \Psi_k(\beta)e^{ik\overline{\phi}}, \qquad\qquad \left[\partial_\beta^2 + k^2 - \lambda_s^2\,\Lambda\,\theta(-\beta)\,e^{2\sqrt{d}\beta}\right]\,\Psi_k(\beta) = 0. \tag{9.40}$$

In the region $\beta > 0$ the potential is vanishing, and the outgoing solution is a superposition of eigenstates of Π_β corresponding to positive and negative frequency modes $\psi_\beta^\pm$, as in case

(d) of Fig. 9.2. In the region $\beta < 0$ the solution is a combination of Bessel functions $J_\nu(z)$, of imaginary index $\nu = \pm ik/\sqrt{d}$ and argument $z = i\lambda_s\sqrt{\Lambda/d}\, e^{\sqrt{d}\beta}$. If we fix the boundary conditions at $\beta \to -\infty$, by imposing that the Universe starts expanding from the string perturbative vacuum,

$$\Psi_{\rm in} = \lim_{\beta\to-\infty} \Psi(\beta, \overline{\phi}) \sim e^{ik(\overline{\phi}-\beta)}, \tag{9.41}$$

then the WDW wave function is uniquely determined as:

$$\begin{aligned} \Psi(\beta, \overline{\phi}) &= N_k J_{-\frac{ik}{\sqrt{d}}}\left(i\lambda_s\sqrt{\frac{\Lambda}{d}}\, e^{\sqrt{d}\beta}\right) e^{ik\overline{\phi}}, \qquad \beta < 0, \\ &= \left[A_+(k)e^{-ik\beta} + A_-(k)e^{ik\beta}\right] e^{ik\overline{\phi}}, \qquad \beta > 0. \end{aligned} \tag{9.42}$$

With the matching conditions at $\beta = 0$ we can finally compute the Bogoliubov coefficients $|c_\pm(k)|^2 = |A_\pm(k)|^2/|N_k|^2$ determining, in the third quantization formalism, the number n_k of universes produced from the vacuum, for each mode k (where k represents a given configuration in the space of the initial parameters).

As shown by an explicit computation [130, 290], the wave function is parametrically amplified (i.e. $n_k \gg 1$), for all $k < \lambda_s\sqrt{\Lambda}$. In that case, the birth of our present post-big bang Universe may proceed efficiently, and may be represented in minisuperspace as the forced production of pairs of Universes from the quantum fluctuations of the string perturbative vacuum. However, for a realistic process occurring at the string scale, with $\dot\beta \sim \lambda_s$ and $e^{\phi/2} \sim g_s$, the condition of parametric amplification can be written as

$$k \sim \left(\frac{\Omega_3}{\lambda_s^3}\right)\frac{1}{g_s^2} \lesssim \lambda_s\sqrt{\Lambda}, \tag{9.43}$$

where $\Omega_3 = a^3 \int d^3x$ is the proper spatial volume emerging from the transition in the post-big bang regime. This implies that the transition is strongly favoured for configurations of small enough spatial volume in string units, large enough coupling g_s, and/or large enough cosmological constant Λ (in string units), just as in the cases discussed in the previous subsection.

For $k \gg \lambda_s\sqrt{\Lambda}$ the wave function does not "hit" the barrier, there is no parametric amplification, and the inital state runs almost undisturbed towards the singularity. Only a small, exponentially suppressed fraction is able to emerge in the post-big bang regime, exactly as in the case of tunnelling (or quantum reflection). In the context of third quantization, this process can still be described as the production of pairs of universes, but the number of pairs is now exponentially damped, $n_k \sim \exp(-k/\lambda_s\sqrt{\Lambda})$, with a Boltzmann factor corresponding to a "thermal bath" of universes, at the effective temperature $T \sim \sqrt{\Lambda}$ in superspace.

It seems possible to conclude, therefore, that the low-energy quantum-cosmology processes considered in this subsection may allow a quantum transition from pre- to post-big

bang configurations, even if such configurations are classically disconnected. With a "realistic" dilaton potential, the "bubbles" of post-big bang phase nucleated out of the string perturbative vacuum seem, however, to be too small to reproduce our present Universe without a further long period of post-big bang inflation. The quantum processes reported in this Section, however, could explain the "foam" of infinitely many "baby Universes" required, for instance, in the context of chaotic inflation [442] and of other, standard, inflationary scenarios.

10 Conclusion

In this last section we will conclude our report by discussing a possible "late-time" consequence of the pre-big bang scenario –the large-scale dominance of the dilatonic dark energy density– and by presenting a list of various problems and aspects of such a scenario that are not included in the previous sections. Finally, we will summarize our personal outlook of string cosmology, together with some speculations about its possible future perspectives.

10.1 Towards the future: a dilaton-dominated Universe?

The dilaton is, undoubtedly, one of the most important ingredients of the pre-big bang scenario illustrated in this paper. Indeed, the dilaton implements the duality symmetry, controls the strength of all interactions, sustains the initial inflationary evolution, contributes to the amplification of the quantum fluctuations (and, in particular, to the production of seeds for the magnetic fields), leading eventually to the formation of a cosmic background of massive scalar particles. All such effects are typical of string cosmology and represent the "imprint" of the pre-big bang scenario with respect to other, more conventional, inflationary scenarios.

Not satisfied with all such effects, however, the dilaton could still be hale and hearty, and still in action even today, so that it would affect in a determinant way not only the very early cosmological past, but also the present (and, possibly, future) state of our Universe. The dilaton potential energy, or a mixture of kinetic and potential energy density, could represent in fact the dark component responsible for the cosmic acceleration observed very recently [535, 516]. In this sense, string theory can automatically provide, with the dilaton, a "non-minimal" model of quintessence [293, 311] (i.e. a model of quintessence based on a scalar field non-minimally coupled to gravity and to elementary matter fields).

There are, basically, two possible scenarios, depending on the (currently uncertain) shape of the non-perturbative dilaton potential $V(\phi)$. In the context of superstring models for grand-unified theories (GUTs), the present value of the dilaton should determine in fact the whole set of gravitational and gauge coupling parameters, which are today constant (or, if they are time-dependent, are nevertheless running very slowly on a cosmological time

scale). This constraint can be implemented in two ways.

The first possibility is a dilaton almost frozen at a minimum of the potential, in the moderate-coupling regime with $-\phi$ of order 1, in such a way that the GUT gauge coupling is fixed to [403]

$$\alpha_{\rm GUT} \sim (M_{\rm s}/M_{\rm P})^2 \sim e^{\phi} \sim 0.1 - 0.01. \tag{10.1}$$

The second possibility is a dilaton monotonically running towards $+\infty$, and the couplings saturated at small values in the strong-coupling regime because of the large number N of fields entering the loop corrections, or the large value of the quadratic Casimir C for unification gauge groups like E_8. Realistic values of $\alpha_{\rm GUT}$ and $M_{\rm s}$ can be obtained at $\phi \gg 1$, for $N \sim C \sim 10^2$, typically as [606, 607]:

$$\alpha_{\rm GUT} \sim \frac{e^{\phi}}{1 + Ce^{\phi}}, \qquad\qquad \frac{M_{\rm s}}{M_{\rm P}} \sim \frac{e^{\phi}}{1 + Ne^{\phi}}. \tag{10.2}$$

In both cases, if the present value $V_0 \equiv V(\phi_0)$ of the potential is of the order of the present Hubble scale H_0^2 (and, possibly, $\dot\phi^2 \sim V_0$), the dilaton can reproduce the observed "dark-energy" effects, playing the role of the so-called "quintessence" [138]. The required fine-tuning of the amplitude of the potential (and the corresponding ultra-light value of the dilaton mass, $m \sim H_0$), seems to be unavoidable. In this context, however, it seems possible to alleviate the problem of the "cosmic coincidence" [572], generated by the observed (approximate) equality $V_0 \sim \rho_0$, where ρ_0 is the present energy density of the dominant dark-matter component. Let us explain how this could happen, in both the weak-coupling and the strong-coupling scenarios of dilatonic quintessence.

In the first case we note that, even assuming that the dilaton gets frozen at a value ϕ_0 after the transition to the post-big bang regime, with a potential energy that is initially negligible ($V_0 \ll H_{\rm eq}^2$), it may keep frozen for the whole duration of the radiation epoch, but it tends to be shifted away from equilibrium as soon as the Universe enters the matter-dominated regime.

Suppose in fact that we add a potential $V(\phi)$ to the low-energy gravidilaton action:

$$S = -\frac{1}{2\lambda_{\rm s}^2} \int d^4x \sqrt{|g|}\; e^{-\phi} \left[R + (\nabla\phi)^2 + V(\phi) \right] + S_m, \tag{10.3}$$

where S_m describes, for simplicity, perfect-fluid sources minimally coupled to the background. For a homogeneous and conformally flat metric, the dilaton equation can then be written in the form

$$\ddot\phi + 3H\dot\phi - \dot\phi^2 + \frac{1}{2}e^{\phi}(\rho - 3p) + V' + V = 0. \tag{10.4}$$

Combining this equation with the standard conservation equation of the matter sources,

$$\dot\rho + 3H(\rho + p) = 0, \tag{10.5}$$

it follows that constant and stable solutions $\phi = \phi_0$ (with $\dot\phi = 0 = \ddot\phi$) are allowed in three cases only: 1) vacuum, $\rho = p = 0$, $V + V' = 0$; 2) cosmological constant, $\rho = -p = \rho_0 =$ const, $V + V' = -2e^{\phi_0}\rho_0 =$ const; 3) radiation, $\rho = 3p$, $V + V' = 0$.

So, even if the dilaton is "sleeping" in the radiation era, it "wakes up" and starts rolling away from the freezing position determined by $V + V' = 0$ just after the equilibrium epoch. We are thus led to the following question: For which values of the potential V_0 may the dilaton bounce back to the minimum, and the matter era be followed by a quintessential, potential-dominated epoch?

If the answer would point *only and precisely* at $V_0 \sim H_0^2$, then the cosmic coincidence would be explained. If the answer would indicate for V_0 a restricted range of values, including H_0^2, the coincidence problem would remain, but it would be alleviated.

The answer to the above question depends on the shape of the potential, and on the effective strength of the dilaton coupling to macroscopic matter. To illustrate this point we may consider the E-frame cosmological equations (6.2), obtained from the action (10.3) through the conformal transformation

$$\widetilde{g}_{\mu\nu} = g_{\mu\nu}e^{-\phi}, \qquad \widetilde{\phi} = \phi, \qquad \widetilde{V} = e^{\phi}V, \qquad \widetilde{\rho} = \rho e^{2\phi}, \qquad \widetilde{p} = pe^{2\phi}. \tag{10.6}$$

In units of $16\pi G = 1$ (and omitting the tilde, for simplicity), the dilaton equation becomes

$$\ddot\phi + 3H\dot\phi + \frac{1}{2}\alpha(\phi)(\rho - 3p) + V' = 0, \tag{10.7}$$

where we have taken into account, through the coupling function $\alpha(\phi)$, a possible loop renormalization of the effective dilaton coupling in the matter action. In the radiation era, $\rho = 3p$ and a stable, frozen dilaton ($\dot\phi = 0$) thus corresponds to an extremum of the E-frame potential, $V' = 0$. In the matter era, when $p = 0$, there is a dilaton acceleration away from the minimum, $\ddot\phi = -\alpha\rho/2$, possibly contrasted by the restoring force $-V'$.

Consider now a typical, supersymmetry-breaking dilaton potential, instantonically suppressed [89] at $\phi \to -\infty$, with a non-trivial structure developing a minimum in the region of moderate coupling, and exponentially growing at $\phi \to +\infty$ because of the conformal transformation to the E-frame. A "minimal" example of such a potential can be simply parametrized (in the E-frame) as follows [398]

$$V = m^2\left[e^{k_1(\phi-\phi_1)} + \beta e^{-k_2(\phi-\phi_1)}\right]e^{-\epsilon\exp[-\gamma(\phi-\phi_1)]}, \tag{10.8}$$

where $k_1, k_2, \phi_1, \epsilon, \beta, \gamma$ are dimensionless numbers of order 1 For an illustrative purpose we will choose here the particular values $k_1 = k_2 = \beta = \gamma = 1$, $\epsilon = 0.1$, $\phi_1 = -3$, in such a way that the minimum is at $\phi_0 = -3.112...$, and that $g_s^2 = \exp(\phi_0) \simeq 0.045$, in agreement with Eq. (10.1).

With the above choice of parameters, the dilaton potential is plotted in Fig. 10.1 for different values of m. It is important to note that the extremum ϕ_0 *is not* separated from

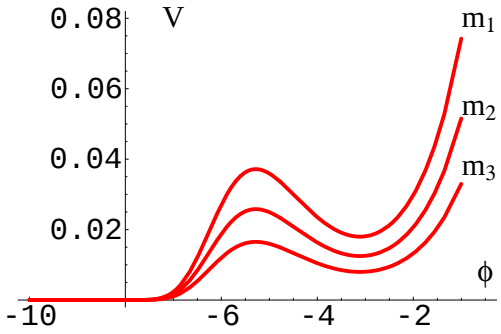

Figure 10.1: *Plot of $V(\phi)$ from Eq. (10.8), with $k_1 = k_2 = \beta = \gamma = 1$, $\epsilon = 0.1$, $\phi_1 = -3$. The three curves, from top to bottom, correspond respectively to $m = 1/10, 1/12, 1/15$ in units of $M_P^2 = 2$.*

the perturbative regime ($\phi \to -\infty$) by an infinite potential barrier, and that the lower $V_0 \sim m^2$, the lower the barrier, the weaker is the restoring force $-V'$ and the easier it is for the dilaton to escape from the minimum and run to $-\infty$.

To match present phenomenology, the value of V_0, in such a context, has to be chosen very small in string units, $V_0 \sim H_0^2 \sim (10^{-33}\ \text{eV})^2$ (see below for a possible justification). It follows that the potential barrier is very low, and the dilaton would certainly escape from the minimum at the beginning of the matter era, *unless* the coupling to matter ($\alpha\rho/2$) is also correspondingly small. Such a coupling, on the other hand, *has* to be very small, because the corresponding scalar force has a very long range, $V''(\phi_0) \sim m^2 \sim V_0 \sim H_0^2$, and the dilaton must be strongly decoupled ($\alpha \ll 1$, at least today) to avoid unacceptable violations of the equivalence principle [197, 198].

It can be shown, as a consequence, that the present values of α allowed by the gravitational phenomenology ($\alpha_0 \lesssim 10^{-3}$) are compatible with a late cosmological phase dominated by the dilaton potential only for a restricted range of V_0, which depends on the value of α at the equilibrium epoch, $\alpha_{\rm eq}$ [293]. Expanding $\alpha(\phi)$ around the minimum, and starting from $\alpha_{\rm eq} = 0.1$, for instance, a numerical analysis shows that the dilaton, after a small shift at the equilibrium epoch, bounces back to the minimum provided $10^{-7} H_{\rm eq} \lesssim m \lesssim H_{\rm eq}$, which includes the realistic case $m \sim V_0^{1/2} \sim H_0 \sim 10^{-6} H_{\rm eq}$. In such a context, therefore, the coincidence problem is not strictly solved, but possibly alleviated (see [293] for a more detailed discussion).

An alternative scenario for a dilatonic interpretation of the observed "quintessential" effects, is based on the assumption that the dilaton never gets trapped in a minimum, and evolves monotonically (and boundlessly) from negative (pre-big bang) to positive (post-big bang) values, with a potential smoothly approaching zero as $\phi \to +\infty$ (see Fig. 2.1). The non-perturbative potential, in this context, is typically characterized by a bell-like shape,

which can be parametrized (in the S-frame) as [311]:

$$V = m^2 \left[e^{-\frac{1}{\beta_1}\exp(-\phi)} - e^{-\frac{1}{\beta_2}\exp(-\phi)} \right], \tag{10.9}$$

where $\beta_1 > \beta_2 > 0$ are dimensionless numbers of order unity and, again, the mass scale m satisfies $m \ll H_{\rm eq}$, to avoid that the dilaton becomes dominant too early.

As before, the potential is instantonically suppressed at $\phi \to -\infty$, but now is also esponentially suppressed in the $\phi \to +\infty$ limit. The mass scale m, in the context of a non-perturbative potential, is naturally related to the string scale by $m \sim \exp\left[-\beta\alpha_{\rm GUT}^{-1}\right] M_{\rm s}$, where β is a model-dependent parameter of order 1. This may explain the smallness of the potential in string units, but it should not hide the fact that the constant parameter β has to be precisely adjusted *ad hoc* if we want to start the accelerated dilaton-dominated phase not much earlier than at redshifts of order one [587], and not later than today.

In the context of a running-dilaton scenario, the gauge couplings and the ratio $M_{\rm s}/M_{\rm P}$ are to be stabilized, in the $\phi \to +\infty$ limit, by the loop corrections. Such corrections, on the other hand, also play a fundamental role in determining the effective dilatonic charge of the elementary fields appearing in the matter action. Thanks to this interplay, a dilatonic charge that switches on at late times, in the (non-baryonic) dark-matter sector, seems able to provide a possible dynamical explanation of the cosmic coincidence [311] (see [195, 196] for possible related violations of free-fall universality, and time variation of the natural constants, induced in this context).

Such a dynamical approach to the coincidence problem is based, in particular, on the following loop-corrected action

$$S = -\frac{1}{2\lambda_{\rm s}^2}\int d^4x\sqrt{-g}\ \left[e^{-\psi(\phi)}R + Z(\phi)\,(\nabla\phi)^2 + 2\lambda_{\rm s}^2 V(\phi)\right] + S_m(\phi, \text{matter}), \tag{10.10}$$

where V is the potential of Eq. (10.9), and the loop form factors ψ and Z, in a minimal "induced-gravity" scenario, are assumed to be parametrized at large ϕ as follows [606, 607]:

$$e^{-\psi(\phi)} = e^{-\phi} + c_1^2\,, \qquad\qquad Z(\phi) = e^{-\phi} - c_2^2, \tag{10.11}$$

where $c_1^2 \sim c_2^2 \sim 10^2$, in agreement with Eq. (10.2). We also assume that the action S_m contains non-baryonic dark-matter whose dilatonic charge density per unit of gravitational mass, σ_d/ρ_d, switches on as $\phi \to +\infty$, and can be parametrized as [311]:

$$q(\phi) = \frac{\sigma_d}{\rho_d} = -\frac{2}{\sqrt{-g}\rho_d}\frac{\delta S_m}{\delta\phi} = q_0\frac{e^{q_0\phi}}{c^2 + e^{q_0\phi}}, \tag{10.12}$$

where, again, $c^2 \sim 10^2$. Note that we are using here a definition of dilatonic charge density different from the one previously introduced in Eqs. (6.16) and (8.41).

The analytical and numerical study of such a model shows that, in a way that is largely independent from the initial conditions, the Universe is eventually driven to an asymptotic

accelerated regime characterized by a frozen ratio between the dark-matter and the dilatonic energy density, $\rho_d/\rho_\phi = \text{const}$. The present approximate equality of ρ_d and V_0 is no longer a coincidence, in this context, but a dynamical property of the asymptotic configuration, as in models of "coupled quintessence" [373, 26].

For a better illustration of the "late-time" post-big bang scenario, described by Eqs. (10.10)–(10.12), it is convenient to consider the E-frame cosmological equations for the tilted variables defined through the conformal transformation

$$a = c_1 \widetilde{a} e^{\psi/2}, \qquad dt = c_1 d\widetilde{t} e^{\psi/2}, \qquad \widetilde{\rho} = c_1^2 \rho e^{2\psi}, \qquad \widetilde{p} = c_1^2 p e^{2\psi}, \qquad \widetilde{\sigma} = c_1^2 \sigma e^{2\psi} \tag{10.13}$$

(we are considering, as before, perfect-fluid sources evolving in a conformally flat metric background). Note that c_1^2, according to Eq. (10.11), controls the asymptotic value of the ratio between the string and the Planck scale, $c_1^2 \lambda_{\rm s}^{-2} = \lambda_{\rm P}^{-2}$. The E-frame equations thus become (omitting the tilde, and in units $2\lambda_{\rm P}^2 = 16\pi G = 1$) [311]

$$6H^2 = \rho + \rho_\phi, \tag{10.14}$$

$$4\dot{H} + 6H^2 = -p - p_\phi, \tag{10.15}$$

$$k^2(\phi)\left(\ddot{\phi} + 3H\dot{\phi}\right) + k(\phi)\, k'(\phi)\, \dot{\phi}^2 + \hat{V}'(\phi) + \frac{1}{2}\left[\psi'(\phi)(\rho - 3p) + q\rho\right] = 0, \tag{10.16}$$

where a dot denotes differentiation with respect to the E-frame cosmic time, and we have defined:

$$k^2(\phi) = 3\psi'^2 - 2e^{\psi} Z, \qquad \hat{V} = c_1^4 e^{2\psi} V, \tag{10.17}$$

$$\rho_\phi = \frac{1}{2} k^2(\phi)\dot{\phi}^2 + \hat{V}(\phi), \qquad p_\phi = \frac{1}{2} k^2(\phi)\dot{\phi}^2 - \hat{V}(\phi). \tag{10.18}$$

The source terms $\rho, p, \sigma = q\rho$ generically include the contribution of radiation, baryonic and non-baryonic matter components, but only for the non-baryonic dark component ρ_d is the dilaton charge assumed to be significantly different from zero, and parametrized as in Eq. (10.12). By analysing the time evolution of the (total) dilaton energy density, ρ_ϕ, we then find that the post-big bang solutions are generally characterized by three dynamical phases [311].

Suppose in fact that the radiation-dominated, post-big bang regime starts with with a negligible dilaton potential, a negligible dark-matter energy density $\rho_d \sim a^{-3}$, and a fully kinetic dilaton energy density ,which is rapidly diluted as $\rho_\phi \sim a^{-6}$. When ρ_ϕ falls below ρ_d one finds that the Universe enters a first "focusing" phase with $\rho_\phi \sim a^{-2}$, so that the dilaton kinetic energy starts growing with respect to radiation, and converges towards the other energy components.

When the matter becomes dominant, the Universe enters a subsequent "dragging" phase, where ρ_ϕ and ρ_d evolve in time with the same behaviour, together with the baryonic matter density (uncoupled to the dilaton), $\rho_b \sim a^{-3}$. Actually, ρ_ϕ and ρ_d are diluted a little bit

faster (like $a^{-3-\epsilon^2}$) than ρ_b, but the deviation is controlled by a parameter, which is constant and very small during the dragging phase,

$$[\epsilon(\phi)]_{\rm drag} \equiv \left[\frac{\psi' + q}{k}\right]_{\rm drag} \sim \frac{q_0 c_1}{c^2 c_2} \sim \frac{1}{c^2} \ll 1. \tag{10.19}$$

At late times (but not later than today) the potential $V(\phi)$ and the dilaton charge $q(\phi)$ of dark-matter come eventually into play, and the Universe enters a "freezing" phase characterized by a constant ratio ρ_ϕ/ρ_d, and by accelerated evolution. In this asymptotic regime, $\phi \to +\infty$, the parameters of the model approach the constant values

$$k^2(\phi) = 2c_2^2/c_1^2 \equiv 2/\lambda^2, \quad \sigma = \sigma_d, \quad \rho = \rho_d, \quad q(\phi) = q_0, \tag{10.20}$$

and the E-frame equations can be written as

$$\dot\rho_d + 3H\rho_d - \frac{q_0}{2}\,\rho_d\dot\phi = 0, \qquad \dot\rho_\phi + 6H\rho_k + \frac{q_0}{2}\,\rho_d\dot\phi = 0, \tag{10.21}$$

$$1 = \Omega_d + \Omega_k + \Omega_V, \qquad 1 + \frac{2\dot H}{3H^2} = \Omega_V - \Omega_k, \tag{10.22}$$

where

$$\rho_d = 6H^2\Omega_d, \qquad \rho_\phi = \rho_k + \rho_V, \tag{10.23}$$

$$\rho_k = 6H^2\Omega_k = \dot\phi^2/\lambda^2, \qquad \rho_V = 6H^2\Omega_V = \hat V. \tag{10.24}$$

They can be solved by an asymptotic configuration with constant Ω_k, Ω_V and Ω_d (acting as late-time attractor [24]), where the dilaton fraction of critical energy density, $\Omega_k + \Omega_V$, the dilaton equation of state, $w_\phi = (\Omega_k - \Omega_V)/(\Omega_k + \Omega_V)$, and the asymptotic acceleration parameter, are given by [311]

$$\Omega_\phi = \frac{12 + q_0(q_0+2)\lambda^2}{(q_0+2)^2\lambda^2}, \qquad w_\phi = -\frac{q_0(q_0+2)\lambda^2}{12 + q_0(q_0+2)\lambda^2}, \tag{10.25}$$

$$\frac{\ddot a}{aH^2} = 1 + \frac{\dot H}{H^2} = \frac{q_0 - 1}{q_0 + 2}. \tag{10.26}$$

Such an asymptotic configuration is illustrated in Fig. 10.2, where we have plotted various curves at $\Omega_\phi = $ const and $w_\phi = $ const in the $\{q_0, \lambda\}$ plane. As shown in the figure, a positive acceleration ($q_0 > 1$) is perfectly compatible with the range of Ω_ϕ and w_ϕ allowed by present phenomenological constraints [617, 49], namely $0.6 \lesssim \Omega_\phi \lesssim 0.7$ and $-1 \le w_\phi \lesssim -0.4$.

We note, finally, that a simple integration of Eq. (10.26) gives the late-time evolution of the dilaton and dark-matter energy density,

$$\rho_\phi \sim \rho_d \sim H^2 \sim a^{-6/(2+q_0)}. \tag{10.27}$$

The baryon dark matter, uncoupled to the dilaton ($q = 0, \rho_b \sim a^{-3}$), is rapidly diluted with respect to ρ_d, and we thus obtain, in the context of such a dilatonic scenario, also a possible explanation of the present smallness of the ratio ρ_b/ρ_d [311] (see also [27]).

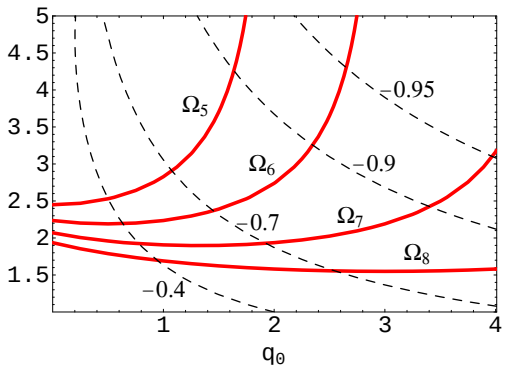

Figure 10.2: *Asymptotic configurations in the plane* $\{q_0, \lambda\}$*, according to Eq. (10.25). The full curves correspond to the constant values* $\Omega_5 = 0.5$*,* $\Omega_6 = 0.6$*,* $\Omega_7 = 0.7$*,* $\Omega_8 = 0.8$*, and the dashed curves to the constant values of* w_ϕ *indicated in the figure.*

10.2 Other open problems

In this report we have focalized our attention on many important aspects of the pre-big bang scenario (in particular, on those mainly studied in the past years), without pretending, of course, to have presented an exhaustive discussion of all possible problems and aspects of such scenario. There are important topics that we have left almost untouched, also because too few results ae available on these in the present literature. Thus, it seems to us appropriate to conclude with a list (and a brief discussion) of such topics, which we hope will be the object of more intensive studies in the next years.

In particular, we have given very little room to a discussion, in the pre-big bang context, of a realistic model of (dynamical?) dimensional reduction of the superstring manifolds, the (possible) compactification and the associated stabilization of the internal dimensions. The best approach to this problem, at present, is probably the model of [109], generalized to a brane-gas in [10, 103], which assumes, however, a toroidal structure of the space-time manifold from the very beginning (as in standard Kaluza–Klein compactification). There are no results, at present, concerning the possible evolution of the pre-big bang phase into a multi-dimensional non-compact structure, eventually characterized by a warped geometry able to confine long-range interactions on a four-dimensional brane [530].

see here

Also, we have not discussed in detail the matching of the pre-big bang regime to the subsequent Friedmann–Robertson–Walker (FRW) phase, the possibility in this context of background oscillations (of the metric and of the dilaton), with related "preheating" and reheating effects, and the associated particle production (for instance gravitino production, which is a potential problem [620] for any inflationary model based on a supersymmetric theory). All these problems, including dimensional reduction, are (more or less directly) related to the fact that there is at present no compelling model of a complete and realistic

exit transition (see Section 8).

It is worth mentioning, in this respect, that the problem of reheating and of possible dangerous relics (moduli and gravitinos) in pre-big bang cosmology has recently been faced in [132]. The main difference from the reheating scenario of standard inflationary models is that, for pre-big bang inflation, the matching to the FRW phase may occur in principle at higher-curvature scales (typically, $H_1 \sim M_{\rm s} \sim 10^{17}$ GeV), where gravitinos and moduli fields should also be copiously produced in scattering processes. Another important difference is that the particles present at the beginning of the FRW phase are produced (with various spectra) directly from the amplification of the quantum fluctuations of the background fields during inflation (as stressed in Subsection 8.4), with typical densities $\rho_i(t) \sim N_i H_1^4 (a_1/a)^4$, where N_i is the number of helicity states in species i.

In such a context, the thermalization scale of scalar and vector particles, charged under the group of the observable sector, has been estimated in [132] by assuming that the dilaton and the internal dimensions are already frozen at the beginning of the FRW phase, when $H = H_1$ and $g = g_1 \equiv M_{\rm s}/M_{\rm P}(t_1)$. The charged particles, with typical energy $\omega \sim H_1 a_1/a$ and number density $n_r \sim \rho_r/\omega$, where

$$\rho_r = \Omega_r \rho_c \sim \frac{N_r}{N_{\rm tot}} \frac{H_1^2 M_{\rm P}^2 a_1^4}{a^4} = \frac{N_r}{N_{\rm tot}} \left(\frac{H_1 M_{\rm s}}{g_1}\right)^2 \left(\frac{a_1}{a}\right)^4 \tag{10.28}$$

($N_{\rm tot}$ is the total number of degrees of freedom present at $t = t_1$), interact with cross section $\sigma \sim \alpha^2/\omega^2$, where $\alpha \sim g_1^2$. The thermalization scale $H_{\rm th} \sim n_r \sigma$ is then given by

$$H_{\rm th} \sim g_1^4 \left(\frac{N_r}{N_{\rm tot}}\right)^2 \frac{M_{\rm s}^4}{H_1^3}. \tag{10.29}$$

If the charged particles are already dominat at $t = t_1$, i.e. $N_r \sim N_{\rm tot}$, $\Omega_r \sim 1$, and if $H_1 \sim M_{\rm s}$ as in minimal pre-big bang models, then the post-big bang Universe thermalizes at a scale $H_{\rm th} \sim g_1^4 H_1 \sim (10^{-4} - 10^{-8})\ M_{\rm s}$, with a corresponding reheating temperature $T_{\rm r} \sim (H_{\rm th} M_{\rm P})^{1/2} \sim g_1^{5/2} M_{\rm P} \sim g_1^{3/2} M_{\rm s}$. If, on the contrary, $\Omega_r < 1$, then reheating is only achieved once the gauge-singlet fields, which carry the remainder of the total energy density, have decayed into radiation.

Such a reheating process generates an entropy density $s \sim \rho_r/T_r$, which is not sufficient, however, to dilute the unwanted relic particles produced gravitationally during inflation, and also in scatterings of the thermal bath for $t > t_1$. For the moduli, in particular, the number-density-to-entropy-density ratio $Y_m = n_m/s_m$ turns out to be [132] $Y_m \sim 0.3/g_*$, where g_* is the number of degrees of freedom in the radiation bath after thermalization. For $g_* \sim 10^2 - 10^3$ this is well above the limit imposed by nucleosynthesis [620], which implies $Y \lesssim 10^{-13}$.

The same is true for gravitinos. Indeed, even if they are not gravitationally produced during pre-big bang inflation because they keep effectively massless [120], gravitinos are

Affleck Dine: Takahashi Fuminobu PDF

produced in scatterings of the high-temperature thermal bath, and their number to entropy ratio can be estimated (for $H_1 \sim M_s$) as [132] $Y_{3/2} \sim g_1^{5/2} g_*^{-7/4}$, which is still in excess of the nucleosynthesis limit (see also [431, 584] for gravitino production in a string cosmology context).

The dilution of the unwanted relics possibly present at the beginning of the post-big bang phase thus requires a strong entropy release, at the level of about ten orders of magnitude for gravitationally produced relics in the context of minimal pre-big bang models (so as to suppress Y from 10^{-3} to 10^{-13}). Such an entropy production may be due to a secondary reheating phase, as already discussed in Subsection 6.3, and a typical possibility corresponds to the oscillations and decay of a scalar (or pseudoscalar) field, which gets a mass in the post-big bang epoch through a symmetry- (or supersymmetry-) breaking mechanism.

A natural string-cosmology candidate for this effect is the axion (as discussed in Subsection 7.5), or even the dilaton, if it is heavy enough. In particular, if the initial value of the oscillating field is at least of order M_s, it is possible to dilute in this way moduli and gravitinos, but not monopoles (possibly produced by GUT symmetry breaking), while an initial value of order M_P is marginally sufficient also for monopoles [132].

It is possible, however, that the monopole problem be independently solved by an efficient monopole–antimonopole annihilation [176]. We should keep in mind, also, that in more complicated, non-minimal pre-big bang backgrounds (see Subsection 5.3), the energy distribution among the produced particles may be drastically altered, with a resulting easier dilution of the relics component, and a smaller required amount of entropy production (see [132] for a detailed discussion).

We note, finally, that the entropy-production process is independently interesting by itself, in such a context, as it can naturally accommodate baryogenesis if the oscillating scalar is identified with the Affleck–Dine condensate, made of squarks and sleptons, whose decay generates the baryon asymmetry [6]. In particular, a "mixed" reheating phase with two oscillating fields (a modulus and a condensate), seems to be able to reproduce both the required dilution of relics and the right baryon asymmetry in the context of pre-big bang cosmology [132]. The oscillations of the Affleck–Dine condensate could be sufficient, by themselves, to solve the moduli and gravitinos problem in the context of non-minimal pre-big bang models, provided the resulting baryon asymmetry were kept small enough by a very small CP-violation parameter, or by a very efficient electroweak mechanism of baryon-number erasure [261].

10.3 Outlook

Although it is always difficult to make forecasts, particularly in theoretical physics, we are at least confident of one thing: research on the implications of string/M-theory on fundamental cosmological questions is not just a momentary fashion. It is going to continue

at an increasing rate as long as new cosmological data come in, and put more and more puzzles in front of theoretical cosmology. Indeed, the last few years have witnessed a sudden jump of interest in this field, triggered partly by the new data on CMB anisotropy and on evidence for an accelerating Universe, partly by the new theoretical developments related to large extra dimensions, the possible lowering of the quantum-gravity scale, and the brane-world ideas.

One line of research has been devoted to the modifications of the standard cosmological equations for observers confined to a brane immersed in a higher-dimensional space-time [88]. Such modifications are in principle sufficiently drastic to make us worry about how to preserve the successes of the standard set-up, on the one hand, and about how to use the deviations in order to improve the situation where the standard description may face difficulties (e.g. during inflation [179] and early cosmology) on the other hand.

ekpyrotic

A second development, which is even more relevant to this report, is the proposal of new cosmologies [412, 413] that share many properties with the pre-big bang scenario. As in the latter, they assume that the big bang singularity is spurious, and that the Universe (and time) had a long existence prior to it. Also, the initial state is assumed to be very perturbative, although not as generic as in our approach. The most important difference, however, is that such models are constructed around the brane-world idea (see Subsection 8.5). Thus, although some of the pictures look very similar, their meaning is drastically different. As an example, the plane-wave collision described in Subsection 3.4 superficially resembles the brane collision of the ekpyrotic scenario; but while in the former our Universe lies in the bulk, in the model of [412, 413] it lies on one of the two colliding branes. Also, the big bang is the moment of collision in [412, 413] while, in the pre-big bang scenario, it emerges sometime after the collision, as a consequence of gravitational collapse.

bouncing U

The model in [413] is even closer to the pre-big bang idea, in the sense that, there, the extra dimension represents the dilaton in the strong-coupling regime, according to the M-theory conjecture [376]. It is thus possible to draw the bouncing-Universe scenario of [413] in the same phase-space plane as was used in Fig. 8.1. The difference between the two scenarios is very simple: while the pre-big bang Universe starts its evolution in the upper-right quadrant of the diagram –and thus inflates, in the S-frame, as it evolves from weak to strong coupling– in the scenario of [413] the Universe starts in the lower-right quadrant and thus contracts, even in the S-frame, while it evolves from strong to weak coupling.

We have seen in Subsections 1.3 and 2.4 that an accelerated contraction is as good as an accelerated expansion for solving the horizon and flatness problems, hence the model of [413] appears a priori as good as the pre-big bang model. It is important, however, to stress some differences, especially regarding the exit problem. If the exit is to occur at weak coupling, it should be possible to study it within the tree-level effective action (although non-perturbatively in α'). This was precisely the framework described in Subsection 8.2, where it was shown that, at least order by order in the α' expansion, a conserved quantity

forbids a transition from contraction to expansion in the S-frame. The possibility remains, however, of non-perturbative (world-sheet instanton) effects for achieving the bounce.

Still at the theoretical level, both the ekpyrotic and the pre-big bang scenario suffer from the criticism of [190] about the generic occurrence of a phase of chaotic oscillations. This is even more crucial for the models of [412, 413], since they need a highly fine-tuned initial state [389, 390] and, again, cannot invoke string-loop corrections for stopping BKL oscillations.

At the more phenomenological level the main difference between the two classes of models concerns the spectrum of (adiabatic) scalar metric perturbations. While these have been claimed to have naturally a blue spectrum in a pre-big bang context [115] (see also [382, 583]), the authors of [412] have claimed to get naturally (although through a certain hypothesis on the shape of a potential) a scale-invariant spectrum, as favoured by observations. Without taking sides, we will mention that this claim has been challenged by several authors [456, 105] (see however [227]). On the other hand, the curvaton mechanism of [459] could still generate scale-invariant adiabatic perturbations even if these were not immediately generated during the pre-big bang (or pre-bounce) phase.

Since string/M-theory is proposed by its fans as a candidate unified theory of all phenomena, it simply cannot allow itself to avoid tackling the fundamental questions that classical and quantum gravity are proposing: What is the fate of the classical singularities –that are so ubiquitous in general relativity– in the context of a consistent quantum theory of gravity?

In this report the pre-big bang scenario was described in such a way that the reader's attention was drawn to the big bang singularity of general relativity and to how its avoidance, in string theory, could open up new possibilities for solving the long-standing problems of classical, hot big bang cosmology. Amusingly, in doing so, we were led to connect this question to the other big puzzle of general relativity: the fate of black-hole singularities, with all its ramifications into the information paradox, the loss of quantum coherence, and the like.

To conclude, let us try to draw some generic lessons from the particular attempt at a new, string-based cosmology that we have illustrated in the previous sections.

(1) Neither our Universe, nor space and time themselves, have to emerge from a singularity: the singularities of classical gravitational theories should signal the lack of finite-size and quantum corrections, or the need for new degrees of freedom in the description of physics at very short distances. Also, the Universe did not have to start from the very beginning as a hot and dense "soup" of particles and radiation: a hot Universe can emerge from a cold one, thanks to the parametric amplification of the quantum fluctuations of the vacuum, during inflation.

(2) Inflation does not need a scalar potential (or, more generally, an effective cosmological

constant), and can naturally appear as the result of the underlying duality symmetries of the cosmological background. Also, inflation can be represented as a contraction (in the appropriate frame), and the study of generic initial conditions can be related, mathematically, to the study of gravitational collapse in general relativity.

(3) The Planck scale does not provide any fundamental impenetrable barrier, which would limit our direct experimental information on what happened before. On the contrary, the phenomenological imprint of the pre-Planckian epoch can be encoded into a rich spectrum of observable relics, reaching us today directly out of the pre-big bang Universe. Pre-big bang physics can thus be the object of dedicated experimental searches, which will be able to tell us, hopefully in a not too far future, whether or not there are chances for present string-cosmology models to provide a successful description of our primordial Universe.

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References

[1] R. B. Abbott, S. M. Barr and S. D. Ellis, Phys. Rev. **D30**, 720 (1984).

[2] R. B. Abbott, B. Bednarz and S. D. Ellis, Phys. Rev. **D33**, 2147 (1986).

[3] L.F. Abbot and M. B. Wise, Nucl. Phys. **B244**, 541 (1984).

[4] A. Abramovici et al., Science **256**, 325 (1992).

[5] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions* (Dover, New York, 1972).

[6] I. Affleck and M. Dine, Nucl. Phys. **B249**, 361 (1985).

[7] E. J. Ahn, M. Cavaglià and A. Olinto, hep-th/0201042.

[8] E. J. Ahn and M. Cavaglià, hep-ph/0205168.

[9] K. Akama and H. Terazawa, Gen. Relat. Gravit. **15**, 201 (1983).

[10] S. Alexander, R. Brandenberger and D. Easson, Phys. Rev. **D62**, 103509 (2000).

[11] S. O. Alexeyev, A. V. Toporensky and V. O. Ustiansky, Class. Quant. Grav. **17**, 2243 (2000).

[12] B. Allen, Phys. Rev. **D37**, 2078 (1988).

[13] B. Allen, in *Astrophysical Sources of Gravitational waves*, Proc. Les Houches School, 1996, eds. J. A. Marck and J. P. Lasota (University Press, Cambridge, 1997), p. 373.

[14] B. Allen and R. Brustein, Phys. Rev. **D55**, 3260 (1997).

[15] B. Allen, E. E. Flanagan and M. A. Papa, Phys. Rev. **D61**, 024024 (2000).

[16] B. Allen and S. Koranda, Phys. Rev. **D50**, 3713 (1994).

[17] B. Allen and J. D. Romano, Phys. Rev. **D59**, 102001 (1999).

[18] E. Alvarez, Phys. Rev. **D31**, 418 (1985).

[19] E. Alvarez and J. Conde, Mod. Phys. Lett. **A17**, 413 (2002).

[20] E. Alvarez and M. Osorio, Int. J. Theor. Phys. **28**, 949 (1989).

[21] U. Amaldi, W. de Boer and H. Furstenau, Phys. Lett. **B260**, 447 (1991).

[22] D. Amati, M. Ciafaloni and G. Veneziano, Int. J. Mod. Phys. **A3**, 1615 (1988).

[23] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. **B216**, 41 (1989).

[24] L. Amendola, Phys. Rev. D **62**, 043511 (2000).

[25] L. Amendola, M. Litterio and F. Occhionero, Phys. Lett. **B237**, 348 (1990).

[26] L. Amendola and D. Tocchini-Valentini, Phys. Rev. D **64**, 043509 (2001).

[27] L. Amendola and D. Tocchini-Valentini, Phys. Rev. **D65**, 063508 (2002).

[28] S. Anderegg and V. N. Mukhanov, Phys. Lett. **B331**, 30 (1994).

[29] C. Angelantonj, L. Amendola, M. Litterio and F. Occhionero, Phys. Rev. **D51**, 1607 (1995).

[30] I. Antoniadis, in *Gravitational Waves and Experimental Gravity*, Proc. XXXIV Rencontres de Moriond, Les Arcs, 1999, eds. J. Tran Thanh Van et al. (World Scientific, Singapore, 2000).

[31] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B436**, 257 (1998).

[32] I. Antoniadis, E. Gava and K. S. Narain, Phys. Lett. **B283**, 209 (1992).

[33] I. Antoniadis, E. Gava and K. S. Narain, Nucl. Phys. **B393**, 93 (1992).

[34] I. Antoniadis, J. Rizos, and K. Tamvakis, Nucl. Phys. **B415**, 497 (1994).

[35] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B429**, 263 (1998).

[36] A. Ashtekar and R. Geroch, Rep. Prog. Phys. **37**, 1211 (1974).

[37] P. Astone, in Proc. *Second Edoardo Amaldi Conference on Gravitational Waves*, CERN, 1997, eds. E. Coccia et al. (World Scientific, Singapore, 1998), p. 192.

[38] P. Astone et al., Astron. Astrophys. **351**, 811 (1999).

[39] P. Astone et al., in Proc. *Second Edoardo Amaldi Conference on Gravitational Waves*, CERN, 1997, eds. E. Coccia et al. (World Scientific, Singapore, 1998), p. 551.

[40] P. Astone, J. Lobo and B. Schutz, Classical Quant. Grav. **11**, 2093 (1994).

[41] P. Astone, G. Pallottino and G. Pizzella, Classical Quant. Grav. **14**, 2019 (1997).

[42] M. Axenides, R. H. Brandenberger and M. S. Turner, Phys. Lett. **B126**, 178 (1983).

[43] D. Babusci, L. Baiotti, F. Fucito and A. Nagar, Phys. Rev. **D64**, 062001 (2001).

[44] D. Babusci, S. Foffa, G. Losurdo, M. Maggiore, G. Matone and R. Sturani, in *Gravitational Waves*, Proc. SIGRAV School, Centre A. Volta, Como, 1999, eds. I. Ciufolini et al. (IOP Publishing, Bristol, 2001), p. 179.

[45] D. Babusci and M. Giovannini, Int. J. Mod. Phys. **D10**, 477 (2001).

[46] D. Babusci and M. Giovannini, Phys. Rev. **D60**, 083511 (1999).

[47] D. Babusci and M. Giovannini, Classical Quant. Grav. **17**, 2621 (2000).

[48] D. Bak and S. Rey, Classical Quant. Grav. **17**, L1 (2000).

[49] A. Balbi, C. Baccigalupi, S. Matarrese, F. Perrotta and N. Vittorio, Astrophys. J. **547**, L89 (2001).

[50] A. J. Banday et al., Astrophys. J. **475**, 393 (1997).

[51] T. Banks, W. Fischler and L. Motl, JHEP **9901**, 019 (1999).

massive dilaton

[52] T. Banks, D. B. Kaplan and A. E. Nelson, Phys. Rev. **D49**, 779 (1994).

[53] J. Bardeen, Phys. Rev. **D22**, 1822 (1980).

[54] R. Bar-Kana, Phys. Rev. **D64**, 7138 (1996).

[55] J. D. Barrow, Phys. Lett. **B180**, 335 (1986).

[56] J. D. Barrow, Phys. Rev. **D48**, 3592 (1993).

[57] J. D. Barrow and M. P. Dabrowski, Phys. Rev. **D55**, 630 (1997).

[58] J. D. Barrow and M. P. Dabrowski, Phys. Rev. **D57**, 7204 (1998).

[59] J. D. Barrow and M. P. Dabrowski, Phys. Rev. **D58**, 103502 (1998).

[60] J. D. Barrow and K. E. Kunze, Phys. Rev. **D55**, 623 (1997).

[61] J. D. Barrow and K. E. Kunze, Phys. Rev. **D56**, 741 (1997).

[62] J. D. Barrow and K. E. Kunze, Phys. Rev. **D57**, 2255 (1998).

[63] J. D. Barrow and K. E. Kunze, Chaos, Soliton Frac. **10**, 257 (1999).

[64] J. D. Barrow, J. P. Mimoso and M. R. de Garcia Maia, Phys. Rev. **D48**, 3630 (1993).

[65] N. Bartolo and A. R. Liddle, Phys. Rev. **D65**, 121301 (2002).

[66] B. A. Bassett, Phys. Rev. D **56**, 3439 (1997).

[67] B. A. Bassett, C. Gordon, R. Maartens and D. I. Kaiser, Phys. Rev. **D61**, 061302 (2000).

[68] R. A. Battye and E. P. S. Shellard, Phys. Rev. **D53**, 1811 (1996).

[69] K. Behrndt and S. Forste, Phys. Lett. **B320**, 253 (1993).

[70] K. Behrndt, S. Forste and S. Schwager, Nucl. Phys. **B508**, 391 (1997).

[71] J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973).

[72] J. D. Bekenstein, Phys. Rev. **D23**, 287 (1981).

[73] J. D. Bekenstein, Int. J. Theor. Phys. **28**, 967 (1989).

[74] J. D. Bekenstein, Phys. Rev. **D49**, 1912 (1994).

[75] V. A. Belinskii and I. M. Khalatnikov, Sov. Phys. JETP **36**, 591 (1973).

[76] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).

[77] C. L. Bennet et al., Astrophys. J. **430**, 423 (1994).

[78] C. L. Bennet et al., Astrophys. J. **464**, L1 (1996).

[79] M. G. Bento and O. Bertolami, Classical Quant. Grav. **12**, 1919 (1995).

[80] P. Bernard, G. Gemme, R. Parodi and E. Picasso, Rev. Sci. Instrum. **72**, 2428 (2001).

[81] O. Bertolami and D. F. Mota, Phys. Lett. **B455**, 96 (1999).

[82] M. Bianchi, M. Brunetti, E. Coccia, F. Fucito and J. A. Lobo, Phys. Rev. D **57**, 4525 (1998).

[83] M. Bianchi, E. Coccia, C. Colacino, V. Fafone and F. Fucito, Classical Quant. Grav. **13**, 2865 (1996).

[84] A. P. Billyard, A. A. Coley and J. E. Lidsey, Phys. Rev. **D59**, 123505 (1999).

[85] A. P. Billyard, A. A. Coley and J. E. Lidsey, J. Math. Phys. **40**, 5092 (1999).

[86] A. P. Billyard, A. A. Coley and J. E. Lidsey, Classical Quant. Grav. **17**, 453 (2000).

[87] A. P. Billyard, A. A. Coley, J. E. Lidsey and U. S. Nilsson, Phys. Rev. **D61**, 043504 (2000).

[88] P. Binétruy, C. Deffayet and D. Langlois, Nucl. Phys. **B565**, 269 (2000).

[89] P. Binétruy and M. K. Gaillard, Phys. Rev. **D34**, 3069 (1986).

[90] N. D. Birrel and P. C. W. Davies, *Quantum fields in curved spaces* (University Press, Cambridge, 1982).

[91] A. K. Bisvas, J. Maharana and R. K. Pradhan, Phys. Lett. **B462**, 243 (1999).

[92] H. J. Blome and W. Priester, Astron. Astrophys. **250**, 43 (1991).

[93] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, Proc. Roy. Soc. Lond. **A269**, 21 (1962).

[94] A. Borde, A. Guth and A. Vilenkin, gr-qc/0110012.

[95] A. Borde and A. Vilenkin, Phys. Rev. Lett. **72**, 3305 (1994).

[96] R. Bousso, JHEP **9906**, 028 (1999).

[97] R. Bousso, JHEP **9907**, 004 (1999).

[98] M. J. Bowick, L. Smolin and L. C. R. Wijewardhana, Gen. Relat. Gravit. **19**, 113 (1987).

[99] V. Bozza, M. Gasperini and G. Veneziano, Nucl. Phys. **B619**, 191 (2001).

[100] V. Bozza, M. Gasperini, M. Giovannini and G. Veneziano, hep-ph/0206131.

[101] V. Bozza and G. Veneziano, JHEP **0010**, 35 (2000).

[102] R. Brandenberger, APCTP Bulletin **6** (2001).

[103] R. H. Brandenberger, D. Easson and D. Kimberly, Nucl. Phys. **B623**, 421 (2002).

[104] R. H. Brandenberger, R. Easther and J. Maia, JHEP **9808**, 007 (1998).

[105] R. Brandenberger and F. Finelli, JHEP **0111**, 056 (2001).

[106] R. Brandenberger, V. F. Mukhanov and T. Prokopec, Phys. Rev. Lett. **69**, 3606 (1992).

[107] R. Brandenberger, V. F. Mukhanov and T. Prokopec, Phys. Rev. **D48**, 2443 (1993).

[108] R. Brandenberger, V. F. Mukhanov and A. Sornborger, Phys. Rev. **D48**, 1629 (1993).

[109] R. Brandenberger and C. Vafa, Nucl. Phys. **B316**, 391 (1989).

[110] M. Brandle, A. Lukas and B. A. Ovrut, Phys. Rev. **D63**, 026003 (2001).

[111] M. Brunetti, E. Coccia, V. Fafone and F. Fucito, Phys. Rev. **D59**, 44027 (1999).

[112] M. Bruni, G. F. R. Ellis and P. K. S. Dunsby, Classical Quant. Grav. **9**, 921 (1992).

[113] R. Brustein, Chaos, Soliton Frac. **10**, 283 (1999).

[114] R. Brustein, S. Foffa and R. Sturani, Phys. Lett. **B471**, 352 (2000).

[115] R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov and G. Veneziano, Phys. Rev. **D51**, 6744 (1995).

[116] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. **B361**, 45 (1995).

[117] R. Brustein, M. Gasperini and G. Veneziano, Phys. Rev **D55**, 3882 (1997).

[118] R. Brustein, M. Gasperini and G. Veneziano, Phys. Lett. **B431**, 277 (1998).

[119] R. Brustein and M. Hadad, Phys. Rev. **D57**, 725 (1998).

[120] R. Brustein and M. Hadad, Phys. Lett. **B477**, 263 (2000).

[121] R. Brustein and R. Madden, Phys. Lett. **B410**, 110 (1997).

[122] R. Brustein and R. Madden, Phys. Rev. **D57**, 712 (1998).

[123] R. Brustein and R. Madden, JHEP **9907**, 006 (1999).

[124] R. Brustein and D. H. Oaknin, Phys. Rev. **D60**, 023508 (1999).

[125] R. Brustein and P. J. Steinhardt, Phys. Lett. **B302**, 196 (1993).

ok [126] R. Brustein and G. Veneziano, Phys. Lett. **B329**, 429 (1994).

[127] R. Brustein and G. Veneziano, Phys. Rev. Lett. **84**, 5695 (2000).

[128] A. Buonanno and T. Damour, Phys. Rev. **D64**, 043501 (2001).

[129] A. Buonanno, T. Damour and G. Veneziano, Nucl. Phys. **B543**, 275 (1999).

[130] A. Buonanno, M. Gasperini, M. Maggiore and C. Ungarelli, Classical Quant. Grav.**14**, L97 (1997).

[131] A. Buonanno, M. Gasperini and C. Ungarelli, Mod. Phys. Lett. **A12**, 1883 (1997).

[132] A. Buonanno, M. Lemoine and K. A. Olive, Phys. Rev. **D62**, 083513 (2000).

[133] A. Buonanno, M. Maggiore and C. Ungarelli, Phys. Rev. **D55**, 3330 (1997).

ok [134] A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, Phys. Rev. **D57**, 2543 (1998). quantum inhomogeneities

[135] A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, JHEP **9801**, 004 (1998).

[136] E. R. Caianiello, M. Gasperini and G. Scarpetta, Classical Quant. Grav. **8**, 659 (1991).

[137] R. R. Caldwell and B. Allen, Phys. Rev. **D45**, 3447 (1992).

[138] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. **80**, 582 (1998).

[139] C. Callan, S. Giddings, J. Harvey and A. Strominger, Phys. Rev. **D45**, R1005 (1992).

[140] C. G. Callan, R. C. Myers and M. J. Perry, Nucl. Phys. **B311**, 673 (1988).

[141] B. A. Campbell, A. Linde and K. A. Olive, Nucl. Phys. **B355**, 146 (1991).

[142] S. Capozziello and R. De Ritis, Int. J. Mod. Phys. **D2**, 367 (1993).

[143] S. Capozziello and R. De Ritis, Int. J. Mod. Phys. **D2**, 373 (1993).

[144] C. Caprini and R. Durrer, Phys. Rev. **D65**, 023517 (2002).

[145] B. Caron et al., Classical Quant. Grav. **14**, 1461 (1997).

[146] B. J. Carr, Astrophys. J. **201**, 1 (1975).

[147] B. J. Carr and S. W. Hawking, Mon. Not. Roy. Astron. Soc. **168**, 399 (1974).

[148] C. Cartier, E. J. Copeland and M. Gasperini, Nucl. Phys. **B607**, 406 (2001).

[149] C. Cartier, E. J. Copeland and R. Madden, JHEP **0001**, 035 (2000).

[150] C. Cartier, J. C. Hwang and E. J. Copeland, Phys. Rev. **D64**, 103504 (2001).

[151] M. Cavaglià, Classical Quant. Grav. **18**, 1355 (2001).

[152] M. Cavaglià and V. De Alfaro, Gen. Relat. Gravit. **29**, 773 (1997).

[153] M. Cavaglià and P. V. Moniz, Classical Quant. Grav. **18**, 95 (2001).

[154] M. Cavaglià and C. Ungarelli, Classical Quant. Grav. **16**, 1401 (1999).

[155] M. Cavaglià and C. Ungarelli, Nucl. Phys. Proc. Suppl. **B88**, 355 (2000).

[156] M. Cerdonio, L. Conti, J.A. Lobo, A. Ortolan and J. P. Zendri, Phys. Rev. Lett. **87**, 031101 (2001).

[157] B. Cheng and A. V. Olinto, Phys. Rev. **D50**, 2421 (1994).

[158] T. Chiba, Phys. Rev. **D59**, 083508 (1999).

[159] G. V. Chibisov and V. N. Mukhanov, Mon. Not. Roy. Astron. Soc. **200**, 535 (1982)

[160] A. Chincarini, G. Gemme, R. Parodi, P. Bernard and E. Picasso, gr-qc/0203024.

[161] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).

[162] N. Christensen, Phys. Rev. **D46**, 5250 (1992).

[163] D. Christodoulou, Commun. Math. Phys. **105**, 337 (1986).

[164] D. Christodoulou, Commun. Math. Phys. **109**, 613 (1987).

[165] D. Christodoulou, Commun. Pure Appl. Math. **44**, 339 (1991).

[166] D. Christodoulou, Commun. Pure Appl. Math. **46**, 11331 (1993).

[167] D. Clancy, A. Feinstein, J. E. Lidsey and R. Tavakol, Phys. Rev. **D60**, 043503 (1999).

[168] D. Clancy, A. Feinstein, J. E. Lidsey and R. Tavakol, Phys. Lett. **B451**, 303 (1999).

[169] D. Clancy, J. E. Lidsey and R. Tavakol, Classical Quant. Grav. **15**, 257 (1998).

[170] D. Clancy, J. E. Lidsey and R. Tavakol, Phys. Rev. **D58**, 44017 (1998).

[171] D. Clancy, J. E. Lidsey and R. Tavakol, Phys. Rev. **D59**, 063511(1999).

[172] E. Coccia, V. Fafone, G. Frossati, J. Lobo and J. Ortega, Phys. Rev. **D57**, 2051 (1998).

[173] E. Coccia, V. Fafone and I. Modena, Rev. Sci. Instrum. **63**, 5432 (1992).

[174] E. Coccia, M. Gasperini and C. Ungarelli, Phys. Rev. **D65**, 067101 (2002).

[175] E. J. Copeland, R. Easther and D. Wands, Phys. Rev. **D56**, 874 (1997).

[176] E. J. Copeland, D. Haws, T. W. Kibble, D. Mitchell and N. Turok, Nucl. Phys. **B298**, 445 (1988).

[177] E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. **D50**, 4868 (1994).

[178] E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. **D51**, 1569 (1995).

[179] E. J. Copeland, A. R. Liddle and J. E. Lidsey, Phys. Rev. **D64**, 023509 (2001).

[180] E. J. Copeland, A. R. Liddle, J. E. Lidsey and D. Wands, Phys. Rev. **D58**, 63508 (1998)

[181] E. J. Copeland, A. R. Liddle, J. E. Lidsey and D. Wands, Gen. Relat. Gravit. **30**, 1711 (1998).

[182] E. J. Copeland, J. E. Lidsey and D. Wands, Nucl. Phys. **B506**, 407 (1997).

[183] E. J. Copeland, J. E. Lidsey and D. Wands, Phys. Rev. **D57**, 625 (1998).

[184] E. J. Copeland, J. E. Lidsey and D. Wands, Phys. Lett. **B443**, 97 (1998).

[185] E. J. Copeland, P. M. Saffin and O. Tornkvist, Phys. Rev. **D61**, 105005 (2000).

ok [186] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. **B76**, 409 (1978).

in DUFF [187] A. M. Cruise, Classical Quant. Grav. **17**, 2525 (2000).

[188] E. Cuoco, G. Curci and E. Beccaria, in Proc. *Second Edoardo Amaldi Conference on Gravitational Waves*, CERN, 1997, eds. E. Coccia et al. (World Scientific, Singapore, 1998), p. 524.

[189] M. Dabrowski and C. Kiefer, Phys. Lett. **B397**, 185 (1997).

[190] T. Damour and M. Henneaux, Phys. Rev. Lett. **85**, 920 (2000).

[191] T. Damour and M. Henneaux, Gen. Relat. Gravit. **32**, 2339 (2000).

[192] T. Damour and M. Henneaux, Phys. Lett. **B488**, 108 (2000).

[193] T. Damour and M. Henneaux, Phys. Rev. Lett. **86**, 4749 (2001).

[194] T. Damour, M. Henneaux, B. Julia and H. Nicolai, Phys. Lett. **B509**, 323 (2001).

[195] T. Damour, F. Piazza and G. Veneziano, gr-qc/0204094.

[196] T. Damour, F. Piazza and G. Veneziano, hep-th/0205111.

[197] T. Damour and A. M. Polyakov, Nucl. Phys. **B423**, 532 (1994).

[198] T. Damour and A. M. Polyakov, Gen. Relat. Gravit. **26**, 1171 (1994).

[199] T. Damour and A. Vilenkin, Phys. Rev. **D53**, 2981 (1996).

[200] T. Damour and A. Vilenkin, Phys. Rev. Lett. **78**, 2288 (1997).

[201] A. Davis, K. Dimopoulos, T. Prokopec and O. Tornkvist, Phys. Lett. **B501**, 165 (2001).

[202] S. P. de Alwis, Phys. Lett. **B289**, 278 (1992).

[203] P. de Bernardis et al., Nature **404**, 955 (2000).

[204] P. de Bernardis et al., Astrophys. J. **564**, 559 (2002).

[205] B. De Carlos, J. A. Casas, F. Quevedo and E. Roulet, Phys. Lett. **B318**, 447 (1993).

O(d,d) non Abelian isometries

[206] X. C. de la Ossa and F. Quevedo, Nucl. Phys. **B403**, 377 (1993).

[207] M. Demianski, in Proc. *It. Conf. on General Relativity and Gravitational Physics*, Capri, 1990, eds. R. Cianci et al. (World Scientific, Singapore, 1991), p. 19.

[208] M. Demianski, A. G. Polnarev and P. Naselski, Phys. Rev. **D47**, 5275 (1993).

[209] J. P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, Phys. Lett. **B271**, 307 (1991).

[210] G. De Risi and M. Gasperini, Phys. Lett. **B503**, 140 (2001).

[211] G. De Risi and M. Gasperini, Phys. Lett. **B521**, 335 (2001).

[212] N. Deruelle, C. Gundlach and D. Polarski, Classical Quant. Grav. **9**, 137 91992).

[213] N. Deruelle, C. Gundlach and D. Langlois, Phys. Rev. **D45**, R3301 (1992).

[214] N. Deruelle and V. F. Mukhanov, Phys. Rev. **D52**, 5549 (1995).

[215] H. J. de Vega, A. L. Larsen and N. Sanchez, Phys. Rev. **D61**, 066003 (2000).

[216] B. S. De Witt, Phys. Rev. **160**, 1113 (1967).

[217] S. Dimopoulos et al., Astrophys. J. **330**, 545 (1988).

[218] A. D. Dolgov, Phys. Rev. **D48**, 2499 (1993).

[219] M. R. Douglas, in *Nonperturbative Aspects of Strings, Branes and Supersymmetry*, Proc. ICTP Spring School, Trieste, 1998 (World Scientific, Singapore, 1999), p. 131.

[220] R. Durrer, Phys. Rev. **D42**, 2533 (1990).

[221] R. Durrer, Fundams. Cosm. Phys., **15**, 209 (1994).

[222] R. Durrer, J. Phys. Stud. **5**, 177 (2001).

ok [223] R. Durrer, M. Gasperini, M. Sakellariadou and G. Veneziano, Phys. Lett. **B436**, 66 (1998).

ok [224] R. Durrer, M. Gasperini, M. Sakellariadou and G. Veneziano, Phys. Rev. **D59**, 43511 (1999).

[225] R. Durrer and J. Laukenmann, Classical Quant. Grav. **13**, 1069 (1996).

[226] R. Durrer and M. Sakellariadou, Phys. Rev. **D62**, 123504 (2000).

[227] R. Durrer and F. Vernizzi, hep-ph/0203275.

[228] D. M. Eardley and S. B. Giddings, gr-qc/0201034.

[229] D. Easson, hep-th/0111055, in Proc. of the *COSMO-01 Workshop*, Rovaniemi, 2001.

[230] D. A. Easson and R. Brandenberger, JHEP **9909**, 003 (1999).

[231] R. Easther and D. A. Lowe, Phys. Rev. Lett. **82**, 4967 (1999).

[232] R. Easther and K. Maeda, Phys. Rev. **D54**, 7572 (1996).

[233] R. Easther, K. Maeda and D. Wands, Phys. Rev. **D53**, 4247 (1996).

[234] G. F. R. Ellis and M. Bruni, Phys. Rev. **D40**, 1804 (1989).

[235] G. F. R. Ellis, D. C. Roberts, D. Solomons and P. K. S. Dunsby, Phys. Rev. **D62**, 084004 (2000).

[236] J. Ellis, K. Enqvist, D. V. Nanopoulos and M. Quiros, Nucl. Phys. **B277**, 231 (1986).

[237] J. Ellis, S. Kalara, K. A. Olive and C. Wetterich, Phys. Lett. **B228**, 264 (1989).

[238] J. Ellis, D. V. Nanopoulos and M. Quiros, Phys. Lett. **B174**, 176 (1986).

[239] J. Ellis, D. V. Nanopoulos and S. Sarkar, Nucl. Phys. **B259**, 175 (1985).

[240] J. Ellis, C. Tsamish and M. Voloshin, Phys. Lett. **B194**, 291 (1987).

[241] K. Enqvist and M. S. Sloth, Nucl. Phys. **B626**, 395 (2002).

[242] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions* (MacGraw-Hill, New York, 1955), Vol. 3.

[243] R. Fabbri and M. Pollock, Phys. Lett. **B125**, 445 (1983).

[244] A. Feinstein, gr-qc/0106001, in *Fourth Mexican School on Gravity and Mathematical Physics*, Huatulco, 2000.

[245] A. Feinstein, K. E. Kunze and M. A. Vázquez–Mozo, Classical Quant. Grav. **17**, 3599 (2000).

[246] A. Feinstein, K. E. Kunze and M. A. Vázquez–Mozo, Phys. Lett. **B491**, 190 (2000).

[247] A. Feinstein, R. Lazkoz and M. A. Vázquez–Mozo, Phys. Rev. **D56**, 5166 (1997).

[248] A. Feinstein and M. A. Vázquez–Mozo, Phys. Lett. **B441**, 40 (1998).

[249] A. Feinstein and M. A. Vázquez–Mozo, Nucl. Phys. **B568**, 405 (2000).

[250] S. Ferrara, C. Kounnas and F. Zwirner, Nucl. Phys. **B429**, 589 (1994).

[251] F. Finelli and R. Brandenberger, Phys. Rev. **D65**, 103522 (2002).

[252] F. Finelli and A. Gruppuso, Phys. Lett. **B502**, 216 (2001).

[253] E. Fischbach and C. Talmadge, Nature **356**, 207 (1992).

[254] W. Fischler and L. Susskind, hep-th/9806039.

[255] E. Flanagan, Phys. Rev. **D48**, 2389 (1993).

[256] E. E. Flanagan, D. Marolf and R. M. Wald, Phys. Rev. **D62**, 084035 (2000).

[257] S. Foffa and M. Maggiore, Phys. Rev. **D58**, 023505 (1998).

[258] S. Foffa, M. Maggiore and R. Sturani, Phys. Rev. **D59**, 043507 (1999).

[259] S. Foffa, M. Maggiore and R. Sturani, Nucl. Phys. **B552**, 395 (1999).

[260] L. H. Ford and L. Parker, Phys. Rev. **D16**, 1601 (1977).

[261] M. Gaillard, H. Murayama and K. Olive, Phys. Lett. **B355**, 71 (1995).

[262] M. Galluccio, F. Litterio and F. Occhionero, Phys. Rev. Lett. **79**, 970 (1997).

[263] W. D. Garretson, G. B. Field and S. M. Carrol, Phys. Rev. **D46**, 5346 (1992);

[264] J. Garriga and E. Verdaguer, Phys. Rev. **D39**, 1072 (1989).

[265] M. Gasperini's web site at `http://www.ba.infn.it/~gasperin`

[266] M. Gasperini, Phys. Lett. **B163**, 84 (1985).

[267] M. Gasperini, Phys. Rev. Lett. **56**, 2873 (1986).

[268] M. Gasperini, Phys. Lett. **B258**, 70 (1991).

[269] M. Gasperini, in Proc. *Advances in Theoretical Physics*, Vietri, 1990, ed. E. R. Caianiello (World Scientific, Singapore 1991), p. 77.

[270] M. Gasperini, Gen. Relat. Gravit. **24**, 219 (1992).

[271] M. Gasperini, Phys. Lett. **B327**, 314 (1994).

[272] M. Gasperini, in Proc. *Second Paris Cosmology Colloquium*, Observatoire de Paris, 1994, eds. H. J. de Vega and N. Sanchez (World Scientific, Singapore, 1995), p. 429.

[273] M. Gasperini, in Proc. *Birth of the Universe and Fundamental Physics*, Rome, 1994, ed. F. Occhionero (Springer, New York, 1995), p. 71.

[274] M. Gasperini, in Proc. *Third Paris Cosmology Colloquium*, Observatoire de Paris, 1995, eds. H. J. de Vega and N. Sanchez (World Scientific, Singapore, 1996), p. 253.

[275] M. Gasperini, in *String Gravity and Physics at the Planck Energy Scale*, Proc. Erice School, 1995, eds. N. Sanchez and A. Zichichi (Kluwer Acad. Publ., Dordrecht, 1996), p. 305.

[276] M. Gasperini, in Proc. *It. Conf. on General Relativity and Gravitational Physics*, Rome, 1996, eds. M. Bassan et al. (World Scientific, Singapore, 1997), p. 181.

[277] M. Gasperini, Phys. Rev. **D56**, 4815 (1997).

[278] M. Gasperini, in *String theory in curved space times*, ed. N. Sanchez (World Scientific, Singapore 1998), p. 333.

[279] M. Gasperini, in Proc. *Fourth Paris Cosmology Colloquium*, Observatoire de Paris, 1997, eds. H. J. de Vega and N. Sanchez (World Scientific, Singapore, 1998), p. 85.

[280] M. Gasperini, in Proc. *Second Edoardo Amaldi Conference on Gravitational Waves*, CERN, 1997, eds. E. Coccia et al. (World Scientific, Singapore, 1998), p. 62.

[281] M. Gasperini, Gen. Relat. Gravit. **30**, 1703 (1998).

[282] M. Gasperini, Int. J. Mod. Phys. **A13**, 4779 (1998).

[283] M. Gasperini, in Proc. *Fifth Paris Cosmology Colloquium*, Observatoire de Paris, 1998, eds. H. J. de Vega and N. Sanchez (Publications Observatoire de Paris, 1999), p. 317.

[284] M. Gasperini, Mod. Phys. Lett. **A14**, 1059 (1999).

[285] M. Gasperini, Phys. Lett. **B470**, 67 (1999).

[286] M. Gasperini, Phys. Rev. **D61**, 87301 (2000).

[287] M. Gasperini, Phys. Lett. **B477**, 242 (2000).

[288] M. Gasperini, Classical Quant. Grav. **17**, R1 (2000).

[289] M. Gasperini, in *Gravitational Waves*, Proc. SIGRAV School, Centre A. Volta, Como, 1999), eds. I. Ciufolini et al. (IOP Publishing, Bristol, 2001), p. 280.

[290] M. Gasperini, Int. J. Mod. Phys. **D10**, 15 (2001).

[291] M. Gasperini, Phys. Rev. **D63**, 047301 (2001).

[292] M. Gasperini, in Proc. *IX Marcel Grossmann Meeting*, Rome, 2000, eds. V. G. Gurzadyan, R. Jantzen and R. Ruffini (World Scientific, Singapore, 2002), in press.

[293] M. Gasperini, Phys. Rev. **D64**, 043510 (2001).

[294] M. Gasperini and M. Giovannini, Phys. Lett. **B287**, 56 (1992).

[295] M. Gasperini and M. Giovannini, Phys. Lett. **B282**, 36 (1992).

[296] M. Gasperini and M. Giovannini, Classical Quant. Grav. **9**, L137 (1992).

[297] M. Gasperini and M. Giovannini, Phys. Lett. **B301**, 334 (1993).

[298] M. Gasperini and M. Giovannini, Classical Quant. Grav. **10**, L133 (1993).

[299] M. Gasperini and M. Giovannini, Phys. Rev. **D47**, 1519 (1993).

[300] M. Gasperini and M. Giovannini, Classical Quant. Grav. **14**, 735 (1997).

[301] M. Gasperini and M. Giovannini, in *String theory in curved space times*, ed. N. Sanchez (World Scientific, Singapore, 1998), p. 249.

[302] M. Gasperini, M. Giovannini, K. A. Meissner and G. Veneziano, Nucl. Phys. (Proc. Suppl.) **B49**, 70 (1996).

[303] M. Gasperini, M. Giovannini, K. A. Meissner and G. Veneziano, in *String theory in curved space times*, ed. N. Sanchez (World Scientific, Singapore, 1998), p. 49.

[304] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. **D48**, R439 (1993).

[305] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75**, 3796 (1995).

[306] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. **D52**, 6651 (1995).

[307] M. Gasperini, M. Maggiore, and G. Veneziano, Nucl. Phys. **B494**, 315 (1997).

[308] M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. **B272**, 277 (1991).

[309] M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. **B296**, 51 (1992).

ok [310] M. Gasperini, J. Maharana, and G. Veneziano, Nucl. Phys. **B472**, 349 (1996).

[311] M. Gasperini, F. Piazza and G. Veneziano, Phys. Rev. **D65**, 023508 (2001).

[312] M. Gasperini and R. Ricci, Classical Quant. Grav. **12**, 677 (1995).

[313] M. Gasperini, R. Ricci and G. Veneziano, Phys. Lett. **B319**, 438 (1993).

ok [314] M. Gasperini, N. Sanchez and G. Veneziano, Int. J. Mod. Phys. **A6**, 3853 (1991)

[315] M. Gasperini, N. Sanchez and G. Veneziano, Nucl. Phys. B **364**, 365 (1991).

[316] M. Gasperini and G. Ummarino, Phys. Lett. **B266**, 275 (1991).

[317] M. Gasperini and C. Ungarelli, Phys. Rev. **D64**, 064009 (2001).

[318] M. Gasperini and G. Veneziano, Phys. Lett. **B277**, 256 (1992).

ok [319] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).

ok [320] M. Gasperini and G. Veneziano, Mod. Phys. Lett. **A8**, 3701 (1993).

[321] M. Gasperini and G. Veneziano, Phys. Rev. **D50**, 2519 (1994).

[322] M. Gasperini and G. Veneziano, Gen. Rel. Grav. **28**, 1301 (1996).

[323] M. Gasperini and G. Veneziano, Phys. Lett. **B387**, 715 (1996).

[324] M. Gasperini and G. Veneziano, Phys. Rev. **D59**, 43503 (1999).

[325] U. Gen, A. Ishibashi and T. Tanaka, hep-th/0110286.

[326] M. Giovannini, *Teoria quantistica delle perturbazioni in cosmologia di stringa*, PhD Thesis (unpublished), University of Turin (1995).

[327] M. Giovannini, Phys. Rev. **D55**, 595 (1997).

[328] M. Giovannini, Phys. Rev. **D56**, 631 (1997).

[329] M. Giovannini, Phys. Rev. **D56**, 3198 (1997).

[330] M. Giovannini, Phys. Rev. **D57**, 7223 (1998).

[331] M. Giovannini, Phys. Rev. **D59**, 063503 (1999).

[332] M. Giovannini, Phys. Rev. **D59**, 083511 (1999).

[333] M. Giovannini, Phys. Rev. **D59**, 123518 (1999).

[334] M. Giovannini, Phys. Rev. **D60**, 123511 (1999).

[335] M. Giovannini, Phys. Rev. **D62**, 123505 (2000).

[336] M. Giovannini, Phys. Rev. **D64**, 064023 (2001).

[337] M. Giovannini, H. Kurki-Suonio and E. Sihvola, astro-ph/0203430.

[338] M. Giovannini and M. Shaposhnikov, Phys. Rev. **D62**, 103512 (2000).

[339] G. Giribet and C. Simeone, Mod. Phys. Lett. **A16**, 19 (2001).

ok [340] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. **244**, 77 (1994).

[341] M. Gleiser and J. G. Taylor, Phys. Lett. **B164**, 36 (1985).

[342] A. S. Goncharov, A. D. Linde and M. I. Vysotsky, Phys. Lett. **B147**, 279 (1984).

[343] A. Ghosh, G. Pollifrone and G. Veneziano, Phys. Lett. **B440**, 20 (1998).

[344] A. Ghosh, R. Madden and G. Veneziano, Nucl. Phys. **B570**, 207 (2000).

[345] D. Grasso and H. R. Rubinstein, Phys. Rep. **348**, 163 (12001).

ok [346] M. B. Green, J. Schwartz and E. Witten, *Superstring theory* (University Press, Cambridge, 1987).

[347] B. Greene, *The Elegant Universe* (W. W. Norton and Co., New York & London, 1999).

[348] L. P. Grishchuk, Sov. Phys. JETP **40**, 409 (1975).

[349] L. P. Grishchuk, Sov. Phys. Usp. **31**, 490 (1988).

[350] L. P. Grishchuk, Sov. Sci. Rev. E. Astrophys. Space Phys. **7**, 267 (1988).

[351] L. P. Grishchuk, in *Squeezed states and uncertainty relations*, eds. D. Han et al. (NASA Conf. Pub. N. 31353, 1992), p. 329.

[352] L. P. Grishchuk, in Proc. *6th Marcel Grossmann Meeting*, Kyoto, 1991, ed. H. Sato (World Scientific, Singapore, 1992).

[353] L. P. Grishchuk, in *Gravitational Waves and Experimental Gravity*, Proc. XXXIV Rencontres de Moriond, Les Arcs, 1999, eds. J. Tran Thanh Van et al. (World Scientific, Singapore, 2000).

[354] L. P. Grishchuk and Y. V. Sidorov, Classical Quant. Grav. **6**, L161 (1989).

[355] L. P. Grishchuk and Y. V. Sidorov, Phys. Rev. **D42**, 3413 (1990).

[356] L. P. Grishchuk and M. Solokhin, Phys. Rev. **D43**, 2566 (1991).

[357] M. McGuigan, Phys. Rev. **D38**, 3031 (1988).

[358] M. McGuigan, Phys. Rev. **D39**, 2229 (1989).

[359] M. McGuigan, Phys. Rev. **D41**, 418 (1990).

[360] A. Guth, Phys. Rev. **D23**, 347 (1981).

[361] W. Hamilton, in Proc. *Second Edoardo Amaldi Conference on Gravitational Waves*, CERN, 1997, eds. E. Coccia et al. (World Scientific, Singapore, 1998), p. 115.

[362] A. Hammesfahr, Classical Quant. Grav. **18**, 4045 (2001).

[363] S. Hanany et al., Astrophys. J. **545**, L5 (2000).

[364] E. R. Harrison, Phys. Rev. **D1**, 2726 (1970).

[365] E. R. Harrison, Phys. Rev. Lett. **30**, 188 (1973).

[366] J. B. Hartle and S. W. Hawking, Phys. Rev. **D28**, 2960 (1983).

[367] S. Hassan and A. Sen, Nucl. Phys. **B375**, 103 (1992).

[368] S. W. Hawking, Astrophys. J. **145**, 544 (1966).

[369] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[370] S. W. Hawking, Nucl. Phys. **B239**, 257 (1984).

[371] S. W. Hawking and G. F. R. Ellis, *The large scale structure of spacetime* (University Press, Cambridge, 1973).

[372] S. W. Hawking and R. Penrose, Proc. Roy. Soc. Lond. **A314**, 529 (1970).

[373] A. Hebecker and C. Wetterich, Phys. Rev. Lett. **85**, 3339 (2000).

[374] I. Heng, D. Blair, E. Ivanov and M. Tobar, in Proc. *Second Edoardo Amaldi Conference on Gravitational Waves*, CERN, 1997, eds. E. Coccia et al. (World Scientific, Singapore, 1998), p. 127.

[375] G. 't Hooft, in *Abdus Salam Festschrift: A Collection of Talks*, eds. A. Ali , J. Ellis and S. Randjbar-Daemi (World Scientific, Singapore, 1993).

[376] P. Horava and E. Witten, Nucl. Phys. **B460**, 506 (1996).

[377] P. Horava and E. Witten, Nucl. Phys. **B475**, 94 (1996).

[378] K. Hotta, K. Kikkawa and H. Kunitomo, Prog. Theor. Phys. **98**, 687 (1997).

[379] J. Hough, in Proc. *Second Edoardo Amaldi Conference on Gravitational waves*, CERN, 1997, eds. E. Coccia et al. (World Scientific, Singapore, 1998), p. 97.

[380] C. D. Hoyle, U. Schmidt, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, D. J. Kapner and H. E. Swanson, Phys. Rev. Lett. **86**, 1418 (2001).

[381] J. H. Hwang, Astrophys. J. **375**, 443 (1991).

[382] J. H. Hwang, Astropart. Phys. **8**, 201 (1998).

[383] J. H. Hwang and H. Noh, Phys. Rev. **D61**, 043511 (2000).

[384] J. H. Hwang and H. Noh, Phys. Rev. **D65**, 124010 (2002).

[385] J. H. Hwang and E. T. Vishniac, Astrophys. J. **382**, 363 (1991).

[386] W. Israel, Nuovo Cimento **B44**, 1 (1966).

[387] W. Johnson and S. Merkowitz, Phys. Rev. Lett. **70**, 2367 (1993).

[388] T. Kajita and Y. Totsuka, Rev. Mod. Phys. **73**, 85 (2001).

[389] R. Kallosh, L. Kofman and A. Linde, Phys. Rev. **D64**, 123523 (2001).

[390] R. Kallosh, L. Kofman, A. Linde and A. Tseytlin, Phys. Rev. **D64**, 123524 (2001).

[391] N. Kaloper, Phys. Rev. **D55**, 3394 (1997).

[392] N. Kaloper and A. Linde, Phys. Rev. **D60**, 103509 (1999).

[393] N. Kaloper, I. I. Kogan and K. A. Olive, Phys. Rev. **D57**, 7340 (1998).

[394] N. Kaloper, A. Linde and R. Bousso, Phys. Rev. **D59**, 043508 (1999).

[395] N. Kaloper, R. Madden and K. A. Olive, Nucl. Phys. **B452**, 677 (1995).

[396] N. Kaloper, R. Madden and K. A. Olive, Phys. Lett. **B371**, 34 (1996).

[397] N. Kaloper and K. A. Meissner, Phys. Rev. **D56**, 7940 (1997).

[398] N. Kaloper and K. A. Olive, Astropart. Phys. **1**, 185 (1993).

[399] N. Kaloper and K. A. Olive, Phys. Rev. **D57**, 811 (1998).

[400] M. Kamionkowski and A. H. Jaffe, Int. J. Mod. Phys. **A16**, 116 (2001).

[401] M. Kamionkowski, A. Kosowsky and A. Stebbins, Phys. Rev. Lett. **78**, 2058 (1997).

[402] P. Kanti, J. Rizos and K. Tamvakis, Phys. Rev. **D59**, 083512 (1999).

[403] V. Kaplunovsky, Phys. Rev. Lett. **55**, 1036 (1985).

[404] V. Kaspi, J. Taylor and M. Ryba, Astrophys. J. **428**, 713 (1994).

[405] K. Kawabe and the TAMA collaboration, Classical Quant. Grav. **14**, 1477 (1997).

[406] S. Kawai, M. Sakagami and J. Soda, Phys. Lett. **B437**, 284 (1998).

[407] S. Kawai and J. Soda, Phys. Rev. **D59**, 063506 (1999).

[408] S. Kawai and J. Soda, Phys. Lett. **B460**, 41 (1999).

[409] M. Kawasaki, N. Sugiyama and T. Tanagida, Phys Rev. **D54**, 2442 (1996).

[410] A. A. Kehagias and A. Lukas, Nucl. Phys. **B477**, 549 (1996).

[411] S. Y. Khlebnikov and I. I. Tkachev, Phys. Rev. **D56**, 653 (1997).

ekpyrotic [412] J. Khouri, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. **D64**, 123522 (2001).

[413] J. Khouri, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, Phys. Rev. **D65**, 086007 (2002).

[414] T. W. Kibble and A. Vilenkin, Phys. Rev. **D52**, 679 (1995).

ok [415] K. Kim, Phys. Rev. Lett. **43**, 103 (1979).

[416] E. Kiritsis and C. Kounnas, Phys. Lett. **B331**, 51 (1994).

[417] E. Kiritsis and C. Kounnas, in Proc. *Second Paris Cosmology Colloquium*, Observatoire de Paris, 1994, eds. H. J. De Vega and N. Sanchez (World Scientific, Singapore, 1995), p. 500.

[418] E. Kohlprath and G. Veneziano, gr-qc/0203093.

[419] E. W. Kolb, D. Lindley and D. Seckel, Phys. Rev. **D30**, 1205 (1984).

[420] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990).

[421] N. Kozimirov and I. I. Tkachev, Mod. Phys. Lett. **A4**, 2377 (1988).

[422] L. M. Krauss and M. White, Phys. Rev. Lett. **69**, 869 (1992).

[423] K. E. Kunze, Classical Quant. Grav. **16**, 3795 (1999).

[424] K. E. Kunze and R. Durrer, Classical Quant. Grav. **17**, 2597 (2000).

[425] L. Landau and E. Lifshitz, *The classical theory of fields* (Pergamon Press, Oxford, 1962).

[426] F. Larsen and F. Wilczek, Phys. Rev. **D55**, 4591 (1997).

[427] A. Lawrence and E. Martinec, Classical Quant. Grav. **13**, 63 (1996).

[428] Y. Leblanc, Phys. Rev. **D38**, 3087 (1988).

[429] J. Levin, Phys. Rev. **D51**, 462 (1995).

[430] D. Lemoine and M. Lemoine, Phys. Rev. **D52**, 1955 (1995).

[431] M. Lemoine, Phys. Rev. **D60**, 103522 (1999).

[432] J. Lesgourgues, D. Polarski, S. Prunet and A. A. Starobinski, Astron. Astrophys. **359**, 414 (2000).

[433] A. R. Liddle and D. H. Lyth, Phys. Rep. **231**, 1 (1993).

[434] J. E. Lidsey, Phys. Rev. **D52**, R5407 (1995).

[435] J. E. Lidsey, Phys. Rev. **D55**, 3303 (1997).

[436] J. E. Lidsey, Classical Quant. Grav. **15**, L77 (1999).

[437] J. E. Lidsey, D. Wands and E. J. Copeland, Phys. Rep. **337**, 343 (2000).

[438] E. M. Lifchitz and I. M. Kalatnikov, Adv. Phys. **12**, 185 (1963).

[439] A. Linde, Phys. Lett. **B129**, 177 (1983).

[440] A. D. Linde, Sov. Phys. JETP **60**, 211 (1984).

[441] A. D. Linde, *Particle physics and inflationary cosmology* (Harwood, New York, 1990).

[442] A. D. Linde, Phys. Lett. **B351**, 99 (1995).

[443] A. Linde, Phys. Rev. **D58**, 083514 (1998).

[444] LIGO MIT/Caltech Groups, *Research and Development program for Advanced LIGO detectors*, LIGO-M970107-00-M (September 1997).

[445] C. Lovelace, Phys. Lett. **B135**, 75 (1984).

[446] H. Lu, S. Mukherji, C. N. Pope and K. W. Xu, Phys. Rev. **D55**, 7926 (1997).

[447] F. Lucchin and S. Matarrese, Phys. Lett. **B164**, 282 (1985).

[448] H. Luck and the GEO600 Team, Classical Quant. Grav. **14**, 1741 (1997).

[449] A. Lukas and B. A. Ovrut, Phys. Lett. **B437**, 291 (1998).

[450] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Phys. Rev. **D59**, 086001 (1999).

[451] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Nucl. Phys. **B552**, 246 (1999).

[452] A. Lukas, B. A. Ovrut and D. Waldram, Phys. Lett. **B393**, 65 (1997).

[453] A. Lukas, B. A. Ovrut and D. Waldram, Nucl. Phys. **B495**, 365 (1997).

[454] A. Lukas and R. Poppe, Mod. Phys. Lett. **A12**, 597 (1997).

[455] V. N. Lukash, Sov. Phys. JETP **52**, 807 (1980).

[456] D. H. Lyth, Phys. Lett. **B526**, 173 (2002).

[457] D. H. Lyth and D. Robert, Phys. Rev. **D57**, 7120 (1998).

[458] D. H. Lyth and E. Stewart, Phys. Rev. **D53**, 1784 (1996).

[459] D. H. Lyth and D. Wands, Phys. Lett. **B524**, 5 (2002).

[460] R. Madden, JHEP **0101**, 23 (2001).

[461] K. Maeda and M. D. Pollock, Phys. Lett. **B173**, 251 (1986).

[462] M. Maggiore, Nucl. Phys. **B525**, 413 (1998).

[463] M. Maggiore, Phys. Rep. **331**, 283 (2000).

[464] M. Maggiore and A. Nicolis, Phys. Rev. **D62**, 024004 (2000).

[465] M. Maggiore and A. Riotto, Nucl. Phys. **B548**, 427 (1999).

ok [466] M. Maggiore and R. Sturani, Phys. Lett. **B415**, 335 (1997).

[467] J. Maharana, S. Mukherji and S. Panda, Mod. Phys. Lett. **A12**, 447 (1997).

[468] J. Maharana, E. Onofri and G. Veneziano, JHEP **9804**, 004 (1998).

[469] J. Maharana and J. H. Schwartz, Nucl. Phys. **B390**, 3 (1992).

[470] MAP's web site at `http://map.gsfc.nasa.gov`

[471] A. L. Maroto, Phys. Rev. **D64**, 083006 (2001).

[472] M. Mars, Phys. Rev. **D51**, 3899 (1995).

[473] X. Martin and A. Vilenkin, Phys. Rev. Lett. **77**, 2879 (1996).

[474] J. Marto and P. V. Moniz, in Proc. *IX Marcel Grossmann Meeting*, Rome, 2000, eds. V. G. Gurzadyan, R. Jantzen and R. Ruffini (World Scientific, Singapore, 2002), in press.

[475] P. Mauskopf et al., Astrophys. J. **536**, L59 (2000).

[476] K. A. Meissner, Phys. Lett. **B392**, 298 (1997).

ok [477] K. A. Meissner and G. Veneziano, Mod. Phys. Lett. **A6**, 3397 (1991).

[478] K. A. Meissner and G. Veneziano, Phys. Lett. **B267**, 33 (1991).

[479] A. Melchiorri, F. Vernizzi, R. Durrer and G. Veneziano, Phys. Rev. Lett. **83**, 4464 (1999).

[480] L. E. Mendes and A. R. Liddle, Phys. Rev. **D60**, 063508 (1999).

[481] S. Merkowitz and W. Johnson, Phys. Rev. **D51**, 2546 (1995).

[482] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. **B293**, 385 (1987).

[483] A. D. Miller et al., Astrophys. J. **524**, L1 (1999).

[484] A. G. Mirzabekian and G. A. Vilkovisky, Classical Quant. Grav. **12**, 2173 (1995).

[485] R. Moessner and M. Trodden, Phys. Rev. **D51**, 2801 (1995).

[486] S. Mollerach, Phys. Rev. **D42**, 313 (1990).

[487] T. Moroi and T. Takahashi, Phys. Lett. **B522**, 215 (2001).

[488] T. Moroi and T. Takahashi, hep-ph/0206026.

[489] M. Mueller, Nucl. Phys. **B337**, 37 (1990).

[490] V. F. Mukhanov, Sov. Phys. JETP **67**, 1297 (1988).

[491] V. F. Mukhanov and R. H. Brandenberger, Phys. Rev. Lett. **68**, 1969 (1992).

[492] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

[493] K. Nakao, T. Harada, M. Shibata, S. Kawamura and T. Nakamura, Phys. Rev. **D63**, 082001 (2001).

[494] Y. Nambu, Phys. Rev. **D63**, 044013 (2001).

[495] Y. Nambu, Phys. Rev. **D65**, 104013 (2002).

[496] Y. Nambu and M. Sasaki, Phys. Rev. **D42**, 3918 (1990).

[497] C. R. Nappi and E. Witten, Phys. Lett. **B293**, 309 (1992).

[498] K. S. Narain, Phys. Lett. **B169**, 41 (1986).

[499] K. S. Narain, M. H. Sarmadi and E. Witten, Nucl. Phys. **B279**, 369 (1987).

[500] M. Narita, T. Torii and K. Maeda, Classical Quant. Grav. **17**, 4597 (2000).

[501] H. B. Nilles, Phys. Rep. **110**, 1 (1984).

[502] A. Notari and A. Riotto, hep-th/0205019.

[503] M. Novello, L. A. R. Oliveira, J. M. Salim and E. Elbaz, Int. J. Mod. Phys. **D1**, 641 (1993).

[504] M. Novello and J. M. Salim, Phys. Rev. **D20**, 377 (1978).

[505] I. D. Novikov and A. G. Polnarev, Astron. Zh. **57**, 250 (1980) [Sov. Astron. **24**, 147 (1980)].

[506] G. Pallottino, in Proc. *Gravitational Waves, Sources and Detectors*, Cascina, 1996, eds. I Ciufolini and F. Fidecaro (World Scientific, Singapore, 1997), p. 159.

[507] A. Papapetrou, Proc. Roy. Soc. Lond. **A209**, 248 (1951).

[508] E. N. Parker, *Cosmical magnetic fields* (Clarendon Press, Oxford, 1979).

[509] J. Peacock and S. Dodds, Mon. Not. Roy. Astron. Soc. **267**, 1020 (1994).

[510] P. J. E. Peebles and A. Vilenkin, Phys. Rev. **D59**, 063505 (1999).

[511] N. Pinto-Neto and R. Colistete Jr., Phys. Lett. **A290**, 219 (2001).

[512] F. Pegoraro, E. Picasso and L. A. Radicati, J. Phys. **A11**, 1949 (1978).

[513] R. Penrose, Proc. Roy. Soc. Lond. **A284**, 159 (1965).

[514] R. Penrose, Phys. Rev. Lett. **14**, 57 (1965).

[515] R. Penrose, *The emperor's new mind* (Oxford University Press, New York, 1989).

[516] S. Perlmutter et al., Astrophys. J. **517**, 565 (1999).

[517] S. Perlmutter et al., Nature **391**, 51 (1999).

[518] P. Peter and M. Pinto-Neto, hep-th/0203013.

[519] L. O. Pimentel, Mod. Phys. Lett. **A14**, 43 (1999).

[520] G. Pizzella, Classical Quant. Grav. **14**, 1481 (1997).

[521] PLANCK's web site at
`http://astro.estec.esa.nl/SA-general/Projects/Planck`

[522] D. Polarski and A. A. Starobinski, Classical Quant. Grav. **13**, 377 (1996).

[523] J. Polchinski, Rev. Mod. Phys. **68**, 1245 (1996).

ok [524] J. Polchinski, *String theory*, (University Press, Cambridge, 1998).

[525] M. D. Pollock, Nucl. Phys. **B324**, 187 (1989).

[526] M. D. Pollock, Int. J. Mod. Phys. **A7**, 4149 (1992).

[527] R. Poppe and S. Schwager, Phys. Lett. **B393**, 51 (1997).

[528] G. A. Prodi et al., in Proc. *Second Edoardo Amaldi Conference on Gravitational Waves*, CERN, 1997, eds. E. Coccia et al. (World Scientific, Singapore, 1998), p. 148.

[529] K. S. Rama, Phys. Lett. **B408**, 91 (1997).

[530] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).

[531] L. Randall and S. Thomas, Nucl. Phys. **B449**, 229 (1995).

[532] B. Ratra, Astrophys. Lett. **391**, L1 (1992).

[533] S. J. Rey, Phys. Rev. Lett. **77**, 1929 (1996).

[534] C. E. Reece, P. J. Reiner and A. C. Melissinos, Phys. Lett. **A104**, 341 (1984).

[535] A. G. Riess et al., Astron. J. **116**, 1009 (1998).

[536] W. Rindler, Mon. Not. Roy. Astron. Soc. **116**, 6 (1956).

[537] A. Riotto, Phys. Rev. **D61**, 123506 (2000).

[538] R. C. Ritter et al., Phys. Rev. **D42**, 977 (1990).

[539] J. Rizos and K. Tamvakis, Phys. Lett. **B326**, 57 (1994).

[540] N. Rosen, Astrophys. J. **297**, 347 (1985).

[541] V. A. Rubakov, Phys. Lett. **B148**, 280 (1984).

[542] V. A. Rubakov, Phys. Lett. **B214**, 503 (1988).

[543] V. A. Rubakov, M. Sazhin and A. Veryaskin, Phys. Lett. **B115**, 189 (1982).

[544] J. G. Russo, L. Susskind and L. Thorlacius, Phys. Rev. **D46**, 3444 (1992).

[545] R. K. Sachs, Proc. Roy. Soc. Lond. **A270**, 103 (1962).

[546] R. K. Sachs and M. Wolfe, Astrophys. J. **147**, 73 (1967).

[547] A. A. Saharian, hep-th/9709188.

[548] V. Sahni, Phys. Rev. **D42**, 453 (1990).

[549] V. Sahni, H. Feldman and A. Stebbins, Astrophys. J. **385**, 1 (1992).

[550] V. Sahni, M. Sami and T. Souradeep, Phys. Rev. **D65**, 023518 (2002).

[551] N. Sanchez and G. Veneziano, Nucl. Phys. **B 333**, 253 (1990).

[552] A. K. Sanyal and B. Modak, Phys. Rev. **D63**, 064021 (2001).

[553] M. Sasaki, Prog. Theor. Phys. **70**, 394 (1980).

[554] M. Sasaki, Prog. Theor. Phys. **76**, 1036 (1986).

[555] B. S. Sathyaprakash, P. Goswami and K. P. Sinha, Phys. Rev. **D33**, 2196 (1986).

[556] J. H. Schwarz, Nucl. Phys. Proc. Suppl. **55B**, 1 (1997).

[557] V. F. Schwarztmann, JETP Letters **9**, 184 (1969).

[558] D. Seckel and M. S. Turner, Phys. Rev. **D32**, 3178 (1985).

[559] U. Seljak and M. Zaldarriaga, Phys. Rev. Lett. **78**, 2054 (1997).

[560] A. Sen, Phys. Lett. **B271**, 295 (1991).

[561] A. Sen, Int. J. Mod. Phys. **A9**, 3707 (1994).

[562] J. M. M. Senovilla, Phys. Rev. Lett. **64**, 2219 (1990).

[563] D. Shadev, Phys. Lett. **B317**, 155 (1984).

[564] M. Shibata, K. Nakao and T. Nakamura, Phys. Rev. **D50**, 7304 (1994).

[565] B. L. Shumaker, Phys. Rep. **135**, 317 (1986).

ok

[566] P. Sikivie, Int. J. Mod. Phys. **D3** Suppl., 1 (1994).

[567] G. F. Smooth et al., Astrophys. J. **396**, 1 (1992).

[568] SNO Collaboration, nucl-ex/0204008.

[569] SNO Collaboration, nucl-ex/0204009.

[570] A. A. Starobinski, JETP Lett. **30**, 682 (1979).

[571] E. D. Stewart and D. H. Lyth, Phys. Lett. **B302**, 171 (1993).

[572] P. Steinhardt, in *Critical problems in physics*, edis. V. L. Fitch and D. R. Marlow (University Press, Princeton, NJ, 1997).

[573] P. J. Steinhardt and N. Turok, hep-th/0111030.

ok [574] L. Susskind, J. Math. Phys. **36**, 6377 (1995).

[575] J. L. Synge, *Relativity: the general theory* (North Holland, Amsterdam, 1960).

[576] P. Szekeres, J. Math. Phys. **13**, 286 (1972).

ok [577] T. R. Taylor and G. Veneziano, Phys. Lett. **B213**, 459 (1988).

[578] R. C. Tolman, *Relativity, thermodynamics and cosmology* (Dover Publications, New York, 1987).

[579] P. Townsend, Phys. Lett. **B350**, 184 (1995).

[580] A. A. Tseytlin, Mod. Phys. Lett. **A6**, 1721 (1991).

[581] A. A. Tseytlin, Classical Quant. Grav. **9**, 403 (1992).

[582] A. A. Tseytlin and C. Vafa, Nucl. Phys. **B372**, 443 (1992).

[583] S. Tsujikawa, Phys. Lett. **B526**, 179 (2002).

[584] S. Tsujikawa and H. Yajima, Phys. Rev. **D64**, 023519 (2001).

[585] M. S. Turner, Phys. Rev. **D55**, 435 (1997).

[586] M. S. Turner, Phys. Rep. **333**, 619 (2000).

[587] M. S. Turner and A. G. Riess, astro-ph/0106051.

[588] M. S. Turner and E. J. Weinberg, Phys. Rev. **D56**, 4604 (1997).

[589] M. S. Turner and L. M. Widrow, Phys. Rev. **D37**, 2473 (1988).

[590] M. S. Turner and F. Wilczek, Phys. Rev. Lett. **65**, 3080 (1990).

[591] N. Turok, Phys. Rev. Lett. **60**, 549 (1988).

[592] N. Turok and S. W. Hawking, gr-qc/9802062.

[593] C. Ungarelli and A. Vecchio, in *Gravitational Waves*, Proc. Third Edoardo Amaldi Conference, CALTECH, Pasadena, 1999, ed. S. Meshkov (AIP Conference Proceedings **523** (2000)) p. 90.

[594] C. Ungarelli and A. Vecchio, Phys. Rev. **D63**, 064030 (2001).

[595] T. Vachaspati, Phys. Lett. **B265**, 258 (1991).

[596] C. van de Bruck, M. Dorca, R. H. Brandenberger and A. Lukas, Phys. Rev. **D62**, 123515 (2000).

[597] G. Veneziano, Europhys. Lett. **2**, 133 (1986).

[598] G. Veneziano, in Proc. *School of Subnuclear Physics*, Erice, 1989, ed. A. Zichichi (Plenum Press, New York, 1990), p. 199.

ok [599] G. Veneziano, Phys. Lett. **B265**, 287 (1991).

[600] G. Veneziano, in Proc. *Second Paris Cosmology Colloquium*, Observatoire de Paris, 1994, eds. H. J. de Vega and N. Sanchez (World Scientific, Singapore, 1995), p. 322.

[601] G. Veneziano in *String Gravity and Physics at the Planck Energy Scale*, Proc. Erice School, 1995, eds. N. Sanchez and A. Zichichi (Kluwer Acad. Publ., Dordrecht, 1996), p. 285.

ok [602] G. Veneziano, in Proc. *Gravitational Waves: Sources and Detectors*, Cascina, 1996, eds. I Ciufolini and F. Fidecaro (World Scientific, Singapore, 1997), p.133.

ok [603] G. Veneziano, Phys. Lett. **B406**, 297 (1997). graceful exit

[604] G. Veneziano, Phys. Lett. **B454**, 22 (1999).

[605] G. Veneziano, in *The primordial Universe*, Proc. LXXI Les Houches Summer School, 1999, eds. P. Binétruy, R. Schaeffer, J. Silk and F. David (Springer-Verlag, Berlin, 2000), p. 581.

[606] G. Veneziano, in *The Geometric Universe*, eds. L. J. Mason, K. P. Tod, S. T. Tsou and N. M. J. Woodhouse (University Press, Oxford, 1998), p. 235.

[607] G. Veneziano, JHEP **0206**, 051 (2002).

axion [608] F. Vernizzi, A. Melchiorri and R. Durrer, Phys. Rev. **D63**, 063501 (2001).

[609] A. Vilenkin, Phys. Rev. **D30**, 509 (1984).

[610] A. Vilenkin, Phys. Rev. **D33**, 3650 (1986).

[611] A. Vilenkin, Phys. Rev. **D37**, 888 (1988).

[612] A. Vilenkin, Phys. Rev. **D46**, 2355 (1992).

[613] A. Vilenkin, in *String Gravity and Physics at the Planck Energy Scale*, Proc. Erice, School, 1995, eds. N. Sanchez and A. Zichichi (Kluwer Acad. Pub., Dordrecht, 1996), p. 345.

[614] S. Vitale, M. Cerdonio, E. Coccia and A. Ortolan, Phys. Rev. **D55**, 17414 (1997).

[615] R. M. Wald, *General relativity* (University Press, Chicago, IL, 1984).

[616] D. Wands, Phys. Rev. **D60**, 023507 (1999).

[617] L. Wang, R. R. Caldwell, J. P. Ostriker and P. J. Steinhardt, Astrophys. J. **530**, 17 (2000).

[618] S. Weinberg, *Gravitation and cosmology* (Wiley, New York, 1971).

[619] S. Weinberg, Phys. Rev. Lett. **40**, 223 (1978).

[620] S. Weinberg, Phys. Rev. Lett. **48**, 1303 (1982).

[621] P. S. Wesson, Astron. Astrophys. **151**, 276 (1985).

[622] J. A. Wheeler, in *Battelle Rencontres*, eds. C. De Witt and J. A. Wheeler (Benjamin, New York, 1968).

[623] F. Wilczeck, Phys. Rev. Lett. **40**, 279 (1978).

[624] E. Witten, Phys. Lett. **B149**, 351 (1984).

[625] E. Witten, Phys. Rev **D44**, 314 (1991).

[626] E. Witten, Nucl. Phys. **B443**, 85 (1995).

[627] E. Witten, Nucl. Phys. **B471**, 135 (1996).

[628] H. Yajima, K. Maeda and H. Ohkubo, Phys. Rev. **D62**, 024020 (2000).

[629] S. S. Yazadjiev, Phys. Rev. **D63**, 063510 (2001).

[630] U. Yurtsever, Phys. Rev. **D37**, 2803 (1988).

[631] Y. B. Zel'dovich, Mon. Not. Roy. Astron. Soc. **160**, 1 (1972).

[632] Y. B. Zeldovich, A. A. Ruzmaikin and D. D. Sokoloff, *Magnetic fields in astrophysics* (Gordon and Breach, New York, 1983).

[633] Y. Zel'dovich and A. A. Starobinski, Sov. Astron. Lett. **10**, 135 (1984).

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Relaxed Bounds on the Dilaton Mass In a String Cosmology Scenario

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Abstract

We discuss bounds on the dilaton mass, following from the cosmological amplification of the quantum fluctuations of the dilaton background, under the assumption that such fluctuations are dominant with respect to the classical background oscillations. We show that if the fluctuation spectrum grows with the frequency the bounds are relaxed with respect to the more conventional case of a flat or decreasing spectrum. As a consequence, the allowed range of masses may become compatible with models of supersymmetry breaking, and with a universe presently dominated by a relic background of dilaton dark matter.

Relaxed bounds on the dilaton mass in a string cosmology scenario.

It is known that the coherent oscillations around the minimum of the potential of a cosmic scalar background [1-4], such as the dilaton [5-7], put severe cosmological bounds on the allowed value of the mass m of that scalar field. Such oscillations, however, become coherent, and then constrain the mass, only when the possible spatial inhomogeneities of the background are negligible.

Spatial inhomogeneities of the dilaton background can arise either because of thermal fluctuations, or because of the gravitational amplification of quantum fluctuations. If the spatial inhomogeneities of thermal origin are diluted by a subsequent inflationary expansion (or if they are absent simply because the maximum temperature scale is lower than the Planckian temperature required for thermal equilibrium), then their contribution to the energy density (in a mode expansion of the background oscillations) becomes negligible with respect to the mass contribution. The coherent oscillations of the background, with frequency m, can then dominate the dilaton energy density beginning at a scale $H \simeq m$.

The same happens for the energy density stored in the quantum fluctuations, if they are amplified with a spectrum which is flat or decreasing with frequency. Also in that case the mass coherent contribution begins to dominate (with respect to spatial inhomogeneities) the fluctuation energy at $H \simeq m$, thus avoiding a significant relaxation of the cosmological bounds, even for negligible amplitude of the classical background oscillations [2].

If, on the contrary, the quantum fluctuations are amplified with a spectrum which grows with frequency, their energy density stays dominated by the small scale inhomogeneities until values of H much lower than m. In that case the bounds on m can be alleviated, provided the classical background solution approaches enough the minimum of the potential before the scale drops to $H = m$.

A growing spectral distribution of the dilaton fluctuations is a typical outcome of the "Pre-big-bang" inflationary scenario [8,9], suggested by the duality properties of the string cosmology equations [10]. In such context, as we shall see in this paper, the relaxation of the bound has two important consequences: the allowed range of dilaton masses becomes compatible a) with model of supersymmetry breaking able to provide a natural resolution of the gauge hierarchy problem [11], and b) with the possibility that our universe is presently dominated by a relic background of dilaton dark matter. (It should be mentioned that a growing scalar perturbation spectrum is also predicted by the "hybrid inflation" model proposed by Linde [12] and recently generalized to the class of "false vacuum inflation" [13]

(see also [14]). Growing spectra, moreover, have been shown to be necessary for a simultaneous fit of the COBE anisotropies and of the observed bulk motion and large voids structures on a $50Mpc$ scale [15]).

In view of its importance, we start by recalling the standard arguments leading to the bounds on the dilaton mass based on the coherent oscillations of the classical background [1-7]. For scales $H < H_1 \leq M_p$, where M_p is the Planck mass and H_1 the Hubble parameter at the time t_1 which marks the end of inflation (and which we assume to coincide with the beginning of the radiation-dominated era), a homogeneous dilaton background must satisfy the equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \tag{1}$$

(we neglect for the moment the friction term due to a possible dilaton decay, with width $\Gamma \simeq m^3/M_p^2$). For values of ϕ sufficiently near to the minimum ϕ_0 of V, we approximate the potential by $V = m^2(\phi - \phi_0)^2/2$, with $m < H_1$. When $H >> m$ one thus finds that ϕ approaches asymptotically a constant value ϕ_1,

$$\phi = \phi_1 + \phi_2(\frac{H}{H_1})^{1/2} \tag{2}$$

so that, for $H << H_1$, the distance from the minimum can be approximated by $|\phi - \phi_0| \simeq |\phi_1 - \phi_0| = const.$ Typically, for interactions of gravitational strength, ϕ_0, ϕ_1 and ϕ_2 are all of order M_p but, without fine tuning, one expects that also $|\phi_1 - \phi_0| \sim M_p$. As a consequence, when the scale H drops to $H = m$, the homogeneous background ϕ begins to oscillate coherently with frequency m and initial amplitude of order M_p, which decreases in time like $a^{-3/2}$ (a is the scale factor of the isotropic metric background). The associated energy density, ρ_ϕ, decreases like a^{-3}, starting from an initial value $\rho_i \simeq m^2 M_p^2$ which is of the same order as the radiation energy density ρ_γ at that epoch. For $H < m$ the universe thus enters a phase of dust-like matter domination, in which $\rho_\gamma/\rho_\phi \sim a^{-1}$. This leaves two possible alternatives open. If

$$m \lesssim H_2 \simeq 10^{-27} eV \tag{3}$$

where H_2 is the usual equilibrium scale corresponding to the matter-radiation transition, then ρ_ϕ stays always smaller than the critical density, and the oscillating dilaton background could survive until tody. This possibility seems to be excluded by the present tests of the equivalence principle, which imply [16,17]

$$m \gtrsim m_0 = 10^{-4} eV \tag{4}$$

(see however [18]). As a consequence, the dilaton must have decayed, i.e. the energy stored in the coherent oscillations must have converted in radiation, at a decay scale

$$H_d = \Gamma \simeq \frac{m^3}{M_p^2} < m \tag{5}$$

However, since the dilaton begins to dominate much before the nucleosynthesis scale H_N, as

$$H_N \simeq \frac{(1MeV)^2}{M_p} << m_0 \lesssim m, \tag{6}$$

one must impose that the reheating temperature T_r associated to the dilaton decay, such that

$$T_r^4 \simeq M_p^2 H_d^2, \tag{7}$$

be large enough to allow a subsequent nucleosynthesis phase. This means $T_r \simeq (m^3/M_p)^{1/2} \gtrsim 1MeV$, namely

$$m \gtrsim 10^4 \ GeV. \tag{8}$$

Moreover, the radiation temperature T_d just before dilaton decay is

$$T_d = T_m(\frac{a_m}{a_d}) \simeq (mM_p)^{1/2}(\frac{H_d}{m})^{2/3} \simeq (\frac{m^{11}}{M_p^5})^{1/6} \tag{9}$$

(the index m means that the variable is to be evaluated at the scale $H = m$). The reheating from T_d to T_r will thus produce an entropy increase

$$\Delta S = (\frac{T_r}{T_d})^3 \simeq \frac{M_p}{m} \tag{10}$$

In order to preserve any pre-existing baryon-antibaryon asymmetry, the condition $\Delta S \lesssim 10^5$ should be satisfied [5-7]. Such a condition implies

$$m \gtrsim 10^{14} \ GeV \tag{11}$$

This last requirement could be alleviated, however, in the case of low-energy (electro-weak, for instance) baryogenesis; in particular, in the case of baryogenesis associated to the dilaton decay itself [7,11], occuring at scales not much distant from nucleosynthesis.

These are the standard arguments, leading to bounds on m which are independent of the inflation scale H_1, and which are crucially grounded on the assumption that the asymptotic value ϕ_1, approached by ϕ during its evolution for $H >> m$,

lies at a distance of order unity (in Planck units) from the minimum ϕ_0 of the potential.

One might thus be led to think that the bounds could be evaded if, owing to some mechanism, the initial amplitude of the classical background oscillations would be lowered to $|\phi-\phi_0| << M_p$. For instance, if the asymptotic value ϕ_1 would be fine-tuned to ϕ_0, then $\phi - \phi_0$ would be no longer constant but would become scale-dependent according to eq.(2). At a scale H, the shift from the minimum would be typically of order $|\phi - \phi_0| \sim M_p(H/H_1)^{1/2}$, thus leading to coherent oscillations for $H \leq m$ with initial energy density $\rho_1 \simeq (mM_p)^2(m/H_1) < (mM_p)^2$.

Even in that case, however, the scenario would not be free of problems. Indeed, as pointed out in Ref.[2], besides the classical oscillations one must always take into account also the quantum fluctuations of the background, $\delta\phi = \chi$, amplificated by the inflationary evolution. The fluctuation modes χ_k satisfy the equation

$$\ddot{\chi}_k + 3H\dot{\chi}_k + (\frac{k^2}{a^2} + m^2)\chi_k = 0 \tag{12}$$

and for $m < H_1$ the spectral distribution of the associated energy density, ρ_χ, is

$$k\frac{d\rho_\chi}{dk} \simeq (\frac{H_1 a_1}{a})^4(\frac{k}{k_1})^{n-1} \tag{13}$$

Here $k_1 = a_1 H_1$ is the maximum amplified comoving frequency, and the spectral index, $n = 3-2\alpha$, is fixed by the power α determining the evolution (in conformal time η) of the scale factor during inflation, $a \sim |\eta|^{-\alpha}$.

Let us discuss, first of all all, the case of the scale-invariant, Harrison-Zeldovich spectrum ($n = 1$), corresponding to de Sitter-like inflation ($\alpha = 1$) with $H = H_1 = const.$ In this case all the modes χ_k contribute to the energy density

$$\rho_\chi(t) \simeq (\frac{H_1 a_1}{a})^4 = H_1^2 H^2 \tag{14}$$

with the same amplitude. (Note the condition $H_1 < M_p$ to be satisfied in order that ρ_χ does not overclose the universe during radiation dominance). A generic mode $k > am$ begins to oscillate at a scale $H_k = k/a_k$, with an amplitude χ_k which is initially of order H_1, and which decreases in time as $\chi_k \simeq H_1(H/H_k)^{1/2}$. When $H \simeq m$, the non-relativistic modes ($k/a < m$) begin to oscillate, with initial amplitude H_1 and frequency $\simeq m$, and they immediately become dominant with respect to the other modes, as their contribution to ρ_χ decreases like a^{-3} instead of a^{-4}. For $H \leq m$ we are thus in a situation where, beside the energy density stored in the possible coherent oscillations of the classical background, $\rho_\phi = \rho_m(a_m/a)^3$,

we must have, necessarily, also some energy stored in the coherent oscillations of the quantum fluctuations, with

$$\rho_\chi \simeq m^2 H_1^2 (\frac{a_m}{a})^3 \tag{15}$$

In this paper we want to discuss how the cosmological bounds on the dilaton mass are relaxed when we move from a scenario in which the fluctuation spectrum is flat or decreasing ($n \leq 1$), to another scenario characterized by a growing ($n > 1$) spectrum. We shall thus assume, henceforth, that the classical oscillations (whose initial amplitude is model-dependent) are always negligible, $\rho_\phi < \rho_\chi$, and that all bounds on m follow from the cosmological amplification of the quantum fluctuations only. We will obtain, in this way, the *maximum* (approximately model-independent) allowed region in parameter space.

The energy density (15) is smaller than the radiation energy ρ_γ when $H = m$, but it grows with respect to ρ_γ as the curvature scale decreases in time, until it equals ρ_γ at an initial scale H_i. If $H_i < H_2$, which means

$$m < H_2 (\frac{M_p}{H_1})^4 \tag{16}$$

(recall that H_2 denotes the usual matter-radiation transition scale of eq.(3)), then ρ_χ stays always smaller than the critical density. If, however, $H_i > H_2$ then

$$H_i = m(\frac{H_1}{M_p})^4 \tag{17}$$

and the dilaton must have already decayed, $H_d > H_0$, to avoid contradictions with the presently observed matter density. This implies

$$m \gtrsim (M_p^2 H_0)^{1/3}, \tag{18}$$

where $H_0 \sim 10^{-61} M_p$ is the present curvature scale. The dilaton decay generates an entropy $\Delta S = (T_r/T_d)^3$, where T_r is the reheating temperature (7) and T_d the radiation temperature at the dilaton decay epoch, namely

$$T_d = T_i(\frac{a_i}{a_d}) = (M_p H_i)^{1/2} (\frac{H_d}{H_i})^{2/3} \simeq (\frac{m^{11}}{M_p H_1^4})^{1/6} \tag{19}$$

This gives

$$\Delta S = \frac{H_1^2}{m M_p} \tag{20}$$

If $m < 10^4 GeV$ the reheating temperature is too low ($< 1MeV$) to allow nucleosynthesis: we must impose that nucleosynthesis already occurred, $H_i < H_N$, and that [5-7] $\Delta S \lesssim 10$, in order not to destroy all light nuclei formed. If, on the contrary, $m > 10^4 GeV$, the nucleosynthesis scale is subsequent to dilaton decay, and the only possible constraint is [5-7] $\Delta S \lesssim 10^5$ in order to preserve primordial baryogenesis.

These are the conditions to be imposed if $m < H_1$. If $m > H_1 = k_1/a_1$, then all modes are always non-relativistic, and the spectral energy distribution becomes [19]

$$k\frac{d\rho_\chi}{dk} \simeq H_1^4 (\frac{a_1}{a})^3 (\frac{m}{H_1})^2 (\frac{k}{k_1})^{n-1} \tag{21}$$

This gives, for $n = 1$,

$$\rho_\chi(t) \simeq m^2 H_1^2 (\frac{a_1}{a})^3 \tag{22}$$

The scale $H = H_1$ marks the beginning of coherent oscillations with frequency m and initial energy $\rho_i = m^2 H_1^2$ (hence $m < M_p$ to avoid overcritical density). There are no further bounds on m, in this case, as the fluctuation energy is dissipated before a possible dilaton dominance. Indeed, the scale $H_i = H_1 (m/M_p)^4$ corresponding to the equality $\rho_\chi = \rho_\gamma$ is always smaller than the decay scale, $H_d = m^3/M_p^2 > H_i$.

The values of (m, H_1) allowed by the previous constraints are shown in **Fig.1**. One can see that the bounds on m are relaxed but, as stressed in [2], too low values of H_1 are in general required to be compatible with an interesting range of masses. The preferred supersymmetry breaking scale $m \sim 1TeV$, for instance, is forbidden unless $H_1 \lesssim 10^{-8} M_p$. Similar values of H_1 ($\sim 10^{-6} - 10^{-9} M_p$) are required to be compatible with the possibility of a not yet decayed, and presently dominating, dilaton.

The situation becomes even worse for quantum fluctuation with a decreasing spectrum ($n < 1$). In that case the initial amplitude χ_i of the coherent oscillations become larger, $\chi_i \simeq H_1 (H_1/m)^{(1-n)/4} > H_1$ and, as a consequence, a lower inflation scale H_1 is required to be compatible with the same given value of m.

String cosmology, however, suggests a scenario in which the standard radiation dominated era is preceeded by a so-called pre-big-bang phase, describing the evolution from a flat, weakly coupled initial state [8]. The universe superinflates, bends up and heats up to a maximum scale H_1, after which curvature and temperature begin to decrease. The dilaton grows up to the strong coupling regime, and its settlement to a constant value marks the beginning of the standard cosmological evolution. The particle production associated with the transition between

pre and post-big-bang regime is characterized by a growing spectrum [8,9], and imposes the constraint $H_1 < M_p$ to prevent an overcritical density of produced massless particles (such as gravitons). In that context, the quantum fluctuations of the dilaton background are also amplified with a growing spectrum, in particular with the same spectrum of tensor perturbations in the case of a vacuum, dilaton-driven pre-big-bang [19]. (One may note that if such a spectrum would apply also to the scalar part of the metric perturbations, it would impossible to explain the anisotropy observed by COBE [20], which should then to be ascribed to other sources. It should be stressed, however, that the dilaton perturbations and the scalar perturbations of the metric background are not necessarily forced to have the same spectrum, if other gravitational sources, beside the dilaton, are present [19]).

Motivated by string cosmology, we shall thus discuss how the previous bounds on the dilaton mass are to be changed if the fluctuation spectrum is growing with frequency. If $n > 1$ in eq.(13), the fluctuation energy (14) is initially dominated by the maximum frequency mode $k_1 = a_1 H_1$, whose amplitude, χ_1, decreases in time as $\chi_1 \simeq (H H_1)^{1/2}$. As a consequence, the coherent oscillations with frequency m, which begin at $H \simeq m$, may become dominant at a scale H_{nr} lower than in the previous case, such that $k_1/a_{nr} = m$, namely

$$H_{nr} = \frac{m^2}{H_1} < m \tag{23}$$

Only for $H \lesssim H_{nr}$ the contribution to ρ_χ of the small scale inhomogeneities becomes negligible with respect to the mass contribution, and the behaviour of ρ_χ becomes that of non-relativistic matter,

$$\rho_\chi \simeq (\frac{m H_1 a_1}{k_1})^4 (\frac{a_{nr}}{a})^3 = m^4 (\frac{a_{nr}}{a})^3 \tag{24}$$

corresponding to coherent oscillations with frequency m and initial amplitude m.

This energy density stays always smaller than the critical one if $H_{nr} < H_2$, and also if ρ_χ equals ρ_γ at a scale $H_i < H_2$, namely for

$$m < (H_1 H_2)^{1/2} (\frac{M_p}{H_1})^2 \tag{25}$$

The same happens if $H_i > H_2$, where

$$H_i = \frac{m^2}{M_p} (\frac{H_1}{M_p})^3 \tag{26}$$

but the dilaton decays before becoming dominant, $H_d > H_i$. If on the contrary, $H_i > H_2$ and $H_d < H_i$, the dilaton must have decayed (to avoid a present over-critical density) when it was dominant and, as before, we are left with two possible alternatives.

If $T_r < 1MeV$ we must impose that nucleosynthesis already occurred, i.e. $H_N > H_i$ (with the scale H_i of eq.(26)), and that $\Delta S = (T_r/T_d)^3 \lesssim 10$. Since

$$T_d = T_i(\frac{a_i}{a_d}) \simeq (M_p H_i)^{1/2}(\frac{H_d}{H_i})^{2/3} = (\frac{m^{10}}{M_p H_1^3})^{1/6} \tag{27}$$

the produced entropy to be bounded is, in this case,

$$\Delta S = (\frac{H_1^3}{m M_p^2})^{1/2} \tag{28}$$

The other possibility, $T_r > 1MeV$, allows a nucleosynthesis phase subsequent to dilaton decay, so that the only bound is imposed by primordial baryogenesis, $\Delta S \lesssim 10^5$. The case $m > H_1$, finally, provides the only bound $m < M_p$ exactly like in the previous case of a flat perturbation spectrum.

The allowed region in the plane (m, H_1), for the case of quantum fluctuations with a growing spectrum, is illustrated in **Fig.2**. The forbidden region describes a sort of funnel wedged between the lower limit $m = m_0 = 10^{-4}eV$ allowed by the equivalence principle, and the upper limit $m = M_p$ allowed by the closure density. The bounds are relaxed with respect to the previous case, in such a way that for $H_1 \lesssim 10^{-5}M_p$ practically all values of m are allowed. In particular, a dilaton mass in the TeV range (required by models of supersymmetry breaking [11]), becomes compatible with an inflation scale $H_1 \simeq 10^{-5}M_p$, which seems to be the value suggested by the observed COBE anisotropy [21]. Moreover, a universe presently dominated by a relic background of not yet decayed dilatons becomes possible for

$$1\ eV \lesssim m \lesssim 100\ MeV \tag{29}$$

and compatible with realistic (in a string cosmology context) inflation scales $H_1 \geq 10^{-5}M_p$.

In conclusion, our qualitative analysis shows that the bounds on the dilaton mass arising from the cosmological amplification of the quantum fluctuations are less constraining when their spectrum is growing, like in a string cosmology context, instead of being flat or decreasing like in other, more conventional inflationary models. The bounds of **Fig.2** define the *maximum* allowed region, corresponding

to the case in which the classical dilaton background seats exactly at the minimum of the effective potential, and it is neither oscillating nor running at the scale $H = m$. Such a region would be of course reduced if classical background oscillations were to be added to the quantum fluctuations. It can be easily verified, however, that as long as one can arrange a scenario in which the distance of the background from the minimum decreases with the scale like $H^{1/2}$ (or faster), the bounds on m in the higher scales sector $H_1 \gtrsim 10^{-5}$ (which is the interesting sector for string cosmology) remain practically unchanged with respect to the bounds reported in **Fig.2**.

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References

1. G. D. Coughlan et al., Phys. Lett. B131 (1983) 59
2. A. S. Goncharov, A. D. Linde and M. I. Vysotsky, Phys. Lett. B147 (1993) 279
3. G. German and G. G. Ross, Phys. Lett. B172 (1986) 305
4. O. Bertolami, Phys. Lett. B209 (1988) 277
5. J. Ellis, D. V. Nanopoulos and M. Quiros, Phys. Lett. B174 (1986) 176
6. J. Ellis, N. C. Tsamish and M. Voloshin, Phys. Lett. B194 (1987) 291
7. B. de Carlos et al., Phys. Lett. B318 (1993) 447
8. M. Gasperini and G. Veneziano, Astropart. Phys. 1 (1993) 317;
   Mod. Phys. Lett. A8 (1993) 3701
9. M. Gasperini and M. Giovannini, Phys. Rev. D47 (1993) 1529
10. G. Veneziano, Phys. Lett. B265 (1991) 287;
    K. A. Meissner and G. Veneziano, Mod. Phys. Lett. A6 (1991) 1721;
    A. A. Tseytlin, Mod. Phys. Lett. A6 (1991) 1721;
    A. A. Tseytlin and C. Vafa, Nucl. Phys. B372 (1992) 443;
    M. Gasperini and G. Veneziano, Phys. Lett. B277 (1992) 256
11. T. Banks, D. B. Kaplan and A. E. Nelson, UCDS/PTH 93-26, RU-37 (July 1993)
12. A. D. Linde, Phys. Lett. B259 (1991) 38;
    Phys. Rev. D49 (1994) (in press)
13. E.J. Copeland et al., SUSSEX-AST 94/1-1, (astro-ph/9401011)
14. S. Mollerach, S. Matarrese and F. Lucchin, astro-ph/9309054
15. T. Piran et al., astro-ph/9305019
16. T. R. Taylor and G. Veneziano, Phys. Lett. B213 (1088) 450
17. J. Ellis et al., Phys. Lett. B228 (1989) 264
18. T. Damour and A. M. Polyakov, hep-th/9401069
19. M. Gasperini and G. Veneziano, Dilaton production in string cosmology (to appear)
20. G. Smoot et al., Astrophys. J. 316 (1992) L1
21. M. White, Phys. Rev. D46 (1992) 4198;
    T. Souradeep and V. Sahni, Mod. Phys. Lett. A7 (1992) 3541;
    A. R. Liddle, SUSSEX-AST 93/7-2 (July 1993).

Figure captions

Fig.1 Maximum allowed region (inside the full lines) for quantum fluctuations amplified with a scale-invariant spectrum, and dominant with respect to the classical background oscillations. The dilaton mass is given in units of $m_0 = 10^{-4} eV$. The lines marked by a, b, c, d, e, f, g correspond respectively to: a) $m = m_0$, lower bound on m from the equivalence principle; b) $H_1 = M_p$, upper bound on H_1 from the closure density; c) $T_r = 1 MeV$, lower bound on the reheating temperature from nucleosynthesis; d) $m = M_p$, upper bound on $m > H_1$ from the closure density; e) $m = H_2 (M_p/H_1)^4$, upper bound on m from the present matter-to-radiation energy density ratio; f) $H_1^2/mM_p = 10^5$, upper limit on entropy production from primordial baryogenesis; g) $H_1^2/mM_p = 10$, upper limit on entropy from nucleosynthesis.

Fig.2 Maximum allowed region (inside the full lines) in the case of quantum fluctuations amplified with a growing spectrum, and dominant with respect to the classical background oscillations. The lines marked by a, b, c, d are the same as in Fig.1. The other lines correspond to: e) $m = (H_1 H_2)^{1/2} (M_p/H_1)^2$, upper bound on m from the present matter-to-radiation energy density ratio; f) $(H_1^3/mM_p^2)^{1/2} = 10^5$, upper limit on entropy production from primordial baryogenesis; g) $(H_1^3/mM_p^2)^{1/2} = 10$, upper limit on entropy production from nucleosynthesis.

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Repulsive gravity in the very early Universe

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Abstract

I present two examples in which the curvature singularity of a radiation-dominated Universe is regularized by *(a)* the repulsive effects of spin interactions, and *(b)* the repulsive effects arising from a breaking of the local gravitational gauge symmetry. In both cases the collapse of an initial, asymptotically flat state is stopped, and the Universe bounces towards a state of decelerated expansion. The emerging picture is typical of the pre-big bang scenario, with the main difference that the string cosmology dilaton is replaced by a classical radiation fluid, and the solutions are not duality-invariant.

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The aim of this Essay is to discuss the possibility of avoiding the initial cosmological singularity through a phase of repulsive gravity occurring in the very early Universe. I will consider two mechanisms of repulsive gravity: spin-torsion interactions and spontaneous breaking of the local $SO(3,1)$ gauge symmetry. I will show that in both cases the condition of geodesic convergence [1] can be violated, and the cosmological equations may admit regular homogeneous and isotropic solutions for which the energy density and the curvature grow up to a maximum (finite) scale, and then decrease, with a smooth joining to the standard decelerated evolution.

The interesting aspect of such models is that they do not require any violation of the strong energy condition [1] in the conventional matter sector. Indeed, in both cases I will simply take a radiation-like equation of state for the sources (no vacuum energy term will be included). In spite of the fact that I will use classical generalization of the Einstein equations, the results obtained might be of some relevance for applications to string cosmology, where the present cosmological phase is expected to emerge from a phase of growing curvature, through a smooth transition that should avoid the initial singularity [2].

I will first discuss the case of spin-torsion interactions. Torsion is a natural ingredient of gauge theories of the Poincarè group [3], as it represents the field strength of local translations, and it is thus the required Yang-Mills partner of the curvature (the field strength of local Lorentz rotations). In addition, torsion couples minimally to the axial current of spinor matter, as required by local supersymmetry: simple supergravity, containing only the graviton and the gravitino, can indeed be formulated as an Einstein-Cartan theory for the Rarita-Schwinger field [4].

The Einstein-Cartan theory [3], which I will consider in this paper, is the simplest example of gravitational theory with torsion. In such a theory torsion does not propagate, and it can be non-vanishing only in the presence of an intrinsic spin density of matter. As a consequence, no significant effect is expected for macroscopic bodies at ordinary densities; torsion interactions may become important, however, in the regime of extremely high density and curvature of the early Universe.

Let us thus consider a cosmological application of the Einstein-Cartan theory, by taking a perfect gas of spinning particles as the effective matter source. In that case the connection is non-symmetric, $\Gamma_{[\mu\nu]}{}^{\alpha} \neq 0$, and besides the equation relating the Einstein tensor and the canonical (non-symmetric) energy-momentum tensor,

$$G_{\mu\nu}(\Gamma) = 8\pi G T_{\mu\nu}, \tag{1}$$

we have an additional algebraic relation [3] between the torsion, $Q_{\mu\nu}{}^{\alpha} = \Gamma_{[\mu\nu]}{}^{\alpha}$, and the canonical spin density tensor, $S_{\mu\nu}{}^{\alpha}$:

$$Q_{\mu\nu}{}^{\alpha} = 8\pi G \left(S_{\mu\nu}{}^{\alpha} + \frac{1}{2}\delta^{\alpha}_{\mu} S_{\nu\beta}{}^{\beta} - \frac{1}{2}\delta^{\alpha}_{\nu} S_{\mu\beta}{}^{\beta} \right). \tag{2}$$

Thanks to the above relation, torsion can be eliminated everywhere in eq. (1). By assuming a convective model of spinning fluid minimally coupled to the geometry of the Riemann-Cartan manifold [5], we can rewrite eq. (1) in the standard Einsteinian form for a symmetric connection, but with additional terms that are linear and quadratic in the spin tensor of the matter sources.

In the absence of some external polarizing field the spins are randomly oriented, and the linear terms are zero after an appropriate space-time averaging, $\langle S_{\mu\nu\alpha} \rangle = 0$; the quadratic terms, however, are non-vanishing also on the average, $\langle S_{\mu\nu\alpha} S^{\mu\nu\alpha} \rangle \neq 0$. Because of the spinning sources we are thus led to a modified set of cosmological equations, even for unpolarized matter, and in the averaged macroscopic limit. For a spatially flat metric $g_{\mu\nu} = \mathrm{diag}\,(1, -a^2\delta_{ij})$, in particular, the averaged cosmological equations can be written as [6]:

$$H^2 = \frac{8\pi G}{3} \left(\rho - 2\pi G \sigma^2 \right), \tag{3}$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3} \left(\rho + 3p - 8\pi G \sigma^2 \right). \tag{4}$$

Their combination gives the conservation equation

$$\dot{\rho} - 2\pi G (\sigma^2)^{\cdot} + 3H \left(\rho + p - 4\pi G \sigma^2 \right) = 0, \tag{5}$$

where $H = \dot{a}/a$, and a dot denotes differentiation with respect to cosmic time. I have defined $\sigma^2 = \langle S_{\mu\nu\alpha} S^{\mu\nu\alpha} \rangle / 2$, and ρ, $p > 0$ are the energy density and the pressure of the fluid in the zero spin limit.

When $8\pi G \sigma^2 > \rho + 3p$ the condition of geodesic convergence is violated,

$$R_{\mu\nu} u^{\mu} u^{\nu} = -3 \left(\dot{H} + H^2 \right) < 0, \tag{6}$$

even if the pressure satisfy the strong energy condition, $\rho + 3p > 0$. In a previous paper this repulsive contribution of the spin density was used to discuss the possibility of spin-dominated inflation [6]. Here it will be used for a possible regularization of the initial curvature singularity.

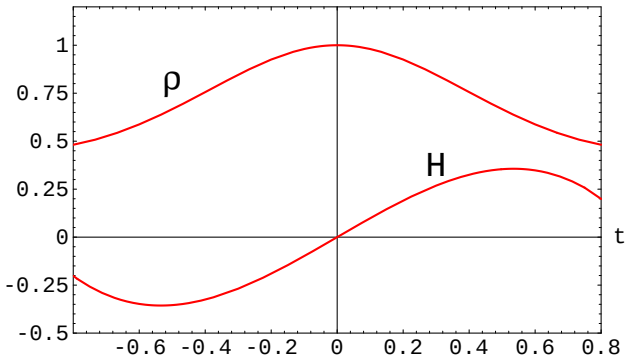

FIG. 1. *Time evolution of the Hubble factor and of the energy density according to eq. (7). I have put* $c_1 = c_2 = 1$, *and time is measured in units of* $(3/8\pi)^{1/2} l_p$.

The spin contribution to the geometry depends, of course, on the particular model of fluid. In order to show that this repulsive interaction can be strong enough to allow a smooth cosmological evolution, I will consider a spinning liquid of unpolarized fermions [7], with equation of state $p = \gamma\rho$, and averaged squared spin tensor $\sigma^2 \propto \rho^{2/(1+\gamma)}$. In this case the equations can be integrated exactly. For relativistic fermions, in particular, we have $\gamma = 1/3$, the conservation equation (5) gives $\rho \propto a^{-4}$, and the integration of eq. (3) leads to

$$\frac{t}{l_p}\sqrt{\frac{8\pi}{3}} = \frac{a}{2}\sqrt{c_1 a^2 - c_2} + \frac{c_2}{2}\ln\left|a + \sqrt{c_1 a^2 - c_2}\right| \tag{7}$$

(c_1, c_2 are dimensionless positive constants, and we are measuring time in Planck length units, with $l_p = \sqrt{G}$).

A plot of the energy density and of the Hubble parameter for this solution is shown in Fig. 1. The curvature is everywhere regular, and the models describes a smooth evolution from a phase of accelerated contraction, growing curvature, to a phase of decelerated expansion, decreasing curvature. The scale factor contracts down to a minimal value $a_m = \sqrt{c_2/c_1}$, and then re-expands (like $a \sim t^{1/2}$, asymptotically). In string cosmology, this behaviour is typical of the pre-big bang scenario represented in terms of the Einstein frame metric [8].

It may be interesting to observe that a similar class of solutions can also be obtained from the string cosmology equations through a duality boost of the flat, two-dimensional Milne metric [9]. Indeed, this fact is more than a coincidence, as the global $O(3,3)$ duality group,

used in [9], introduces a non-trivial antisymmetric tensor background, $H_{[\mu\nu\alpha]} \neq 0$, which is known to have a geometric interpretation as the torsion of an appropriate connection. The main difference is that in string cosmology the "matter source" is the scalar dilaton field, while in this example matter is more conventionally represented as a perfect fluid, and the duality symmetry of string theory is lost.

A second, possible mechanism for the generation of repulsive interactions in the early Universe is associated to the breaking of the local $SO(3,1)$ symmetry of the gravitational interaction [10]. This symmetry is part of the local gauge group of gravity: in the gauge approach to general relativity, the anholonomic Ricci connection ω_μ^{ij} represents in fact the Yang-Mills potential of local Lorentz rotations, which transforms as a covariant vector in the index μ under general reparametrizations, and as an antisymmetric tensor in the two "internal" indices i, j, under the action of the local $SO(3,1)$ group.

Like every gauge symmetry, also this local Lorentz symmetry can be broken spontaneously when an appropriate (geometric) potential, generated by a self-interacting antisymmetric tensor, appears in the action [11]. This breaking leads to an effective "quasi-riemannian" theory [12], namely to a gauge theory of gravity invariant under general reparametrizations, but with a local tangent space group other than the Lorentz group. From a phenomenological point of view, the main consequences of such a breaking are the possible appearance of repulsive forces, [10,11], and the possible violation of the equivalence principle [13,14].

The violation of the weak equivalence principle, however, is not a necessary consequence of any Lorentz symmetry breaking. If we consider, for instance, a four-dimensional quasi-riemannian theory with local $SO(3)$ invariance, we find that the most general model contains four independent parameters in the gravitational part of the action, and three parameters in the matter action. By imposing four conditions on these seven parameters it is always possible to preserve the covariant conservation of the energy momentum tensor, in such a way that the motion of test particles remains geodesic [13].

In that case the causal structure of space-time is still determined by the metric tensor, the classical singularity theorems [1] still can be applied, and the violation of geodesic convergence is still a necessary condition for singularity prevention. Because of the modified dynamical equations, however, geodesic convergence and strong energy condition are no longer equivalent [15], so that a smooth and complete model of cosmological evolution can be

implemented even with conventional matter sources, satisfying the strong energy condition.

As a particular example of this possibility I will consider here a one-parameter, $SO(3)$-invariant quasi-riemannian model of gravity, which for a closed, homogeneous and isotropic manifold is described by the action

$$S = 16\pi G S_m - \int dt a^3 \left[(1+\epsilon)\frac{6H^2}{N} - \frac{6k}{a^2}N \right]. \tag{8}$$

Here S_m is the action for perfect fluid matter, N is the lapse function, k is the spatial curvature (in Planck length units), and ϵ is a dimensionless constant parametrizing the breaking of the local Lorentz symmetry. All the other parameters have been fixed in such a way as to preserve the geodesic motion of the cosmological fluid [15]. In the limit $\epsilon \to 0$ the action reduces to the standard, general relativistic action.

The variation with respect to N and a, in the cosmic time gauge $N = 1$, leads to the equations

$$(1+\epsilon)H^2 + \frac{k}{a^2} = \frac{8\pi}{3}G\rho, \tag{9}$$

$$(1+\epsilon)\left(2\dot{H} + 3H^2\right) + \frac{k}{a^2} = -8\pi G p, \tag{10}$$

and their combination gives

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{11}$$

in agreement with the weak equivalence principle, $\nabla_\nu T_\mu{}^\nu = 0$. Note that in the absence of spatial curvature this particular breaking of the gauge symmetry has no effect on a cosmological metric, apart from a trivial renormalization of the gravitational coupling constant.

The value of ϵ depends on the parameters of the antisymmetric tensor potential [11] that breaks spontaneously $SO(3,1)$ down to $SO(3)$. Today, and at a macroscopical level, a breaking of local Lorentz symmetry is strongly constrained by many experimental data [13,14]. In the regime of extremely high temperature and density of the very early Universe, however, such phenomenological constraints do not necessarily apply, and for $\epsilon < -1$ gravity may become repulsive enough to prevent the singularity, even if $\rho + 3p > 0$.

Consider in fact a radiation fluid, $p = \rho/3$, so that, from eq. (11), $\rho = \rho_0 a^{-4}$. The integration of eq. (9), for $k = +1$ and $\epsilon < -1$, gives then

$$a(t) = \left[\frac{8\pi}{3}\rho_0 l_p^4 + \frac{1}{|1+\epsilon|}\left(\frac{t}{l_p}\right)^2 \right]^{1/2}, \tag{12}$$

where ρ_0 is a positive integration constant. For this solution, the plot of the Hubble parameter

$$H = \frac{t}{t^2 + |1+\epsilon|\frac{8\pi}{3}\rho_0 l_p^6} \tag{13}$$

and of the energy density is qualitatively the same as the plot of Fig. 1: the initial collapse of an asymptotically flat state is stopped, and the Universe bounces to a state of curvature-dominated, linear expansion. Note however that, unlike the Einstein-Cartan solution of the previous example, in this case the Universe does not become asymptotically radiation-dominated.

In conclusion, I would like to stress the fundamental role played by antisymmetric tensors in these two examples of regular cosmological models. In the first case the repulsive forces stopping the collapse are due to the coupling between the spin and the antisymmetric torsion field, in the second case they are due to a self-interacting antisymmetric tensor that provides the right "Higgs potential" for the breaking of the local $SO(3,1)$ symmetry. This suggests that a successful, singularity-free pre-big bang scenario might require a non-trivial antisymmetric tensor background, arising either from the NS (Neveu-Schwartz) or the RR (Ramond-Ramond) sector of the underlying string theory (or M-theory) effective action [16].

REFERENCES

[1] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time* (Cambridge Univ. Press, Cambridge, 1973).

[2] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).

[3] F. W. Hehl, P. von der Heyde, G. D. Kerlick and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).

[4] P. van Nieuwenhuizen, Phys. Rep. **68**, 189 (1981).

[5] J. R. Ray and L. L. Smalley, Phys. Rev. D **27**, 1383 (1983).

[6] M. Gasperini, Phys. Rev. Lett. **56**, 2873 (1986).

[7] S. Nurgaliev and V. N. Ponomariev, Phys. Lett. B **130**, 378 (1983).

[8] M. Gasperini and G. Veneziano, Phys. Rev. D **50** 2519 (1994).

[9] M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B **272**, 277 (1991); Phys. Lett. B **296**, 51 (1992).

[10] M. Gasperini, Phys. Lett. B **163**, 84 (1985).

[11] M. Gasperini, Phys. Rev. D **33**, 3594 (1986).

[12] S. Weinberg, Phys. Lett. B **138**, 47 (1984).

[13] M. Gasperini, in Proc. of the Int. School on *"Gravitational measurements, fundamental metrology and constants"* (Erice, May 1987), ed. by V. N. Melnikov (Kluver Acad. Pub., Dordrecht, 1988), p. 181.

[14] S. Coleman and S. L. Glashow, Phys. Lett. B **405**, 249 (1997); S. L. Glashow et al., Phys. Rev. D **56**, 2433 (1997).

[15] M. Gasperini, Class. Quantum Grav. **4**, 485 (1987).

[16] E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. D **51**, 1569 (1995); A. Lukas, B. A. Ovrut and D. Waldram, Phys. Lett. B **393**, 65 (1997); Nucl. Phys. B **495**, 365 (1997); H. Lu, S. Mukherji, C. N. Pope and K. W. Xu, Phys. Rev. D **55**, 7926 (1997);

R. Poppe and S. Schwager, Phys. Lett. B **393**, 51 (1997); E. J. Copeland, J. H. Lidsey and D. Wands, Phys. Rev. D **57**, 625 (1998); N. Kaloper, I. I. Kogan and K. A. Olive, hep-th/9711027.

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SELF-SUSTAINED INFLATION AND DIMENSIONAL REDUCTION FROM FUNDAMENTAL STRINGS

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ABSTRACT

In a previous paper we have shown that an ideal gas of fundamental strings is not able to sustain, by itself, a phase of isotropic inflation of the Universe. We show here that fundamental strings can sustain, instead, a phase of anisotropic inflation - accompanied by the contraction of a sufficient number of internal dimensions . The conditions to be met for the existence of such a solution to the Einstein and string equations are derived, and the possibility of a successful resolution of the standard cosmological problems in the context of this model is discussed.

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1. Introduction

It is well known that the classical problems of standard cosmology may be solved by a primordial phase of accelerated expansion (inflation). Many inflation scenarios, involving just the minimal number of spatial dimensions, have been proposed with various degrees of success (see [1] for a recent review). In most cases, a large and positive cosmological constant (vacuum energy) is needed, during a long enough period of time. Since present bounds on the cosmological constant are very tight, one is faced with the problem of fine-tuning a potential so that the ratio of vacuum energy density during and after inflation is enormously large.

In view of this problem it seems worthwhile to consider inflationary scenarios in which no cosmological constant is actually required in order to drive inflation in the three physical spatial dimensions. A possible mechanism of this kind is provided by the simultaneous contraction of some extra dimensions, as first pointed out in some pioneer papers on Kaluza-Klein (KK) cosmology [2].

Consider indeed a cosmological manifold which is the direct product of a $(d+1)$-dimensional space-time and an n dimensional space $M_D = M_{d+1} \bigotimes M_n$. We shall denote by $D = d + n + 1$ the total number of dimensions. Suppose that, in the cosmic-time "gauge", this space is described by the metric

$$G_{AB} = diag(1, -R^2(t)\delta_{ij}, -r^2(t)\delta_{ab}) \tag{1.1}$$

where $R(t)$ and $r(t)$ are, respectively, the scale factors of M_d and M_n. (Conventions: $A, B = 0, ...D-1$; $i, j = 1, ...d$; $a, b = d+1, ...d+n$). In this metric, the $(0,0)$ component of the Einstein equations reads

$$\frac{d}{R}\frac{d^2R}{dt^2} + \frac{n}{r}\frac{d^2r}{dt^2} = -8\pi G_D(T_0{}^0 - \frac{T}{D-2}) \tag{1.2}$$

where G_D is the D-dimensional gravitational constant, and T is the trace of the energy-momentum of the matter sources . It may be possible, therefore, to realize

an inflationary expansion in d dimensions (i.e. $d^2R/dt^2 > 0$) even if the right-hand side of eq.(1.2) is negative, provided the evolution of the other n dimensions is characterized by a negative acceleration ($d^2r/dt^2 < 0$).

The consistency, and the efficiency, of a mechanism of dynamical dimensional reduction depends crucially, of course, on the number of dimensions and on the equation of state characterizing the gravitational sources, as shown in detail by a recent phenomenological discussion [3]. As possible models of source, compatible with inflation during the decoupling of internal and external dimensions, the examples so far considered include a relativistic gas of massless particles [2], a perfect fluid with phenomenological equation of state [3-5], the antisymmetric tensor of supergravity theories [6] and, more recently, scalar fields at finite temperature [7].

The problem with KK cosmology is that there is no consistent quantum theory of gravitational or gauge interactions in $D > 4$. The only exception to this rule appears to reside in string theory. Since string matter may behave differently from point particles it seems worthwhile to reconsider these KK inflationary scenarios within string theory. Also, in a realistic string unification theory, the scale of compactification of the extra dimensions is expected to be not much different from the string mass scale [8-10]. It may be possible, therefore, for the multi-dimensional phase of our universe to be necessarily a "stringy" phase so that, as stressed in [8], dimensional reduction cannot be adequately described in the field theory limit, but requires instead the direct application of string theory in non-trivial backgrounds.

In this paper we present a particular solution of the Einstein- plus-string equations, showing that it is possible to have a self-sustained string plus gravity system which exhibits inflationary expansion of three-dimensional space, and simultaneous contraction of the other n dimensions, provided $n > 10$.

This solution describes a phase dominated by unstable strings [11,12], i.e. non-

oscillating string configurations whose proper amplitude tends to evolve, asymptotically, like the scale factor (while the co-moving amplitude becomes "frozen"). Their effective pressure is negative in physical three-dimensional space, and positive (but negligible) in internal space. With this source, the internal dimensions contract with a negative acceleration, while the three-dimensional spatial expansion turns out to be of the super-inflationary type [13], i.e. characterized by $d^2R/dt^2 > 0$ and $dH/dt > 0$, where $H = R^{-1}dR/dt$ is the Hubble parameter.

Consequently, the string energy density grows with R and approaches a singularity in a finite (cosmic time) interval. However, this picture of the early universe cannot be extended above a maximal density, where string and other quantum corrections to the Einstein equations cannot be neglected, and where, in any case, a transition to the standard, radiation-dominated scenario may be expected. Nevertheless it seems possible, according to our mechanism, to obtain a significative inflation of the causal horizon of physical space-time already before reaching these maximal (Planckian) densities.

The content of the paper is as follows. In Section 2, considering the equations of motion for a string embedded in an isotropic Friedman-Robertson-Walker (FRW) background, we review the general form of the leading order solution valid, in the large R limit, in the case of inflationary expansion. In Section 3 we present a new approximate solution to the same string equations, valid in the small R limit for a negatively accelerated contraction ($dR/dt < 0, d^2R/dt^2 < 0$). In Section 4 we discuss the values of D and n for which the Einstein and string equations can be simultaneously and consistently satisfied by anisotropic configurations in which d dimensions inflate, and the remaining n dimensions contract. The possibility that this scenario , in the particular case of $D = 4 \bigoplus 22$ dimensions, may represent a viable mechanism to solve at least some of the standard cosmological problems is finally discussed in Section 5.

2. Asymptotic string configurations in inflationary expanding backgrounds

The equations of motion of a string, coupled to a D-dimensional background metric G_{AB}, can be written [11,14] (in the gauge in which the world-sheet metric is conformally flat)

$$\ddot{X}^A - X''^A + \Gamma_{BC}{}^A(\dot{X}^B + X'^B)(\dot{X}^C - X'^C) = 0 \tag{2.1}$$

where Γ is the Christoffel connection for the metric G_{AB}; a dot and a prime denote, respectively, differentiation with respect to the world-sheet time and space variables, τ and σ. The variation of the action with respect to the world-sheet metric provides, in addition, the two constraints [11,14]

$$G_{AB}(\dot{X}^A\dot{X}^B + X'^A X'^B) = 0$$

$$G_{AB}\dot{X}^A X'^B = 0 \tag{2.2}$$

We are interested, in this Section, in the case of a homogeneous and isotropic cosmological background, with flat spatial sections, described in the cosmic time gauge ($X^0 = t$) by the FRW metric

$$G_{AB} = diag(1, -R^2(t)\delta_{ij}) \ , \ \ i,j = 1, ..., D-1 \tag{2.3}$$

(see [14] for a discussion of the fact that such a manifold does not provide a conformally invariant σ-model, at the quantum level, and hence it is not a candidate string vacuum). The string equations and constraints thus become, explicitly,

$$\ddot{t} - t'' + R\frac{dR}{dt}[(\dot{X}^i)^2 - (X'^i)^2] = 0 \tag{2.4}$$

$$\ddot{X}^i - X''^i + \frac{2}{R}\frac{dR}{dt}(\dot{t}\dot{X}^i - t'X'^i) = 0 \tag{2.5}$$

$$\dot{t}^2 + t'^2 = R^2[(\dot{X}^i)^2 + (X'^i)^2] \tag{2.6}$$

$$\dot{t}t' = R^2 \dot{X}^i X'^i \tag{2.7}$$

Consider, in particular, an inflationary expanding background ($dR/dt > 0$, $d^2R/dt^2 > 0$), parametrized by the following scale factor

$$R(t) = [k(1-\alpha)(t_c - t)]^{-\alpha/(1-\alpha)} \tag{2.8}$$

where $\alpha > 0$ and k, t_c are positive constant parameters. We have, in particular, super-inflation ($dH/dt > 0$) for $0 < \alpha < 1$, power-law inflation ($dH/dt < 0$) for $\alpha > 1$, while de Sitter (i.e. exponential) inflation ($dH/dt = 0$) corresponds to $\alpha = 1$. The $R \to \infty$ limit corresponds to $t \to t_c$ for super-inflation, and to $t \to \infty$ in the other cases.

For this background, the exact solution to the string equations (2.4-2.7), in the large R limit, can be expanded [11,15] as a power series in τ around a configuration described by ($\tau < 0$)

$$t(\sigma, \tau) = t_c - \frac{(-kL\tau)^{1-\alpha}}{k(1-\alpha)} \tag{2.9}$$

$$X^i(\sigma, \tau) = A^i(\sigma) + \frac{\tau^2 A''^i}{2(1-2\alpha)} + \tau^{1+2\alpha} B^i(\sigma) \tag{2.10}$$

where

$$L^2 = A'^i A'^i \qquad , \qquad A'^i B^i = 0 \tag{2.11}$$

and, by exploiting σ reparametrization invariance, we have imposed the "gauge" $A'^i A''^i = 0$, which implies $L' = 0$. This configuration, which depends on $2D-4$ arbitrary functions of σ, represents the general solution of eqs.(2.4-2.7), to leading order in τ, as τ goes to zero. (Note that, according to eq.(2.9), the scale factor can be rewritten, in this approximation,

$$R(\tau) = (-kL\tau)^{-\alpha} \tag{2.12}$$

so that the $\tau \to 0$ limit corresponds indeed to $R \to \infty$).

In the case $\alpha = 1$ (de Sitter), eq.(2.9) is to be replaced by [11]

$$t(\sigma,\tau) = -k^{-1} ln(-kL\tau) \tag{2.13}$$

while, for $\alpha = 1/2$, eq.(2.10) is to be replaced by [11]

$$X^i(\sigma,\tau) = A^i + \frac{\tau^2}{2}[B^i + A''^i ln(-\tau)] \tag{2.14}$$

We also recall that the general solution for X^i, without the gauge fixing condition $L' = 0$, has been reported in [11].

It is important to stress that, for these string configurations, the world-sheet time τ turns out to be proportional, asymptotically, to the conformal time coordinate η of the background manifold, defined by $R = dt/d\eta$. Indeed, from eqs.(2.9) and (2.12), $\dot{t} = RL$, so that

$$\eta = \tau L \tag{2.15}$$

Moreover we note, for future reference, that for these configurations RX'^i behaves asymptotically like R (as $\dot{X}^i \to 0$ for $\tau \to 0$), and that the configurations are characterized by the general properties

$$|\dot{t}| >> |t'| \qquad , \qquad |\dot{X}^i| << |X'^i| \tag{2.16}$$

As a consequence, their net contribution to the effective pressure of the gravitational sources is negative. Indeed, by using the constraints (2.2), the string energy-momentum tensor T^{AB} obtained by varying the action with respect to the background metric, i.e.

$$T^{AB}(x) = \frac{1}{\pi\alpha'\sqrt{-G}} \int d\sigma d\tau (\dot{X}^A \dot{X}^B - X'^A X'^B)\delta^D(X - x) \tag{2.17}$$

can be shown to satisfy the identity [11]

$$\pi\alpha'\sqrt{-G}[G_{00}T^{00}(x) - G_{ij}T^{ij}(x)] =$$

$$= -2 \int d\sigma d\tau \delta^D(X - x)[G_{00}(X'^0)^2 + G_{ij}\dot{X}^i\dot{X}^j] \tag{2.18}$$

($(\alpha')^{-1}$ is the string tension, and $G = detG_{AB}$). On the other hand, for a perfect fluid in the FRW background (2.3), one has

$$T_{00} = \rho, \;\; T_{ij} = -pG_{ij} = pR^2\delta_{ij} \tag{2.19}$$

where ρ and p are functions of the cosmic time only. According to eq.(2.15), the equation of state of an ideal gas of strings described asymptotically by eqs.(2.9), (2.10) can thus be approximated by [11]

$$\rho = -p(D-1) \tag{2.20}$$

which implies $p < 0$.

3. Asymptotic string configurations in backgrounds with accelerated contraction

Consider again a string coupled to the FRW metric (2.3), with a scale factor, however, which parametrizes a negatively accelerated ($d^2R/dt^2 < 0$) contraction ($dR/dt < 0$), i.e.

$$R(t) = [k(1-\delta)(t_c - t)]^{\delta/(1-\delta)} \tag{3.1}$$

where $0 < \delta < 1/2$ and $t < t_c$.

In the small R (i.e. $t \to t_c$) limit, the exact solution to eqs.(2.4 -2.7) can be expanded again as a power series in τ, around the following leading order approximation ($\tau < 0$)

$$t(\sigma, \tau) = t_c - \frac{(-kL\tau)^{1-\delta}}{k(1-\delta)} \tag{3.2}$$

$$X^i(\sigma, \tau) = A^i(\sigma) + \frac{\tau^2}{2}D^i(\sigma) + \tau^{1-2\delta}B^i(\sigma) \tag{3.3}$$

where

$$L^2(\sigma) = [(1-2\delta)^2 B^i B^i k^{4\delta}]^{1/(1-2\delta)} \tag{3.4}$$

$$D^i(\sigma) = \frac{1}{1+2\delta}(A''^i + 2\delta\frac{L'}{L}A'^i) \tag{3.5}$$

and A^i, B^i are arbitrary functions of σ satisfying the constraint $A'^i B^i = 0$ (note that, according to (3.2), the $\tau \to 0$ limit corresponds now to $R \to 0$).

Since, for the general solution, only $2D-4$ arbitrary functions are required, we can further restrict the leading order approximation (3.2), (3.3) by imposing the convenient gauge condition $B^i B'^i = 0$ (which implies, like before, $L' = 0$). In any case this approximate solution describes, in the $R \to 0$ limit, string configurations with shrinking proper amplitude, for which RX'^i behaves asymptotically like R, while $R\dot{X}^i$ behaves like R^{-1}. These configurations may be regarded as unstable, in the sense of Refs. [11,14], since they are not oscillating in τ as if the string oscillators would develop, in the small R regime, imaginary frequencies. Such configurations, moreover, are characterized by the asymptotic properties

$$|\dot{t}| >> |t'| \qquad , \qquad |\dot{X}^i| >> |X'^i| \tag{3.6}$$

which lead, in the perfect fluid approximation, to a radiation-like equation of state.

The identity (2.18) can be indeed be rewritten as

$$\pi\alpha'\sqrt{-G}[G_{00}T^{00}(x) + G_{ij}T^{ij}(x)] =$$

$$= -2\int d\sigma d\tau \delta^D(X-x)[G_{00}(X'^0)^2 + G_{ij}X'^i X'^j] \tag{3.7}$$

For an ideal gas of strings satisfying approximately eq. (3.6) at small R we thus have, after using (2.19)

$$\rho = p(D-1) \tag{3.8}$$

which is just the equation of state for a gas of massless particles.

4. String-driven inflation and dimensional reduction

Consider now a D-dimensional anisotropic background, $M_D = M_{d+1} \bigotimes M_n$, parametrized in the cosmic time gauge by $X^A = (t, X^i, X^a)$ and by the metric (1.1) (we recall that i, j range from 1 to d, while a, b from $d+1$ to $d+n$). In this background the stress tensor for a perfect fluid has the general form

$$T_A{}^B = diag(\rho, -p\delta_i{}^j, -q\delta_a{}^b) \tag{4.1}$$

where ρ, p, q are functions of t only (we have called q the pressure in the "internal" n-dimensional space M_n). The Einstein equations,

$$R_A{}^B = 8\pi G_D (T_A{}^B - \frac{T}{D-2}\delta_A{}^B) \tag{4.2}$$

with a perfect fluid as source, become explicitly

$$\frac{d}{R}\frac{d^2R}{dt^2} + \frac{n}{r}\frac{d^2r}{dt^2} = -\frac{8\pi G_D}{D-2}[\rho(D-3) + pd + nq] \tag{4.3}$$

$$\frac{1}{R}\frac{d^2R}{dt^2} + \frac{d-1}{R^2}(\frac{dR}{dt})^2 + \frac{n}{rR}\frac{dR}{dt}\frac{dr}{dt} = \frac{8\pi G_D}{D-2}[\rho - p(1-n) - nq] \tag{4.4}$$

$$\frac{1}{r}\frac{d^2r}{dt^2} + \frac{n-1}{r^2}(\frac{dr}{dt})^2 + \frac{d}{rR}\frac{dr}{dt}\frac{dR}{dt} = \frac{8\pi G_D}{D-2}[\rho - pd + q(d-1)] \tag{4.5}$$

We take, in particular, a perfect gas of strings as the dominant source of gravity. According to the constraints (2.2), the string stress tensor must satisfy the identity

$$\pi\alpha'\sqrt{-G}(G_{00}T^{00} - G_{ij}T^{ij} + G_{ab}T^{ab}) =$$

$$= -2\int d\sigma d\tau \delta^D(X-x)[G_{00}(X'^0)^2 + G_{ij}\dot{X}^i\dot{X}^j + G_{ab}X'^aX'^b] \tag{4.6}$$

where T^{AB} is given in eq.(2.17), and X^A satisfies the string equations (2.1) and constraints (2.2), which in this background become, respectively,

$$\ddot{t} - t'' + R\frac{dR}{dt}[(\dot{X}^i)^2 - (X'^i)^2] + r\frac{dr}{dt}[(\dot{X}^a)^2 - (X'^a)^2] = 0 \tag{4.7}$$

$$\ddot{X}^i - X''^i + \frac{2}{R}\frac{dR}{dt}(\dot{t}\dot{X}^i - t'X'^i) = 0 \tag{4.8}$$

$$\ddot{X}^a - X''^a + \frac{2}{r}\frac{dr}{dt}(\dot{t}\dot{X}^a - t' X'^a) = 0 \tag{4.9}$$

$$\dot{t}^2 + t'^2 = R^2[(\dot{X}^i)^2 + (X'^i)^2] + r^2[(\dot{X}^a)^2 + (X'^a)^2] \tag{4.10}$$

$$\dot{t}t' = R^2 \dot{X}^i X'^i + r^2 \dot{X}^a X'^a \tag{4.11}$$

We look for consistent solutions to the coupled Einstein-string equations representing, asymptotically, an accelerated expansion of M_d ($dR/dt > 0, d^2R/dt^2 > 0$ for $R \to \infty$), and an accelerated contraction of M_n ($dr/dt < 0, d^2r/dt^2 < 0$ for $r \to 0$). Consequently we expect, as shown in the previous two Sections, a solution of the string equations characterized by the asymptotic properties

$$|\dot{t}| >> |t'| \quad , \quad |\dot{X}^i| << |X'^i| \quad , \quad |\dot{X}^a| >> |X'^a| \tag{4.12}$$

corresponding, according to eq.(4.6), to a perfect fluid with "equation of state"

$$\rho + pd - nq = 0 \tag{4.13}$$

By making the ansatz

$$q = \gamma p \tag{4.14}$$

where γ is a constant parameter, we can eliminate q and p from the Einstein equations (4.3-4.5), and we find then the particular exact solution

$$\begin{aligned} R(t) &= [k(t_c - t)]^\beta \qquad , \qquad r = R^{-\epsilon} \\ \rho &= \rho_0 [k(t_c - t)]^{-2} \end{aligned} \tag{4.15}$$

where k and t_c are positive integration constants, ρ_0 is a positive constant which can be expressed in terms of d, n, k, and

$$\epsilon = -\frac{2d - \gamma(D-2)}{D - 2n} \tag{4.16}$$

$$\beta = \frac{2(d - n\gamma)}{d(d - 1 - n\gamma) - n\epsilon(d - \gamma - n\gamma)} \tag{4.17}$$

This solution describes, in the $t \to t_c$ limit, inflationary expansion of M_d, and simultaneous contraction of M_n, provided

$$\epsilon > 0 \qquad\qquad \beta < 0 \tag{4.18}$$

(Note that the expansion is always of the super-inflationary type, since

$$\frac{d}{dt}(\frac{1}{R}\frac{dR}{dt}) > 0 \tag{4.19}$$

for $\beta < 0$). Moreover, the solution is consistent with a positive energy density provided

$$\frac{a + b(d - n\epsilon)}{n\epsilon(1+\epsilon) - (d - n\epsilon)(d - 1 - n\epsilon)} < 0 \tag{4.20}$$

where

$$\begin{aligned} a &= \frac{(D-4)(d-n\gamma) - 2n\gamma}{d - n\gamma} \\ b &= \frac{D - 2n}{d - n\gamma} \end{aligned} \tag{4.21}$$

In order to discuss for which values of ϵ, β, γ the string equations are satisfied, when $t \to t_c$, by configurations consistent with the assumed properties (4.12-4.14), we consider separately the three possibilities $|\gamma| = \infty$, $0 < |\gamma| < \infty$ and $\gamma = 0$. Such possibilities define, respectively, a string gas with the asymptotic properties $|p| << q, |p| \sim q$ and $|p| >> q$, and correspond, respectively , to string configurations characterized, in addition to eq.(4.12), by the following asymptotic behaviours

$$|RX'^i| << |r\dot{X}^a| \tag{4.22}$$

$$|RX'^i| \sim |r\dot{X}^a| \tag{4.23}$$

$$|RX'^i| >> |r\dot{X}^a| \tag{4.24}$$

(recall eqs.(4.1)and (2.17)). We also know, from our previous discussion of the isotropic case, that RX'^i behaves like R for $R \to \infty$, while $r\dot{X}^a$ behaves like r^{-1}

for $r \to 0$. The three cases above are thus consistent with the solution (4.15) provided, respectively, $\epsilon > 1, \epsilon = 1$ and $\epsilon < 1$.

In the first case, in which the internal pressure q tends to dominate, only the trivial (flat space) solution is asymptotically consistent, since $\beta \to 0$ for $|\gamma| \to \infty$. In the second case in which γ is finite (and non-vanishing) the consistency condition (4.23), which implies $\epsilon = 1$, fixes the value of γ according to eq. (4.16):

$$\gamma = \frac{3D - 4n - 2}{D - 2} \tag{4.25}$$

The constraint $q = \gamma p$, which can also be rewritten

$$G_{ab} T^{ab} = \frac{n\gamma}{d} G_{ij} T^{ij} \tag{4.26}$$

is not compatible, however, with the general form of the leading solution to the string equations; it can be satisfied only by a particular solution, determined by the asymptotic condition

$$r^2 (\dot{X}^a)^2 = -\frac{n\gamma}{d} R^2 (X'^i)^2 \tag{4.27}$$

For the last case ($\gamma = 0, \epsilon < 1$) we can find instead a general solution which satisfies the required properties. In this case the coefficients ϵ and β become

$$\epsilon = -\frac{2d}{D - 2n} \qquad\qquad \beta = \frac{2(D - 2n)}{(D - n)(D - 2)} \tag{4.28}$$

and the conditions $\beta < 0, \;\; 0 < \epsilon < 1$ imply

$$D < 2n \quad , \quad 3D - 4n - 2 < 0 \tag{4.29}$$

which are both fulfilled iff $n > 3d + 1$ (of course $n, d > 0$). For this background the general form of the approximate solution to eqs. (4.7-4.11), in the limit $t \to t_c$, is (to leading order in $\tau, \tau < 0$)

$$t = t_c - \frac{1}{k} [-kL\tau(1 - \beta)]^{1/(1-\beta)} \tag{4.30}$$

$$X^i = A^i + \frac{\tau^2}{2} D^i + B^i \tau^{(1-3\beta)/(1-\beta)} \tag{4.31}$$

$$X^a = A^a + \frac{\tau^2}{2} D^a + B^a \tau^{(1-\beta+2\epsilon\beta)/(1-\beta)} \tag{4.32}$$

where A and B are functions of σ, $\quad L^2 = A'^i A'^i$ and, in the gauge $L' = 0$,

$$D^i = \frac{1-\beta}{1+\beta} A''^i, \qquad\qquad D^a = \frac{1-\beta}{1-\beta-2\epsilon\beta} A''^a \tag{4.33}$$

Moreover, A and B must satisfy the constraint

$$(\frac{1-3\beta}{1-\beta})[-kL(1-\beta)]^{\frac{2\beta}{1-\beta}} B^i A'^i + (\frac{1-\beta+2\epsilon\beta}{1-\beta})[-kL(1-\beta)]^{-\frac{2\epsilon\beta}{1-\beta}} B^a A'^a = 0 \tag{4.34}$$

imposed by eq.(4.11).

In this approximation, the scale factors can be rewritten in terms of τ as

$$R = [-kL\tau(1-\beta)]^{\beta/(1-\beta)}, \qquad\qquad r = R^{-\epsilon} \tag{4.35}$$

and one can easily check that, when the conditions (4.29) are satisfied, the properties (4.12) and (4.24), as well as the energy condition (4.20), and the condition $1 - \beta + 2\epsilon\beta > 0$ required by eq. (4.32), are satisfied. The string configurations (4.30-4.32) lead then, in the perfect fluid approximation, to the equation of state

$$p = -\frac{\rho}{d} \qquad\qquad q = 0 \tag{4.36}$$

which can sustain the inflation-dimensional-reduction scenario of eq. (4.15). For a manifold with $D - n = 4$, in particular, this happens, according to the dimensional bound (4.29), provided $n > 10$.

We note, finally, that the proper amplitude of these string configurations , $\chi^i = RX^i, \chi^a = rX^a$, grows in the external space, and shrinks in the internal one, as $\tau \to 0$. In both cases, however, the time evolution of χ tends to follow, asymptotically, that of the scale factor, so we may regard these strings as a generalization to the anisotropic case of the unstable configurations recently discussed

for isotropic backgrounds [11,14]. In this sense we can say, therefore, that inflation, and the asymptotic decoupling of internal and external dimensions ($r \to 0$ for $R \to \infty$), are induced by unstable strings.

5. Discussion and conclusions

A gas of unstable strings, which can form and develop at large R in inflationary expanding backgrounds, is characterized by an effective negative pressure. If the expansion is isotropic, the pressure is not negative enough (at least in the perfect fluid approximation) to sustain inflation by itself, as pointed out in a previous paper [11].

It is possible, however, that unstable strings may sustain simultaneously inflation and dimensional reduction, as discussed in the previous Section, provided the number of contracting dimensions is large enough, i.e. $n > 10$ for $D = 4 + n$. In this case, the asymptotic evolution of a $M_4 \bigotimes M_n$ background is described in particular by these scale factors (according to eqs.(4.15, 4.28))

$$R = [k(t_c - t)]^{(4-n)/(4+2n)} \quad , \quad r = [k(t_c - t)]^{6/(4+2n)} \tag{5.1}$$

The four-dimensional expansion is then of the super-inflationary type ($dH/dt > 0$), while the internal contraction has, as expected, a negative acceleration ($d^2r/dt^2 < 0$).

In order to discuss whether this model can provide, during the dimensional decoupling, a significative amount of inflation, let us consider explicitly the interesting case $D = 26$ (i.e.$\beta = -3/8, \epsilon = 1/3$). In this case our solution (4.15) of the Einstein equations implies

$$8\pi G_D \rho (t_c - t)^2 = 15/16 \tag{5.2}$$

and, by denoting by "f" and "i" the end and the beginning of the phase dominated

by these unstable strings, we have the relations

$$\frac{r_f}{r_i} = (\frac{R_f}{R_i})^{-1/3} \quad , \quad \frac{\rho_f}{\rho_i} = (\frac{R_f}{R_i})^{16/3} \tag{5.3}$$

Let us assume, moreover, that the typical initial size of causally connected regions, $d_i \simeq (t_c - t_i)$, coincides with the proper length of a typical string, i.e.

$$t_c - t_i = LR_i \tag{5.4}$$

For the string solution (4.30) we have indeed $t_c - t = (11/8)|\tau|LR$, so that a larger size $d_i > LR_i$ would imply $|\tau_i| > 1$ (or, in conformal time, $|\eta_i| > L$), in contrast with the conditions required for the validity of our approximation, which represent an expansion in the $\tau \to 0$ limit.

For $t_c - t << t_c - t_i$ the proper size $d(t)$ of the particle horizon in four-dimensional space grows like R,

$$d(t) = R(t) \int_{t_i}^{t} dt' R^{-1}(t') =$$

$$= \frac{(t_c - t)^\beta}{1 - \beta}[(t_c - t_i)^{1-\beta} - (t_c - t)^{1-\beta}] \simeq \frac{R(t)}{R_i} d_i \tag{5.5}$$

($\beta = -3/8$), as is typical of inflationary backgrounds. The proper size $\ell(t)$ of the event horizon, on the contrary, shrinks as $t \to t_c$,

$$\ell(t) = R(t) \int_{t}^{t_c} dt' R^{-1}(t') = \frac{t_c - t}{1 - \beta} \tag{5.6}$$

(typical behaviour of super-inflation only). As a consequence of this shrinking, unstable configurations may develop [11,12], until finally the Planck density is reached,

$$\rho_f \simeq \rho_p = L_p^{-D} \tag{5.7}$$

at a Planck curvature scale

$$[R(d^2R/dt^2)^{-1}]_{t=t_f}^{1/2} \simeq t_c - t_f = L_p \tag{5.8}$$

Here we suppose that the whole scenario breaks down and the standard adiabatic evolution begins. For a drastic solution of the horizon problem [1,16] one may thus require that the final size of the particle horizon in four-dimensional space, $d_f = d_i(R_f/R_i) = LR_f$, be as large as the present size $d_0 \sim 10^{61} L_p$, rescaled down at the Planck era:

$$LR_f \geq d_0 \frac{R_f}{R_0} = d_0 \frac{T_0}{T_p} \sim 10^{29} L_p \tag{5.9}$$

(T_p and T_0 are, respectively, the Planck temperature and the present temperature of the cosmic microwave background, $T_0 \sim 10^{-13} GeV$).

With the four constraints (5.4), (5.7-5.9) the model is completely determined. Indeed, using eqs. (5.7), (5.8) one gets from (5.2) $G_D = L_p^{D-2}$. Moreover, combining eqs.(5.3),(5.4),

$$\frac{LR_i}{L_p} = (\frac{LR_f}{L_p})^{8/11} \tag{5.10}$$

The condition (5.9) is thus satisfied provided

$$LR_i \geq 10^{22} L_p \tag{5.11}$$

The requirement (5.9) is strong enough * to accomodate the present values of the entropy S_0 and of the density parameter Ω_0 [1]. To be viable, this scenario needs four-dimensionally causally connected regions whose size, at the Planck curvature scale, is very large in natural (Planckian) units. However, such an otherwise "unnaturally large" value is not put in by hand, but is the dynamical consequence of a previous super-inflationary expansion. Thus, the basic difference from conventional inflationary models is that the Planck era is the final state of inflation rather than its ancestor.

It is true that the Universe has to be very large in its initial state (see eq.(5.11)), as it is the case in the models of ref.[2]. However, unlike in ref.[2],

* A primordial inflation at the Planck scale cannot dilute, however, a possible monopole production during the GUT phase transition.

in the context of string theory a classical, low curvature regime is naturally expected to occur before the Planck era as a consequence of a symmetry (duality [17]) connecting it to the present epoch.

It should be noted, moreover, that a condition much less stringent than (5.9) is required if one only wants to explain the homogeneity of the cosmic microwave background. In such a case, in fact, one only needs the present size of the visible universe (d_0), rescaled down at the time of last scattering t_{rec} of the cosmic radiation [16] (at a temperature $T_{rec} \sim 10^4 T_0$), to be not larger than the particle horizon at the time. Starting at the Planck scale with a causal region of size $d_f = LR_f$, and assuming radiation-dominated evolution (i.e. $R \sim t^{1/2}$) down to t_{rec}, the homogeneity is thus explained provided

$$d_0 \frac{R_{rec}}{R_0} = d_0 \frac{T_0}{T_{rec}} \leq LR_f(\frac{t_{rec}}{t_p}) = LR_f(\frac{T_p}{T_{rec}})^2 \tag{5.12}$$

(we have used the fact that in the standard decelerated scenario the particle horizon grows like t, while the scale factor like T^{-1}). Eq.(5.12) is satisfied for

$$LR_f \geq 10 L_p \tag{5.13}$$

which implies, using (5.10), $LR_i \geq 10^{8/11} L_p$.

It seems possible, therefore, to reproduce the desired amount of inflation if one supposes that the universe, before the Planck era, was in a primordial phase dominated by a classical gas of very massive fundamental strings ($M_i \simeq LR_i/\alpha' \geq 10^{22} M_p$ if we impose, for example, the condition (5.11)). Note that although, initially, these strings are closely packed (with an averaged (distance/size) ratio of order $(LR_i/L_p)^{(4-D)/(D-1)} \leq 10^{-19}$), they only have a tiny energy density in Planck units, namely (from eq.(5.3))

$$\rho_i \leq 10^{-44} \rho_p \tag{5.14}$$

(i.e. they are a highly diluted gas, from a gravitational point of view). This density grew up to ρ_p, after which some violent transition to local physics ("big bang") led to the standard scenario in which the density started to decrease.

This picture of the early universe fits well with the scenario described in [17], where it is suggested that, because of $R \leftrightarrow R^{-1}$ duality, a "stringy" phase before the radiation-dominated expansion is to be expected, and in this phase the temperature should grow (instead of decreasing with R), up to some maximal value of order T_p.

The transition occurring at the Planck scale cannot be described, of course, in the context of the model considered in this paper, since at $T = T_p$ the Einstein equations are no longer valid and moreover, because of the string Hagedorn temperature, a complete thermodynamical treatment is certainly needed (see [18,19]; see also [20] for a formal regularization procedure which avoids the occurrence, in cosmological backgrounds, of curvatures larger than the Planck one).

The physical interpretation of such a transition as a possible model for the big bang was also previously suggested in [21]. The growth of the density and of the temperature during the string phase were obtained, however, by means of an isotropic contraction of the whole universe (with all the spatial dimensions on the same footing). In our model, on the contrary, the density and the energy scale (i.e. the inverse of the curvature radius) grow together with the size of the particle horizon, which is expanding in the physical four-dimensional spacetime. We stress that it is just because of this unconventional property of superinflation that it becomes possible for the universe to emerge at the Planck time with causally connected regions whose size is much larger than the Planck length (see eqs.(5.9),(5.13)).

In conclusion, we have shown that the dominance of unstable configurations, in a multidimensional string phase, can drive an anisotropic evolution of the back-

ground and realize dynamically an effective decoupling between different spatial dimensions. Unfortunately, this mechanism seems to be incapable of choosing, by itself, both the preferred total dimensionality, and the number of contracting dimensions. However, once these parameters are assigned, one has a model in which the universe reaches the Planck density through a super-inflationary evolution and thus with horizons which may be large enough to avoid causality problems after the transition to the standard scenario.

References

1. K.A.Olive, Phys.Rep.190(1990)307
2. D.Shadev, Phys.Lett.137B(1984)155;
   R.B.Abbott, S.M.Barr and S.D.Ellis, Phys.Rev.D30(1984)720;
   E.W.Kolb, D.Lindley and D.Seckel, Phys.Rev.D30(1984)1205
3. M.Demianski and A.Polnarev, Phys.Rev.D41(1990)3003
4. Y.Tosa, Phys.Rev.D30(1984)2054
5. D.Shadev,Phys.Rev.D30(1984)2495
6. R.G.Moorhouse and J.Nixon, Nucl.Phys.B261(1985)172
7. E.I.Guendelman, The inflation-compactification mechanism, LA-UR-90-2124 (Los Alamos Preprint,1990)
8. V.S.Kaplunovsky, Phys.Rev.Lett.55(1985)1036
9. M. Dine and N. Seiberg, Phys.Rev.Lett.55(1985)366
10. R.Petronzio and G.Veneziano, Mod.Phys.Lett.A2(1987)707
11. M.Gasperini, N.Sánchez and G.Veneziano, Highly unstable fundamental strings in inflationary cosmologies, CERN Preprint TH.5893/90 (to appear in Int.J.Mod.Phys.A)
12. M.Gasperini, Kinematic interpretation of string instability in a background gravitational field, Torino University Preprint DFTT- 38/90 (to appear in Phys.Lett.B)
13. F.Lucchin and S.Matarrese, Phys.Lett.164B(1985)282
14. N.Sánchez and G.Veneziano, Nucl.Phys.B333(1990)253
15. Nguyen Suan Han and G. Veneziano, Inflation-driven string instabilities: towards a systematic large-R expansion, CERN preprint TH.6009/91
16. See for example R.H.Brandenberger, in "Physics of the early universe", ed. by J.A.Peacock, A.F.Heavens and A.T.Davies (SUSSP Publications, Edinburgh University, 1990) p.281

17. R.H.Brandenberger and C.Vafa, Nucl.Phys.B316(1989)391

18. Y.Aharonov, F.Englert and J.Orloff, Phys.Lett.B199(1987)366

19. Y.Leblanc, Cosmological aspects of the heterotic string above the Hagedorn temperature, CTP-1588 Preprint (Cambridge, Ma, April 1988)

20. M.Gasperini, A geometric regularization procedure for the curvature of cosmological backgrounds, in Proc. of the Workshop "Advances in theoretical physics" (Vietri, 1990) (World Scientific, Singapore) (in press)

21. E.Alvarez, Phys.Rev.D31(1985)418

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String Cosmology versus Standard and Inflationary Cosmology

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Abstract

This paper presents a review of the basic, model-independent differences between the pre-big bang scenario, arising naturally in a string cosmology context, and the standard inflationary scenario. We use an unconventional approach in which the introduction of technical details is avoided as much as possible, trying to focus the reader's attention on the main conceptual aspects of both scenarios. The aim of the paper is not to conclude in favour either of one or of the other scenario, but to raise questions that are left to the reader's meditation. Warnings: the paper does not contain equations, and is not intended as a complete review of all aspects of string cosmology.

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String cosmology versus standard and inflationary cosmology

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Abstract. This paper presents a review of the basic, model-independent differences between the pre-big bang scenario, arising naturally in a string cosmology context, and the standard inflationary scenario. We use an unconventional approach in which the introduction of technical details is avoided as much as possible, trying to focus the reader's attention on the main conceptual aspects of both scenarios. The aim of the paper is not to conclude in favour either of one or of the other scenario, but to raise questions that are left to the reader's meditation. Warnings: the paper does not contain equations, and is not intended as a complete review of all aspects of string cosmology.

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1. Introduction

The standard cosmological scenario [1, 2], rightfully one of the best celebrated conquests of the physics of the XX century, cannot be extrapolated to arbitrarily high energy and curvature scales without clashing with the singularity problem.

A singularity, on the other hand, often represents a signal that the physical laws we are applying have been extrapolated outside their domain of validity. As a well known example, we may quote here the case of the spectral energy distribution of radiation in thermal equilibrium. By applying the laws of classical physics one finds indeed the Rayleigh-Jeans spectrum, that diverges like ω^3 at high frequency. By taking into account instead the appropriate quantum corrections, this classical singularity is regularized by the bell-like Planck distribution, $\omega^4 \left(e^{\omega/T} - 1\right)^{-1}$, as illustrated in Fig. 1.

String theory suggests that the initial curvature singularity of the standard cosmological scenario could be similarly regularized by a bell-shaped curve, as qualitatively illustrated in Fig. 2. As we go back in time the curvature, instead of blowing up, could reach a maximum controlled by the string scale, and then decrease towards a state of very low curvature and weak coupling, approaching asymptotically the so-called string perturbative vacuum. This behaviour, indirectly suggested by duality and thermodynamical arguments [3], as well as, independently, by the motion of

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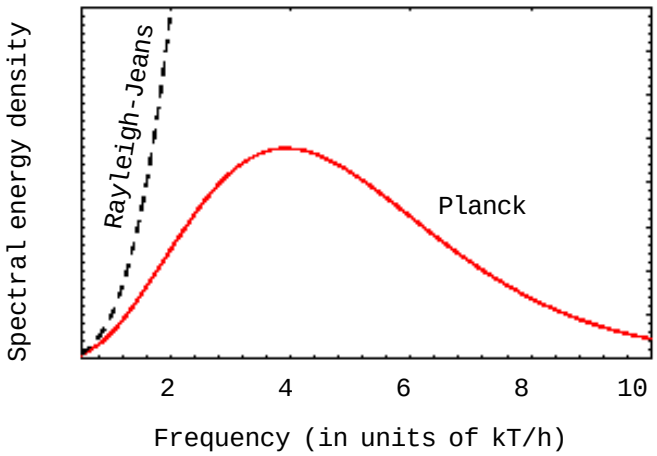

Figure 1. *The Planck distribution (full line) regularizes the classical Rayleigh-Jeans prediction (dashed line), for the spectral energy density of radiation in thermal equilibrium. The classical distribution is only valid at low enough frequency scales.*

strings in rolling backgrounds [4], is naturally grounded on the duality symmetry of the cosmological string effective action [5], and its possible implementation in the context of a realistic model of the early Universe has a lot of dynamical and phenomenological consequences [6].

In the string comology scenario of Fig. 2, the big bang singularity is replaced by a phase of high but finite curvature. It comes thus natural, in such a context, to call "pre-big bang" the initial phase in which the curvature is growing, in contrast to the subsequent "post-big bang" phase, with decreasing curvature and standard decelerated evolution.

The conventional inflationary scenario [2, 7], on the other hand, suggests a picture of the early Universe which is approximately intermediate between the standard and

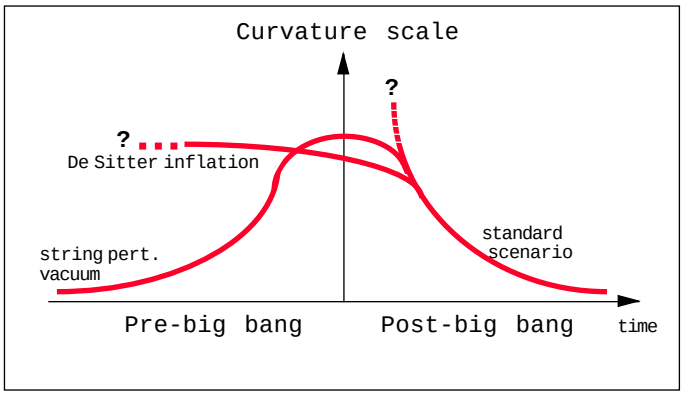

Figure 2. *Curvature scale versus time for standard, inflationary (de Sitter), and string cosmology models of the Universe. In the pre-big bang scenario the classical curvature singularity is regularized, and the standard cosmological evolution is only valid at late enough time scales.*

the string cosmology one. In that context the curvature, instead of blowing up, is expected to approach a nearly constant value, typical of a de Sitter-like geometry (see Fig. 2). However, a de Sitter phase with exponential expansion at constant curvature, implemented in the context of the conventional, potential-dominated inflation, cannot be extended back in time for ever, as first discussed in [8]. Indeed, quoting Alan Guth's recent survey of inflationary cosmology [9]:

"... Nevertheless, since inflation appears to be eternal only into the future, but not to the past, an important question remains open. How did all start? Although eternal inflation pushes this question far into the past, and well beyond the range of observational tests, the question does not disappear."

A possible answer to the above question, suggested by string cosmology and represented graphically in Fig. 2, is that all started from a state approaching, asymptotically, the flat, cold and empty string perturbative vacuum. Even if this starting point is infinitely far into the past, however, the initial state of our Universe might be non completely beyond the range of present observations, in contrast to the sentence quoted above, because the starting point may affect the dynamics of the subsequent inflationary evolution. The initial curvature scale, for instance, is constant or decreasing according to the conventional inflationary scenario, while it is growing according to the pre-big bang scenario, and this may lead to important phenomenological consequences.

In spite of the existence of a few particular examples [10], an unambiguous regularization of the curvature singularity (to all orders in the string effective action), toghether with a complete description of the transition from the pre- to the post-big bang regime, is not an easy achievement (see however [11] for recent encouraging results). Also, it is fair to say that the pre-big bang models, at present, are not free from other (more or less important) difficulties, that some aspects of the pre-big bang scenario are still unclear, and that further work is certainly needed for a final answer to all the difficulties.

Assuming that all the problems can (and will) be solved in a satisfactory way, string cosmology will provide eventually a model of the early Universe somewhat different, however, from the standard inflationary picture. The aim of this paper, therefore, is to present in a compact form a comparison (and a short discussion) of the way in which a phase of inflation could be implemented within a cosmology based on the string effective action, with respect to the standard cosmology based on the Einstein equations. Basically all the differences arise, as we will see, from the fact that in string cosmology the Universe starts evolving from an initial state at very low curvature and weak coupling, while in conventional inflation the initial state is assumed to approach the Planckian, quantum gravity regime.

The paper is organized as follows. We will discuss kinematical and dynamical differences in Sect. 2, quantum cosmology differences in Sect. 3, and phenomenological differences in Sect. 4. Sect. 5 is devoted to some concluding remarks. We will avoid

as much as possible the introduction of technical details – all contained in the papers quoted in the bibliography – trying to emphasize the ideas and the main physical aspects of the different inflationary scenarios.

As already stressed in the Abstract, it seems appropriate to recall that this paper is *not* intended as a complete review of all aspects of string cosmology. Rather, the paper is narrowly focused on those aspects where there is an important overlapping of methods and objectives and, simultaneously, a strong contrast of basic assumptions, for the string cosmology and the standard scenario. For a more exhaustive and systematic approach, the interested reader is referred to the excellent review paper devoted to a presentation and a technical discussion of all presently existing (super)string cosmology models [12], as well as to two recent introductory lectures on the pre-big bang scenario [13].

2. Kinematical and dynamical differences

The idea of inflation hystorically was born [14] to solve the problem of monopoles that could be largely produced in the early Universe, at the energy scale of grand unified theories (GUT). More generally, inflation is now understood as a period of accelerated evolution that can explain why the present Universe is so flat and smooth over a so large scale of distances [2, 7].

For hystorical reasons, i.e. because inflation was first implemented as a period of supercooling of the Universe trapped in a "false" vacuum state, in the context of GUT phase transitions, inflation was first associated [14] to a phase of exponential expansion (in cosmic time) of the scale factor $a(t)$, corresponding to a de Sitter (or "quasi-de Sitter") geometrical state. But it was soon realized that any type of accelerated expansion ($\dot{a} > 0$, $\ddot{a} > 0$, where the dots denote differentiation with respect to cosmic time), can in principle solve the kinematic problems of the standard scenario [15].

Actually, even accelerated contraction ($\dot{a} < 0$, $\ddot{a} < 0$) is effective to this purpose [16], as expected from the fact that accelerated expansion can be transformed into accelerated contraction through an appropriate field redefinition, like the one connecting the String and the Einstein frame. Indeed, physical effects such as the dilution of inhomogeneities should be independent from the choice of the frame. So, inflation can be generally identified, in a frame-independent way, as a period of accelerated evolution of the scale factor ($\mathrm{sign}\ \dot{a} = \mathrm{sign}\ \ddot{a}$).

There are, however, two classes of accelerated evolution very different from a dynamical point of view, and depending on the behaviour – growing or decreasing in time – of the curvature of the space-time manifold. A first important point to be stressed is thus the fact that the pre-big bang scenario corresponds to a phase of accelerated evolution characterized by growing (or non-decreasing) curvature [5, 6], while the phase that we shall call "standard inflation " [2, 7] is characterized by a curvature scale which tends to be constant – in the limiting case of a de Sitter metric – or slightly decreasing in time.

Before proceeding further, two remarks are in order. The first is that a phase of accelerated evolution and growing curvature, also called superinflation (or pole-inflation) [17], is not a peculiarity of string cosmology, but is possible even in general relativity: in higher-dimensional backgrounds, for instance, in the context of dynamical dimensional reduction. The important difference is that, in string cosmology, superinflation does not necessarily requires neither the shrinking of the internal dimensions, nor some exotic matter source [18] or symmetric breaking mechanism [19]: it can be simply driven, even in three spatial dimensions, by the kinetic energy of the rolling dilaton field ϕ [5, 6], parametrizing the growth of the string coupling $g = \exp(\phi/2)$ from zero (the string perturbative vacuum), to the strong coupling regime, $g \sim 1$.

The second remark is that the words "pre-big bang" should be referred to the complete cosmological evolution from the initial state approaching the string perturbative vacuum, up to the beginning of the hot, radiation-dominated phase of the standard scenario. Altough in most of this paper we shall restrict our discussion to the low-energy part of the pre-big bang phase, appropriately described in terms of the low-energy string effective action, the complete pre-big bang history necessarily includes a high-curvature "stringy" phase [20], which may also be of the inflationary type [21], and which has a curvature expected to be, on the average, non decreasing.

For a power-law, accelerated, conformally flat background, the time-behaviour of the curvature scale follows the behaviour of the absolute value of the Hubble parameter, $|H| = |\dot{a}/a|$. In such a background, on the other hand, the inverse of the Hubble parameter (i.e. the Hubble horizon $|H|^{-1}$) also controls the (finite) proper distance between the surface of the event horizon and the wordline of a comoving observer. Such a distance is shrinking for pre-big bang inflation (where $|H|$ is growing), while it is non-decreasing in standard inflation.

As a consequence of the fact that the horizon is shrinking, and the curvature is growing, it turns out that the initial state of the phase of inflation, in the pre-big bang scenario, is characterized by a curvature which is very small in Planck (or string) units, and by a Hubble horizon very large in the same units (for the sake of simplicity we may identify, at the end of inflation, the present value of the string length L_s and of the Planck lenght L_p; indeed, at tree-level, they are related by [22] $L_s = \langle g \rangle L_p = \langle \exp \phi/2 \rangle L_p$, with a present dilaton expectation value $\langle g \rangle \sim 0.3 - 0.03$). The initial state is also characterized by another very small dimensionless number, the initial string coupling $g \ll 1$, corresponding to a dilaton approaching the perturbative vacuum value, $\phi \to -\infty$.

By contrast, the standard inflationary scenario is characterized by a dilaton already settled to its present vacuum expectation value; the coupling is always strong (g is of order one), and the size of the initial horizon may be of order one in Planck units, if the initial curvature approaches the Planck scale. These kinematical and dynamical differences between standard and pre-big bang inflation are summarized in Table I.

As evident from the pre-big bang curve of Fig. 2, the smaller is the value of the initial curvature, the longer is the duration of the phase of pre-big bang inflation. The request that the inflation phase be long enough to solve the horizon and flatness

	Standard Inflation	**Pre-big bang Inflation**
Time evolution	*accelerated*	*accelerated*
Driving energy	*inflaton potential*	*dilaton kinetic energy*
Curvature	*constant or decreasing*	*growing*
Event horizon	*constant or growing*	*shrinking*
Initial curvature scale	*model-dependent*	*arbitrarily small*
Initial coupling	*strong, non-perturbative*	*arbitrarily weak, perturbative*

Table 1. Kinematical and dynamical differences between standard and pre-big bang inflation.

problems [14] thus imposes bounds on the dimensionless parameters of the initial state, controlled by the curvature and by the string coupling. Parameters such as the typical size of an initial homogeneous domain, in Planck units, must be by far greater than one for inflation to be successful. This aspect of string cosmology was pointed out already in the first papers on the pre-big bang scenario [5, 6] and, even before, also in the context of string-driven superinflation [4].

The fact that the initial curvature and coupling are small, and that the initial state is characterized by very large dimensionless numbers, may be interpreted however as a possible fine-tuning of the pre-big bang models [23], or even as a serious drawback, preventing the solution of the flatness and homogeneity problems, and supporting the conclusion that "the current version of the pre-big bang scenario cannot replace usual inflation" [24].

Consider, for instance, the horizon/homogeneity problem. At the onset of pre-big bang inflation, as usual in an inflationary context, the maximal allowed homogeneity scale is bounded by the size of the horizon. The point is that the initial horizon size H^{-1} is very large, instead of being of Planckian order. The basic question thus becomes: is an initial homogeneity scale of the order of the maximal scale H^{-1} necessarily unnatural if the initial curvature is small and, consequently, the initial horizon is large in Planck (or string) units? In other words, which basic length scale has to be used to measure the naturalness of the initial homogeneous domain, which subsequently inflates to reproduce the presently observed Universe? The Planck length or the radius of the causal horizon?

The choice of the Planck length, emphasized in [24], is certainly appropriate when the initial conditions are imposed in a state approaching the high-curvature, quantum

gravity regime, like in models of chaotic inflation [25], for instance. In the pre-big bang scenario, on the contrary, the initial conditions are to be imposed when the Universe is deeply inside the low-curvature, weak coupling, classical regime. In that regime the Universe does not know about the Planck length, and the only available classical scale of distance, the horizon, should not be discarded "a priori" as unnatural [26].

This does not mean, of course, that the horizon should be always *assumed* as the natural scale of homogeneity. This suggests, however, that the naturalness of homogeneity over a large horizon scale should be discussed on the grounds of some quantitative and objective criterium, as attempted for instance in [27], taking into account also the effects of quantum fluctuations [28] that could destroy the initial, classical homogeneity (see also [29] for a discussion of "generic" initial conditions in a string cosmology context).

Another point concerns the flatness problem. In order to explain the precise fine-tuning of the present density to the critical one, the initial state of pre-big bang inflation must be characterized by large dimensionless parameters, thus reintroducing from the beginning the large numbers that one would like to explain.

This may seem to be quite unsatisfactory, as emphasized in [24]. It should be pointed out, however, that if one accepts the point of view that large numbers are always to be avoided at the onset of inflation, then should also accept the fact that natural initial conditions are only possible in the context of models in which inflation starts at the Planck scale, in order to have, for the curvature, an initial dimensionless ratio of order one. This rules out, as a satisfactory explanation of our present cosmological state, not only the pre-big bang scenario, but any model in which inflation starts at scales smaller than Planckian (unless we imagine a scenario with different stages of inflation, each of them responsible for solving different problems, and occurring at different scales: but, again, is this a natural cosmological configuration?)

Even for a single stage of inflation, occurring very near to the Planck scale, we are not free from problems, however, as we are led eventually to the following question: can we trust the naturalness of inflation models in which classical general relativity is applied to set up initial conditions at Planckian curvature scales, i.e. deeply inside the non-perturbative, quantum gravity regime? (in string cosmology, the Planckian regime directly affects the exit from the inflationary phase, and only indirectly set constraints on the initial conditions, through the finite duration of the low energy phase).

Assuming that the answer be positive, we are led to a situation that can be graphically summarized, in a qualitative way, as in Fig. 3.

- Case (a) represents a standard inflationary model of the Universe in which inflation starts at the Planck scale. The time arrow points from bottom to top, and the shaded area at the time t_0 represents a spatial section of our present homogeneous Universe, of size fixed by the present Hubble radius H_0^{-1}. As we go back in time, according to the solutions of the standard cosmological model, the horizon shrinks linearly in cosmic time, while the proper size of the present homogeneous region, controlled by the scale factor $a(t)$, shrinks slowly. When we reach the Planck scale,

at the time t_f, the causal horizon is smaller than the homogeneous region, roughly by the factor 10^{-30}.

To solve this problem, the phase of standard evolution is preceeded by a phase of exponential de Sitter inflation, long enough in time from t_i to t_f, during which the curvature and the horizon stay frozen at the Planck scale, and our present portion of the Universe may "re-enter" inside the causal horizon.

It should be stressed that, in a realistic inflationary model, the horizon has to be slightly increasing from t_i to t_f, because the scale corresponding to our present Hubble radius has to cross the horizon, during inflation, at a curvature scale H_1 smaller than Planckian. We must require, in particular, that $H_1/M_p \lesssim 10^{-5}$ in order to avoid too much amplification of gravitational perturbations, that would contradict the present degree of homogeneity observed by COBE [30] at large angular scales. However, for the sake of graphical simplicity, we shall ignore this complication that is not essential for our present discussion.

- Case (b) represents a string cosmology model of the Universe, in which the inflationary pre-big bang phase, from t_i to t_f, is represented in terms of the contracting metric of the Einstein frame [16], in order to emphasize (graphically) the underlying duality and time-reversal symmetry of the scenario (there is no need, of course, that in the Einstein frame the pre-big bang scale factor exactly coincide with the time-reversal of the post-big bang solution).

  The main difference from case (a) is that in the pre-big bang epoch the curvature is growing, and the event horizon shrinks linearly in cosmic time, from t_i to t_f, instead of being constant. Since the scale factor shrinks at a slower rate, however, it is still possible for the initial homogeneous domain to be "pushed out" of the horizon, and for the Universe to emerge at the Planck scale, at the time t_f, in the same configuration as in case (a). The subsequent evolution from t_f to t_0 is the same as in the standard scenario.

- Case (c), finally, represents a string cosmology model in which the period of pre-big bang inflation corresponds in part to a phase of growing curvature, growing dilaton and shrinking horizon (from t_i to t_s), and in part to a phase in which the curvature, the horizon, and eventually the dilaton, are frozen at the Planck scale (from t_s to t_f). The initial horizon is still large in Planck units, but it is no longer reflection-symmetric to t_0, depending on the duration of the high curvature phase from t_s to t_f.

According to [24], the model (a) represents an acceptable solution to the horizon and flatness problem because inflation starts at the Planck scale, and all the dimensionless parameters characterizing the initial configuration are of order one. When the phase at constant curvature of the model (c) extends in time like in case (a) the two models pratically coincide for what concerns the naturalness of the initial conditions, as in both cases our Universe emerges at the Planck scale from a single domain of Planckian size, and we loose any observational tracks of what happened before.

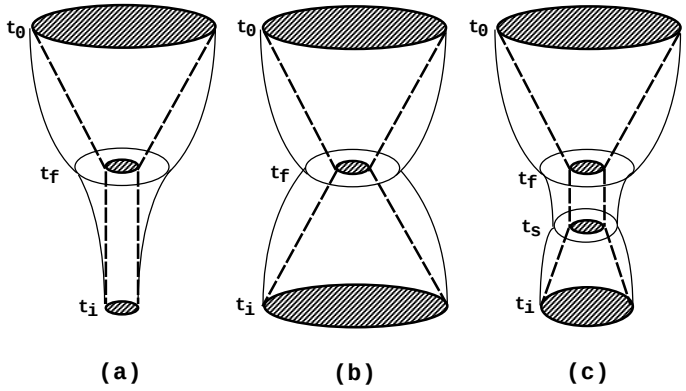

Figure 3. *Qualitative evolution of the horizon scale and of the proper size of a homogeneous region for (a) standard de Sitter inflation with constant Hubble horizon, (b) pre-big bang inflation (in the Einstein frame) with shrinking horizon, and (c) pre-big bang including a phase of high curvature inflation at the string scale. The time direction coincides with the vertical axis. The three horizontal spatial sections t_0, t_f and t_i corresponds, from top to bottom, to the present time, to the end, and to the beginning of inflation. The shaded area represents the horizon, and the dashed lines its time evolution. The full curves represent the time evolution of the proper size of the homogeneous region, controlled by the scale factor.*

The aim of the pre-big bang scenario, on the other hand, is to attempt a description of the possible cosmological evolution *before* the Planck epoch. The main difference between case (a) and (c) is that, if the duration of the phase of inflation at constant curvature is shorter than the minimal duration required for a solution of all the standard kinematic problems, what happened before the Planck scale may then become visible. In other words, there are phenomenological consequences that can be ascribed to the phase of pre-Planckian evolution, and that can be tested (at least in principle) even today (see Section 4). The model of case (b), in particular, is the limiting case in which the duration of the high curvature phase shrinks to a point, and the Universe emerges at the Planck curvature scale with a homogeneous domain large enough to fill our present Hubble radius through the subsequent standard evolution.

The above discussion refers to the curvature in Planck units, but the same arguments can also be applied to the initial value of the string coupling. If the coupling is of order one at the beginning of inflation then it is natural, in the sense of [24], if it is much smaller than one, then the inflationary growth of the coupling may have observable consequences.

To conclude this discussion it seems difficult, in our opinion, to discard a model of pre-Planckian evolution, like those illustrated in case (b), (c), only on the grounds of the large parameters characterizing its initial conditions. Such an argument could be applied in the impossibility of observational tests, namely in the absence of any phenomenological evidence about the cosmological evolution before the Planck era. But, as pointed out before, the pre-Planckian epoch becomes invisible only in the limiting case (a), in which the effective models reduces to standard inflation starting at the

Planck scale, with no unnatural initial conditions.

If, on the contrary, the high curvature phase is shorter than in case (a), then the model requires an initial horizon larger than Planckian, and a small initial coupling – which are possibly unnatural according to standard criteria [24]. In that case, however, such initial conditions are in principle accessible to present observations, so why do not try to test the scenario observationally, and try to analyze the naturalness in terms of a Bayesan approach, as attempted in [27]? In that case, the computation of "a posteriori" probabilities suggests that the observation of a large initial horizon and a small initial coupling may become "a posteriori" natural [27], because of the duality symmetry intrinsic to the pre-big bang scenario.

Finally, even if the initial state should require a certain degree of fine-tuning, this does not necessarily implies that the pre-big bang cosmology described by string models is to be discarded (after all, the description of our late-time Universe given by the standard cosmological model is rather satisfactory, in spite of the fine-tuning required in such a model if the initial state is extrapolated back in time until the Planck epoch).

Usually, the need for fine-tuning in the initial conditions means that the model is incomplete, and that a more general dynamical mechanism is required, to explain the particular initial conditions. Thus, it might well be that the pre-big bang picture provided by string cosmology does not represents the whole story of our Universe, and that only an earlier evolution can explain why, at a certain instant of time, the Universe is lead to a state so similar to the string perturbative vacuum.

3. Quantum cosmology differences

In the standard inflationary scenario the phase of exponential, de Sitter-like expansion at constant curvature cannot be infinitely extended towards the past, for a well known reason of geodesic completeness. A complete manifold requires an earlier contracting phase: in that case, however, it seems impossible, in models dominated by the inflaton potential, to stop the collapse and to bounce towards the expanding phase [8]. One has thus the problem of explaining how the Universe could emerge at the Planck scale in the initial state appropriate to exponential expansion.

At curvature scales of order one in Planck units we are in the full quantum gravity regime, and the use of the quantum cosmology approach seems to be appropriate. In this approach the Universe is represented by a wave function satisfying the Wheeler-De Witt (WDW) equation [31], and evolving in the so-called superspace, whose points represent all possible spatial geometric configurations. For pratical applications, however, the evolution of the WDW wave function is usually studied in a "minisuperspace" context, where only a finite numbers of coordinates is chosen to parametrize the different geometrical configurations.

With an appropriate choice of the boundary conditions it is then possible to obtain cosmological solutions of the WDW equation describing the "birth of the Universe" as an effect of quantum tunnelling [32, 33]. In that case, if the geometric state of the

Universe is characterized by a cosmological constant Λ (due, for instance, to the vacuum energy-density induced by a scalar field potential), the tunnelling probability is found to be proportional to $\exp(-\Lambda^{-1})$, where Λ is measured in Planck units. The Universe tends thus to emerge in a state of big vacuum energy, just appropriate to the onset of inflation. The quantum cosmology approach seems thus to provide a natural mechanism to explain the formation of "baby universes", emerging at the Planck scale, and ready to inflate according to the standard inflationary scenario [34].

The minisuperspace approach to quantum cosmology is known to be affected by various problems of technical nature: the probabilistic interpretation, the unambiguous determination of an appropriate time parameter, the semiclassical limit, the ordering of quantum operators, and so on. The most unsatisfactory aspect of this approach, in our opinion, is however the fact that the boundary conditions for the tunnelling process are to be chosen "ad-hoc". They are by no means compelling, and it is possible indeed to impose different boundary conditions, for instance according to the "no boundary" criterium [35], leading to a completely different result for the probability of creation of universes – results that are not always appropriate to inflationary initial conditions.

The source of this problem is the fact that, in a quantum description of the birth of the Universe, the final cosmological state (i.e. the Universe that we want to obtain) is well known, while the initial cosmological state (before the quantum transition) is completely unknown, at least in the context of the standard inflationary scenario. Indeed, the cosmological tunnelling is usually referred as a proces of tunnelling "from nothing" [32], just to stress the ignorance about the initial vacuum state. The classical theory of the standard cosmological scenario cannot help, because the initial state, in that context, is the big bang singularity, i.e. just what the quantum approach would like to avoid.

In a string cosmology context, the quantum approach based on minisuperspace can be implemented in a straightforward way, with the only difference that the differential WDW equation represents the Hamiltonian constraint following not from the Einstein action, but from the low-energy string effective action [36, 37, 38]. As a consequence, the "minimal" minisuperspace is at least two-dimensional, because the action always contains the dilaton, besides the metric. The formal problems related to the minisuperspace approach remains, with the possible exception of the operator-ordering problem, as the quantum ordering is unambiguosly fixed by the global, pseudo-orthogonal $O(d,d)$ symmetry of the low-energy string effective action [36, 38]. Another possible exception is the identification of the time-like coordinate in minisuperspace [39].

There is, however, a radical difference for what concerns boundary conditions. In the context of the pre-big bang scenario the initial, asymptotic state of the Universe is unambiguosly prescribed – the string perturbative vacuum – and cannot be chosen "ad-hoc". Such initial state is perfectly appropriate to a low-energy normalization of the WDW wave function, and the transition probability of string cosmology only depends on the dynamics, i.e. on the effective potential appearing in the WDW equation.

It is now interesting to observe that if we compute, in the context of the pre-

big bang scenario, the transition probability from the perturbative vacuum to a final, post-big bang configuration characterized by a non-vanishing cosmological constant, we obtain [37] a probability distribution $P(\Lambda)$ very similar to the one of the conventional quantum cosmology, computed with tunneling boundary conditions. The reason is that, by imposing the perturbative vacuum as the boundary condition to the WDW equation, the WDW solutions contain only outgoing waves at the singular spacetime boundary, just like in the case of tunnelling boundary conditions [32]. In this sense, we can say that the "ad-hoc" prescription of tunnelling boundary conditions simulates, in a phenomenological way, the birth of the Universe from the string perturbative vacuum. This suggests that, instead of "tunnelling from nothing", we should speak of "tunnelling from the string perturbative vacuum" or, even better, of quantum instability and "decay" of the perturbative vacuum [40].

A further, important difference should be mentioned. The transition from the pre- to the post-big bang phase induced by the cosmological constant (or, more generally, by an appropriate dilaton potential), is represented, in the minisuperspace of string cosmology, not like a tunnelling effet, but like a quantum reflection of the WDW wave function, over an effective potential barrier. The correct description that we obtain in string cosmology for the birth of the Universe, therefore, is that of a "quantum scattering" effect [36, 37, 39].

The various differences between quantum inflationary cosmology and quantum string cosmology are summarized in Table II. Besides the formal aspects (such as tunnelling versus reflection), the basic difference is that in the standard inflationary scenario the Universe, because of quantum cosmology effects, is expected *to enter* in the inflationary regime, while in the pre-big bang scenario the Universe is expected *to exit* from the inflationary regime (or at least from the phase of growing curvature). So, in standard inflation, quantum effects at the Planck scale are expected to be responsible for inflationary initial conditions. In string cosmology, on the contrary, initial conditions are to be imposed in the opposite, low energy quantum regime, where quantum effects are negligible.

It seems appropriate, at this point, to comment on the fact that the initial state of pre-big bang inflation seems to be characterized by a large entropy S, if one assumes, as in [24], that the de Sitter relation between entropy and horizon area remains valid also when the horizon is not constant in time (or, in other words, if one assumes a saturation of the bound provided by the holographic principle [41], applied however to the Hubble horizon [42]). If S is large in Planck units, the probability that such a configuration be obtained through a process of quantum tunnelling, $\exp(-S)$, is exponentially small, as emphasized in [24]. However, as stressed above, in string cosmology quantum effects such as tunnelling or reflection are expected to be effective *at the end* of inflation, and not *at the beginning*, i.e. *not* to explain the origin of the initial state. A large entropy of the initial state, in the weakly-coupled, highly-classical regime, can only correspond to a large probability of such configuration, which is proportional to $\exp(+S)$, like for every classical and macroscopic configurations (not arising from quantum tunnelling).

	Quantum cosmology	**Quantum string cosmology**
Formal approach	*WDW, minisuperspace*	*WDW, minisuperspace*
Quantum ordering	*arbitrary*	*fixed by duality*
Boundary conditions	*arbitrary*	*string perturbative vacuum*
Outgoing waves	*tunnelling from nothing*	*reflection from pre-big bang*
Quantum transition	*beginning of inflation*	*exit from inflation*

Table 2. Quantum cosmology differences between standard and pre-big bang inflation.

Let us stress, finally, another important difference between conventional quantum cosmology and quantum string cosmology. In string cosmology quantum geometrical effects cannot be fully accounted for, as fas as we limit ourself to a WDW equation obtained from the low-energy string effective action. Indeed, when approaching the Planck scale, the string theory action acquires (even at small coupling, i.e. at tree-level in the quantum loop expansion) higher curvature correction [43], weighed by the inverse string tension α'. They are to be included into the Hamiltonian constraint, and lead in general to a higher-derivative WDW equation. This problem has been discussed in [44], and it has been shown in [45] that when the higher curvature corrections appear in the form of an Eulero density, then the WDW approach can only be applied to a dimensionally reduced version of the theory.

The quantum cosmology results reported in this section refer, in this sense, only to a model of "low-energy" quantum string cosmology [40]. In the full quantum gravity regime, in order to include all the higher-derivative contributions, the correct WDW equation should follow not from the effective action, but possibly from a conformal, sigma-model action [46], which automatically takes into account all orders in α'. We note, however, that duality transformations in toroidal moduli space have recently suggested [47] that the Lorenztian structure of the low-energy minisuperspace may have an exact meaning also in an M-theory context, even if the exact M-theory equations are expected to be in general different.

In fact, when the curvature is large in string units ($\alpha' H^2 > 1$), and also the string coupling is large ($g^2 = e^{\phi} > 1$), we necessarily enter the M-theory regime where new quantum effects are possible, such as a copious production of higher-dimensional D-branes [48]. If the curvature is small enough, the strong coupling regime of string cosmology is then expected to be described by 11-dimensional supergravity theory, and

the dilaton to be interpreted as the radius (i.e., the modulus field) of the 11-th dimension [49]. In this context string cosmology becomes U-duality covariant [12, 50, 51], and the presence in the action of Ramond-Ramond fields may be helpful to evade the problem of the curvature singularity [52, 53, 54]. The singularity, in addition, could also disappear as a result of the embedding of the low-energy solutions of string theory into a higher-dimensional ($d = 11$) manifold [51, 55].

These results seem to suggests that an appropriate quantum description of the birth of the Universe will be probably achieved only within a full M-theory approach to the strong coupling regime, in which the pre-big bang acceleration is damped, the curvature is regularized, and the Universe bounces back to the phase of standard evolution.

4. Phenomenological differences

One of the most important (and probably also most spectacular) phenomenological predictions of inflation is the parametric amplification of metric (and of other different types of) perturbations [56], and the corresponding generation of primordial inhomogeneity spectra, directly from the quantum fluctuations of the background fields in their vacuum state (see [57] for a review). Such fluctuations, when decomposed in Fourier modes, satisfy a canonical Schrodinger-like equation, whose effective potential is determined by the so-called" pump field", which depends in its turn on the background geometry.

It is then evident that different backgrounds lead to different pump fields, to a different evolution of perturbations, and thus to different spectra. In string cosmology, in particular, there are two main properties of the background that can affect the final form of the perturbation spectra. They are:

- (A) the growth of the curvature scale;
- (B) the scalar-tensor (i.e. gravi-dilaton) nature of the background.

Property (A) has two important consequences. The first, that we will call (A1), is that the pre-big bang scenario leads to metric perturbation spectra growing with frequency [58] (instead of being flat, or decreasing, like in standard inflation), because the spectral distribution of metric perturbations tends to follow the behaviour of the curvature scale at the time of the first horizon crossing. The second, that we will call (A2), is that the growth of the curvature can also force the comoving amplitude of perturbations to grow (instead of being frozen) outside the horizon. This effect, implicitly contained in the earlier, pioneer studies [59], was first explicitly pointed out in [60], and only later independently re-discovered in a string cosmology context [61]. As a further consequence of property (A) we should mention, finally, the fact that perturbations are amplified in a final "squeezed vacuum" state, and not in a "squeezed thermal vacuum" [62].

Let us first discuss the second effect (A2). This effect, on one hand, is interesting because it may lead to an amplification of perturbations more efficient than in the standard inflationary scenario. On the other hand, however, it is dangerous, because

the perturbation amplitude could grow too much, during the pre-big bang phase, so as to prevent the application of the standard linearized approach, which neglects effects of back-reaction [57].

Such an "anomalous" growth of perturbations cannot be eliminated by a change of frame, because the associated physical (i.e. observable) energy density spectrum is obviously frame-independent. However, the breakdown of the linear approximation *is*, in general, *gauge-dependent*. For the particular case of scalar metric perturbations, in three isotropic dimensions, the linear approximation breaks down in the standard longitudinal gauge, but is restored in a more appropriate "off-diagonal" gauge [63], also called "uniform-curvature" gauge [64]. Moreover, as a consequence of a particolar form of duality invariance that appears explicitly in the Hamiltonian approach to perturbation theory [65], the final energy-density spectrum can always be correctly estimated by neglecting the growing mode, provided one includes in the full Hamiltonian both the contribution of the amplitude and of its conjugate momentum.

There are backgrounds, however, in which the growth of perturbations remains too strong even after the elimination of all unphysical gauge effects, and we have to limit ourselves to a restricted portion of parameter space for the linear approximation to be valid. Even in this case, however, the effect (A1) has two interesting consequences.

The first is that a growing spectrum leads to the formation of relic backgrounds whose amplitude is higher at higher frequency, where in general the backgrounds are also more easily detectable. A typical example is the formation of a relic background of cosmic gravitons which, in the frequency range of present resonant-mass and interferometric detectors ($10^2 - 10^3$ Hz), could be up to eight orders of magnitude stronger than expected in the context of standard inflation [58, 66]. Thus in principle detectable, in a not-so-far future, by the (planned) advanced version of the interferometric gravitational antennas, or by spherical resonant detectors.

The second consequence is that the normalization of the peak of the spectrum, at high frequency, is automatically controlled by the string scale [67]. The peak amplitude may be high enough to support a picture in which all the radiation, that becomes dominant at the beginning of the standard era, is produced through a process of parametric amplification, directly from the quantum fluctuations of the vacuum during the pre-big bang epoch [67, 68, 69]. Indeed, for the background fields that interact more strongly than gravitationally, the amplified fluctuations are expected to thermalize, and their energy-density is expected to grow in time with respect to the dilaton kinetic energy that was driving the background during the phase of pre-big bang inflation. This possibility is absent in the standard inflationary scenario, where the spectrum of perturbations is decreasing, and the normalization of the spectrum is determined at low frequency by the observation of the large scale CMB anisotropy [30]: the resulting energy-density of the fluctuations, in that case, is by far too low to dominate, eventually, the post-inflationary background.

Up to now we have reported some phenomenological consequences of the property (A). For what concerns the property (B), i.e. the gravi-dilaton nature of the background,

we will quote here a peculiar string cosmology effect, the amplification of the quantum fluctuations of the electromagnetic field due to their direct coupling to the dilaton, according to the effective Lagrangian density $\sqrt{-g}e^{-\phi}F_{\mu\nu}F^{\mu\nu}$.

In general relativity the dilaton is absent, the Lagrangian is invariant under a conformal rescaling of the metric, and the coupling of the electromagnetic field to a conformally flat metric, typical of inflation, can always be eliminated (unless the coupling is non-minimal and/or violates $U(1)$ gauge invariance [70]). As a consequence, the inflationary evolution of the metric background is unable to amplify the electromagnetic fluctuations. In string cosmology, on the contrary, such fluctuations are amplified by the accelerated growth of the dilaton (acting as the pump field) during the pre-big bang phase. If the high-curvature string phase is long enough, it is then possible to produce in this way the "seeds", required for instance by the galactic dynamo, for the generation of cosmic magnetic fields on a large scale [71]. String cosmology thus provides a possible solution to a longstanding astrophysical "puzzle", i.e. the generation of the primordial seed fields, through a mechanism which is uneffective in the standard inflationary scenario.

This certainly represents an advantage with respect to the standard scenario. The different amplification of perturbations, however, is also asssociated to possible drawbacks. In particular, the fact that the metric perturbation spectrum, in string cosmology, grows with a very steep slope, and it is rigidly normalized at high frequency, makes problematic the matching to the anisotropy observed at the present horizon scale [30]. The generation of the observed CMB anisotropy, with the right spectrum, is instead one of the most celebrated results of standard inflation [2, 7, 57].

A possible solution to this problem, in string cosmology, comes from the amplification of the fluctuations of the Kalb-Ramond axion, which is one of the fundamental fields appearing in the string effective action, already at low energy. Indeed, unlike metric perturbations, the axion perturbations can be amplified with a rather flat spectrum [72] and, through the (integrated) Sachs-Wolfe effect [73], they can induce the temperature anisotropy observed in the CMB radiation on a large scale, both in case of massless [74] and massive [75] axion fluctuations.

It is important to observe, in that case, that the slope of the spectrum is no longer arbitrary, but rigidly determined by the COBE normalization (imposed at low frequency), and by the string normalization (imposed at the opposite, high-frequency end of the spectrum). The resulting slope turns out to be slightly increasing [74, 75], but still in agreement with the observational limits at the horizon scale [76]. At higher frequency scales, however, important differences from standard inflation may appear in the peak structure of the spectrum [77]. The possible axionic origin of the fluctuations of the CMB temperature is thus expected to be confirmed (or disproved) in a very near future, by the planned satellite observations.

The main differences between standard and string cosmology inflation, for what concerns cosmological perturbations, are summarized in Table III. The last entry of the Table refers to the amplification of the fluctuations of the dilaton background [78],

	Standard Inflation	**Pre-big bang Inflation**
Pump field	*metric*	*metric and dilaton*
Spectrum (vs frequency)	*flat or decreasing*	*increasing*
Amplitude outside horizon	*frozen*	*increasing*
Normalization	*low frequency (COBE)*	*high frequency (string scale)*
Electromagnetic fields	*unaffected*	*amplified by the dilaton*
CMB anisotropy	*metric fluctuations*	*axion fluctations*
Dilaton fluctuations	*absent*	*dilaton productions*

Table 3. Amplification of vacuum fluctuations in the inflationary and pre-big bang scenario.

another peculiar effect of string cosmology, because the dilaton is absent in the standard scenario. This effect, analogous to the amplification of tensor metric perturbations, can be interpreted (in a second quantization approach) as a process of dilaton production, which leads to the formation of a relic background of cosmic dilatons. The background is subject to various constraints, depending on the slope of the spectrum and on the mass of the dilaton, but if dilatons are light enough [79] (namely $m \lesssim 10$ KeV), they are not yet decayed and could represent today a significant fraction of the dark matter, that seems required to match various astrophysical observations.

Detecting such a background, through the gravity-like interactions of the dilatons at low-energy scales, is however a challenge that seems beyond the possibilities of present technology [80], unless the dilaton couples universally to macroscopic bodies, represents a significant fraction of dark matter, and its mass lies within the sensitivity band of gravitational detectors [81].

5. Conclusion

The pre-big bang scenario provides a model, suggested and supported by string theory, of the possible cosmological evolution before our Universe emerged at the Planck scale.

If the subsequent post-Planckian evolution follows the standard inflationary scenario, then initial conditions can be imposed at the Planck scale, and are probably natural in the sense of [24], but any track of what happened before disappears from our

observational range. If, on the contrary, inflation at the Planck scale is not too long, and inside our present Hubble radius there are comoving length scales that crossed the horizon during the low-curvature pre-big bang phase, then pre-Plackian initial conditions are in principle accessible to present observations, and their naturaleness can be discussed in terms of a Bayesan analysis [27], based on "a posteriori" probabilities. Quantum cosmology methods can also be applied, taking into account however that quantum effects are possibly important at the end, and not at the beginning, of inflation.

We believe that the possibility of looking back in the past before the Planck era is the most distinctive aspect of string cosmology, with respect to the standard inflationary cosmology. Concerning the possible tracks of the pre-Planckian Universe, we have emphasized , in particular, three effects, referring to observations to be performed 1) in a not-so-far future, 2) in a near future, and 3) to observations already (in part) performed. These effects are, respectively: 1) the production of a cosmic gravity wave background, 2) the axion-induced anisotropy of the CMB radiation, and 3) the production of seeds for the cosmic magnetic fields.

These effects are not necessarily compatible among them (at least in the "minimal" version of the pre-big bang models), and it seems thus possible to test, and eventually exclude (or confirm) the pre-big bang scenario on the grounds of its phenomenological consequences. Even if, as recently stressed [82], the idea itself of inflation "cannot be falsified", the particular models can (and must) be tested, and the pre-big bang scenario, after all, can be regarded as a particular, unconventional model of primordial inflation.

It seems thus appropriate to stress, in conclusion, that the pre-big bang scenario is not alternative to the idea of inflation, but only alternative to a more conventional realization of inflation which, for hystorical reason, is still deeply anchored to the standard big bang picture, where the initial state must necessarily represent a very small, curve and dense Universe. The effort of this paper aims at stimulating the reader's meditation on the fact that this standard picture is a possibility, not a necessity, and that quite different initial conditions are possible, and not necessarily unlikely.

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References

[1] S. Weinberg, *Cosmology and gravitation* (Wiley, New York, 1972).

[2] E. W. Kolb and M. S. Turner, *The Early Universe*, (Addison Wesley, Redwood City, Ca, 1990).

[3] Y. Leblanc, Phys. Rev. D **38**, 3087 (1988); R. Brandenberger and C. Vafa, Nucl. Phys. B **316**, 391 (1989); E. Alvarez and M. A. R. Osorio, Int. J. Theor. Phys. **28**, 949 (1989); A. A. Tseytlin, Class. Quantum Grav. **9**, 979 (1992); A. A. Tseytlin and C. Vafa, Nucl. Phys. B **372**, 443 (1992).

[4] M. Gasperini, N. Sanchez and G. Veneziano, Nucl. Phys. B **364**, 365 (1991).

[5] G. Veneziano, Phys. Lett. B **265**, 287 (1991); K. A. Meissner and G. Veneziano, Phys. Lett. B **267**, 33 (1991); Mod. Phys. Lett. A **6**, 3397 (1991); M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B **272**, 277 (1991).

[6] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993); Mod. Phys. Lett. A **8**, 3701 (1993); Phys. Rev. D **50**, 2519 (1994). An updated collections of papers on the pre-big bag scenario is available at `http://www.to.infn.it/~gasperin`

[7] A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, Chur, Switzerland, 1990).

[8] A. Vilenkin, Phys. Rev. D **46**, 2355 (1992); A. Borde, A. Vilenkin, Phys. Rev. Lett. **72**, 3305 (1994).

[9] A. Guth, *The inflationary Universe*, (Vintage, London, 1998), p. 271.

[10] Last paper in Ref. [5]; first paper in Ref. [6]; R. Brustein and R. Madden, Phys. Rev. D **57**, 712 (1998); S. Foffa, M. Maggiore and R. Sturani, Nucl. Phys. B **552**, 395 (1999); D. A. Easson and R. Brandenberger, JHEP **9909**, 003 (1999). G. F. R. Ellis, D. C. Roberts, D. Solomons and P. K. S. Dunsby, *Using the dilaton potential to obtain string cosmology solutions*, hep-th/9912005.

[11] C. Cartier, E. J. Copeland and R. Madden, JHEP **0001**, 035 (2000).

[12] J. E. Lidsey, D. Wands and E. J. Copeland, *Superstring Cosmology*, hep-th/9909061 (Phys. Rep., in press).

[13] M. Gasperini, *Elementary introduction to pre-big bang cosmology and to the relic graviton background*, hep-th/9907067, in Proc. of the Second SIGRAV School (Center A. Volta, Como, April 1999), ed. by V. Gorini et al (IOP Publishing, Bristol, 2000); G. Veneziano, *String cosmology: the pre-big bang scenario*, hep-th/0002094, in Proc. of the Les Houches School (July 1999), in press.

[14] A. Guth, Phys. Rev. D **23**, 347 (1981).

[15] F. Lucchin and S. Matarrese, Phys. Lett. B **164**, 282 (1985).

[16] See, in particular, the second and the third paper of Ref. [6].

[17] D. Shadev, Phys. Lett. B **317**, 155 (1984); R. B. Abbott, S. M. Barr and S. D. Ellis, Phys. Rev. D **30**, 720 (1984); E. W. Kolb, D. Lindley and D. Seckel, Phys. Rev. D **30**, 1205 (1984).

[18] M. Gasperini, Phys. Rev. Lett. **56**, 2873 (1986).

[19] M. Gasperini, Phys. Lett. B **163**, 164 (1985); Class. Quantum Grav. **4**, 485 (1987).

[20] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B **494**, 315 (1997).

[21] M. Maggiore and R. Sturani, Phys. Lett. B **415**, 335 (1997).

[22] V. Kaplunovsky, Phys. Rev. Lett. **55**, 1036 (1985).

[23] M. S. Turner and E. J. Weinberg, Phys. Rev. D **56**, 4604 (1997).

[24] N. Kaloper, A. Linde and R. Bousso, Phys. Rev. D **59** 043508 (1999).

[25] A. D. Linde, Phys. Lett. B **175**, 395 (1986).

[26] M. Gasperini, Phys. Rev. D61, 87301 (2000).

[27] A. Buonanno, T. Damour and G. Veneziano, Nucl. Phys. B **543**, 275 (1999).

[28] A. Gosh, G. Pollifrone and G. Veneziano, Phys. Lett. B **440**, 20 (1998).

[29] D. Clancy, J. E. Lidsey and R. Tavakol, Phys. Rev. D **59**, 063511 (1999); Phys. Rev. D **58**, 44017 (1998); D. Clancy, A. Feinstein, J. E. Lidsey and R. Tavakol, Phys. Lett. B **451**, 303 (1999); A. Feinstein, K. E. Kunze and M. A. Vazquez-Mozo, *Initial conditions and the structure of the singularity in pre-big bang cosmology*, hep-th/0002070.

[30] A. J. Banday et al., Ap. J. **475**, 393 (1997).

[31] B. S. De Witt, Phys. Rev. **160**, 1113 (1967); J. A. Wheeler, in *Battelle Rencontres*, eds. C. De Witt and J. A. Wheeler (Benjamin, New York, 1968).

[32] A. Vilenkin, Phys. Rev. D **30**, 509 (1984); D **33**, 5560 (1986); D **37**, 888 (1988).

[33] A. D. Linde, Sov. Phys. JEPT **60**, 211 (1984); Lett. Nuovo Cimento **39**, 401 (1984); Y. Zel'dovich and A. A. Starobinski, Sov. Astron. Lett. **10**, 135 (1984); V. A. Rubakov, Phys. Lett. B **148**, 280 (1984).

[34] A. Linde, Phys. Lett. B **351**, 99 (1995).

[35] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983); S. W. Hawking, Nucl. Phys. B **239**, 257 (1984); S. W. Hawking and D. N. Page, Nucl. Phys. B **264**, 185 (1986).
[36] M. Gasperini, J. Maharana and G. Veneziano, Nucl. Phys. B **472**, 394 (1996).
[37] M. Gasperini and G. Veneziano, Gen. Rel. Grav. **28**, 1301 (1996).
[38] A. A. Kehagias and A. Lukas, Nucl. Phys. B **472**, 549 (1996).
[39] M. Cavaglià and V. De Alfaro, Gen. Rel. Grav. **29**, 773 (1997); M. Cavaglià and C. Ungarelli, Class. Quantum Grav. **16**, 1401 (1999).
[40] M. Gasperini, *Birth of the Universe in string cosmology*, in Proc. of the Euroconference "Fourth Paris Cosmology Colloquium" (Observatoire de Paris, June 1997), eds. H. J. De Vega and N. Sanchez (World Scientific, Singapore, 1998), p. 85; M. Gasperini, Int. J. Mod. Phys. A **13**, 4779 (1998).
[41] G. 't Hooft, in "Salam Festschrift" , eds. A. Ali et al. (World Scientific, Singapore, 1993), p. 284; L. Susskind, J. Math. Phys. **36**, 6377 (1995).
[42] G. Veneziano, Phys. Lett. B **454**, 22 (1999).
[43] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B **293**, 385 (1987).
[44] M. D. Pollock, Nucl. Phys. B **324**, 187 (1989).
[45] M. D. Pollock, Int. J. Mod. Phys. A **7**, 4149 (1992).
[46] E. Kiritsis and C. Kounnas, *Dynamical topology change, compactification and waves in a stringy early Universe*, in Proc. of the Euroconference "Second Paris Cosmology Colloquium" (Observatoire de Paris, June 1994), eds. H. J. De Vega and N. Sanchez (World Scientific, Singapore, 1995), p. 500
[47] T. Banks, W. Fischler and L. Motl, JHEP **9901**, 019 (1999).
[48] M. Maggiore and A. Riotto, Nucl. Phys. B **548**, 427 (1999).
[49] E. Witten, Nucl. Phys. B **471**, 136 (1996); P. Horawa and E. Witten, Nucl. Phys. B **475**, 94 (1996).
[50] A. Lukas and B. A. Ovrut, Phys. Lett. B **437**, 291 (1998).
[51] A. Feinstein and M. A. Vazquez-Mozo, *M-theory resolution of four-dimensional cosmological singularities via U-duality*, hep-th/9906006.
[52] A. Lukas, B. A. Ovrut and D. Waldram, Nucl. Phys. B **495**, 365 (1997).
[53] K. Behrndt, S. Forste and S. Schwager, Nucl. Phys. B **508**, 391 (1997); R. Poppe and S. Schwager, Phys. Lett. B **393**, 51 (1997); N. Kaloper, Phys. Rev. D **55**, 3304 (1997).
[54] F. Larsen and F. Wilczek, Phys. Rev. D **55**, 4591 (1997).
[55] N. Kaloper, I. I. Kogan and K. A. Olive, Phys. Rev. D **57**, 7340 (1998).
[56] L. P. Grishchuk, Sov. Phys. JEPT **40**, 409 (1975); A. A. Starobinski, HEPT Letters **30**, 682 (1979).
[57] V. F. Mukhanov, H. A. Feldman and R. Brandenberger, Phys. Rep. **215**, 203 (1992).
[58] M. Gasperini and M. Giovannini, Phys. Lett. B **282**, 36 (1992); Phys. Rev. D **47**, 1529 (1993).
[59] Second paper of Ref. [56].
[60] R. B. Abbott, B. Bednarz and S. D. Ellis, Phys. Rev. D **33**, 2147 (1986).
[61] Second paper of Ref. [6].
[62] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. D **48**, R439 (1993).
[63] R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov and G. Veneziano, Phys. Rev. D **51**, 6744 (1995).
[64] J. Hwang, Ap. J **375**, 442 (1991).
[65] R. Brustein, M. Gasperini and G. Veneziano, Phys. Lett. B **431**, 277 (1998).
[66] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B **361**, 45 (1995).
[67] R. Brustein, M. Gasperini and G. Veneziano, Phys. Rev. D **55**, 3882 (1997).
[68] See also M. Gasperini, *Status of string cosmology: phenomenological aspects*, in Proc. of the "Fourth Course of the Int. School of Astrophysics D. Chalonge" (Erice, 1995), eds. N. Sanchez and A. Zichichi (Kluwer, Dordrecht, 1996), p. 305.
[69] A. Buonanno, K.A. Meissner, C. Ungarelli and G. Veneziano, JHEP 01, 004 (1998).

[70] M. S. Turner and L. M. Widrow, Phys. Rev. D **37**, 2743 (1988).
[71] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75**, 3796 (1995).
[72] E. J. Copeland, R. Easther and D. Wands, Phys. Rev. D **56**, 874 (1997); E. J. Copeland, J. Lidsey and D. Wands, Nucl. Phys. B **506**, 407 (1997).
[73] R. K. Sachs and A. M. Wolfe, Ap. J. **147**, 73 (1967).
[74] R. Durrer, M. Gasperini, M. Sakellariadou and G. Veneziano, Phys. Lett. B **436**, 66 (1998); Phys. Rev. D **59**, 043511 (1999).
[75] M. Gasperini and G. Veneziano, Phys. Rev. D **59**, 043503 (1999).
[76] C. Bennet et al, Astrophys. Lett. **464**, L1 (1996).
[77] A. Melchiorri, F. Vernizzi, R. Durrer and G. Veneziano, Phys. Rev. Lett. **83** , 4464 (1999).
[78] M. Gasperini, Phys. Lett. B **327**, 214 (1994); see also the third paper of Ref. [6].
[79] M. Gasperini, *Relic dilatons in string cosmology*, in Proc. of the "12th Italian Conference on Gen. Rel. and Grav. Phys." (Roma 1996), eds. M. Bassan et al. (World Scientific, Singapore, 1997), p. 181.
[80] M. Gasperini, Phys. Lett. B **470**, 67 (1999).
[81] M. Gasperini, Phys. Lett. B **477**, 242 (2000).
[82] A. Linde, *Present status of inflationary theory*, talk given at the "First Cosmology and Particle Physics Workshop" (CERN, TH Division, June 1998).

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String Cosmology versus Standard and Inflationary Cosmology

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Abstract

This paper presents a review of the basic, model-independent differences between the pre-big bang scenario, arising naturally in a string cosmology context, and the standard inflationary scenario. We use an unconventional approach in which the introduction of technical details is avoided as much as possible, trying to focus the reader's attention on the main conceptual aspects of both scenarios. The aim of the paper is not to conclude in favour either of one or of the other scenario, but to raise questions that are left to the reader's meditation. Warnings: the paper does not contain equations, and is not intended as a complete review of all aspects of string cosmology.

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String cosmology versus standard and inflationary cosmology

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Abstract. This paper presents a review of the basic, model-independent differences between the pre-big bang scenario, arising naturally in a string cosmology context, and the standard inflationary scenario. We use an unconventional approach in which the introduction of technical details is avoided as much as possible, trying to focus the reader's attention on the main conceptual aspects of both scenarios. The aim of the paper is not to conclude in favour either of one or of the other scenario, but to raise questions that are left to the reader's meditation. Warnings: the paper does not contain equations, and is not intended as a complete review of all aspects of string cosmology.

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1. Introduction

The standard cosmological scenario [1, 2], rightfully one of the best celebrated conquests of the physics of the XX century, cannot be extrapolated to arbitrarily high energy and curvature scales without clashing with the singularity problem.

A singularity, on the other hand, often represents a signal that the physical laws we are applying have been extrapolated outside their domain of validity. As a well known example, we may quote here the case of the spectral energy distribution of radiation in thermal equilibrium. By applying the laws of classical physics one finds indeed the Rayleigh-Jeans spectrum, that diverges like ω^3 at high frequency. By taking into account instead the appropriate quantum corrections, this classical singularity is regularized by the bell-like Planck distribution, $\omega^4 \left(e^{\omega/T} - 1\right)^{-1}$, as illustrated in Fig. 1.

String theory suggests that the initial curvature singularity of the standard cosmological scenario could be similarly regularized by a bell-shaped curve, as qualitatively illustrated in Fig. 2. As we go back in time the curvature, instead of blowing up, could reach a maximum controlled by the string scale, and then decrease towards a state of very low curvature and weak coupling, approaching asymptotically the so-called string perturbative vacuum. This behaviour, indirectly suggested by duality and thermodynamical arguments [3], as well as, independently, by the motion of

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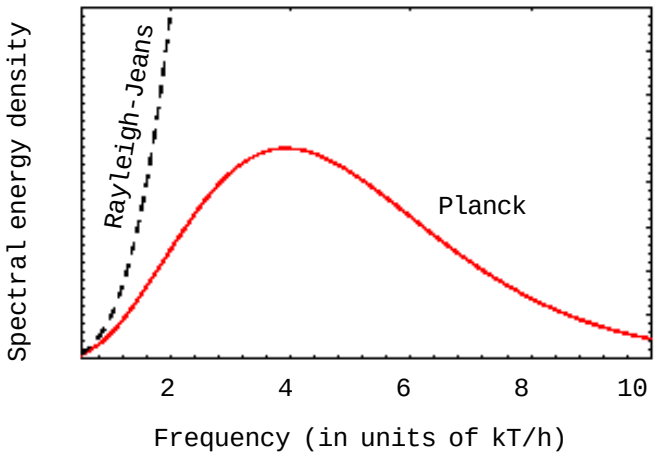

Figure 1. *The Planck distribution (full line) regularizes the classical Rayleigh-Jeans prediction (dashed line), for the spectral energy density of radiation in thermal equilibrium. The classical distribution is only valid at low enough frequency scales.*

strings in rolling backgrounds [4], is naturally grounded on the duality symmetry of the cosmological string effective action [5], and its possible implementation in the context of a realistic model of the early Universe has a lot of dynamical and phenomenological consequences [6].

In the string comology scenario of Fig. 2, the big bang singularity is replaced by a phase of high but finite curvature. It comes thus natural, in such a context, to call "pre-big bang" the initial phase in which the curvature is growing, in contrast to the subsequent "post-big bang" phase, with decreasing curvature and standard decelerated evolution.

The conventional inflationary scenario [2, 7], on the other hand, suggests a picture of the early Universe which is approximately intermediate between the standard and

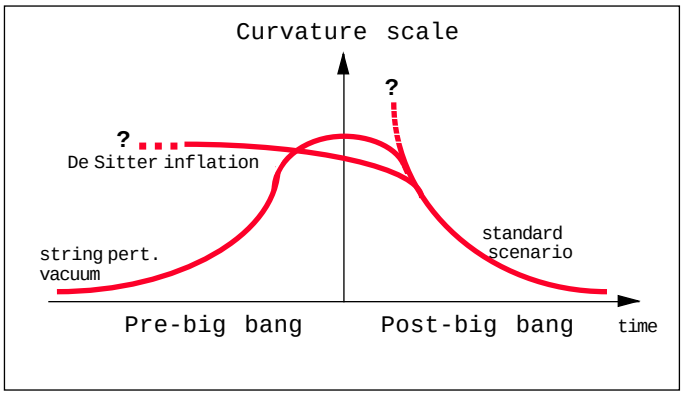

Figure 2. *Curvature scale versus time for standard, inflationary (de Sitter), and string cosmology models of the Universe. In the pre-big bang scenario the classical curvature singularity is regularized, and the standard cosmological evolution is only valid at late enough time scales.*

the string cosmology one. In that context the curvature, instead of blowing up, is expected to approach a nearly constant value, typical of a de Sitter-like geometry (see Fig. 2). However, a de Sitter phase with exponential expansion at constant curvature, implemented in the context of the conventional, potential-dominated inflation, cannot be extended back in time for ever, as first discussed in [8]. Indeed, quoting Alan Guth's recent survey of inflationary cosmology [9]:

"... Nevertheless, since inflation appears to be eternal only into the future, but not to the past, an important question remains open. How did all start? Although eternal inflation pushes this question far into the past, and well beyond the range of observational tests, the question does not disappear."

A possible answer to the above question, suggested by string cosmology and represented graphically in Fig. 2, is that all started from a state approaching, asymptotically, the flat, cold and empty string perturbative vacuum. Even if this starting point is infinitely far into the past, however, the initial state of our Universe might be non completely beyond the range of present observations, in contrast to the sentence quoted above, because the starting point may affect the dynamics of the subsequent inflationary evolution. The initial curvature scale, for instance, is constant or decreasing according to the conventional inflationary scenario, while it is growing according to the pre-big bang scenario, and this may lead to important phenomenological consequences.

In spite of the existence of a few particular examples [10], an unambiguous regularization of the curvature singularity (to all orders in the string effective action), toghether with a complete description of the transition from the pre- to the post-big bang regime, is not an easy achievement (see however [11] for recent encouraging results). Also, it is fair to say that the pre-big bang models, at present, are not free from other (more or less important) difficulties, that some aspects of the pre-big bang scenario are still unclear, and that further work is certainly needed for a final answer to all the difficulties.

Assuming that all the problems can (and will) be solved in a satisfactory way, string cosmology will provide eventually a model of the early Universe somewhat different, however, from the standard inflationary picture. The aim of this paper, therefore, is to present in a compact form a comparison (and a short discussion) of the way in which a phase of inflation could be implemented within a cosmology based on the string effective action, with respect to the standard cosmology based on the Einstein equations. Basically all the differences arise, as we will see, from the fact that in string cosmology the Universe starts evolving from an initial state at very low curvature and weak coupling, while in conventional inflation the initial state is assumed to approach the Planckian, quantum gravity regime.

The paper is organized as follows. We will discuss kinematical and dynamical differences in Sect. 2, quantum cosmology differences in Sect. 3, and phenomenological differences in Sect. 4. Sect. 5 is devoted to some concluding remarks. We will avoid

as much as possible the introduction of technical details – all contained in the papers quoted in the bibliography – trying to emphasize the ideas and the main physical aspects of the different inflationary scenarios.

As already stressed in the Abstract, it seems appropriate to recall that this paper is *not* intended as a complete review of all aspects of string cosmology. Rather, the paper is narrowly focused on those aspects where there is an important overlapping of methods and objectives and, simultaneously, a strong contrast of basic assumptions, for the string cosmology and the standard scenario. For a more exhaustive and systematic approach, the interested reader is referred to the excellent review paper devoted to a presentation and a technical discussion of all presently existing (super)string cosmology models [12], as well as to two recent introductory lectures on the pre-big bang scenario [13].

2. Kinematical and dynamical differences

The idea of inflation hystorically was born [14] to solve the problem of monopoles that could be largely produced in the early Universe, at the energy scale of grand unified theories (GUT). More generally, inflation is now understood as a period of accelerated evolution that can explain why the present Universe is so flat and smooth over a so large scale of distances [2, 7].

For hystorical reasons, i.e. because inflation was first implemented as a period of supercooling of the Universe trapped in a "false" vacuum state, in the context of GUT phase transitions, inflation was first associated [14] to a phase of exponential expansion (in cosmic time) of the scale factor $a(t)$, corresponding to a de Sitter (or "quasi-de Sitter") geometrical state. But it was soon realized that any type of accelerated expansion ($\dot{a} > 0$, $\ddot{a} > 0$, where the dots denote differentiation with respect to cosmic time), can in principle solve the kinematic problems of the standard scenario [15].

Actually, even accelerated contraction ($\dot{a} < 0$, $\ddot{a} < 0$) is effective to this purpose [16], as expected from the fact that accelerated expansion can be transformed into accelerated contraction through an appropriate field redefinition, like the one connecting the String and the Einstein frame. Indeed, physical effects such as the dilution of inhomogeneities should be independent from the choice of the frame. So, inflation can be generally identified, in a frame-independent way, as a period of accelerated evolution of the scale factor ($\mathrm{sign}\ \dot{a} = \mathrm{sign}\ \ddot{a}$).

There are, however, two classes of accelerated evolution very different from a dynamical point of view, and depending on the behaviour – growing or decreasing in time – of the curvature of the space-time manifold. A first important point to be stressed is thus the fact that the pre-big bang scenario corresponds to a phase of accelerated evolution characterized by growing (or non-decreasing) curvature [5, 6], while the phase that we shall call "standard inflation " [2, 7] is characterized by a curvature scale which tends to be constant – in the limiting case of a de Sitter metric – or slightly decreasing in time.

Before proceeding further, two remarks are in order. The first is that a phase of accelerated evolution and growing curvature, also called superinflation (or pole-inflation) [17], is not a peculiarity of string cosmology, but is possible even in general relativity: in higher-dimensional backgrounds, for instance, in the context of dynamical dimensional reduction. The important difference is that, in string cosmology, superinflation does not necessarily requires neither the shrinking of the internal dimensions, nor some exotic matter source [18] or symmetric breaking mechanism [19]: it can be simply driven, even in three spatial dimensions, by the kinetic energy of the rolling dilaton field ϕ [5, 6], parametrizing the growth of the string coupling $g = \exp(\phi/2)$ from zero (the string perturbative vacuum), to the strong coupling regime, $g \sim 1$.

The second remark is that the words "pre-big bang" should be referred to the complete cosmological evolution from the initial state approaching the string perturbative vacuum, up to the beginning of the hot, radiation-dominated phase of the standard scenario. Altough in most of this paper we shall restrict our discussion to the low-energy part of the pre-big bang phase, appropriately described in terms of the low-energy string effective action, the complete pre-big bang history necessarily includes a high-curvature "stringy" phase [20], which may also be of the inflationary type [21], and which has a curvature expected to be, on the average, non decreasing.

For a power-law, accelerated, conformally flat background, the time-behaviour of the curvature scale follows the behaviour of the absolute value of the Hubble parameter, $|H| = |\dot{a}/a|$. In such a background, on the other hand, the inverse of the Hubble parameter (i.e. the Hubble horizon $|H|^{-1}$) also controls the (finite) proper distance between the surface of the event horizon and the wordline of a comoving observer. Such a distance is shrinking for pre-big bang inflation (where $|H|$ is growing), while it is non-decreasing in standard inflation.

As a consequence of the fact that the horizon is shrinking, and the curvature is growing, it turns out that the initial state of the phase of inflation, in the pre-big bang scenario, is characterized by a curvature which is very small in Planck (or string) units, and by a Hubble horizon very large in the same units (for the sake of simplicity we may identify, at the end of inflation, the present value of the string length L_s and of the Planck lenght L_p; indeed, at tree-level, they are related by [22] $L_s = \langle g \rangle L_p = \langle \exp \phi/2 \rangle L_p$, with a present dilaton expectation value $\langle g \rangle \sim 0.3 - 0.03$). The initial state is also characterized by another very small dimensionless number, the initial string coupling $g \ll 1$, corresponding to a dilaton approaching the perturbative vacuum value, $\phi \to -\infty$.

By contrast, the standard inflationary scenario is characterized by a dilaton already settled to its present vacuum expectation value; the coupling is always strong (g is of order one), and the size of the initial horizon may be of order one in Planck units, if the initial curvature approaches the Planck scale. These kinematical and dynamical differences between standard and pre-big bang inflation are summarized in Table I.

As evident from the pre-big bang curve of Fig. 2, the smaller is the value of the initial curvature, the longer is the duration of the phase of pre-big bang inflation. The request that the inflation phase be long enough to solve the horizon and flatness

	Standard Inflation	**Pre-big bang Inflation**
Time evolution	*accelerated*	*accelerated*
Driving energy	*inflaton potential*	*dilaton kinetic energy*
Curvature	*constant or decreasing*	*growing*
Event horizon	*constant or growing*	*shrinking*
Initial curvature scale	*model-dependent*	*arbitrarily small*
Initial coupling	*strong, non-perturbative*	*arbitrarily weak, perturbative*

Table 1. Kinematical and dynamical differences between standard and pre-big bang inflation.

problems [14] thus imposes bounds on the dimensionless parameters of the initial state, controlled by the curvature and by the string coupling. Parameters such as the typical size of an initial homogeneous domain, in Planck units, must be by far greater than one for inflation to be successful. This aspect of string cosmology was pointed out already in the first papers on the pre-big bang scenario [5, 6] and, even before, also in the context of string-driven superinflation [4].

The fact that the initial curvature and coupling are small, and that the initial state is characterized by very large dimensionless numbers, may be interpreted however as a possible fine-tuning of the pre-big bang models [23], or even as a serious drawback, preventing the solution of the flatness and homogeneity problems, and supporting the conclusion that "the current version of the pre-big bang scenario cannot replace usual inflation" [24].

Consider, for instance, the horizon/homogeneity problem. At the onset of pre-big bang inflation, as usual in an inflationary context, the maximal allowed homogeneity scale is bounded by the size of the horizon. The point is that the initial horizon size H^{-1} is very large, instead of being of Planckian order. The basic question thus becomes: is an initial homogeneity scale of the order of the maximal scale H^{-1} necessarily unnatural if the initial curvature is small and, consequently, the initial horizon is large in Planck (or string) units? In other words, which basic length scale has to be used to measure the naturalness of the initial homogeneous domain, which subsequently inflates to reproduce the presently observed Universe? The Planck length or the radius of the causal horizon?

The choice of the Planck length, emphasized in [24], is certainly appropriate when the initial conditions are imposed in a state approaching the high-curvature, quantum

gravity regime, like in models of chaotic inflation [25], for instance. In the pre-big bang scenario, on the contrary, the initial conditions are to be imposed when the Universe is deeply inside the low-curvature, weak coupling, classical regime. In that regime the Universe does not know about the Planck length, and the only available classical scale of distance, the horizon, should not be discarded "a priori" as unnatural [26].

This does not mean, of course, that the horizon should be always *assumed* as the natural scale of homogeneity. This suggests, however, that the naturalness of homogeneity over a large horizon scale should be discussed on the grounds of some quantitative and objective criterium, as attempted for instance in [27], taking into account also the effects of quantum fluctuations [28] that could destroy the initial, classical homogeneity (see also [29] for a discussion of "generic" initial conditions in a string cosmology context).

Another point concerns the flatness problem. In order to explain the precise fine-tuning of the present density to the critical one, the initial state of pre-big bang inflation must be characterized by large dimensionless parameters, thus reintroducing from the beginning the large numbers that one would like to explain.

This may seem to be quite unsatisfactory, as emphasized in [24]. It should be pointed out, however, that if one accepts the point of view that large numbers are always to be avoided at the onset of inflation, then should also accept the fact that natural initial conditions are only possible in the context of models in which inflation starts at the Planck scale, in order to have, for the curvature, an initial dimensionless ratio of order one. This rules out, as a satisfactory explanation of our present cosmological state, not only the pre-big bang scenario, but any model in which inflation starts at scales smaller than Planckian (unless we imagine a scenario with different stages of inflation, each of them responsible for solving different problems, and occurring at different scales: but, again, is this a natural cosmological configuration?)

Even for a single stage of inflation, occurring very near to the Planck scale, we are not free from problems, however, as we are led eventually to the following question: can we trust the naturalness of inflation models in which classical general relativity is applied to set up initial conditions at Planckian curvature scales, i.e. deeply inside the non-perturbative, quantum gravity regime? (in string cosmology, the Planckian regime directly affects the exit from the inflationary phase, and only indirectly set constraints on the initial conditions, through the finite duration of the low energy phase).

Assuming that the answer be positive, we are led to a situation that can be graphically summarized, in a qualitative way, as in Fig. 3.

- Case (a) represents a standard inflationary model of the Universe in which inflation starts at the Planck scale. The time arrow points from bottom to top, and the shaded area at the time t_0 represents a spatial section of our present homogeneous Universe, of size fixed by the present Hubble radius H_0^{-1}. As we go back in time, according to the solutions of the standard cosmological model, the horizon shrinks linearly in cosmic time, while the proper size of the present homogeneous region, controlled by the scale factor $a(t)$, shrinks slowly. When we reach the Planck scale,

at the time t_f, the causal horizon is smaller than the homogeneous region, roughly by the factor 10^{-30}.

To solve this problem, the phase of standard evolution is preceeded by a phase of exponential de Sitter inflation, long enough in time from t_i to t_f, during which the curvature and the horizon stay frozen at the Planck scale, and our present portion of the Universe may "re-enter" inside the causal horizon.

It should be stressed that, in a realistic inflationary model, the horizon has to be slightly increasing from t_i to t_f, because the scale corresponding to our present Hubble radius has to cross the horizon, during inflation, at a curvature scale H_1 smaller than Planckian. We must require, in particular, that $H_1/M_p \lesssim 10^{-5}$ in order to avoid too much amplification of gravitational perturbations, that would contradict the present degree of homogeneity observed by COBE [30] at large angular scales. However, for the sake of graphical simplicity, we shall ignore this complication that is not essential for our present discussion.

- Case (b) represents a string cosmology model of the Universe, in which the inflationary pre-big bang phase, from t_i to t_f, is represented in terms of the contracting metric of the Einstein frame [16], in order to emphasize (graphically) the underlying duality and time-reversal symmetry of the scenario (there is no need, of course, that in the Einstein frame the pre-big bang scale factor exactly coincide with the time-reversal of the post-big bang solution).

  The main difference from case (a) is that in the pre-big bang epoch the curvature is growing, and the event horizon shrinks linearly in cosmic time, from t_i to t_f, instead of being constant. Since the scale factor shrinks at a slower rate, however, it is still possible for the initial homogeneous domain to be "pushed out" of the horizon, and for the Universe to emerge at the Planck scale, at the time t_f, in the same configuration as in case (a). The subsequent evolution from t_f to t_0 is the same as in the standard scenario.

- Case (c), finally, represents a string cosmology model in which the period of pre-big bang inflation corresponds in part to a phase of growing curvature, growing dilaton and shrinking horizon (from t_i to t_s), and in part to a phase in which the curvature, the horizon, and eventually the dilaton, are frozen at the Planck scale (from t_s to t_f). The initial horizon is still large in Planck units, but it is no longer reflection-symmetric to t_0, depending on the duration of the high curvature phase from t_s to t_f.

According to [24], the model (a) represents an acceptable solution to the horizon and flatness problem because inflation starts at the Planck scale, and all the dimensionless parameters characterizing the initial configuration are of order one. When the phase at constant curvature of the model (c) extends in time like in case (a) the two models pratically coincide for what concerns the naturalness of the initial conditions, as in both cases our Universe emerges at the Planck scale from a single domain of Planckian size, and we loose any observational tracks of what happened before.

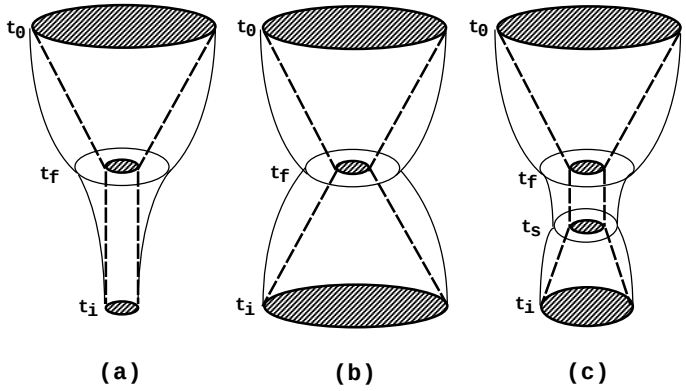

Figure 3. *Qualitative evolution of the horizon scale and of the proper size of a homogeneous region for (a) standard de Sitter inflation with constant Hubble horizon, (b) pre-big bang inflation (in the Einstein frame) with shrinking horizon, and (c) pre-big bang including a phase of high curvature inflation at the string scale. The time direction coincides with the vertical axis. The three horizontal spatial sections t_0, t_f and t_i corresponds, from top to bottom, to the present time, to the end, and to the beginning of inflation. The shaded area represents the horizon, and the dashed lines its time evolution. The full curves represent the time evolution of the proper size of the homogeneous region, controlled by the scale factor.*

The aim of the pre-big bang scenario, on the other hand, is to attempt a description of the possible cosmological evolution *before* the Planck epoch. The main difference between case (a) and (c) is that, if the duration of the phase of inflation at constant curvature is shorter than the minimal duration required for a solution of all the standard kinematic problems, what happened before the Planck scale may then become visible. In other words, there are phenomenological consequences that can be ascribed to the phase of pre-Planckian evolution, and that can be tested (at least in principle) even today (see Section 4). The model of case (b), in particular, is the limiting case in which the duration of the high curvature phase shrinks to a point, and the Universe emerges at the Planck curvature scale with a homogeneous domain large enough to fill our present Hubble radius through the subsequent standard evolution.

The above discussion refers to the curvature in Planck units, but the same arguments can also be applied to the initial value of the string coupling. If the coupling is of order one at the beginning of inflation then it is natural, in the sense of [24], if it is much smaller than one, then the inflationary growth of the coupling may have observable consequences.

To conclude this discussion it seems difficult, in our opinion, to discard a model of pre-Planckian evolution, like those illustrated in case (b), (c), only on the grounds of the large parameters characterizing its initial conditions. Such an argument could be applied in the impossibility of observational tests, namely in the absence of any phenomenological evidence about the cosmological evolution before the Planck era. But, as pointed out before, the pre-Planckian epoch becomes invisible only in the limiting case (a), in which the effective models reduces to standard inflation starting at the

Planck scale, with no unnatural initial conditions.

If, on the contrary, the high curvature phase is shorter than in case (a), then the model requires an initial horizon larger than Planckian, and a small initial coupling – which are possibly unnatural according to standard criteria [24]. In that case, however, such initial conditions are in principle accessible to present observations, so why do not try to test the scenario observationally, and try to analyze the naturalness in terms of a Bayesan approach, as attempted in [27]? In that case, the computation of "a posteriori" probabilities suggests that the observation of a large initial horizon and a small initial coupling may become "a posteriori" natural [27], because of the duality symmetry intrinsic to the pre-big bang scenario.

Finally, even if the initial state should require a certain degree of fine-tuning, this does not necessarily implies that the pre-big bang cosmology described by string models is to be discarded (after all, the description of our late-time Universe given by the standard cosmological model is rather satisfactory, in spite of the fine-tuning required in such a model if the initial state is extrapolated back in time until the Planck epoch).

Usually, the need for fine-tuning in the initial conditions means that the model is incomplete, and that a more general dynamical mechanism is required, to explain the particular initial conditions. Thus, it might well be that the pre-big bang picture provided by string cosmology does not represents the whole story of our Universe, and that only an earlier evolution can explain why, at a certain instant of time, the Universe is lead to a state so similar to the string perturbative vacuum.

3. Quantum cosmology differences

In the standard inflationary scenario the phase of exponential, de Sitter-like expansion at constant curvature cannot be infinitely extended towards the past, for a well known reason of geodesic completeness. A complete manifold requires an earlier contracting phase: in that case, however, it seems impossible, in models dominated by the inflaton potential, to stop the collapse and to bounce towards the expanding phase [8]. One has thus the problem of explaining how the Universe could emerge at the Planck scale in the initial state appropriate to exponential expansion.

At curvature scales of order one in Planck units we are in the full quantum gravity regime, and the use of the quantum cosmology approach seems to be appropriate. In this approach the Universe is represented by a wave function satisfying the Wheeler-De Witt (WDW) equation [31], and evolving in the so-called superspace, whose points represent all possible spatial geometric configurations. For pratical applications, however, the evolution of the WDW wave function is usually studied in a "minisuperspace" context, where only a finite numbers of coordinates is chosen to parametrize the different geometrical configurations.

With an appropriate choice of the boundary conditions it is then possible to obtain cosmological solutions of the WDW equation describing the "birth of the Universe" as an effect of quantum tunnelling [32, 33]. In that case, if the geometric state of the

Universe is characterized by a cosmological constant Λ (due, for instance, to the vacuum energy-density induced by a scalar field potential), the tunnelling probability is found to be proportional to $\exp(-\Lambda^{-1})$, where Λ is measured in Planck units. The Universe tends thus to emerge in a state of big vacuum energy, just appropriate to the onset of inflation. The quantum cosmology approach seems thus to provide a natural mechanism to explain the formation of "baby universes", emerging at the Planck scale, and ready to inflate according to the standard inflationary scenario [34].

The minisuperspace approach to quantum cosmology is known to be affected by various problems of technical nature: the probabilistic interpretation, the unambiguous determination of an appropriate time parameter, the semiclassical limit, the ordering of quantum operators, and so on. The most unsatisfactory aspect of this approach, in our opinion, is however the fact that the boundary conditions for the tunnelling process are to be chosen "ad-hoc". They are by no means compelling, and it is possible indeed to impose different boundary conditions, for instance according to the "no boundary" criterium [35], leading to a completely different result for the probability of creation of universes – results that are not always appropriate to inflationary initial conditions.

The source of this problem is the fact that, in a quantum description of the birth of the Universe, the final cosmological state (i.e. the Universe that we want to obtain) is well known, while the initial cosmological state (before the quantum transition) is completely unknown, at least in the context of the standard inflationary scenario. Indeed, the cosmological tunnelling is usually referred as a proces of tunnelling "from nothing" [32], just to stress the ignorance about the initial vacuum state. The classical theory of the standard cosmological scenario cannot help, because the initial state, in that context, is the big bang singularity, i.e. just what the quantum approach would like to avoid.

In a string cosmology context, the quantum approach based on minisuperspace can be implemented in a straightforward way, with the only difference that the differential WDW equation represents the Hamiltonian constraint following not from the Einstein action, but from the low-energy string effective action [36, 37, 38]. As a consequence, the "minimal" minisuperspace is at least two-dimensional, because the action always contains the dilaton, besides the metric. The formal problems related to the minisuperspace approach remains, with the possible exception of the operator-ordering problem, as the quantum ordering is unambiguosly fixed by the global, pseudo-orthogonal $O(d,d)$ symmetry of the low-energy string effective action [36, 38]. Another possible exception is the identification of the time-like coordinate in minisuperspace [39].

There is, however, a radical difference for what concerns boundary conditions. In the context of the pre-big bang scenario the initial, asymptotic state of the Universe is unambiguosly prescribed – the string perturbative vacuum – and cannot be chosen "ad-hoc". Such initial state is perfectly appropriate to a low-energy normalization of the WDW wave function, and the transition probability of string cosmology only depends on the dynamics, i.e. on the effective potential appearing in the WDW equation.

It is now interesting to observe that if we compute, in the context of the pre-

big bang scenario, the transition probability from the perturbative vacuum to a final, post-big bang configuration characterized by a non-vanishing cosmological constant, we obtain [37] a probability distribution $P(\Lambda)$ very similar to the one of the conventional quantum cosmology, computed with tunneling boundary conditions. The reason is that, by imposing the perturbative vacuum as the boundary condition to the WDW equation, the WDW solutions contain only outgoing waves at the singular spacetime boundary, just like in the case of tunnelling boundary conditions [32]. In this sense, we can say that the "ad-hoc" prescription of tunnelling boundary conditions simulates, in a phenomenological way, the birth of the Universe from the string perturbative vacuum. This suggests that, instead of "tunnelling from nothing", we should speak of "tunnelling from the string perturbative vacuum" or, even better, of quantum instability and "decay" of the perturbative vacuum [40].

A further, important difference should be mentioned. The transition from the pre- to the post-big bang phase induced by the cosmological constant (or, more generally, by an appropriate dilaton potential), is represented, in the minisuperspace of string cosmology, not like a tunnelling effet, but like a quantum reflection of the WDW wave function, over an effective potential barrier. The correct description that we obtain in string cosmology for the birth of the Universe, therefore, is that of a "quantum scattering" effect [36, 37, 39].

The various differences between quantum inflationary cosmology and quantum string cosmology are summarized in Table II. Besides the formal aspects (such as tunnelling versus reflection), the basic difference is that in the standard inflationary scenario the Universe, because of quantum cosmology effects, is expected *to enter* in the inflationary regime, while in the pre-big bang scenario the Universe is expected *to exit* from the inflationary regime (or at least from the phase of growing curvature). So, in standard inflation, quantum effects at the Planck scale are expected to be responsible for inflationary initial conditions. In string cosmology, on the contrary, initial conditions are to be imposed in the opposite, low energy quantum regime, where quantum effects are negligible.

It seems appropriate, at this point, to comment on the fact that the initial state of pre-big bang inflation seems to be characterized by a large entropy S, if one assumes, as in [24], that the de Sitter relation between entropy and horizon area remains valid also when the horizon is not constant in time (or, in other words, if one assumes a saturation of the bound provided by the holographic principle [41], applied however to the Hubble horizon [42]). If S is large in Planck units, the probability that such a configuration be obtained through a process of quantum tunnelling, $\exp(-S)$, is exponentially small, as emphasized in [24]. However, as stressed above, in string cosmology quantum effects such as tunnelling or reflection are expected to be effective *at the end* of inflation, and not *at the beginning*, i.e. *not* to explain the origin of the initial state. A large entropy of the initial state, in the weakly-coupled, highly-classical regime, can only correspond to a large probability of such configuration, which is proportional to $\exp(+S)$, like for every classical and macroscopic configurations (not arising from quantum tunnelling).

	Quantum cosmology	**Quantum string cosmology**
Formal approach	*WDW, minisuperspace*	*WDW, minisuperspace*
Quantum ordering	*arbitrary*	*fixed by duality*
Boundary conditions	*arbitrary*	*string perturbative vacuum*
Outgoing waves	*tunnelling from nothing*	*reflection from pre-big bang*
Quantum transition	*beginning of inflation*	*exit from inflation*

Table 2. Quantum cosmology differences between standard and pre-big bang inflation.

Let us stress, finally, another important difference between conventional quantum cosmology and quantum string cosmology. In string cosmology quantum geometrical effects cannot be fully accounted for, as fas as we limit ourself to a WDW equation obtained from the low-energy string effective action. Indeed, when approaching the Planck scale, the string theory action acquires (even at small coupling, i.e. at tree-level in the quantum loop expansion) higher curvature correction [43], weighed by the inverse string tension α'. They are to be included into the Hamiltonian constraint, and lead in general to a higher-derivative WDW equation. This problem has been discussed in [44], and it has been shown in [45] that when the higher curvature corrections appear in the form of an Eulero density, then the WDW approach can only be applied to a dimensionally reduced version of the theory.

The quantum cosmology results reported in this section refer, in this sense, only to a model of "low-energy" quantum string cosmology [40]. In the full quantum gravity regime, in order to include all the higher-derivative contributions, the correct WDW equation should follow not from the effective action, but possibly from a conformal, sigma-model action [46], which automatically takes into account all orders in α'. We note, however, that duality transformations in toroidal moduli space have recently suggested [47] that the Lorenztian structure of the low-energy minisuperspace may have an exact meaning also in an M-theory context, even if the exact M-theory equations are expected to be in general different.

In fact, when the curvature is large in string units ($\alpha' H^2 > 1$), and also the string coupling is large ($g^2 = e^{\phi} > 1$), we necessarily enter the M-theory regime where new quantum effects are possible, such as a copious production of higher-dimensional D-branes [48]. If the curvature is small enough, the strong coupling regime of string cosmology is then expected to be described by 11-dimensional supergravity theory, and

the dilaton to be interpreted as the radius (i.e., the modulus field) of the 11-th dimension [49]. In this context string cosmology becomes U-duality covariant [12, 50, 51], and the presence in the action of Ramond-Ramond fields may be helpful to evade the problem of the curvature singularity [52, 53, 54]. The singularity, in addition, could also disappear as a result of the embedding of the low-energy solutions of string theory into a higher-dimensional ($d = 11$) manifold [51, 55].

These results seem to suggests that an appropriate quantum description of the birth of the Universe will be probably achieved only within a full M-theory approach to the strong coupling regime, in which the pre-big bang acceleration is damped, the curvature is regularized, and the Universe bounces back to the phase of standard evolution.

4. Phenomenological differences

One of the most important (and probably also most spectacular) phenomenological predictions of inflation is the parametric amplification of metric (and of other different types of) perturbations [56], and the corresponding generation of primordial inhomogeneity spectra, directly from the quantum fluctuations of the background fields in their vacuum state (see [57] for a review). Such fluctuations, when decomposed in Fourier modes, satisfy a canonical Schrodinger-like equation, whose effective potential is determined by the so-called" pump field", which depends in its turn on the background geometry.

It is then evident that different backgrounds lead to different pump fields, to a different evolution of perturbations, and thus to different spectra. In string cosmology, in particular, there are two main properties of the background that can affect the final form of the perturbation spectra. They are:

- (A) the growth of the curvature scale;
- (B) the scalar-tensor (i.e. gravi-dilaton) nature of the background.

Property (A) has two important consequences. The first, that we will call (A1), is that the pre-big bang scenario leads to metric perturbation spectra growing with frequency [58] (instead of being flat, or decreasing, like in standard inflation), because the spectral distribution of metric perturbations tends to follow the behaviour of the curvature scale at the time of the first horizon crossing. The second, that we will call (A2), is that the growth of the curvature can also force the comoving amplitude of perturbations to grow (instead of being frozen) outside the horizon. This effect, implicitly contained in the earlier, pioneer studies [59], was first explicitly pointed out in [60], and only later independently re-discovered in a string cosmology context [61]. As a further consequence of property (A) we should mention, finally, the fact that perturbations are amplified in a final "squeezed vacuum" state, and not in a "squeezed thermal vacuum" [62].

Let us first discuss the second effect (A2). This effect, on one hand, is interesting because it may lead to an amplification of perturbations more efficient than in the standard inflationary scenario. On the other hand, however, it is dangerous, because

the perturbation amplitude could grow too much, during the pre-big bang phase, so as to prevent the application of the standard linearized approach, which neglects effects of back-reaction [57].

Such an "anomalous" growth of perturbations cannot be eliminated by a change of frame, because the associated physical (i.e. observable) energy density spectrum is obviously frame-independent. However, the breakdown of the linear approximation *is*, in general, *gauge-dependent*. For the particular case of scalar metric perturbations, in three isotropic dimensions, the linear approximation breaks down in the standard longitudinal gauge, but is restored in a more appropriate "off-diagonal" gauge [63], also called "uniform-curvature" gauge [64]. Moreover, as a consequence of a particolar form of duality invariance that appears explicitly in the Hamiltonian approach to perturbation theory [65], the final energy-density spectrum can always be correctly estimated by neglecting the growing mode, provided one includes in the full Hamiltonian both the contribution of the amplitude and of its conjugate momentum.

There are backgrounds, however, in which the growth of perturbations remains too strong even after the elimination of all unphysical gauge effects, and we have to limit ourselves to a restricted portion of parameter space for the linear approximation to be valid. Even in this case, however, the effect (A1) has two interesting consequences.

The first is that a growing spectrum leads to the formation of relic backgrounds whose amplitude is higher at higher frequency, where in general the backgrounds are also more easily detectable. A typical example is the formation of a relic background of cosmic gravitons which, in the frequency range of present resonant-mass and interferometric detectors ($10^2 - 10^3$ Hz), could be up to eight orders of magnitude stronger than expected in the context of standard inflation [58, 66]. Thus in principle detectable, in a not-so-far future, by the (planned) advanced version of the interferometric gravitational antennas, or by spherical resonant detectors.

The second consequence is that the normalization of the peak of the spectrum, at high frequency, is automatically controlled by the string scale [67]. The peak amplitude may be high enough to support a picture in which all the radiation, that becomes dominant at the beginning of the standard era, is produced through a process of parametric amplification, directly from the quantum fluctuations of the vacuum during the pre-big bang epoch [67, 68, 69]. Indeed, for the background fields that interact more strongly than gravitationally, the amplified fluctuations are expected to thermalize, and their energy-density is expected to grow in time with respect to the dilaton kinetic energy that was driving the background during the phase of pre-big bang inflation. This possibility is absent in the standard inflationary scenario, where the spectrum of perturbations is decreasing, and the normalization of the spectrum is determined at low frequency by the observation of the large scale CMB anisotropy [30]: the resulting energy-density of the fluctuations, in that case, is by far too low to dominate, eventually, the post-inflationary background.

Up to now we have reported some phenomenological consequences of the property (A). For what concerns the property (B), i.e. the gravi-dilaton nature of the background,

we will quote here a peculiar string cosmology effect, the amplification of the quantum fluctuations of the electromagnetic field due to their direct coupling to the dilaton, according to the effective Lagrangian density $\sqrt{-g}e^{-\phi}F_{\mu\nu}F^{\mu\nu}$.

In general relativity the dilaton is absent, the Lagrangian is invariant under a conformal rescaling of the metric, and the coupling of the electromagnetic field to a conformally flat metric, typical of inflation, can always be eliminated (unless the coupling is non-minimal and/or violates $U(1)$ gauge invariance [70]). As a consequence, the inflationary evolution of the metric background is unable to amplify the electromagnetic fluctuations. In string cosmology, on the contrary, such fluctuations are amplified by the accelerated growth of the dilaton (acting as the pump field) during the pre-big bang phase. If the high-curvature string phase is long enough, it is then possible to produce in this way the "seeds", required for instance by the galactic dynamo, for the generation of cosmic magnetic fields on a large scale [71]. String cosmology thus provides a possible solution to a longstanding astrophysical "puzzle", i.e. the generation of the primordial seed fields, through a mechanism which is uneffective in the standard inflationary scenario.

This certainly represents an advantage with respect to the standard scenario. The different amplification of perturbations, however, is also asssociated to possible drawbacks. In particular, the fact that the metric perturbation spectrum, in string cosmology, grows with a very steep slope, and it is rigidly normalized at high frequency, makes problematic the matching to the anisotropy observed at the present horizon scale [30]. The generation of the observed CMB anisotropy, with the right spectrum, is instead one of the most celebrated results of standard inflation [2, 7, 57].

A possible solution to this problem, in string cosmology, comes from the amplification of the fluctuations of the Kalb-Ramond axion, which is one of the fundamental fields appearing in the string effective action, already at low energy. Indeed, unlike metric perturbations, the axion perturbations can be amplified with a rather flat spectrum [72] and, through the (integrated) Sachs-Wolfe effect [73], they can induce the temperature anisotropy observed in the CMB radiation on a large scale, both in case of massless [74] and massive [75] axion fluctuations.

It is important to observe, in that case, that the slope of the spectrum is no longer arbitrary, but rigidly determined by the COBE normalization (imposed at low frequency), and by the string normalization (imposed at the opposite, high-frequency end of the spectrum). The resulting slope turns out to be slightly increasing [74, 75], but still in agreement with the observational limits at the horizon scale [76]. At higher frequency scales, however, important differences from standard inflation may appear in the peak structure of the spectrum [77]. The possible axionic origin of the fluctuations of the CMB temperature is thus expected to be confirmed (or disproved) in a very near future, by the planned satellite observations.

The main differences between standard and string cosmology inflation, for what concerns cosmological perturbations, are summarized in Table III. The last entry of the Table refers to the amplification of the fluctuations of the dilaton background [78],

	Standard Inflation	**Pre-big bang Inflation**
Pump field	*metric*	*metric and dilaton*
Spectrum (vs frequency)	*flat or decreasing*	*increasing*
Amplitude outside horizon	*frozen*	*increasing*
Normalization	*low frequency (COBE)*	*high frequency (string scale)*
Electromagnetic fields	*unaffected*	*amplified by the dilaton*
CMB anisotropy	*metric fluctuations*	*axion fluctations*
Dilaton fluctuations	*absent*	*dilaton productions*

Table 3. Amplification of vacuum fluctuations in the inflationary and pre-big bang scenario.

another peculiar effect of string cosmology, because the dilaton is absent in the standard scenario. This effect, analogous to the amplification of tensor metric perturbations, can be interpreted (in a second quantization approach) as a process of dilaton production, which leads to the formation of a relic background of cosmic dilatons. The background is subject to various constraints, depending on the slope of the spectrum and on the mass of the dilaton, but if dilatons are light enough [79] (namely $m \lesssim 10$ KeV), they are not yet decayed and could represent today a significant fraction of the dark matter, that seems required to match various astrophysical observations.

Detecting such a background, through the gravity-like interactions of the dilatons at low-energy scales, is however a challenge that seems beyond the possibilities of present technology [80], unless the dilaton couples universally to macroscopic bodies, represents a significant fraction of dark matter, and its mass lies within the sensitivity band of gravitational detectors [81].

5. Conclusion

The pre-big bang scenario provides a model, suggested and supported by string theory, of the possible cosmological evolution before our Universe emerged at the Planck scale.

If the subsequent post-Planckian evolution follows the standard inflationary scenario, then initial conditions can be imposed at the Planck scale, and are probably natural in the sense of [24], but any track of what happened before disappears from our

observational range. If, on the contrary, inflation at the Planck scale is not too long, and inside our present Hubble radius there are comoving length scales that crossed the horizon during the low-curvature pre-big bang phase, then pre-Plackian initial conditions are in principle accessible to present observations, and their naturaleness can be discussed in terms of a Bayesan analysis [27], based on "a posteriori" probabilities. Quantum cosmology methods can also be applied, taking into account however that quantum effects are possibly important at the end, and not at the beginning, of inflation.

We believe that the possibility of looking back in the past before the Planck era is the most distinctive aspect of string cosmology, with respect to the standard inflationary cosmology. Concerning the possible tracks of the pre-Planckian Universe, we have emphasized , in particular, three effects, referring to observations to be performed 1) in a not-so-far future, 2) in a near future, and 3) to observations already (in part) performed. These effects are, respectively: 1) the production of a cosmic gravity wave background, 2) the axion-induced anisotropy of the CMB radiation, and 3) the production of seeds for the cosmic magnetic fields.

These effects are not necessarily compatible among them (at least in the "minimal" version of the pre-big bang models), and it seems thus possible to test, and eventually exclude (or confirm) the pre-big bang scenario on the grounds of its phenomenological consequences. Even if, as recently stressed [82], the idea itself of inflation "cannot be falsified", the particular models can (and must) be tested, and the pre-big bang scenario, after all, can be regarded as a particular, unconventional model of primordial inflation.

It seems thus appropriate to stress, in conclusion, that the pre-big bang scenario is not alternative to the idea of inflation, but only alternative to a more conventional realization of inflation which, for hystorical reason, is still deeply anchored to the standard big bang picture, where the initial state must necessarily represent a very small, curve and dense Universe. The effort of this paper aims at stimulating the reader's meditation on the fact that this standard picture is a possibility, not a necessity, and that quite different initial conditions are possible, and not necessarily unlikely.

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References

[1] S. Weinberg, *Cosmology and gravitation* (Wiley, New York, 1972).

[2] E. W. Kolb and M. S. Turner, *The Early Universe*, (Addison Wesley, Redwood City, Ca, 1990).

[3] Y. Leblanc, Phys. Rev. D **38**, 3087 (1988); R. Brandenberger and C. Vafa, Nucl. Phys. B **316**, 391 (1989); E. Alvarez and M. A. R. Osorio, Int. J. Theor. Phys. **28**, 949 (1989); A. A. Tseytlin, Class. Quantum Grav. **9**, 979 (1992); A. A. Tseytlin and C. Vafa, Nucl. Phys. B **372**, 443 (1992).

[4] M. Gasperini, N. Sanchez and G. Veneziano, Nucl. Phys. B **364**, 365 (1991).

[5] G. Veneziano, Phys. Lett. B **265**, 287 (1991); K. A. Meissner and G. Veneziano, Phys. Lett. B **267**, 33 (1991); Mod. Phys. Lett. A **6**, 3397 (1991); M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B **272**, 277 (1991).

[6] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993); Mod. Phys. Lett. A **8**, 3701 (1993); Phys. Rev. D **50**, 2519 (1994). An updated collections of papers on the pre-big bag scenario is available at `http://www.to.infn.it/~gasperin`

[7] A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, Chur, Switzerland, 1990).

[8] A. Vilenkin, Phys. Rev. D **46**, 2355 (1992); A. Borde, A. Vilenkin, Phys. Rev. Lett. **72**, 3305 (1994).

[9] A. Guth, *The inflationary Universe*, (Vintage, London, 1998), p. 271.

[10] Last paper in Ref. [5]; first paper in Ref. [6]; R. Brustein and R. Madden, Phys. Rev. D **57**, 712 (1998); S. Foffa, M. Maggiore and R. Sturani, Nucl. Phys. B **552**, 395 (1999); D. A. Easson and R. Brandenberger, JHEP **9909**, 003 (1999). G. F. R. Ellis, D. C. Roberts, D. Solomons and P. K. S. Dunsby, *Using the dilaton potential to obtain string cosmology solutions*, hep-th/9912005.

[11] C. Cartier, E. J. Copeland and R. Madden, JHEP **0001**, 035 (2000).

[12] J. E. Lidsey, D. Wands and E. J. Copeland, *Superstring Cosmology*, hep-th/9909061 (Phys. Rep., in press).

[13] M. Gasperini, *Elementary introduction to pre-big bang cosmology and to the relic graviton background*, hep-th/9907067, in Proc. of the Second SIGRAV School (Center A. Volta, Como, April 1999), ed. by V. Gorini et al (IOP Publishing, Bristol, 2000); G. Veneziano, *String cosmology: the pre-big bang scenario*, hep-th/0002094, in Proc. of the Les Houches School (July 1999), in press.

[14] A. Guth, Phys. Rev. D **23**, 347 (1981).

[15] F. Lucchin and S. Matarrese, Phys. Lett. B **164**, 282 (1985).

[16] See, in particular, the second and the third paper of Ref. [6].

[17] D. Shadev, Phys. Lett. B **317**, 155 (1984); R. B. Abbott, S. M. Barr and S. D. Ellis, Phys. Rev. D **30**, 720 (1984); E. W. Kolb, D. Lindley and D. Seckel, Phys. Rev. D **30**, 1205 (1984).

[18] M. Gasperini, Phys. Rev. Lett. **56**, 2873 (1986).

[19] M. Gasperini, Phys. Lett. B **163**, 164 (1985); Class. Quantum Grav. **4**, 485 (1987).

[20] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B **494**, 315 (1997).

[21] M. Maggiore and R. Sturani, Phys. Lett. B **415**, 335 (1997).

[22] V. Kaplunovsky, Phys. Rev. Lett. **55**, 1036 (1985).

[23] M. S. Turner and E. J. Weinberg, Phys. Rev. D **56**, 4604 (1997).

[24] N. Kaloper, A. Linde and R. Bousso, Phys. Rev. D **59** 043508 (1999).

[25] A. D. Linde, Phys. Lett. B **175**, 395 (1986).

[26] M. Gasperini, Phys. Rev. D61, 87301 (2000).

[27] A. Buonanno, T. Damour and G. Veneziano, Nucl. Phys. B **543**, 275 (1999).

[28] A. Gosh, G. Pollifrone and G. Veneziano, Phys. Lett. B **440**, 20 (1998).

[29] D. Clancy, J. E. Lidsey and R. Tavakol, Phys. Rev. D **59**, 063511 (1999); Phys. Rev. D **58**, 44017 (1998); D. Clancy, A. Feinstein, J. E. Lidsey and R. Tavakol, Phys. Lett. B **451**, 303 (1999); A. Feinstein, K. E. Kunze and M. A. Vazquez-Mozo, *Initial conditions and the structure of the singularity in pre-big bang cosmology*, hep-th/0002070.

[30] A. J. Banday et al., Ap. J. **475**, 393 (1997).

[31] B. S. De Witt, Phys. Rev. **160**, 1113 (1967); J. A. Wheeler, in *Battelle Rencontres*, eds. C. De Witt and J. A. Wheeler (Benjamin, New York, 1968).

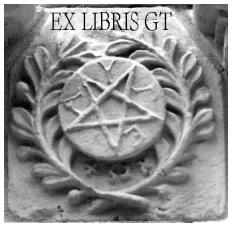
[32] A. Vilenkin, Phys. Rev. D **30**, 509 (1984); D **33**, 5560 (1986); D **37**, 888 (1988).

[33] A. D. Linde, Sov. Phys. JEPT **60**, 211 (1984); Lett. Nuovo Cimento **39**, 401 (1984); Y. Zel'dovich and A. A. Starobinski, Sov. Astron. Lett. **10**, 135 (1984); V. A. Rubakov, Phys. Lett. B **148**, 280 (1984).

[34] A. Linde, Phys. Lett. B **351**, 99 (1995).

[35] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983); S. W. Hawking, Nucl. Phys. B **239**, 257 (1984); S. W. Hawking and D. N. Page, Nucl. Phys. B **264**, 185 (1986).
[36] M. Gasperini, J. Maharana and G. Veneziano, Nucl. Phys. B **472**, 394 (1996).
[37] M. Gasperini and G. Veneziano, Gen. Rel. Grav. **28**, 1301 (1996).
[38] A. A. Kehagias and A. Lukas, Nucl. Phys. B **472**, 549 (1996).
[39] M. Cavaglià and V. De Alfaro, Gen. Rel. Grav. **29**, 773 (1997); M. Cavaglià and C. Ungarelli, Class. Quantum Grav. **16**, 1401 (1999).
[40] M. Gasperini, *Birth of the Universe in string cosmology*, in Proc. of the Euroconference "Fourth Paris Cosmology Colloquium" (Observatoire de Paris, June 1997), eds. H. J. De Vega and N. Sanchez (World Scientific, Singapore, 1998), p. 85; M. Gasperini, Int. J. Mod. Phys. A **13**, 4779 (1998).
[41] G. 't Hooft, in "Salam Festschrift" , eds. A. Ali et al. (World Scientific, Singapore, 1993), p. 284; L. Susskind, J. Math. Phys. **36**, 6377 (1995).
[42] G. Veneziano, Phys. Lett. B **454**, 22 (1999).
[43] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B **293**, 385 (1987).
[44] M. D. Pollock, Nucl. Phys. B **324**, 187 (1989).
[45] M. D. Pollock, Int. J. Mod. Phys. A **7**, 4149 (1992).
[46] E. Kiritsis and C. Kounnas, *Dynamical topology change, compactification and waves in a stringy early Universe*, in Proc. of the Euroconference "Second Paris Cosmology Colloquium" (Observatoire de Paris, June 1994), eds. H. J. De Vega and N. Sanchez (World Scientific, Singapore, 1995), p. 500
[47] T. Banks, W. Fischler and L. Motl, JHEP **9901**, 019 (1999).
[48] M. Maggiore and A. Riotto, Nucl. Phys. B **548**, 427 (1999).
[49] E. Witten, Nucl. Phys. B **471**, 136 (1996); P. Horawa and E. Witten, Nucl. Phys. B **475**, 94 (1996).
[50] A. Lukas and B. A. Ovrut, Phys. Lett. B **437**, 291 (1998).
[51] A. Feinstein and M. A. Vazquez-Mozo, *M-theory resolution of four-dimensional cosmological singularities via U-duality*, hep-th/9906006.
[52] A. Lukas, B. A. Ovrut and D. Waldram, Nucl. Phys. B **495**, 365 (1997).
[53] K. Behrndt, S. Forste and S. Schwager, Nucl. Phys. B **508**, 391 (1997); R. Poppe and S. Schwager, Phys. Lett. B **393**, 51 (1997); N. Kaloper, Phys. Rev. D **55**, 3304 (1997).
[54] F. Larsen and F. Wilczek, Phys. Rev. D **55**, 4591 (1997).
[55] N. Kaloper, I. I. Kogan and K. A. Olive, Phys. Rev. D **57**, 7340 (1998).
[56] L. P. Grishchuk, Sov. Phys. JEPT **40**, 409 (1975); A. A. Starobinski, HEPT Letters **30**, 682 (1979).
[57] V. F. Mukhanov, H. A. Feldman and R. Brandenberger, Phys. Rep. **215**, 203 (1992).
[58] M. Gasperini and M. Giovannini, Phys. Lett. B **282**, 36 (1992); Phys. Rev. D **47**, 1529 (1993).
[59] Second paper of Ref. [56].
[60] R. B. Abbott, B. Bednarz and S. D. Ellis, Phys. Rev. D **33**, 2147 (1986).
[61] Second paper of Ref. [6].
[62] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. D **48**, R439 (1993).
[63] R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov and G. Veneziano, Phys. Rev. D **51**, 6744 (1995).
[64] J. Hwang, Ap. J **375**, 442 (1991).
[65] R. Brustein, M. Gasperini and G. Veneziano, Phys. Lett. B **431**, 277 (1998).
[66] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B **361**, 45 (1995).
[67] R. Brustein, M. Gasperini and G. Veneziano, Phys. Rev. D **55**, 3882 (1997).
[68] See also M. Gasperini, *Status of string cosmology: phenomenological aspects*, in Proc. of the "Fourth Course of the Int. School of Astrophysics D. Chalonge" (Erice, 1995), eds. N. Sanchez and A. Zichichi (Kluwer, Dordrecht, 1996), p. 305.
[69] A. Buonanno, K.A. Meissner, C. Ungarelli and G. Veneziano, JHEP 01, 004 (1998).

[70] M. S. Turner and L. M. Widrow, Phys. Rev. D **37**, 2743 (1988).
[71] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75**, 3796 (1995).
[72] E. J. Copeland, R. Easther and D. Wands, Phys. Rev. D **56**, 874 (1997); E. J. Copeland, J. Lidsey and D. Wands, Nucl. Phys. B **506**, 407 (1997).
[73] R. K. Sachs and A. M. Wolfe, Ap. J. **147**, 73 (1967).
[74] R. Durrer, M. Gasperini, M. Sakellariadou and G. Veneziano, Phys. Lett. B **436**, 66 (1998); Phys. Rev. D **59**, 043511 (1999).
[75] M. Gasperini and G. Veneziano, Phys. Rev. D **59**, 043503 (1999).
[76] C. Bennet et al, Astrophys. Lett. **464**, L1 (1996).
[77] A. Melchiorri, F. Vernizzi, R. Durrer and G. Veneziano, Phys. Rev. Lett. **83** , 4464 (1999).
[78] M. Gasperini, Phys. Lett. B **327**, 214 (1994); see also the third paper of Ref. [6].
[79] M. Gasperini, *Relic dilatons in string cosmology*, in Proc. of the "12th Italian Conference on Gen. Rel. and Grav. Phys." (Roma 1996), eds. M. Bassan et al. (World Scientific, Singapore, 1997), p. 181.
[80] M. Gasperini, Phys. Lett. B **470**, 67 (1999).
[81] M. Gasperini, Phys. Lett. B **477**, 242 (2000).
[82] A. Linde, *Present status of inflationary theory*, talk given at the "First Cosmology and Particle Physics Workshop" (CERN, TH Division, June 1998).



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Inflation, Deflation, and Frame-Independence in String Cosmology

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Abstract

The inflationary scenarios suggested by the duality properties of string cosmology in the Brans-Dicke (or String) frame are shown to correspond to accelerated contraction (deflation) when Weyl-transformed to the Einstein frame. We point out that the basic virtues of inflation (solving the flatness and horizon problems, amplifying vacuum fluctuations, etc.) have physically equivalent counterparts in the deflationary (Einstein-frame) picture. This could be the answer to some objections recently raised to superstring cosmology.

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1. Introduction

A potential source of difficulty for extended-inflation models [1] based on a Brans-Dicke theory of gravity [2] is the choice of the correct frame (metric) in which to describe the space-time geometry at a cosmological level. One may wonder, in particular, in which frame the metric should be of the inflationary type, and satisfy the conditions required to avoid the problems of the standard cosmological scenario.

While the choice of the Einstein (E) frame (in which the Einstein-Hilbert term takes the General-Relativity form) usually simplifies calculations and is quite popular, there are physical motivations for choosing instead the Brans-Dicke (BD) frame, in which matter couples to the metric-tensor in the standard way [3]. Arguments in favour of the BD choice can also be given in string theory [4], where the BD frame metric coincides with the σ-model metric to which test strings are directly coupled. Thus free string motions follow geodesic surfaces with respect to the BD (not the E) metric.

The physical observable properties of a given model should be independent, of course, from the field redefinition (Weyl rescaling) connecting BD and E frames. And indeed, in the case of extended inflation, the metric describing a phase of power-law inflation (with variable Newton constant) in the BD frame, is transformed into a metric, which is still describing power inflation (of the slow-roll type, with exponential potential) in the E frame, as discussed for instance in [5].

In a string theory context, the role of the BD scalar is played by the dilaton field. In such case, as pointed out in [6], there appear to be serious difficulties in arranging a successful phase of dilaton-driven, power-law, extended inflation, at least if theoretically motivated dilaton potentials are used. On the other hand, the cosmological equations obtained from the low-energy string effective action show that the dilaton can drive (even in the absence of a potential) a phase of accelerated expansion. This phase, supposedly describing the Universe before the big-bang (so-called "pre-big-bang" [7]), is characterized by being just the "dual" counterpart (in the sense of ref. [8]) of the "post-big-bang" standard cosmology.

The "pre-big-bang" phase corresponds, in the BD frame, to a superinflationary expansion. When transformed to the E frame, however, the same metric describes, as we shall see, a contracting Universe. Apparently, this represents a difficulty for the whole scenario, since the presence or absence of inflation (and of its bonuses) would seem to become frame-dependent.

In this paper we shall show that, on the contrary, even in the E frame the

solutions of the string cosmology equations provide an adequate description of the inflationary phase, provided we generically mean, by "inflation", a phase of cosmological evolution that is able to avoid the problems (see for instance [9]) related to the decelerated kinematics of the standard cosmological model.

At the same time, and irrespectively of strings and/or BD theory, we shall argue that the solution of many of the standard-cosmology problems achieved by inflation is also possible through the introduction of an early phase of accelerated contraction, that we shall call deflation. This will be the content of the following section.

2. Inflation vs. deflation

It is well known that there are three possible classes of inflationary evolution [10], corresponding to a curvature scale that is constant (De Sitter inflation), decreasing (power inflation) or increasing (superinflation). Less known, however, seems to be the fact that in a phase of growing curvature the solution of the standard cosmological puzzles can be realized in two ways, namely by a metric describing *either* accelerated expansion, $\dot{a} > 0, \ddot{a} > 0$, *or* accelerated contraction, $\dot{a} < 0, \ddot{a} < 0$ (a is the scale factor of a homogeneous and isotropic model, and a dot denotes differentiation with respect to cosmic time).

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1flatness problem

A possible equivalence of superinflation and accelerated contraction is clearly pointed out by an elementary analysis of the so-called flatness problem. If we want the contribution of the spatial curvature k to be suppressed with respect to the other terms of the cosmological equations, then the ratio

$$r_1 = \frac{k}{a^2 H^2} = \frac{k}{\dot{a}^2} \quad , \quad H \equiv \dot{a}/a \, , \tag{2.1}$$

must tend to zero during the inflationary era. Such a condition is clearly satisfied by a metric that behaves, for $t \to +\infty$, as

$$a \sim t^\alpha \quad , \quad t > 0 \quad , \quad \alpha > 1 \, , \tag{2.2}$$

but also by a metric, which, for $t \to 0_-$, behaves as

$$a \sim (-t)^\beta \quad , \quad t < 0 \quad , \quad \beta < 1 \, . \tag{2.3}$$

The case (2.2) corresponds to power inflation, and includes the standard De Sitter exponential inflation in the limit $\alpha \to \infty$. The second case, (2.3), corresponds, for $\beta < 0$, to the well-known case of pole inflation (superinflationary

expansion, $\dot{a}, \ddot{a}, \dot{H}$ all positive). For $0 < \beta < 1$ it describes instead an accelerated contraction, or deflation ($\dot{a}, \ddot{a}, \dot{H}$ all negative). In both cases the curvature scale is growing, and $H, \dot{H}$ diverge as $t \to 0_-$.

2 Horizon problem

A deflationary phase (2.3), with $0 < \beta < 1$, may also provide a solution to the so-called horizon problem. The presently observed large-scale homogeneity and isotropy requires the proper size of the particle horizon to become large enough during the inflationary era, and to go to infinity in the limiting case in which inflation extends for ever in the past. This means that the integral

$$d_p(t) = a(t) \int_{t_1}^{t} dt' a^{-1}(t') \tag{2.4}$$

must diverge, if a is the inflationary scale factor, when t_1 approaches the maximal past extension of the cosmic time coordinate for the given cosmological manifold.

For the metric (2.3) such a limiting time is $-\infty$, and $d_p \to \infty$ for $t_1 \to -\infty$, so that there are no particle horizons in a phase of accelerated contraction.

As a consequence of accelerated contraction, causally connected regions are pushed out of the event horizon, just as in the standard inflationary expansion. It is true that the proper size of a causally connected region tends to contract, asymptotically, like the scale factor. For a patch of initial size $d_1 \sim (-t_1)$ one finds in fact, from eqs. (2.3) and (2.4), that $d_p \to [a(t)/a(t_1)]d_1$ for $|t| << |t_1|$. However, the proper size of the event horizon, defined by

$$d_e(t) = a(t) \int_{t}^{t_2} dt' a^{-1}(t') \tag{2.5}$$

(t_2 is the maximal allowed future extension of the cosmic time coordinate), contracts always faster than d_p. Indeed, $t_2 = 0$ for the metric (2.3), and one finds that $d_e(t) \sim (-t)$ for $t \to 0$. The ratio of the two proper sizes at small t

$$r_2(t) = \frac{d_p(t)}{d_e(t)} \sim (-t)^{\beta - 1} \tag{2.6}$$

shows that the causally connected regions will always cross the horizon, asymptotically, not only in the case of superinflationary expansion ($\beta < 0$), but even in the deflationary case ($0 < \beta < 1$).

We note, for later convenience, that the conditions for a successful resolution of the horizon and flatness problem, when expressed in terms of the conformal time coordinate η ($a = dt/d\eta$), are exactly the same for both superinflationary expansion and accelerated contraction. Moreover, if the contracting phase is long

enough to solve the horizon problem, then also the flatness problem is automatically solved (and vice versa), as in standard inflation.

Indeed the ratio r_2 scales in conformal time like η^{-1}, while the ratio r_1 scales like η^2. The horizon problem is solved if $r_2(\eta_f)$, evaluated at the end of the accelerated evolution ($\eta = \eta_f$), is larger than the present value $r_2(\eta_0) \simeq 1$, rescaled down at η_f. This implies

$$\frac{|\eta_i|}{|\eta_f|} \gtrsim \frac{|\eta_0|}{|\eta_f|} \simeq 10^2 (\frac{T_{rh}}{eV}) \, . \tag{2.7}$$

Here η_i denotes the beginning of the contracting (or expanding) accelerated evolution, T_{rh} the final reheating temperature at $\eta = \eta_f$, and the last equality holds in the hypothesis of standard, adiabatic, radiation-dominated and matter-dominated expansion from η_f down to the present time η_0.

The solution of the flatness problem, on the other hand, is obtained if the ratio r_1 at the end of the accelerated phase is tuned to a value that is small enough, so that the subsequent decelerated evolution leads to a present value of r_1 satisfying the condition $r_1(\eta_0) \lesssim 1$. This means

$$(\frac{\eta_f}{\eta_i})^2 \lesssim (\frac{\eta_f}{\eta_0})^2 \, , \tag{2.8}$$

which is clearly equivalent to eq. (2.7), and which implies a resolution of the flatness and horizon problems (as well as of their rephrasing in terms of the entropy [9]) for both expanding and contracting metrics of the type (2.3).

Besides solving the kinematical problems, a phase of successful inflation is also expected to efficiently amplify the vacuum fluctuations of the metric background. We shall conclude this section by noting that such an amplification can also be provided by a long period of deflation.

Consider, for instance, the amplification of tensor perturbations h^ν_μ (similar arguments hold for the scalar case also). In a four-dimensional conformally flat background, the wave equation for each Fourier component of h can be written in terms of the rescaled variable $\psi = ah$ as [11]

$$\psi'' + (k^2 - \frac{a''}{a})\psi = 0 \tag{2.9}$$

(a prime denotes differentiation with respect to conformal time). In a realistic case, the phase of accelerated evolution is followed by the standard radiation-dominated expansion, with $a \sim \eta$, and the amplification of the fluctuations can be

parametric amplification of VAC fluctuations = graviton prod from VAC

described as a process of graviton production from the vacuum (such an approach will be used in Section 3). Equivalently, in a Schrödinger-like language, the process corresponds to a parametric amplification of the perturbation wave function [11], which is oscillating at $\eta \to \pm\infty$, and evolves with a power-law behaviour in the regions where the co-moving frequency k is negligible with respect to the effective potential a''/a of eq. (2.9).

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By inserting into (2.9) a generic parametrization (in conformal time) of the accelerated metric, $a(\eta) = (-\eta)^{-\delta}$, one finds indeed that the solution behaves like

$$h \sim A_\pm \frac{e^{\pm ik\eta}}{a} \quad , \quad k\eta >> 1 \tag{2.10}$$

$$h \sim A + B\frac{(-\eta)}{a^2} = A + B(-\eta)^{1+2\delta} \quad , \quad k\eta << 1 \tag{2.11}$$

($A_\pm, A, B$ are integration constants). In the case of accelerated expansion ($\delta > 0, a \to \infty$ for $\eta \to 0_-$), the perturbations are amplified because their amplitude tends to stay constant in the $\eta \to 0$ limit, instead of decreasing adiabatically as in the oscillating regime (2.10).

In the case of deflation ($\delta < 0, a \to 0$ for $\eta \to 0_-$), the amplification process is even more efficient than in the previous case, as the amplitude of h grows (with respect to the adiabatic red-shift of the subsequent radiation-dominated expansion) even in the oscillating regime. Moreover, as shown by eq. (2.11), h may even grow asymptotically (instead of being constant) provided $\delta < -1/2$. As we shall see in Section 3, this condition is satisfied in particular, in the E frame, by a 3-dimensional phase driven by stretched strings.

Note that the amplification coefficient corresponding to a phase of accelerated contraction is different, in general, from the one corresponding to a phase of accelerated expansion. It is just because of this difference that the perturbation spectrum may remain unchanged, when an inflationary background is transformed into a deflationary one through a conformal rescaling, as we shall see in the following Section.

3. Pre-big-bang cosmology in the Brans-Dicke and Einstein frames

In a string cosmology context [7,12], a global (at least semi-quantitative) description of the evolution and symmetries of the early Universe is expected to be provided by the low-energy string effective action, possibly supplemented by the action S_m for macroscopic matter sources:

1 ACTION

$$S = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{|g|}e^{-\phi}[R + (\partial_\mu\phi)^2 - \frac{1}{12}H^2_{\mu\nu\alpha} + V] + S_m \tag{3.1}$$

Here $H_{\mu\nu\alpha}$ is the antisymmetric tensor field strength, and V a (possibly non-zero) dilaton potential.

CASE:

In this paper we will consider a $(d+1)$-dimensional, anisotropic metric background of the Bianchi I type, with time-dependent dilaton,

$$g_{00} = 1 \quad , \quad g_{ij} = -a_i^2 \delta_{ij} \quad , \quad \phi = \phi(t) \quad , \quad i,j = 1,2,...,d \tag{3.2}$$

2 conditions

and with vanishing $H_{\mu\nu\alpha}$ and $V(\phi)$. The additional matter sources, which are decoupled from the dilaton in this frame, will be represented by a perfect fluid with anisotropic pressure:

$$T_0^0 = \rho \quad , \quad T_i^j = -p_i \delta_i^j = -\gamma_i \rho \delta_i^j \ . \tag{3.3}$$

By defining as usual [8,7,12]

$$\overline{\phi} = \phi - \ln\sqrt{|g|} \quad , \quad \overline{\rho} = \rho\sqrt{|g|} \quad , \quad \overline{p} = p\sqrt{|g|} \tag{3.4}$$

3 field Eqs

the field equations following from the variation of the action (3.1) can be written in the form [8]

$$\dot{\overline{\phi}}^2 - 2\ddot{\overline{\phi}} + \sum_i H_i^2 = 0 \tag{3.5}$$

$$\dot{\overline{\phi}}^2 - \sum_i H_i^2 = \overline{\rho} e^{\overline{\phi}} \tag{3.6}$$

$$2(\dot{H}_i - H_i \dot{\overline{\phi}}) = \overline{p}_i e^{\overline{\phi}} \tag{3.7}$$

where $H_i = \dot{a}_i/a_i$, and we use units in which $8\pi G = 1$. Their combination gives the usual conservation equation

$$\dot{\overline{\rho}} + \sum_i H_i \overline{p}_i = 0 \ . \tag{3.8}$$

By applying the general procedure illustrated in [7], the background field variables can be separated, and the equations can be integrated exactly, by introducing a suitable time-like coordinate x such that

4 COND of intregrability:

$$\overline{\rho} = \frac{1}{L}\frac{dx}{dt} \tag{3.9}$$

(L is a constant with dimensions of length, in such a way that x is dimensionless). For constant γ_i we obtain the following general exact solution of eqs.(3.5–3.7) (a similar problem was first solved in a different context in [13]):

5solutions:

$$a_i = a_{0i} |(x - x_+)(x - x_-)|^{\gamma_i/\alpha} \left|\frac{x - x_+}{x - x_-}\right|^{\alpha_i} \tag{3.10}$$

$$e^{\overline{\phi}} = e^{\phi_0}|(x-x_+)(x-x_-)|^{-1/\alpha}|\frac{x-x_+}{x-x_-}|^{-\sigma} \tag{3.11}$$

$$\overline{\rho} = \frac{\alpha}{4L^2}e^{\phi_0}|(x-x_+)(x-x_-)|^{(\alpha-1)/\alpha}|\frac{x-x_+}{x-x_-}|^{-\sigma} \tag{3.12}$$

where

$$\alpha = 1 - \sum_i \gamma_i^2 \quad , \quad \sigma = \sum_i \alpha_i \gamma_i \quad , \quad \alpha_i = \frac{\alpha x_i + \gamma_i(\sum_i \gamma_i x_i - x_0)}{\alpha[(\sum_i \gamma_i x_i - x_0)^2 + \alpha(\sum_i x_i^2 - x_0^2)]^{1/2}}$$

$$x_\pm = \frac{1}{\alpha}\{\sum_i \gamma_i x_i - x_0 \pm [(\sum_i \gamma_i x_i - x_0)^2 + \alpha(\sum_i x_i^2 - x_0^2)]^{1/2}\} \tag{3.13}$$

and a_0, ϕ_0, x_0, x_i are integration constants.

This solution has various interesting properties, which we shall discuss elsewhere [14]. Here we only note that there are two curvature singularities at $x = x_\pm$, and that the region between the singularities is unphysical, in the sense that the critical density parameter

$$\Omega(x) \equiv \frac{\rho e^\phi}{(d-1)\sum_i H_i^2} = \frac{(x+x_0)^2 - \sum_i(\gamma_i x + x_i)^2}{(d-1)\sum_i(\gamma_i x + x_i)^2} \tag{3.14}$$

becomes negative. This parameter tends to zero at the singularities, and in this limit the metric (3.10) goes over to the vacuum solutions of string cosmology [15,8]. For $x \to x_\pm$ one finds indeed

$$a_i(t) \sim |t - t_\pm|^{\beta_i^\pm} \ , \tag{3.15}$$

where

$$\beta_i^\pm = \frac{x_i \pm \gamma_i x_\pm}{x_0 + x_\pm} \quad , \quad \sum_i (\beta_i^\pm)^2 = 1 \ . \tag{3.16}$$

However, because of the neglect in the original action (3.1) of truly "stringy" contributions (such as α' and loop corrections), this solution is not expected to provide a reliable description of the very high curvature regime. The appropriate range of validity of the solution is instead the large $|x|$ limit, and in particular $x \to -\infty$, where it provides a typical example of pre-big-bang evolution, characterized by acceleration and growing curvature scale [7].

CASE 2:

If we consider, in particular, the isotropic case with negative pressure ($a_i = a, \gamma_i = \gamma < 0$ for all d spatial directions), then at large negative x we have $|x| \sim |t|^{\alpha/(2-\alpha)}$, and the solution (3.10-3.12) becomes, in this limit,

$$a(t) \sim (-t)^{2\gamma/(1+d\gamma^2)} \quad , \quad \overline{\phi}(t) \sim -\frac{1}{\gamma}\ln a$$

validity range

$$\phi = \overline{\phi} + d \ln a \sim \frac{d\gamma - 1}{\gamma} \ln a \quad , \quad \overline{\rho} \sim a^{-d\gamma} \, . \tag{3.17}$$

For $\gamma = -1/d$, which is the typical equation of state for a perfect gas of stretched (or unstable) strings [16], one thus recovers the particular solution already considered in [7,12] ("string-driven" pre-big-bang). More generally, however, the background (3.17) describes a phase of superinflationary expansion, $H > 0, \ddot{a}/a > 0$, and growing curvature scale, $\dot{H} > 0$, for all $\gamma < 0$.

This is the picture in the BD frame, which may be regarded as the natural one in a string theory context [3]. The passage to the E frame, defined as the frame in which the graviton and dilaton kinetic terms are diagonalized and the action takes the standard form,

"string frame"

$$S_E = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{|\tilde{g}|}[-\tilde{R}(\tilde{g}) + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}] + S_m \, , \tag{3.18}$$

Einstein frame

is obtained through the conformal rescaling

Veneziano rescaling

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} e^{-2\phi/(d-1)} \quad , \quad \tilde{\phi} = \sqrt{\frac{2}{d-1}}\phi. \tag{3.19}$$

The E-transformed scale factor, $\tilde{a}$, and cosmic time coordinate, $\tilde{t}$, are thus related to the original BD ones by

$$\tilde{a} = a e^{-\phi/(d-1)} \quad , \quad d\tilde{t} = dt e^{-\phi/(d-1)}. \tag{3.20}$$

The pre-big-bang configuration (3.17) becomes, in the E frame,

$$\tilde{a}(\tilde{t}) \sim (-\tilde{t})^\beta \quad , \quad \tilde{\phi} \sim \sqrt{\frac{2}{d-1}} \frac{(d-1)(1-d\gamma)}{(\gamma-1)} \ln \tilde{a}$$

$$\tilde{\rho} \sim \tilde{a}^{-2/\beta} \quad , \quad \beta = \frac{2(1-\gamma)}{(d-1)(1+d\gamma^2) - 2(d\gamma - 1)} \tag{3.21}$$

where $\tilde{\rho}$ is conformally related to the original density ρ as

$$\tilde{\rho} = \rho \frac{\sqrt{|g|}}{\sqrt{|\tilde{g}|}} = \rho e^{\phi(d+1)/(d-1)} \tag{3.22}$$

(see for instance [17]). For all $d > 1$ and $\gamma < 0$, the transformed metric (3.21) satisfies

$$\frac{\ddot{\tilde{a}}}{\tilde{a}} < 0 \quad , \quad \tilde{H} < 0 \quad , \quad \dot{\tilde{H}} < 0 \, , \tag{3.23}$$

BD "superinflation" = accelerated CONTRACTION in E frame!!!!!!!!!!!!!!!

where $\tilde{H} = \dot{\tilde{a}}/\tilde{a}$, and the dot denotes here differentiation with respect to $\tilde{t}$. The BD superinflation thus becomes an accelerated contraction of the type (2.3).

This result is a consequence of the non-trivial evolution of the dilaton background that determines the transformation between the two frames, and it is of crucial importance. It implies that, if inflation is long enough in the BD frame to solve the kinematical problems of the standard model, then such problems are also solved in the E frame. Indeed, according to eq. (3.20), the two frames have the same conformal time

$$d\tilde{\eta} = \frac{d\tilde{t}}{\tilde{a}(\tilde{t})} = \frac{dt}{a(t)} = d\eta \tag{3.24}$$

and we have shown in Section 2 that the conditions to be satisfied for solving the kinematical problems, when expressed in conformal time, are the same for both superinflationary expansion and accelerated contraction.

Moreover, the spectrum of the metric perturbations amplified in the course of the background evolution is also the same in both frames. This can be easily shown by considering , for instance, the case of tensor perturbations, and assuming a generic model of background evolution characterized by the transition (at $\eta = \eta_1$) from the accelerated phase to the standard radiation-dominated one. In conformal time, such evolution can be parametrized as

$$a \sim (-\eta)^{-\delta} \quad , \quad \phi \sim \epsilon \ln a \quad , \quad \eta << -\eta_1$$

$$a \sim \eta \quad , \quad \phi \sim const \quad , \quad \eta >> -\eta_1 \ . \tag{3.25}$$

In order to verify the equality of the spectral behaviour, it is crucial to take into account the fact that not only the background solutions, but also the perturbation equations are different, when the frame is changed. In the BD frame, the tensor perturbation equation contains explicitly the contribution of the dilaton background, and for each component of h_μ^ν the equation can be written [7,17]

$$\psi'' + (k^2 - V)\psi = 0 \ , \tag{3.26}$$

where [7,17]

$$\psi = h a^{(d-1)/2} e^{-\phi/2}$$

$$V = \frac{(d-1)a''}{2a} - \frac{\phi''}{2} + \frac{(d-1)(d-3)a'^2}{4a^2} + \frac{\phi'^2}{4} - \frac{(d-1)a'\phi'}{2a} \ . \tag{3.27}$$

By matching the solutions of (3.26) corresponding to the two phases of background evolution, one can compute the Bogoliubov coefficients relating $|in\rangle$ and

The Great BOGO !!!!!!!!!!!!!!!

$|out\rangle$ vacua, and describing the associated graviton production. For co-moving frequencies k that are small enough with respect to the height of the effective potential barrier ($k\eta_1 << 1$), the modulus of the Bogoliubov coefficient is [7,18]

$$|c_-(k)| \simeq (k\eta_1)^{-|\nu|-1/2} \tag{3.28}$$

where

$$\nu = \frac{\delta}{2}(d-1-\epsilon) + \frac{1}{2} \tag{3.29}$$

and the corresponding spectral distribution of gravitons is determined as $\rho(k) = k^4|c_-|^2$. In the case of four-dimensional exponential inflation ($\delta = 1, d = 3, \epsilon = 0$) one thus finds, in particular, the flat Harrison-Zeldovich spectrum.

In the more general case of the background (3.17), one finds that, in conformal time, the kinematics is parametrized according to eq. (3.25) by

$$\delta = -\frac{2\gamma}{1-2\gamma+d\gamma^2} \quad , \quad \epsilon = \frac{d\gamma-1}{\gamma} \, . \tag{3.30}$$

The coefficient $|\nu|$ determining the pre-big-bang graviton spectrum in the BD frame is thus

$$|\nu| = \frac{1}{2}\left|\frac{d\gamma^2-1}{1-2\gamma+d\gamma^2}\right| \, . \tag{3.31}$$

In the E frame, there is no explicit dilaton contribution to the perturbation equation for h, which is exactly the same equation as that satisfied by a minimally coupled scalar field [11] (the dilaton contribution, however, is implicitly contained in the rescaled metric background). Such an equation can still be written in the form (3.26), (3.27), but with $\phi = const$. As a consequence, the spectral coefficient $|\nu|$ of eq. (3.28) is determined by the metric background only, and becomes

$$\nu = \frac{\tilde{\delta}}{2}(d-1) + \frac{1}{2} \, , \tag{3.32}$$

where $\tilde{\delta}$ is the exponent parametrizing, in conformal time, the evolution of the contracting E metric (3.21):

$$\tilde{\delta} = \frac{2(\gamma-1)}{(d-1)(1-2\gamma+d\gamma^2)} \, . \tag{3.33}$$

This value, when inserted into eq. (3.32), provides exactly the same expression for $|\nu|$ as in eq. (3.31), and thus the same graviton spectrum as in the BD frame.

We want to stress, finally, that the same results hold in the case of conformal vacuum backgrounds, namely for solutions of eqs. (3.5-3.7) with $\rho = p = 0$ [8,15]

(the general vacuum solution for the action (3.1) with non-zero $H_{\mu\nu\alpha}$ is given in [19]).

In the vacuum case the analogous of the isotropic, d-dimensional solution (3.17) is, in the BD frame,

$$\begin{aligned} a_\mp(t) &\sim |t|^{\mp 1/\sqrt{d}} \\ \phi_\mp(t) &\sim -(1 \pm \sqrt{d}) \ln|t| = \pm(\sqrt{d} \pm d) \ln a_\mp \end{aligned} \tag{3.34}$$

The two signs correspond to the two duality-related solutions [8], and the upper sign describes a "dilaton-driven", pre-big-bang, superinflationary expansion for t ranging from $-\infty$ to 0.

In the E frame the solution (3.34) becomes (in conformal time)

$$\tilde{a}(\tilde{\eta}) = |\tilde{\eta}|^{1/(d-1)} \quad , \quad \tilde{\phi}(\tilde{\eta}) = \mp\sqrt{2d(d-1)} \ln \tilde{a} \tag{3.35}$$

and it always describes an accelerated contraction of the type (2.3), independently of the choice of sign in eq. (3.34). It is interesting to note that the duality transformation, which is represented in the BD frame as an inversion of the scale factor and a related dilaton shift,

$$a_+ \to a_- = a_+^{-1} \quad , \quad \phi_+ \to \phi_- = \phi_+ - 2d \ln a_+ \tag{3.36}$$

becomes, in the E frame, a transformation between what we may call a strong-coupling and a weak-coupling regime, $\tilde{\phi} \to -\tilde{\phi}$, without changing the metric background described by $\tilde{a}$.

4. Conclusions

The main goal of this paper has been to show that, for what concerns the solution of the kinematical problems (horizon, flatness) of the standard model, and the amplification of the vacuum fluctuations, an accelerated contraction of the metric is equally good as an accelerated expansion.

This observation was motivated by the fact (also discussed in this paper) that accelerated contraction is the behaviour of the metric in a general pre-big-bang cosmological string scenario, when seen in the Einstein frame. Indeed, as already stressed in [7], there are only two ways of implementing a phase of cosmic acceleration and simultaneous growth of the curvature scale: accelerated contraction and superinflationary (or pole-like) expansion. The latter corresponds to the pre-big-bang picture in the conformally related Brans-Dicke frame.

Obviously, a contracting phase cannot dilute the relic abundance of some unwanted remnants, such as the monopoles of the GUT phase transition. However, the same is true for the pre-big-bang scenario in the BD frame, as well as for all models in which the phase of inflationary expansion occurs at some higher fundamental (near Planckian) scale, which is indeed what is expected in a string cosmology context. In this respect, we recall [7] that a pre-big-bang phase should be regarded not necessarily as an alternative, but possibly as a complement to the more conventional inflationary models, which cannot be extended (at least semiclassically) beyond the Planck era.

Moreover, it is clear that deflationary contraction is adiabatic for what concerns radiation, just like the usual inflationary expansion. Therefore, as recently stressed also in [20], a kinematical modification of the standard model can explain the large present value of the cosmic black-body entropy, only if the accelerated evolution is matched to the standard one through a phase dominated by some non-adiabatic process (the so-called "reheating" era).

In the BD picture of the pre-big-bang scenario (see eq. (3.17)), the radiation is supercooled and diluted with respect to the sources that drive inflation. The conservation equation (3.8) leads in fact to an effective source temperature $T_s \sim a^{-d\gamma}$, which grows together with the scale factor for $\gamma < 0$, and satisfies

$$\frac{T_s}{T_r} = \frac{\rho_\gamma}{\rho_r} \sim a^{1-d\gamma} \tag{4.1}$$

(r corresponds here to the radiation-like equation of state, $\gamma = 1/d$). The reheating process is thus expected to represent, in this frame, a sort of non-adiabatic conversion of the hot sources into radiation, such as a possible isothermal decay of the highly excited states of a gas of stretched strings [7].

In the E frame (see eq. (3.21)) the fluid sources satisfy a modified conservation equation,

$$\dot{\tilde{\rho}} + d\tilde{H}(\tilde{\rho} + \tilde{p}) - \frac{\dot{\tilde{\phi}}}{\sqrt{2(d-1)}}(\tilde{\rho} - d\tilde{p}) = 0 \ . \tag{4.2}$$

Radiation still evolves adiabatically, now with a blue-shifted temperature because of the contraction, $\tilde{T}_r \sim \tilde{a}^{-1}$. The effective temperature of the pre-big-bang sources is also blue-shifted, however, since, in the perfect fluid approximation, eq. (4.2) leads to

$$\tilde{T}_s \sim \tilde{a}^{(d^2\gamma^2+1-d\gamma-d\gamma^2)/(\gamma-1)} \tag{4.3}$$

and thus

$$\frac{\tilde{T}_s}{\tilde{T}_r} = \frac{\tilde{\rho}_s}{\tilde{\rho}_r} \sim \tilde{a}^{\gamma(d-1)(d\gamma-1)/(\gamma-1)} \ . \tag{4.4}$$

For $\tilde{a} \to 0$ the temperature of the sources that drive the acceleration ($\gamma < 0, d > 1$) is always growing, even with respect to the radiation temperature. The physical picture of reheating as a non-adiabatic decay of the hot sources is still valid, therefore, also in the Einstein frame.

We would like to stress, finally, that the absence of problems related to some "preferred frame" description of a string cosmology inflation is to be ascribed, to a large extent, to the crucial role played by the dilaton field, which transforms conformally a superinflationary expansion into a deflationary contraction. This is to be traced back to the duality properties of the string effective action [8,12,19,21], and thus gives support to the consistency of an approach to string cosmology based on the effective action (3.1).

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References

1. D. La and P.J. Steinhardt, Phys. Rev. Lett. 62 (1989) 376.
2. C. Brans and C.H. Dicke, Phys. Rev. 124 (1961) 925.
3. See e.g. Ya.B. Zeldovich and I.D. Novikov, "The Structure and the Evolution of the Universe", University of Chicago Press (1982).
4. N. Sanchez and G. Veneziano, Nucl. Phys. B333 (1990) 253;
   L.J. Garay and J. Garcia-Bellido, Nucl. Phys. B400 (1993) 416.
5. E.W. Kolb, D.S. Salopek and M.S. Turner, Phys. Rev. D42 (1990) 3925;
   E.W. Kolb, Physica Scripta T36 (1991) 199.
6. B.A. Campbell, A. Linde and K.A. Olive, Nucl. Phys. B335 (1991) 146;
   R. Brustein and P.J. Steinhardt, Phys. Lett. B302 (1993) 196.
7. M. Gasperini and G. Veneziano, Astropart. Phys. 1 (1993) 317.
8. G. Veneziano, Phys. Lett. B265 (1991) 287.
9. A. H. Guth, Phys. Rev. D23 (1981) 347.
10. F. Lucchin and S. Matarrese, Phys. Lett. B164 (1985) 282.
11. L.P. Grishchuk, Sov. Phys. JETP 40 (1974) 409.
12. M. Gasperini and G. Veneziano, Phys. Lett. B277 (1992) 256.
13. V.A. Ruban and A.M. Finkelstein, Gen. Rel. Grav. 6 (1975) 601.
14. M. Gasperini, N. Sanchez and G. Veneziano, in preparation.
15. M. Mueller, Nucl. Phys. B337 (1990) 37.
16. M. Gasperini, N. Sanchez and G. Veneziano, Nucl. Phys. B364 (1991) 365.
17. S. Kalara, N. Kaloper and K.A. Olive, Nucl. Phys. B341 (1990) 252.
18. M. Gasperini and M. Giovannini, Phys. Rev. D47 (1993) 1529.
19. K.A. Meissner and G. Veneziano, Mod. Phys. Lett. A6 (1991) 3397.
20. Y. Hu, M.S. Turner and E.J. Weinberg, FERMILAB-Pub-92/363-A.
21. A.A. Tseytlin, Mod. Phys. Lett. A6 (1991) 1721;
    A. Sen, Phys. Lett. B271 (1991) 295;
    M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B272 (1991) 277;
    ibid. B296 (1992) 51;
    S.F. Hassan and A. Sen, Nucl. Phys. B375 (1992) 103;
    A.A. Tseytlin and C. Vafa, Nucl. Phys. B372 (1992) 443;
    A.A. Tseytlin, Class. Quantum Grav. 9 (1992) 979;
    E. Kiritsis, "Exact duality symmetries in CFT and string theory",
    CERN-TH.6797/93.

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hep-th/0703055

String Theory and Pre-big bang Cosmology

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Summary. In string theory, the traditional picture of a Universe that emerges from the inflation of a very small and highly curved space-time patch is a possibility, not a necessity: quite different initial conditions are possible, and not necessarily unlikely. In particular, the duality symmetries of string theory suggest scenarios in which the Universe starts inflating from an initial state characterized by very small curvature and interactions. Such a state, being gravitationally unstable, will evolve towards higher curvature and coupling, until string-size effects and loop corrections make the Universe "bounce" into a standard, decreasing-curvature regime. In such a context, the hot big bang of conventional cosmology is replaced by a "hot big bounce" in which the bouncing and heating mechanisms originate from the quantum production of particles in the high-curvature, large-coupling pre-bounce phase. Thanks to the strong coupling there is also an associate production of higher-dimensional branes, which could prepare (and provide the initial conditions for) a subsequent phase of brane-dominated inflation.

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String theory and pre-big bang cosmology

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Summary. In string theory, the traditional picture of a Universe that emerges from the inflation of a very small and highly curved space-time patch is a possibility, not a necessity: quite different initial conditions are possible, and not necessarily unlikely. In particular, the duality symmetries of string theory suggest scenarios in which the Universe starts inflating from an initial state characterized by very small curvature and interactions. Such a state, being gravitationally unstable, will evolve towards higher curvature and coupling, until string-size effects and loop corrections make the Universe "bounce" into a standard, decreasing-curvature regime. In such a context, the hot big bang of conventional cosmology is replaced by a "hot big bounce" in which the bouncing and heating mechanisms originate from the quantum production of particles in the high-curvature, large-coupling pre-bounce phase. Thanks to the strong coupling there is also an associate production of higher-dimensional branes, which could prepare (and provide the initial conditions for) a subsequent phase of brane-dominated inflation.

1 Introduction

The standard cosmological model, formulated and brought to completion by various authors during the second half of the last century (see for instance [1, 2]), provides us with an excellent description of the various important stages of our past cosmological history (such as the radiation era, the nucleosynthesis, the recombination era, the epoch of matter domination, ...). At early enough times, however, such a model is to be modified by the introduction of a "non-standard" epoch of accelerated cosmic evolution, called "inflation", which is needed in order to solve the horizon, flatness and entropy problems [3] implied by the extrapolation, back in time, of the present state according to the standard cosmological equations.

The introduction of a sufficiently long inflationary phase not only "explains" the otherwise unnatural initial conditions required by the subsequent

"standard" evolution, but also provides a natural (and very efficient) mechanism for amplifying the quantum fluctuations of the fundamental cosmic fields in their ground state. This mechanism produces inhomogeneous "seeds" for structure formation and for the fluctuations of the temperature of the Cosmic Microwave Background (CMB) radiation (see e.g. [4]), in remarkable agreement with recent observations (see e.g. COBE [5] and WMAP [6] results).

The most popular (and presently most successful) version of the inflationary scenario is probably the model of "slow-roll" inflation [7], in which the cosmological evolution is dominated by the potential energy of a cosmic scalar field (the so-called "inflaton"). The simplest and most conventional version of such model, however, is affected by various difficulties of conceptual nature. First of all, the peculiar properties of the inflaton (mass, couplings, potential energy, ...) prevent a simple identification of this field in the context of known models of fundamental interactions: the inflaton field has to be introduced *ad hoc*, and its properties are the result of a suitable fine-tuning of the relevant parameters.

Another difficulty is associated with the kinematic properties of the phase of slow-roll inflation, implying that the spatial size of the Hubble horizon cH^{-1}, proportional to the causal event horizon, becomes smaller and smaller as we go back in time (here c is the speed of light and H is the so-called Hubble parameter). It can be shown, in such a context, that the proper (time-dependent) wavelength of the fluctuations presently observed on a large scale (for instance, the temperature fluctuations of the CMB radiation), when rescaled down at the beginning of inflation, has to be smaller than the Planck length, $\lambda_{\rm P} = (8\pi G\hbar/c^3)^{1/2} \sim 10^{-33}$ cm, if inflation lasts long enough to solve the above mentioned problems (here G and $\hbar$ are the Newton and Planck constant, respectively). As a consequence, the initial conditions on the fluctuations of the matter fields and of the geometry are to be imposed inside an energy range which is presently unexplored, and in which the extrapolation of standard physics is – at least – questionable. This is the so-called "trans-Planckian" problem [8, 9].

In addition, if the Hubble horizon is decreasing as we go back in time, then its inverse, the spacetime curvature-scale H/c, is necessarily increasing, and the extrapolation of the model unavoidably leads to an initial singularity – or, at least, to the quantum gravity regime $cH^{-1} \sim \lambda_{\rm P}$ where we have to deal not only with an unknown fluctuation dynamics, but also with an unknown dynamics of the background itself. A proof of the unavoidable presence of the singularity, in the context of potential-driven inflationary cosmology based on the Einstein equations, is given in [10, 11].

All these problems, as well as other important problems of primordial cosmology (why is our Universe four-dimensional? why does a small vacuum energy density seem to survive until today after inflation? ...), should find a satisfactory solution in the context of a truly unified theory of all fundamental interactions. The best candidate for such theory, at present, is string theory, which is based on the assumption that the fundamental components

of all matter and force fields existing in nature are one-dimensional extended objects, called "strings". These objects are characterized by their tension (i.e., energy per unit of length) T and, when quantized, have a characteristic size $\lambda_{\rm s}$ given by:

$$\lambda_{\rm s} = \sqrt{c\hbar/T}. \tag{1}$$

As it will discussed in the following section, the consistency of string theory requires the existence of extra spatial dimensions besides the three ones we are familiar with. In such a context, two possibilities arise. In the most conventional unification scenario, in which the extra spatial dimensions are of size comparable to the string length scale [12], the parameter $\lambda_{\rm s}$ turns out to be extremely small, $\lambda_{\rm s} \sim 10\,\lambda_{\rm P}$. As a result, all effects pertaining to the finiteness of the string size only come into play at energy scales $M_{\rm s}c^2 = \hbar c/\lambda_{\rm s}$, so high to be well outside the reach of present (direct) high-energy experiments. In such a case the predictive power of the theory is low, as what can be tested by accelerator experiments are only the predictions of the so-called low-energy string effective action. Unfortunately, at the present stage of our knowledge, such an action can only be derived within perturbative computational techniques, and its precise form is unknown.

This discouraging conclusion can be avoided, at least in principle, if our Universe contains extra spatial dimensions compactified on length scales which are small if compared to macroscopic standards, but are nevertheless "large" with respect to the parameter $\lambda_{\rm s}$ of string theory [13, 14]. In this case the energy scale $M_{\rm s}c^2$ may be lowered well below the Planck scale $M_{\rm P}c^2 = \hbar c/\lambda_{\rm P}$, even by many orders of magnitude, thus possibly approaching the energy range explorable by near future accelerator experiments.

In any case, the energy scales typical of string theory (whether near to, or far from, the Planck scale) should have been reached during the primordial evolution of our Universe: string theory can then be properly applied to cosmology, to ask if (and how) inflation is naturally predicted, in such a context. The hope is to obtain, on one hand, a solution to the open problems of the conventional inflationary scenario and, on the other hand, a possible phenomenological signature of string theory, according to the historical tradition teaching us that fundamental gravitational theories have always been confirmed by astrophysical observations (as, for instance, Newton's and Einstein's theories of gravity).

In this paper, after a short presentation of string theory in Sect. 2, and of the mechanism of brane-inflation in Sect. 3, we shall concentrate our discussion on the so-called "pre-big bang" scenario [15], based on the scale-factor duality symmetry [16] typical of string theory. It will be shown, in particular, that the Universe (thanks to a fundamental string-theory field, the dilaton) may start inflating from an initial configuration characterized by a very small curvature and a very large Hubble horizon, thus avoiding the singularity and trans-Planckian problems, and yet satisfying the properties required for a successful scenario.

2 String theory: a few basic concepts

String theory is, at present, the only serious candidate for a fully quantum (as well as finite and unified) theory of gravity and of gauge interactions. It is therefore the natural framework for formulating and discussing gravitational problems in which quantum effects are expected to be non-negligible or, even worse, dominant, as in the case in which a spacetime singularity is approached. No surprise, therefore, that natural arenas for string theory are the physics of quantum black holes (in particular the end-point of their evaporation) and that of the cosmological (big bang) singularity. Before discussing the possible applications of string theory, however, it seems appropriate to explain why this theory includes gravity, in a natural and compelling way, unlike in the conventional models of gauge unification.

2.1 Quantization and gravity

We should recall, to this purpose, that an extended one-dimensional object like a string corresponds, dynamically, to a constrained system. For a complete and consistent description of the string motion we must satisfy, in fact, not only the Euler-Lagrange equations, but also a set of dynamical constraints (at any point along the string trajectory), expressed by the vanishing of appropriate functionals. In addition, we have to impose the required boundary conditions, different for the case of open strings (with non-coincident ends) and closed strings (without free ends, like a loop).

When the model is quantized, the coordinates and the momentum of the string are promoted to operators satisfying the canonical commutation relations, and also the constraints are represented by operators (the so-called Virasoro operatos L_m). The Hilbert subspace containing the physical states of the system (after the elimination of all negative-norm states, called "ghosts") is then formed by (all and only) those states satisfying the constraints, that are annihilated by the application of the Virasoro operators L_m appropriately ordered and regularized (see e.g. [17, 18]). Any string models contains an infinite discrete sequence of such operators L_m, and the constraint of lowest order ($L_0 = 0$) corresponds to the mass-shell condition $p_\mu p^\mu = M^2 c^2$ determining the allowed energy levels of the string spectrum (here $p_\mu p^\mu$ is the square of the energy-momentum vector, and M the rest mass, of the string).

A quantized bosonic string, on the other hand, corresponds to a system of infinite harmonic oscillators vibrating along the spatial directions orthogonal to the string itself. The various states of the discrete string spectrum can be obtained (as in the theory of the elementary quantum oscillator) by applying the appropriate number of creation operators (along different spatial directions, in general) to the lowest energy level. As a consequence, the physical string states can be ordered as a tower of states of growing (discrete) mass and angular-momentum eigenvalues. And here we find the "miracle" connecting strings to gravitational interactions.

Looking at the subset of massless eigenstates, in fact, we find that the open string spectrum contains (even in the simplest case) a vector A_μ which is transverse (i.e., it has a vanishing divergence, $\partial^\mu A_\mu = 0$), and which can be associated with an Abelian interaction of vector type, like the electromagnetic interaction. The closed string spectrum, instead, contains – besides a scalar ϕ, the dilaton, and a second-rank antisymmetric tensor $B_{\mu\nu}$, – a symmetric tensor field $h_{\mu\nu}$ which is transverse and traceless ($\partial^\nu h_{\mu\nu} = 0 = h_\mu{}^\mu$), and which has all the required physical properties of the graviton. Thus, unified models of fundamental interactions based on strings must *necessarily* include a tensor interaction of gravitational type.

2.2 Supersymmetry

But the virtues of string theory as basis for a unified model of all interactions are not limited to this result. We should recall, indeed, that the spectrum of the bosonic string that we have considered contains, even after the elimination of the ghost states, other states of "tachyonic" type (i.e., with $M^2 < 0$). To avoid such states (sources of instabilities in a quantum theory context) the model of bosonic string has to be "supersymmetrized". The standard procedure is to associate to the coordinates X^μ, determining the position of the string in the external "target" space in which the string is embedded, the fermionic partners ψ^μ, transforming as two-component Majorana spinors on the (two-dimensional) world-sheet spanned by the string motion, and as Lorentz vectors (with index μ) in the target space manifold. This leads us to the so-called superstring models, which can be consistently formulated (from a quantum point of view) only in five different versions [17, 18].

This generalization of the string model not only eliminates tachyons from the physical spectrum – in a supersymmetric theory, the lowest allowed energy level corresponds to $M^2 = 0$, and negative values are forbidden – but also automatically introduces in the model the spinor fields required to describe the fundamental matter fields (quarks and leptons). In addition, if we consider a model with open and closed superstrings, or a model of heterotic superstrings (i.e., closed strings in which only right-moving modes are supersymmetrized, while left-moving modes are not), we can directly include into the quantum spectrum also non-Abelian gauge fields, with gauge group $SO(32)$ or $E_8 \times E_8$ [17, 18]. These superstring models, when quantized in a flat space-time manifold, are only consistent for a critical number $D = 10$ of dimensions. Remarkably, when $D - 4 = 6$ spatial dimensions are appropriately compactified, the above gauge groups could possibly contain a realistic (low-energy) standard-model phenomenology.

2.3 Conformal invariance

But the most impressive aspect of string theory, for a physicist used to work with the standard (classical or quantum) field theory, is probably the ability

of fixing in a complete and unique way the equations of motion of all fields (bosonic and fermionic, massless and massive) present in the spectrum. This means, in other words, that if we accept superstrings as realistic models of all existing fundamental interactions, these models not only tell us that in Nature must exist, for instance, gravitational fields, Abelian and non-Abelian gauge fields, but also give us *the equations satisfied by these fields* – and, to lowest order, these equations miraculously reduce to the Einstein equations for the gravitational field, and to the Maxwell and Yang-Mills equations for the gauge fields.

This property of string theory probably represents the most revolutionary aspect with respect to theoretical models based on the notion of elementary particle: the motion of a point-like test body, even if quantized, does not impose in fact any restriction on the external fields in which the body is embedded and with which it interacts. Such background fields can satisfy arbitrarily prescribed equations of motion, usually chosen on the grounds of phenomenological indications: we can think, for instance, to the Maxwell equations, empirically constructed from the laws of Gauss, Lenz, Faraday and Ampere. It would be possible, in principle, to formulate different sets of equations still preserving the Lorentz covariance and other symmetry properties (such as the $U(1)$ gauge symmetry) typical of the electromagnetic interactions. Such different equations would be possibly discarded only for their disagreement with experimental results. In a string theory context, on the contrary, such alternative equations must be discarded *a priori*, as they would be inconsistent with the quantization of a charged string interacting with an external electromagnetic field.

The above property of string-theory is grounded on the fact that the string action functional, representing the area of the two-dimensional world-sheet surface spanned by the string motion, is classically invariant under conformal transformations, (i.e., local deformations of the two-dimensional world-sheet metric γ_{ab}, $a, b = 0, 1$). Thanks to this invariance we can always introduce the so-called "conformal gauge" (where the intrinsic metric of the world sheet is the flat Minkowski metric, $\gamma_{ab} = \eta_{ab}$), and we can always eliminate the "longitudinal" string modes, leaving as a complete set of independent degrees of freedom only the modes describing oscillations transverse to the string. The conformal invariance plays thus a crucial role in the process of determining the correct quantum spectrum of physical string states.

When we have a test string interacting with any one of the fields present in its spectrum (for instance the dilaton field, or the gravitational field, or a gauge field if the string is charged), we must then require, for consistency, that the conformal invariance (determining the string spectrum) be preserved by the given interaction, not only at the classical but also at the quantum level. This means, in other words, that the quantization of a string model including background interactions must avoid the presence of "conformal anomalies", i.e. of quantum violations of the conformal invariance already present at the classical level. It follows that the only background-field configurations admis-

sible in a string theory context are those satisfying the conditions of conformal invariance. Such conditions are represented by a set of differential equations corresponding, in every respect, to the equations of motion of the field we are considering.

Unfortunately, however, such equations can hardly be derived in closed and exact form for any given model of interacting string. In practice, we have to adopt a perturbative approach: the action of the string interacting with the background fields (also called "sigma model" action) is quantized by expanding in loops the quantum corrections (as in the standard field-theory context, but with the difference that we are dealing with a two-dimensional conformal field theory). The absence of conformal anomalies is then imposed at any order of the loop expansion, determining the corresponding differential conditions. As a result, the exact background field equations are approximated by an infinite series of equations, containing higher and higher derivative terms as we consider loops of higher and higher order.

To lowest order we then recover the second-order differential equations already well known for the classical fields (the Maxwell, Einstein, Yang-Mills equations, but also the Dirac equations for the fermion fields). To higher orders there are quantum corrections to these equations, in the form of higher derivatives of the fields, appearing as an expansion in powers of the parameter λ_s^2. The corrections to the equations of motion of order λ_s^2 and higher are a typical effect of the theory due to the finite extension of strings: indeed, such corrections disappear in the point-particle limit $\lambda_s \to 0$, while they become important in the strong field/higher curvature limit in which the length scale of a typical process (for instance, the space-time curvature scale) becomes comparable with the string scale λ_s. In the context of pre-big bang cosmology such corrections can play an important role in the transition to the phase of standard decelerated evolution, as we shall see in Sect. 4.

2.4 Gravitational and cosmological applications

Let us now come back to one of the main application of string theory in a gravitational context, concerning the problem of spacetime singularities. Unfortunately, technical difficulties have prevented, so far, to reach clear-cut conclusions on the fate of the cosmological singularity (unlike other kinds of singularities). By contrast, considerable progress was made on quantum black hole physics, in particular a microscopic (statistical mechanics) understanding of their Bekenstein-Hawking entropy was achieved for a particular class of black holes [19]. Also, the study of (gedanken) superplanckian string collisions [20] has led to the conclusion that, in string theory, black holes have a minimal size corresponding to a maximal Hawking temperature of the order of string theory's maximal temperature [21, 22]. An obvious interpretation of this result stems from the simple observation that fundamental quantum strings, unlike their classical analogs, have a minimal optimal size, λ_s. This physical argument strongly suggests a cosmological analogue of the above conclusion. Because of

their finite size quantum strings cannot occupy a vanishing volume, suggesting an upper limit to the energy density and the spacetime curvature. If so, string theory should be able to avoid – or reinterpret – the big bang singularity predicted by classical general relativity.

Other properties of quantum strings that could play an important role in a cosmological context are:

1) Some new symmetries characterizing string theory as opposed to (classical or quantum) field theory.
2) The necessity of new dimensions of space.
3) The existence of new light fields, at least within perturbation theory.

Let us comment briefly on each one of them in turn.

1. An example of a genuine stringy symmetry is the so-called target-space duality (or T-duality) [23]. It stems from quantum mechanics since it is due to the possibility of interchanging the "winding" modes of a closed string (winding number being classically an integer, counting the number of times a string is wrapped around a compact spatial dimension) with its momentum modes (which in a compact direction are discrete, thanks to quantum mechanics). A cosmological variant of T-duality is heavily used to motivate an example of pre-big bang cosmology, as we shall see in Sect. 4.

2. If we wish our string theory to allow for smooth, Minkowski-like solutions, we have to allow for more than three dimensions of space. At weak coupling six extra dimensions are needed, while at strong coupling the dilaton field (that controls the coupling) can be reinterpreted as a seventh extra dimension of space (for a total of $D = 11$ spacetime dimensions). Although one would like to have all the extra dimensions to be small and static during the recent history of our Universe, it is all but inconceivable that extra dimensions may have played some role in the very early (e.g. pre-big bang) Universe. An example of making good use of them in order to produce a realistic spectrum of density perturbations will be given in Sect. 4

3. String theory does *not* automatically give general relativity at large scales. Besides the presence of the extra dimensions, string theory also contains, at least in perturbation theory (i.e. at weak coupling), many massless scalar particles in the spectrum that may induce unobserved long range forces of gravitational (or even stronger) strength. These particles, and their interactions, are described by the low-energy string effective action which, as already stressed, controls the predictive power of the theory at the experimentally accessible energy scales (at least if the string and Planck scale are of comparable magnitude).

One of these particles is the above-mentioned dilaton. Its massless nature reflects the fact that, in perturbation theory, the so-called string coupling g_s is a free parameter (see below for a short explanation of the connection between the string coupling and the dilaton). Other (perturbatively) massless particles are those controlling the sizes and shapes of the extra dimensions. All these unwanted massless particles are generically called "moduli", and the

problem associated with them is called the "moduli stabilization problem". It is generally felt that the mechanism of spontaneous supersymmetry breaking, when correctly implemented, will produce a non-perturbative mass for the moduli and thus stabilize their value. If, however, as suggested by T-duality, the universe started at very weak coupling, the moduli may have been all but static in the early Universe, and may have affected its evolution in an essential way. Consequences of such an evolution should be observable even today, as we shall discuss in Sect. 4.

2.5 Topological expansion

It seems finally appropriate, also in view of the subsequent cosmological applications, to explain the close connection between the string coupling parameter g_s and the dilaton. Let us recall, to this purpose, that the higher-derivative expansion in powers of λ_s^2, introduced previously, is not the only perturbative approximation conceivable in a string theory context. Another (very useful) expansion of the equations of motion of the background fields concerns the topology of the two-dimensional world-sheet surface spanned by the string: it is the so-called "higher genus" expansion.

Consider for instance a closed string, whose propagation in the target-space manifold describes a cylindrical world-sheet surface. If the string splits into two strings, which subsequently recombine to form again the initial string (with a process analogous to the one described by a one-loop Feynman graph in quantum field theory), the world-sheet will acquire the topology of a torus (see Fig. 1). A process with n loops will correspond, in general, to a two-dimensional Riemann surface of genus n, i.e. to a manifold with n "handles". The interactions among strings can then be described, within a perturbative approach, by a partition function which can be expanded in world-sheet configurations of higher and higher genus.

Fig. 1. One-loop graph for a point particle (left) and a closed string (right). In the left picture the world-line of a physical point particles (solid curve) splits into a "world-loop" (dashed curve) representing a virtual particle-antiparticle pair generated by the quantum fluctuations of the vacuum. In the right picture, where the world-lines are replaced by cylindrical "world-sheet" surfaces, the same process is illustrated for the case of a closed string.

For a two-dimensional, closed and orientable manifold, on the other hand, the genus n is determined by the so-called Euler characteristic $\chi = 2 - 2n$. This quantity is a topological invariant which (by virtue of the Gauss-Bonnet theorem) can be expressed as an integral over the intrinsic scalar curvature of the world-sheet surface. The dilaton, by definition, appears in the action S as multiplying such a scalar curvature; hence, for a constant dilaton, $S = (\phi/2)\chi$. The partition function, on the other hand, contains $\exp(-S)$. An expansion of the partition function in a series of higher-genus world sheets thus becomes an expansion in powers *of the exponential of the dilaton field*, $\exp\phi$. But, by definition, the loop approximation is an expansion in powers of the coupling constant g_s^2: this gives the (perturbative) relation between a constant dilaton and the string coupling parameter, $g_s^2 = \exp\phi$. Even if ϕ is not a constant, $\exp\phi$ still plays the role of a *local* effective coupling.

3 String cosmology, inflation and branes

As discussed in the previous section, superstring theory seems to provide complete and consistent models for all the fundamental components of matter and all interactions, valid at all energy scales: thus, it can be appropriately applied to describe the evolution of the primordial Universe in the regime in which all interactions were unified and possible quantized (gravity included). We can then investigate, in this context, the possible (spontaneous?) occurrence of a phase of accelerated inflationary evolution, able to provide the correct initial conditions for the subsequent, standard regime. We can ask, in particular, whether string theory has a natural candidate for the role of the inflaton, and for driving a successful phase of slow-roll inflation, without inventing *ad hoc* a new field characterized by the required properties.

3.1 Inflation

The answer to the above questions would seem to depend on the particular type of superstring model chosen to describe the primordial Universe: there are indeed five different models (see e.g. [17, 18]), related by duality transformations, which probably describe different physical regimes of the same underlying theory (the so-called M-theory), and which are characterized by a significantly different field content. All these models, however, contain a fundamental scalar field, the dilaton, which is coupled to gravity always in the same way, and which leads to an effective model of scalar-tensor gravitational interactions. This scalar field is expected to acquire a non-perturbative potential in the strong coupling regime $g_s^2 \sim 1$: any superstring model thus automatically contains, in principle, all the required ingredients for the formulation of an inflationary scenario based on the dynamics of a self-interacting scalar field.

Unfortunately, a typical string theory potential (without contrived and/or *ad hoc* modifications) seems to be unsuitable for implementing a successful model of slow-roll inflation, as shown by various studies at the beginning of the Nineties [24, 25]. We are thus left with two possible alternatives:

i) look for an inflationary scenario based on the dilaton, different from the conventional slow-roll inflation; or
ii) look for another mechanism, independent of the dilaton, and able to implement a phase of slow-roll inflation with some other typical ingredient of string theory.

The first approach (also the first to be investigated, from a historical point of view) has lead to the pre-big bang scenario that we shall present in the following section. The second approach, which only recently seems to have shown positive results, has lead to scenarios where the inflaton field corresponds to the distance between two three-dimensional membranes (called 3-branes), propagating through a higher-dimensional space-time manifold.

This section is devoted to a brief illustration of this second possibility, considering in particular the interaction between a brane and an anti-brane, which attract each other because of their opposite "charges" [26] (see [27] for a detailed introduction). It should be mentioned, however, that other examples of brane-cosmology exist, in which the two 3-branes are "domain walls" representing the space-time boundaries: we are referring to the so-called "ekpyrotic" [28, 29] or "new ekpyrotic" [30] scenario, where the brane interaction (and their eventual collision) is associated to the shrinking of an extra spatial dimension orthogonal to the branes. For that scenario we refer to the presentation of Paul Steinhardt and Neil Turok in this volume. One should also mention the so-called models of "multi-brane" inflation [31], where several M-theory branes combine their steep interactions to produce an acceptable (i.e., flat enough) inflaton potential as in models of "assisted" inflation [32].

3.2 Dirichlet branes

We should explain, first of all, why higher-dimensional extended objects like branes naturally (and unavoidably, in some cases) appear in the context of strings and superstring models. We recall, to this purpose, that there are two types of strings (open and closed), and that when studying the propagation of an open string in a (higher-dimensional) space-time manifold we have to specify what happens to the ends of the string, imposing appropriate boundary conditions [17, 18]. There are two types of such conditions.

- *Neumann* boundary conditions, in which the ends of the string move in such a way that there is no momentum flowing through the boundaries;
- *Dirichlet* boundary conditions, in which the ends of the string are kept fixed.

In particular, if the string is propagating through a background manifold with D spacetime dimensions, then the position of the ends of the string, with coordinates X_1^μ and X_2^μ, with $\mu = 0, 1, \ldots, D-1$, can be determined so as to satisfy Neumann conditions along $p+1$ space-time dimensions, and Dirichlet conditions along the remaining $(D-p-1)$ (spacelike) orthogonal directions. In this way the ends of an open string are localized on two p-dimensional hyperplanes at fixed positions X_1^i, X_2^i, with $i \in [p+1, D-1]$ (the two hyperplanes can also be coincident, i.e. $X_1^i = X_2^i$). Such p-dimensional hyperplanes are called p-dimensional Dirichlet membranes, or D_p-branes.

It is important to stress that the ends of the open string are fixed along the Dirichlet directions, but can move freely along the (orthogonal) $p+1$ Neumann directions, spanning the "world-hypervolume" of the brane. The string ends, on the other hand, can bring charges, sources of non-Abelian gauge fields: this gives us a natural implementation of the so-called "braneworld" scenario [33], in which the fundamental gauge interactions are strictly localized on a $(p+1)$-dimensional hypersurface which is only a "slice" of the higher-dimensional "bulk" manifold in which the D_p-brane is embedded. A D_3-brane, in particular, could represent a simple model of our four-dimensional Universe, able to explain the "invisibility" – i.e. the inaccessibility through fundamental gauge interactions – of the extra $(D-4)$ spatial dimensions. Such dimensions are necessarily required by a consistent theory of quantum strings, but cannot be probed by interactions confined to live in four dimensions.

In the context of superstring models, on the other hand, an extended object like a D_p-brane acts as a source of a totally antisymmetric tensor field, just as a point-like object (for instance, an electric charge) is a source for the vector field A_μ in the conventional electromagnetic theory. The field A_μ (represented by a 1-form in the language of differential exterior forms) is coupled in fact to the world-line $X^\mu(\xi^0)$ spanned by the point-like source, with an interaction proportional to $\dot{X}^\mu A_\mu$, where $\dot{X} = \partial X/\partial\xi^0$. In the same way, a one-dimensional object is a source for an antisymmetric field (or 2-form) $A_{\mu\nu} = -A_{\nu\mu}$, which is coupled to the two-dimensional "world-sheet" $X^\mu(\xi^0, \xi 1)$ spanned by its evolution: the interaction is proportional to $\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu A_{\mu\nu}$, where $\partial \equiv \partial/\partial\xi^a$, $a = 0, 1$, and ϵ^{ab} is the Levi-Civita antisymmetric symbol in two space-time dimensions. In general, the $(p+1)$-dimensional hypersurface $X^\mu(\xi^a)$, $a = 0, \ldots, p$, spanned by the time-evolution of a D_p-brane, is coupled to a $(p+1)$-form (i.e. to an antisymmetric tensor $A_{\mu\nu\alpha\ldots}$ of rank $p+1$), with interaction

$$\epsilon^{abc\cdots}\left(\partial_a X^\mu \partial_b X^\nu \partial_c X^\alpha \cdots\right) A_{\mu\nu\alpha\cdots} \quad . \tag{2}$$

Thanks to these couplings, extended objects like D_p-branes can interact among themselves not only gravitationally, but also through the field $A_{\mu\nu\alpha\ldots}$. D_3-branes, in particular, interact with the four-form $A_{\mu\nu\alpha\beta}$ present in the low-energy spectrum of the type IIB superstring model [17, 18]. Such an interaction, as in the case of point-like electromagnetic sources, is repulsive for sources of the same sign (for instance, two identical particles or two identical

branes), and attractive for sources of opposite sign (for instance, a particle-antiparticle system, or a brane-antibrane system). In particular, for a system of two identical, static, parallel and supersymmetric branes, initially arranged in the so-called BPS configuration, one finds that the gravitational attraction is exactly balanced by the repulsion due to the antisymmetric-field interaction, and the system remains static (modulo non-perturbative corrections).

3.3 Brane-antibrane inflation

An inflationary model can be obtained, in that context, if we consider instead the interaction of a D_3-brane with an anti-D_3-brane: in that case there is no cancellation between the various types of forces, and the net result is an attractive interaction [26] whose non-constant part is associated with the potential energy $V(Y) \sim Y^{2-n}$, where Y is the modulus field controlling the interbrane distance, and $n = D - p - 1$ is the number of spacelike dimensions orthogonal to the branes. Thus, $V = \beta Y^{-4}$ for $p = 3$ branes embedded in the $D = 10$ target space of superstring theory (β is a dimensionless coefficient of order one in natural units).

After an appropriate dimensional reduction (based on the assumption that the n "orthogonal" dimensions are stabilized in a compact configuration) one arrives, in this way, to a four-dimensional effective action for the scalar modulus Y. This modulus behaves like a canonical scalar field, self-interacting, minimally coupled to gravity, and is thus a possible candidate to sustain a phase of inflationary expansion of the dimensionally-reduced bulk manifold. Unfortunately, however, the effective potential generated by the brane-antibrane interaction is not flat enough to satisfy the requirements of slow-roll inflation – at least if the branes interact through flat and topologically trivial orthogonal dimensions [27].

Two ways out of this difficulty have been suggested in the present literature. A first possibility relies on the assumption that the compact transverse manifold has the topology of a n-dimensional torus, with spatial sections of uniform radius r [26]. When the separation of the brane-antibrane pair is of order $Y \sim r$, the effective potential experienced, say, by the antibrane must be estimated including the contribution of all the topological "images" of the other brane, forming an n-dimensional lattice. The total effective interaction is then obtained by summing over all the contributions of the lattice sites occupied by the brane images. The resulting effective potential is quartic in the displacement of the antibrane from the centre of the hypercubic cell of the lattice, and can satisfy the slow-roll conditions – at least, as long as the interbrane separation remains in the range $Y \sim r$.

A second possibility (which seems to be preferred, at present, in view of a possible stabilization of the extra compact dimensions) is based on the assumption that the section of space orthogonal to the D_3-branes is not flat, but "warped" by a geometry of the anti-de Sitter (AdS) type (with constant curvature scale $R \gg \lambda_s$), representing a solution of the low-energy effective

action of type IIB superstring theory [34]. One can also assume, for simplicity, that the antibrane $\overline{D}_3$ is frozen at the fixed position $y = y_0 \ll R$, located near the infrared end of a five-dimensional bulk manifold AdS_5 (the remaining five spatial dimensions are assumed to be compactified at a constant scale). The D_3-brane, on the contrary, is mobile along the orthogonal direction y, driven by the attractive force towards the antibrane, and has a time-dependent position $y = y_1 \gg y_0$. The coordinate distance of the two branes is then $Y = y_1 - y_0 \simeq y_1$. The Y-dependent part of the interbrane potential is given, again, by the inverse of the distance modulus to the fourth power, but – for reasons of general covariance – the interaction has to expressed in terms of the proper distance of the two branes, given by $y_1\sqrt{|g_{yy}|} = y_1(R/y)$, where (R/y) is the "scale factor" of the AdS_5 geometry along the fifth dimension y orthogonal to the brane. The non-constant part of the potential experienced by the $\overline{D}_3$-brane at $y = y_0$ is then $V \sim (y_0/R)^4 Y^{-4}$, and the slow-roll conditions can now be easily satisfied thanks to the suppression factor $y_0/R \ll 1$ generated by the warped geometry of the bulk manifold.

3.4 The strong coupling regime

All models of brane-antibrane inflation – either based on a non-trivial topology or on a non-trivial geometry of the transverse dimensions – are consistent provided that shape and volume of the extra (non-inflationary) spatial dimensions are stabilized, using some appropriate mechanism [35] which does not affect the relative motion of the two branes. The stabilization mechanism, in its turn, seems to require the presence of antisymmetric tensor fields, whose "fluxes" are associated to branes "wrapping" around around the compact dimensions and "warping" the compactified geometry [36]. This enforces the conclusion that the above realizations of slow-roll inflation are only possible when the Universe is in a phase of "brane-domination".

On the other hand, in a string theory context, the tension of a D-brane (i.e. the mass per unit of spatial (hyper)volume) is proportional to the inverse of the string coupling parameter g_s: it follows that such branes become light, and can be copiously produced, only in the strong coupling limit $g_s \gg 1$. A phase of brane-antibrane inflation seems thus to be naturally (and unavoidably) associated to a phase of strong coupling.

We are then lead to the question: how to "prepare" the strong-coupling (and, possibly high-curvature) regime where the mechanism of brane inflation can be implemented? In other words, what happens "before"? what about the cosmological evolution before the beginning of the inflationary regime? We know, in fact, that slow-roll inflation cannot be past-eternal: going back in time, at fixed value of the gravitational coupling, a phase of slow-roll inflation necessarily leads to a spacetime (big bang) singularity in a finite amount of proper time, and the model of cosmological evolution remains incomplete. Is it possible to arrive at the brane-dominated regime, where the Universe

possibly undergoes a phase of slow-roll inflation, without starting from an initial singularity?

The pre-big bang scenario, introduced in the following section, may provide answers to these questions.

4 The pre-big bang scenario

If we look at the "specialized" literature for a description of the birth and of the first moments of life of our Universe, we may read, in the (probably) most ancient and authoritative book, that

> *"In the beginning God created the Heaven and the Earth,*
> *and the Earth was without form, and void;*
> *and the darkness was upon the face of the deep.*
> *And the Breath of God*
> *moved upon the face of the water."*
>
> (Genesis, The Holy Bible).

The most impressive aspect of these verses, for a modern cosmologist, is probably the total absence of any reference to the hot, kinetic, explosive state that one could expect at (or immediately after) the "big bang" deflagration. What is described, instead, is a somewhat quiet, dark, empty state: indeed, we can read about "void", "darkness", and "the deep" gives us the idea of something enormously desert and empty. In this static configuration there is at most some small fluctuation (the "Breath", inducing a ripple on the surface of this vacuum).

It is amusing to note that a state of this type, flat, cold and vacuum, only ruffled by quantum fluctuations, can be obtained as the initial state of our Universe in a string cosmology context, under the hypothesis that the Universe evolves in a "self-dual" way with respect to a particular symmetry of the low-energy string effective action [15, 16].

4.1 Scale-factor duality

The symmetry we have in mind is the so-called scale-factor duality [16, 37], which generalizes to time-dependent backgrounds the T-duality symmetry mentioned in Sect. 2, typical of closed strings in static manifolds with compact spatial dimensions [23]. According to the T-duality symmetry the energy spectrum of a string, quantized in the presence of compact dimensions of radius R, turns out to be invariant with respect to the transformation $R \to \lambda_s^2/R$, and the simultaneous exchange of the number of times the string is wounded around the compact dimensions with the (discrete) level-number of the momentum operator associated with the string motion along such compact directions.

In the context of a homogenous, isotropic, spatially flat background, described by the invariant space-time interval

$$ds^2 = c^2 dt^2 - a^2(t)\, dx_i dx^i, \tag{3}$$

the radius R is replaced by the scale factor $a(t)$, and the invariance under the transformation $a \to a^{-1}$ is still valid provided the dilaton field ϕ is also simultaneously transformed, according to the (scale-factor duality) transformation

$$a \to \widetilde{a} = a^{-1}, \qquad \phi \to \widetilde{\phi} = \phi - 2d \ln a, \tag{4}$$

where d is the total number of spatial dimensions. There are, however, two important differences between T-duality and scale-factor duality symmetry: *i*) the above transformations represent a symmetry of the (tree-level) equations of motion of the background fields (not necessarily of the quantum string spectrum); *ii*) there is no need of compact spatial dimensions.

The invariance of the string cosmology equations under the transformations (4) can be extended to include the presence of matter sources, as we shall see below. Such an invariance is only a particular case of a more general class of symmetries associated to the global transformations of the $O(d,d)$ group, for backgrounds with Abelian isometries, both without [38, 39] and with [40] matter sources. For the purpose of this paper, however, it will be enough to limit ourselves to the transformations of Eq. (4), and to note that new classes of solutions can be obtained simply by transforming known solutions. These new solutions have kinematic properties representing the "dual counterpart" of the initial ones.

For an explicit illustration of this duality symmetry, and of its applications, we need the cosmological equations obtained from the low-energy string effective action, describing the dynamics of the graviton and dilaton field to the lowest non-perturbative order. For the metric (3), and for a homogeneous dilaton field $\phi = \phi(t)$, in $d = 3$ spatial dimensions, such equations are (see e.g. [41]):

$$\begin{aligned}
&\dot{\phi}^2 - 6H\dot{\phi} + 6H^2 = \frac{2\lambda_{\rm s}^2 c}{\hbar} e^{\phi} \rho, \\
&\dot{H} - H\dot{\phi} + 3H^2 = \frac{\lambda_{\rm s}^2 c}{\hbar} e^{\phi} p, \\
&2\ddot{\phi} - \dot{\phi}^2 + 6H\dot{\phi} - 6\dot{H} - 12H^2 = 0.
\end{aligned} \tag{5}$$

We are working in the so-called string frame (where the motion of a test string is geodesics), and the dot denotes differentiation with respect to the cosmic time t (we have defined $H = \dot{a}/a$). We have also taken into account the possibility of matter sources, in the form of a perfect fluid with stress tensor $T_0{}^0 = \rho$ and $T_i{}^j = -p\delta_i^j$. Note that the combination $(2\lambda_{\rm s}^2 c/\hbar)\exp\phi$ plays the role of the effective gravitational coupling constant $16\pi G/c^2 = 2\lambda_{\rm P}^2 c/\hbar$. Thus, the dilaton controls the relative string-to-Planck length ratio, to this approximation, as

$$\left(\frac{\lambda_{\rm P}}{\lambda_{\rm s}}\right)^2 = \left(\frac{M_{\rm s}}{M_{\rm P}}\right)^2 = e^{\phi} = g_s^2. \tag{6}$$

The above set of cosmological equations is clearly invariant (as in the case of the Einstein equations) under the time-reflection transformation $t \to -t$. In addition, however, the equations are invariant under the duality transformations (4), possibly accompanied by the following transformations of the matter sources [16]:

$$\rho \to \widetilde{\rho} = \rho\, a^6, \qquad p \to \widetilde{p} = -p\, a^6. \tag{7}$$

Thanks to this symmetry we can always associate to any decelerated solution, typical of the standard cosmological model, a "dual partner" describing accelerated (i.e. inflationary) evolution. We stress that this is impossible in the context of the cosmological Einstein equations, where there is no dilaton, and no duality symmetry.

Consider, for instance, a radiation-dominated Universe, characterized by the equation of state $p = \rho/3$, and by a constant dilaton, $\dot{\phi} = 0$. In this case the set of Eqs. (5) is identically satisfied by the particular exact solution

$$a = \left(\frac{t}{t_0}\right)^{1/2}, \qquad \phi = \text{const}, \qquad \rho = \frac{\rho_0}{a^4}, \qquad p = \frac{\rho}{3}, \qquad t > 0, \tag{8}$$

where t_0 and ρ_0 are positive integration constants. This is also the exact solution of the conventional Einstein equations which, for $t > 0$, properly describes the radiation-dominated phase of the standard cosmological scenario [1]. This solution describes a phase of decelerated expansion, decreasing curvature and constant dilaton for $0 < t \leq \infty$,

$$\dot{a} > 0, \qquad \ddot{a} < 0, \qquad \dot{H} < 0, \qquad \dot{\phi} = 0, \tag{9}$$

namely a Universe evolving from the past big bang singularity (approached as $t \to 0$ from positive values) towards a state asymptotically flat and empty, approached as $t \to +\infty$.

By applying a combined scale-factor duality and time-reversal transformation on the solution (8) we can now obtain a new exact solution of the low-energy string cosmology equations (5) which has the form

$$\widetilde{a} = \left(-\frac{t}{t_0}\right)^{-1/2}, \quad \widetilde{\phi} = -3\ln\left(-\frac{t}{t_0}\right), \quad \widetilde{\rho} = \frac{\rho_0}{\widetilde{a}^2}, \quad \widetilde{p} = -\frac{\widetilde{\rho}}{3}, \quad t < 0. \tag{10}$$

This new solution is defined on the negative time semi-axis, $-\infty \leq t < 0$, and describes an (inflationary) phase of accelerated expansion, growing curvature and growing dilaton,

$$\dot{a} > 0, \qquad \ddot{a} > 0, \qquad \dot{H} > 0, \qquad \dot{\phi} > 0. \tag{11}$$

The initial cosmological state is asymptotically flat, $H \to 0$, with esponentially suppressed string coupling, $g_s^2 = \exp\phi \to 0$, as $t \to -\infty$ (the Universe approaches, asymptotically, the so-called "string perturbative vacuum"). Starting from this configuration, the evolution leads towards a future singularity approached as $t \to 0$ from negative values.

4.2 Examples of smooth pre-big bang backgrounds

The two exact solutions (8) and (10) cannot be directly connected to one another, being physically separated by a curvature singularity at $t = 0$. Assuming however that the singularity may disappear in a more realistic model, after including other fields and/or higher-order corrections, we obtain from the duality invariance of the equations a clear suggestion about a possible "temporal completion" of the standard cosmological phase, based on a "self-duality principle".

Indeed, if the scale factor would satisfy (even approximately) the condition $a(t) = a^{-1}(-t)$, then the standard cosmological phase would be preceded and completed by a dual phase which is automatically of the inflationary type (see for instance Eq. (10)). The space-time curvature scale (growing during inflation, decreasing during the decelerated evolution) would have a specular behavior with respect to a critical "bouncing epoch" centered around $t = 0$ (namely would be reflection-symmetric in the $\{t, H\}$ plane with respect to the $t = 0$ axis, marking the transition from the inflationary to the standard regime; see e.g. Fig. 2). Such a bouncing transition, occurring at high but finite curvature, would replace the big bang singularity of the low-energy equations: thus, one is naturally lead to call "pre-big bang" the initial phase (at growing curvature and string coupling), with respect to the subsequent "post-big bang" phase describing standard decelerated evolution.

Regular, self-dual solutions cannot be obtained [42] in the context of the low-energy string-cosmology equations (5). Such equations, however, only represent a zeroth-order approximation to the exact field equations expected in a string theory context, as already stressed in Sect. 2. On the other hand, if the curvature and the string coupling keep growing without any damping during the phase described by the low-energy pre-big bang solutions, then the Universe is necessarily driven to configurations where the curvature and the coupling are large in string units ($\lambda_s^2 H^2/c^2 \sim 1$, $g_s^2 \sim 1$): for those configurations both higher-derivative and higher-loop corrections are to be added (possibly to all orders) to the tree-level equations. All studies performed up to now [43, 44, 45] (see also [41]) have shown that such corrections can indeed contribute to damp the accelerated evolution of the background, and trigger the transition to the post-big bang phase.

Without introducing complicate computations we will present here a phenomenological example of self-dual solutions based on a special class of dilaton potential, depending on ϕ through the "shifted" variable $\overline{\phi} = \phi - 3\ln a$. Potential of that type are motivated by the invariance of $\overline{\phi}$ under the transforma-

tions (4) [16, 38, 39, 40]; in addition, they could be generated by dilaton loop corrections in manifold with compact spatial sections, and can be obtained in the homogeneous limit of appropriate general-covariant (but non-local) actions [46, 47]. With such potential $V = V(\overline{\phi})$ the string cosmology equations can be rewritten in terms of a, $\overline{\phi}$, $\overline{\rho} = \rho a^3$ and $\overline{p} = pa^3$ as follows [38, 39, 40]:

$$\begin{aligned}
&\dot{\overline{\phi}}^2 - 3H^2 - V(\overline{\phi}) = \frac{2\lambda_s^2 c}{\hbar} e^{\overline{\phi}}\,\overline{\rho},\\
&\dot{H} - H\dot{\overline{\phi}} = \frac{\lambda_s^2 c}{\hbar} e^{\overline{\phi}}\,\overline{p},\\
&2\ddot{\overline{\phi}} - \dot{\overline{\phi}}^2 - 3H^2 + V(\overline{\phi}) - \frac{\partial V}{\partial \overline{\phi}} = 0.
\end{aligned} \tag{12}$$

These equations are still invariant under the duality transformations (4) and (7) but, differently from Eq. (5), they admit regular and self-dual solutions. We can also obtain exact analytical integrations for appropriate forms of the potential $V(\overline{\phi})$, and for equations of state such that p/ρ can be written as integrable function of a suitable time parameter [15].

Let us consider, as a simple example, the exponential potential $V = -V_0 \exp(2\overline{\phi})$ (with $V_0 > 0$), to be regarded here only as an effective, low-energy description of the quantum-loop backreaction, possibly computable at higher orders. Let us use, in addition, an equation of state (motivated by analytical simulations concerning the equation of state of a string gas in backgrounds with rolling horizons [48]) evolving between the asymptotic values $p = -\rho/3$ at $t \to -\infty$ and $p = \rho/3$ at $t \to +\infty$, so as to match the low-energy pre-and post-big bang solutions (10) and (8), respectively. The plot of the corresponding solution (see [15] for the exact analytic form) is illustrated in Fig. 2.

The solution smoothly interpolates between the string perturbative vacuum at $t \to -\infty$ and the standard, radiation-dominated phase at constant dilaton (described by Eq. (8)) at $t \to +\infty$, after a pre-big bang phase of growing curvature and growing dilaton described by Eq. (10). The dashed curves of Fig. 2 represent the unbounded growth of H in the low-energy solutions (8), (10), obtained in the absence of the dilaton potential. The plot of V in Fig. 2 shows that the potential contribution dominates the background evolution around the transition epoch $|t| \to 0$, and rapidly disappears at large values of $|t|$.

An even simpler (almost trivial) example is obtained, with the same potential, in the case $p = 0$. The system of equations (12) is then satisfied by a flat geometry ($H = 0 = \dot{H}$) and by a "bell-like" time evolution of the dilaton, sustained by a constant energy density [49]:

$$a = a_0, \qquad \rho = \rho_0, \qquad p = 0, \qquad e^{\phi} = \frac{e^{\phi_0}}{1 + (t/t_0)^2} \tag{13}$$

(the integration constants a_0, ϕ_0, ρ_0, t_0 are related by $\rho_0 \exp\phi_0 = V_0 \exp(2\phi_0) = 4/t_0^2$). For $t \to -\infty$ one has a globally flat initial Universe, filled with a con-

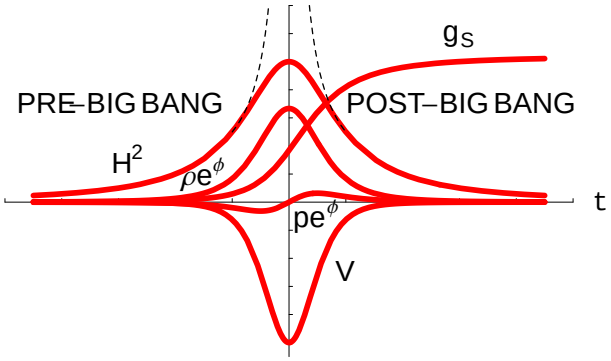


Fig. 2. Example of smooth transition between a phase of pre-big bang inflation and the standard radiation-dominated evolution. The figure gives a qualitative illustration of the evolution in time (from $-\infty$ to $+\infty$) of the string coupling parameter $g_s = \exp(\phi)$, of the spacetime curvature scale $H^2 = \dot{a}^2/a^2$, of the effective energy density $\rho \exp(\phi)$ and pressure $p \exp(\phi)$ of the gravitational sources, and of the effective dilaton potential V. Note that the pre-big bang phase is characterized by growing curvature and negative pressure, the post-big bang phase by decreasing curvature and positive pressure.

stant and uniform distribution of "dust" dark matter, very reminiscent of the "Earth without form" mentioned in the Genesis. And only the dilaton is moving "upon the face of the water" ...

The solution is less trivial, however, when transformed in the Einstein-frame (E-frame) where the graviton and the dilaton are canonically normalized, and where the metric and the energy density are no longer constant. Using the conformal time parameter η, the E-frame version of the solution (13) takes the form [47]

$$\begin{aligned} a_E &= a_0 e^{-\phi_0/2} \left(1 + \frac{\eta^2}{\eta_0^2}\right)^{1/2}, \\ e^{\phi} &= e^{\phi_0} \left(1 + \frac{\eta^2}{\eta_0^2}\right)^{-1}, \\ \rho_E &= \rho_0 a_0^3 \left(1 + \frac{\eta^2}{\eta_0^2}\right)^{-3/2}. \end{aligned} \tag{14}$$

In Fig. 3 we have plotted the exponential of the dilaton field, the E-frame scale factor a_E, the energy density ρ_E, and the Hubble parameter $H_E = a'_E/a_E^2$, obtained from Eq. (14). The background smoothly evolves from growing to decreasing curvature (and dilaton), but the pre-big bang regime corresponds, in the E-frame, to a phase of *accelerated contraction* ($H_E < 0$, $\dot{H}_E < 0$), with a final bounce of the scale factor to the phase of post-big bang (decelerated) expansion.

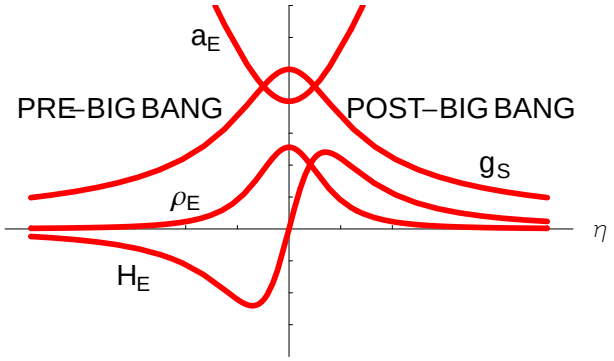


Fig. 3. Example of pre-big bang evolution represented in the E-frame, where the scale factor a_E is shrinking and the Hubble parameter H_E is negative, unlike in the string-frame representation of Fig. 2. The figure also illustrates the evolution (with respect to the conformal time parameter η) of the E-frame energy density ρ_E and of the string coupling parameter $g_s = \exp(\phi)$. The plots are obtained from Eq. (14) with $a_0 = 0.8$, $\phi_0 = 0$, $\rho_0 = 1$, $\eta_0 = 1$.

As clearly illustrated by the above examples, according to the pre-big bang scenario inflation starts in a perturbative regime of very small coupling ($g_s \to 0$), very small curvature ($H \to 0$), and very large space-time horizons cH^{-1}. The trans-Planckian problem (mentioned in Sect. 1) is evaded, and the initial cosmological evolution can be consistently described by a classical, low-energy effective action. This represents a complete overturning of the traditional picture – typical of standard models of inflation [50] – where the initial evolution takes place at ultra-high density and curvature scales, small horizon scale, strong coupling, in a marked quantum regime (see Fig. 4).

Actually, it is possible to characterize these new initial conditions in terms of a principle of "Asymptotic Past Triviality" [51] (APT) according to which the Universe becomes empty, flat, and interaction-free as one goes towards the asymptotic past, $t \to -\infty$. It was argued in [51] that, thanks to gravitational instability, "generic" initial conditions satisfying APT unavoidably lead to the formation of black holes of many different masses/sizes. It was also shown that the spacetime inside such black holes precisely mimics the pre-bounce phase of a pre-big bang cosmology. If the singularity inside the black hole horizon is avoided, a sufficiently large black hole can then give rise to a successful pre-big bang cosmology.

In conclusion, an epoch of pre-big bang inflation is able to solve the kinematical problems of the standard scenario starting from different initial conditions which are not necessarily unnatural or unlikely [51, 52] (see also [53] for a detailed comparison of the pre-big bang versus post-big bang inflationary kinematics). A possible exception concerns the presence of primordial "shear", which is not automatically inflated away during the phase of pre-big

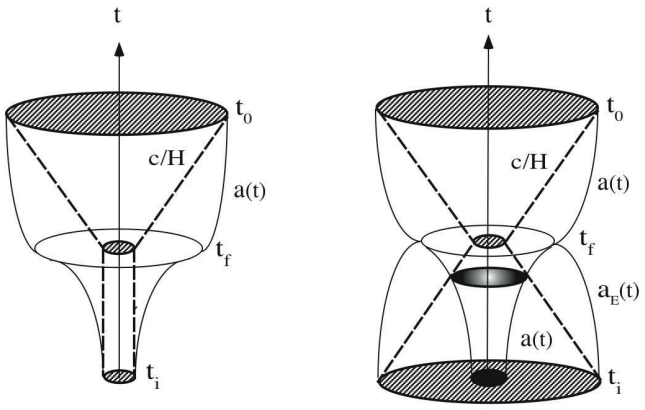


Fig. 4. Comparison between the time evolution of the Hubble horizon c/H (dashed lines) and of the scale factor $a(t)$ (solid curves) in the conventional inflationary scenario (left) and in models of pre-big bang inflation (right). For the pre-big bang phase we have plotted the evolution of both the expanding string-frame scale factor $a(t)$ and the contracting E-frame scale factor $a_E(t)$. The vertical axis is the time axis, and the shaded areas represent causally connected spatial sections of Hubble size c/H at various epochs. The evolution from the end of inflation, t_f, to the present epoch, t_0, is the same in both cases. However, during inflation (i.e. from t_i to t_f) the Hubble horizon is constant (or slightly increasing) in conventional models (left), while it is shrinking in pre-big bang models (right). As a consequence, the size of the initial inflationary patch may be very large (in string or Planck units) for a phase of pre-big bang inflation, but *not* larger than the horizon itself, as illustrated in the figure.

bang evolution: the isotropization of the three-dimensional spatial sections might require some specific post-big bang mechanism (see e.g. the discussion of [54]), unlike in the standard inflationary scenario where the dilution of shear is automatic.

4.3 Phenomenological consequences

Quantum effects, in the pre-big bang scenario, can become important towards the end of the inflationary regime. We can say, in particular, that the monotonic growth of the curvature and of the string coupling automatically "prepares" the onset of a typically "stringy" epoch at strong coupling. This epoch could be characterized by the production of a gas of heavy objects (such as winding strings [55, 56] or mini-black holes [57]) as well as light, higher-dimensional branes [58] (see [59] for a recent review of the so-called phase of "string gas" cosmology). In such a context the interaction (and/or the eventual collision) of two branes can drive a phase of slow-roll inflation [26], as discussed in Sect. 3.

At this point of the cosmological evolution there are two possible alternatives.

i) The phase of string/brane dominated inflation is long enough to dilute all effects of the preceding phase of dilaton inflation, and to give rise to an epoch of slow-roll inflation able to prepare the subsequent evolution, according to the conventional inflationary picture.

ii) The back-reaction of the quantum fluctuations, amplified by the phase of pre-big bang inflation, induces a bounce as soon as the Universe reaches the strong coupling regime; almost immediately, there is the beginning of the phase of decelerated evolution dominated by the (rapidly thermalized) radiation produced by the bouncing transition.

These two possibilities have different impact on the phenomenology of the cosmic background of gravitational radiation and on the anisotropy of the CMB radiation, as we shall now discuss.

In the first case we recover the same phenomenology of conventional models of slow-roll inflation, but with the important difference that the inflationary initial conditions are now "explained" by the preceding pre-big bang evolution, driving the Universe from the string perturbative vacuum to the appropriate configuration for starting inflation. The slow-roll singularity is eliminated and the model becomes geodesically complete, being extended over the whole cosmic-time axis. We can still talk of "birth of the Universe from the string perturbative vacuum", as pointed out (in a quantum cosmology context) in [60, 61, 62].

Here we shall concentrate our discussion on the second possibility, in which there is only one inflationary phase of unconventional (pre-big bang) type. In that case the effects of the higher-curvature/quantum gravity phase are not "washed out" by a subsequent inflationary epoch: the phenomenological predictions are possibly less robust, as may depend on (still unknown) details of string/Planck scale physics. However, the observable consequences of this scenario should bear the direct imprint of the pre-big bang epoch, thus opening a window on time scales and energy scales not accessible to conventional model of inflation.

In particular, because of the peculiar kinematic properties of pre-big bang inflation, the quantum fluctuations of the metric tensor tend to be amplified with an unconventional spectrum which is growing with frequency (instead of being flat, or decreasing, as in the conventional inflationary scenario). A growing spectrum, on the other hand, simultaneously represents an advantage and a difficulty of pre-big bang models with respect to conventional models of inflation.

Tensor metric perturbations

The (phenomenological) advantage concerns the amplification of the (transverse, traceless) tensor part of the metric fluctuations, and the corresponding

formation of a cosmic background of relic gravitational radiation. In models of pre-big bang inflation, in fact, the spectral energy of such background – $\Omega_g h^2$, where Ω_g is the energy density in critical units, and h the present value of the Hubble parameter in units of 100 km s^{-1}Mpc^{-1} – is expected to growth with frequency [63, 64, 65], with a peak value [66] of about $\Omega_g h^2 \sim 10^{-6}$, reached around the cut-off frequency of the spectrum ($\omega \sim 100$ GHz, roughly corresponding to the position of the peak of the electromagnetic CMB radiation). This behaviour is in sharp contrast with the gravitational background produced in models of slow-roll inflation, which is characterized by a peak value of order $\Omega_g h^2 \sim 10^{-10}$ at the frequency scale of the present Hubble radius ($\omega \sim 10^{-18}$ Hz), and which is monotonically decreasing with the frequency (see Fig. 5).

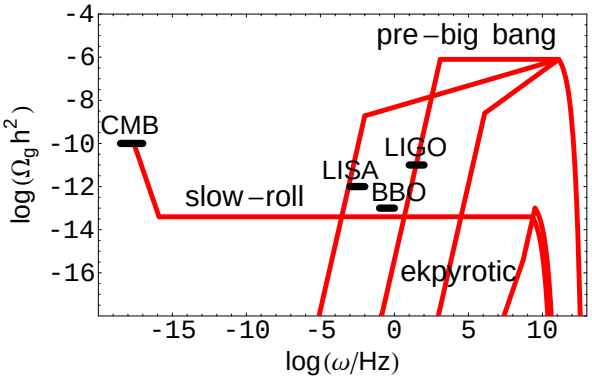


Fig. 5. The figure shows, on a logarithmic scale, the typical spectral energy density $\Omega_g h^2$ of a cosmic background of primordial gravitational waves produced in the context of (*a*) minimal models of pre-big bang inflation (top spectra), (*b*) standard models of slow-roll inflation (flat, intermediate spectrum), and (*c*) ekpyrotic/cyclic models of inflation (bottom spectrum). See [41, 68] for technical details on the properties of these spectra. The figure also shows the maximal background intensity allowed by present CMB data, and the planned sensitivities expected to be reached in the near future by Earth-based gravitational detector such as Advanced LIGO and by space interferometers such as LISA and BBO.

Thanks to these properties, the direct detection of a cosmic background of pre-big bang gravitons is expected to be much more likely and easier than the detection of backgrounds produced in other models of inflation. Indeed, the presently operating, Earth-based gravitational antennas (such as NAUTILUS, AURIGA, LIGO, VIRGO), and those planned to be operative in space (LISA, BBO, DECIGO), cover together a sensitivity band roughly ranging from the mHz to the kHz (see e.g. [41]), and in that range the expected signal of pre-big bang gravitons is possibly much higher than from other types of primordial gravitational backgrounds (see Fig. 5). Conversely, the presence of pre-big

bang gravitons is extremely suppressed at the low frequency scales which are relevant to the observed anisotropy of the CMB radiation, so that their possible contribution to CMB polarization effects [67] is totally negligible.

Thus, a combined non-observation of tensor polarization effects on the CMB radiation, and a direct detection of relic gravitons at high frequencies, could represent a strong signal in favor of the pre-big bang scenario [68]. In the cyclic/ekpyrotic scenario, on the contrary, the primordially produced background of cosmic gravitational waves is expected to be quite negligible, today, both in the low-frequency and high-frequency range [69] (see Fig. 5).

Scalar metric perturbations

The difficulty of a growing spectrum concerns, instead, the scalar part of the metric fluctuations. These fluctuations, amplified by inflation, should represent the primordial "seeds" responsible for the peak structure of the temperature anisotropies $\Delta T/T$, observed today in the CMB radiation: this requires, however, a nearly flat (or "scale-invariant") spectral distribution, as confirmed by most recent observations [6]. In models of pre-big bang inflation, on the contrary, the primordial scalar spectrum tends to be growing [70] like the tensor spectrum, in disagreement with present experimental evidence.

In the absence of a brane-dominated regime, with an associated phase of slow-roll inflation able to produce the required spectrum of scalar perturbations, this problem can be solved by the so-called "curvaton mechanism" [71, 72, 73]. Thanks to this mechanism, a scale-invariant, adiabatic spectrum of scalar curvature perturbations can also be produced by the post-big bang decay of a suitable background field (other than the metric), whose scalar fluctuations have been amplified by inflation with a flat spectrum.

In the context of the pre-big bang scenario the role of the curvaton could be played by the fundamental string theory axion σ, associated by spacetime duality to the four-dimensional components of the antisymmetric tensor $B_{\mu\nu}$ present in the string spectrum [74, 75]. The quantum fluctuations of the axion, in fact, can be amplified by a phase of pre-big bang inflation with a flat spectrum [76, 77], in sharp contrast with the quantum fluctuations of the metric tensor (see Appendix A).

Assuming that the axion fluctuations are amplified by a flat spectrum, it follows that the standard post-big bang phase, on large, super-horizon scales, is initially characterized by a sea of "isocurvature" axion fluctuations, with a negligible component of metric perturbations (which are amplified by inflation with a growing spectrum). These fluctuations may act as "seeds" (i.e. as quadratic sources to the perturbation equations) for the generation of isocurvature metric perturbations [78] (which are possibly allowed only as a sub-dominant component of the total observed anisotropy). However, if the post-big bang axion background acquires a mass and decays, the axion and metric fluctuations become linearly coupled. Thanks to this coupling new scalar (curvature) perturbations are automatically generated with the same

spectral slope as the axion one, and with a spectral amplitude of the same order as the primordial axion amplitude [72, 74, 75]. The net result of this "curvaton" mechanism is a (non-primordial) spectrum of adiabatic metric perturbations which, if sufficiently flat, may successfully reproduce the observed anisotropies of the CMB radiation.

By applying the above mechanism, and assuming such an axion origin of the large-scale anisotropy, it becomes possible to extract from present CMB measurements direct information on models of pre-big bang inflation and on their fundamental parameters, such as the inflation scale in string units [74], or the kinematics of the extra dimensions. According to recent WMAP data [6], for instance, the spectral slope of scalar perturbations (averaged over all scales) is given by

$$n_s \equiv 3 + 2\alpha = 0.951^{+0.015}_{-0.019}. \tag{15}$$

This slope slightly (but unambiguously) deviates from an exactly scale-invariant spectrum, corresponding to $n_s = 1$ (or $2 + 2\alpha = 0$). The comparison with Eq. (A.9) then gives $\beta_0 \simeq -0.355$, which implies (using Eq. (A.6)) $\sum_i \beta_i^2 \simeq 0.62$. A flat spectrum associated to a nine-dimensional isotropic background would give, instead, $\beta_0 = -1/\sqrt{9} \simeq -0.33$, and $\sum \beta_i^2 = 6/9 \simeq 0.66$. The result (15) may thus be interpreted, in $d = 9$, as the indication of a small anisotropy between the time evolution of internal and external spatial dimensions. The experimental value (15), however, is also compatible with $\beta_0 = -1/\sqrt{8} \simeq -0.3535$, associated to a model describing the isotropic expansion of only 8 spatial dimensions!

Primordial magnetic fields

We should add, finally, that the (number and) kinematics of the extra spatial dimensions, together with the dilaton kinematics, play a fundamental role also in the pre-big bang amplification of the quantum fluctuations of the electromagnetic fields, for a possible "primordial" production of large-scale magnetic fields [79, 80, 81]. Remarkably, the values of β_0 and β_i compatible with a (nearly) flat axion spectrum are also compatible with an efficient production of "seeds" for the cosmic magnetic fields, for various types of photon-dilaton coupling (as in discussed in [82] for the specific examples of the heterotic and type I superstring models).

5 Conclusion

We wish to conclude this short review with a comment concerning a possible "cyclic" extension of the pre-big bang scenario (see also [68]). Such an extension is in principle possible because, as in the ekpyrotic case [83, 84], the big bang is not regarded here as a singular and unique event marking the beginning of everything, but only as a transition (even if dramatic, in

various respects) between two different (possibly duality-related) cosmological regimes. The big bang is only one among many other (high-energy, but smooth) processes of our past cosmological history, induced by the gravi-dilaton dynamics, and repeatable, in principle, whenever appropriate initial conditions are given.

A cyclic extension of the pre-big bang scenario might correspond, in particular, to a cyclic alternation of the phase of standard evolution and of its string-theory dual. However, as the dilaton is monotonically growing during the pre-big bang evolution, a cyclic scenario would require – to close the cycle, and prepare the system to new big bang transition – a late-time phase of decreasing dilaton, in order to lead back the system to the perturbative regime at the beginning of a new, self-dual cycle (see e.g. Fig. 3).

Such a bouncing back of the dilaton in the post-big bang regime could require the dominant contribution of a suitable (duality breaking? non-perturbative?) potential. We thus naturally arrive at the question: could such a process of dilaton bouncing be associated to the phase of (low-energy) cosmic acceleration that we are presently experiencing?

Appendix A. The axion fluctuation spectrum

The quantum fluctuations of the axion σ are described by the canonical variable $v = \sigma z$, whose Fourier components v_k satisfy the equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0. \tag{A.1}$$

The prime denotes differentiation with respect to the conformal time η, and the "pump field" $z(\eta)$ is the external field responsible for the cosmological amplification, which in our case depends on the metric and on the dilaton. In an inflationary context we can assume for z a power-like behavior in the negative range of values of the conformal time parameter, i.e. $z = (M_{\rm P}/\sqrt{2})(-\eta/\eta_1)^\alpha$, with $\alpha \leq 1/2$ (the factor $M_{\rm P}/\sqrt{2}$ is due to the canonical normalization). In this case, considering the asymptotic regime "outside the horizon" where the kinetic term k^2 becomes negligible, we can find for Eq. (A.1) the approximate solution

$$v_k \simeq \frac{z}{z_{hc}}\,(v_k)_{hc}, \tag{A.2}$$

where the subscript "hc" means that the quantity is to be evaluated at the horizon crossing epoch $|k\eta| = 1$. Using the canonical normalization of the quantum vacuum fluctuations, $|v_k| = (2k)^{-1/2}$, we obtain

$$v_k \sim \frac{1}{\sqrt{2k}}\,(k\eta)^\alpha, \qquad\qquad \sigma_k = \frac{v_k}{z} \sim \frac{1}{M_{\rm P}\sqrt{k}}(k\eta_1)^\alpha, \tag{A.3}$$

for a resulting (dimensionless) spectrum

$$|\delta_\sigma|^2 = k^3 |\sigma_k|^2 \sim k^{2+2\alpha}. \tag{A.4}$$

The same result is also valid for the metric perturbation spectrum, with the only difference that the power α is referred to the graviton pump filed, different in general from the axion pump field even in the same given background.

Consider now a typical low-energy pre-big bang background, represented by (the negative-time branch of) a higher-dimensional vacuum solution of the tree-level gravi-dilaton equations. We will suppose, in particular, that the geometry can be described by a homogeneous Bianchi-I-type metric (namely by a cosmological metric which is homogeneous but not isotropic along all spatial directions), with three inflationary and isotropically expanding dimensions, with scale factor $a(\eta)$, and n "internal" dimensions, with scale factor $b_i(\eta)$, $i = 1, \ldots, n$. In conformal time, and in the string frame, such solution can be parametrized for $\eta < 0$ as [49]

$$\begin{aligned} &a \sim (-\eta)^{\beta_0/(1-\beta_0)}, \qquad\qquad b_i \sim (-\eta)^{\beta_i/(1-\beta_0)}, \\ &\phi \sim \frac{\sum_i \beta_i + 3\beta_0 - 1}{1-\beta_0} \ln(-\eta), \end{aligned} \tag{A.5}$$

where the constant coefficients β_0, β_i satisfy the Kasner-like condition

$$3\beta_0^2 + \sum_i \beta_i^2 = 1 \tag{A.6}$$

(the isotropy of the three-dimensional space has to be assumed *ad hoc*, as already stressed). For the graviton pump field [64, 65, 70] one then finds

$$z_g \sim a \left(\prod_i b_i \right)^{1/2} e^{-\phi/2} \sim (-\eta)^{1/2}, \tag{A.7}$$

and always obtains, according to Eq. (A.4), a growing spectrum of metric perturbations, $|\delta_g|^2 \sim k^3$ (modulo logarithmic corrections). For the axion pump field [76, 77], instead,

$$z_\sigma \sim a \left(\prod_i b_i \right)^{-1/2} e^{\phi/2} \sim (-\eta)^{(5\beta_0-1)/2(1-\beta_0)}, \tag{A.8}$$

with a corresponding spectral slope

$$2 + 2\alpha = \frac{1+3\beta_0}{1-\beta_0}. \tag{A.9}$$

The axion spectrum, in particular, turns out to be scale-invariant for $\beta_0 = -1/3$ (and $\sum_i \beta_i^2 = 2/3$, according to Eq. (A.6)). Note that if $d = 3 + n = 9$, as prescribed by critical superstring theory [17, 18], and if the external six-dimensional space is isotropic ($\beta_i = \beta$ for all directions), then the value

of β_0 required by a flat spectrum is automatically obtained if the internal dimensions expand or contract with the same kinematic power as the external ones, i.e. $\beta_i = \beta = \pm 1/3$. This seems to suggest a close (but still mysterious) connection between the background isotropy and the scale-invariance of the spectrum.

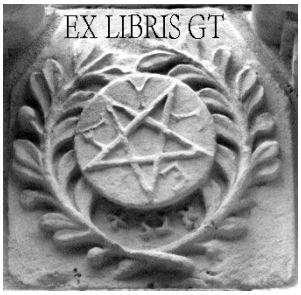
References

1. S. Weinberg (1971): *Gravitation and cosmology*. Wiley, New York.
2. E. W. Kolb, M. S. Turner (1990): *The Early Universe*. Addison-Wesley, Redwood City CA.
3. A. Guth (1981): The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. Phys. Rev. D **23**, 347-356.
4. V. F. Mukhanov (2005): *Physical Foundations of Cosmology*. Cambridge University Press, Cambridge.
5. C. L. Bennet et al (1996): Four year COBE DMR cosmic microwave background observations: Maps and basic results. Astrophys. J. **464**, L1-L4.
6. D. N. Spergel et al (2006): Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology. astro-ph/0603449.
7. A. D. Linde (1983): Chaotic Inflation. Phys. Lett. B **129**, 177-181.
8. R. H. Brandenberger, J. Martin (2001): The Robustness of inflation to changes in superPlanck scale physics. Mod. Phys. Lett. A **16**, 999-1006.
9. R. H. Brandenberger, J. Martin (2001): The TransPlanckian problem of inflationary cosmology. Phys. Rev. D **63**, 123501.
10. A. Vilenkin (1992): Did the universe have a beginning? Phys. Rev. D **46**, 2355-2361.
11. A. Borde, A. Vilenkin (1994): Eternal inflation and the initial singularity. Phys. Rev. Lett. **72**, 3305-3309.
12. V. Kaplunovsky (1985): Mass Scales of the String Unification. Phys. Rev. Lett. **55**, 1036.
13. I. Antoniadis (1990): A Possible new dimension at a few TeV. Phys. Lett. B **246**, 377-384.
14. N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali (1998): The Hierarchy problem and new dimensions at a millimeter. Phys. Lett. B **429**, 263-272.
15. M. Gasperini, G. Veneziano (1993): Pre-big bang in string cosmology. Astropart. Phys. **1**, 317-339.
16. G. Veneziano (1991): Scale factor duality for classical and quantum strings. Phys. Lett. B **265**, 287-294.
17. M. B. Green, J. Schwarz, E. Witten (1987): *Superstring theory*. Cambridge University Press, Cambridge.
18. J. Polchinski (1998): *String theory*. Cambridge University Press, Cambridge.
19. A. Strominger, C. Vafa (1996): Phys. Lett. B **379**, 99
20. D. Amati, M. Ciafaloni, G. Veneziano (1989): Can Space-Time Be Probed Below the String Size? Phys. Lett. B **216**, 41.
21. G. Veneziano (1986): A stringy nature needs just two constants. Europhys. Lett. **2**, 199.
22. L. Susskind (1993): Some speculations about black hole entropy in string theory. hep-th/9309145.

23. K. Kikkawa, M. Y. Yamasaki (1984): Casimir Effects in Superstring Theories. Phys. Lett. B **149** 357.
24. B. A. Campbell, A. Linde, K. A. Olive (1991): Does string theory lead to extended inflation? Nucl. Phys. B **355**, 146-161.
25. R. Brustein, P. J. Steinhardt (1993): Challenges for superstring cosmology. Phys. Lett. B **302**, 196-201.
26. C. Burgess et al (2001): The Inflationary brane anti-brane universe. JHEP **0107**, 047
27. F. Quevedo (2002): Lectures on string/brane cosmology. Class. Quantum Grav. **19**, 5721-5779.
28. J. Khoury et al (2001): The Ekpyrotic universe: Colliding branes and the origin of the hot big bang. Phys. Rev. D **64**, 123522.
29. J. Khoury et al (2002): From big crunch to big bang. Phys. Rev. D **65**, 086007.
30. E. I. Buchbinder, J. Khoury, B. A. Ovrut (2007): New Ekpyrotic cosmology. hep-th/0702154.
31. K. Becker, M. Becker, A. Krause (2005): M-theory inflation from multi M5-brane dynamics. Nucl. Phys. B **715**, 349-371.
32. A. R. Liddle, A. Mazumdar, F. E. Schunk (1998): Assisted inflation, Phys. Rev. D **58**, 061301.
33. V. A. Rubakov, M. Shaposhnikov (1983): Do We Live Inside a Domain Wall? Phys. Lett. B **215**, 136-138.
34. S. Kachru et al (2003): Towards inflation in string theory. JCAP **0310**, 013.
35. S. Kachru et al (2003): De Sitter vacua in string theory. Phys. Rev. D **68**, 0460005.
36. S. Giddings, S. Kachru, J. Polchinski (2002): Hierarchies from fluxes in string compactifications. Phys. Rev. D **66**, 106006.
37. A. A. Tseytlin (1991): Duality and dilaton. Mod. Phys. Lett. A **6** 1721-1732.
38. K. A. Meissner, G. Veneziano (1991): 9) Manifestly O(d,d) invariant approach to space-time dependent string vacua. Mod. Phys. Lett. A **6**, 3397-3404.
39. K. A. Meissner, G. Veneziano (1991): Symmetries of cosmological superstring vacua. Phys. Lett. B **267**, 33-36.
40. M. Gasperini, G. Veneziano (1992): O(d,d) covariant string cosmology. Phys. Lett. B **277**, 256-264.
41. M. Gasperini, G. Veneziano (2003): The Pre-big bang scenario in string cosmology. Phys. Rep. **373**, 1-212.
42. R. Brustein, G. Veneziano (1994): The Graceful exit problem in string cosmology. Phys. Lett. B **329**, 429-434.
43. M. Gasperini, M. Maggiore, G. Veneziano (1997): Towards a nonsingular pre-big bang cosmology. Nucl. Phys. B **494**, 315-330.
44. R. Brustein, R. Madden (1998): A model of graceful exit in string cosmology. Phys. Rev. D. **57**, 712-724.
45. C. Cartier, E. J. Copeland, R. Madden (2000): The Graceful exit in string cosmology. JHEP **0001**, 035.
46. M. Gasperini, M. Giovannini, G. Veneziano (2003): Perturbations in a nonsingular bouncing universe. Phys. Lett. B **569**, 113-122.
47. M. Gasperini, M. Giovannini, G. Veneziano (2004): Cosmological perturbations across a curvature bounce. Nucl. Phys. B **694**, 206-238.
48. M. Gasperini, M. Giovannini, K. A. Meissner, G. Veneziano (1996): Evolution of strings in cosmological backgrounds. Nucl. Phys. (Proc. Suppl.) B **49**, 70-74.

49. M. Gasperini, G. Veneziano (1994): Dilaton production in string cosmology. Phys. Rev. D **50**, 2519-2540.
50. N. Kaloper, A. Linde, R. Bousso (1999): Pre-big bang requires the universe to be exponentially large from the very beginning. Phys. Rev. D **59**, 043508.
51. A. Buonanno, T. Damour, G. Veneziano (1999): Pre-big bang bubbles from the gravitational instability of generic string vacua. Nucl. Phys. B **543**, 275-320.
52. M. Gasperini (2000): Inflation and initial conditions in the pre-big bang scenario. Phys. Rev. D **61**, 87301.
53. M. Gasperini (2000): String cosmology versus standard and inflationary cosmology. Classical Quant. Grav. **17**, R1-R18.
54. M. Gasperini (1999): Looking back in time beyond the big bang. Mod. Phys. Lett. A **14**, 1059-1066.
55. R. Brandenberger, C. Vafa (1989): Superstrings in the Early Universe. Nucl. Phys. B **316**, 391.
56. A. A. Tseytlin, C. Vafa (1992): Elements of string cosmology. Nucl. Phys. B **372**, 443-466.
57. G. Veneziano (2004): A Model for the big bounce. JCAP **0403**, 004.
58. S. Alexander, R. Brandenberger, D. Easson (2000): Brane gases in the early universe. Phys. Rev. D **62**, 103509.
59. T. Battenfeld, S. Watson (2006): String gas cosmology. Rev. Mod. Phys. **78**, 435-454.
60. M. Gasperini, G. Veneziano (1996): Birth of the universe as quantum scattering in string cosmology. Gen. Rel. Grav. **28**, 1301-1307.
61. M. Gasperini, J. Maharana, G. Veneziano (1996): Graceful exit in quantum string cosmology. Nucl. Phys. B **472**, 349-360.
62. M. Gasperini (2001): Birth of the universe as anti-tunneling from the string perturbative vacuum. Int. J. Mod. Phys. D **10**, 15-24.
63. M. Gasperini, M. Giovannini (1992): Constraints on inflation at the Planck scale from the relic graviton spectrum. Phys. Lett. B **282**, 36-43.
64. M. Gasperini, M. Giovannini (1993): Dilaton contributions to the cosmic gravitational wave background. Phys. Rev. D **47**, 1519-1528.
65. R. Brustein, M. Gasperini, M. Giovannini, G. Veneziano (1995): Relic gravitational waves from string cosmology. Phys. Lett. B **361**, 45-51.
66. R. Brustein, M. Gasperini, G. Veneziano (1997): Peak and endpoint of the relic graviton background in string cosmology. Phys. Rev. D **55**, 3882-3885.
67. M. Kamionkowski, A. Kosowski, A. Stebbins (1997): A Probe of primordial gravity waves and vorticity. Phys. Rev. Lett. **78**, 2058-2061.
68. M. Gasperini (2007): *Elements of string cosmology*. Cambridge University Press, Cambridge.
69. L. A. Boyle, P. J. Steinhardt, N. Turok (2004): The Cosmic gravitational wave background in a cyclic universe. Phys. Rev. D **69**, 127302.
70. R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov, G. Veneziano (1995): Metric perturbations in dilaton driven inflation. Phys. Rev. D **51**, 6744-6756.
71. K. Enqvist, M. Sloth (2002): Adiabatic CMB perturbations in pre-big bang string cosmology. Nucl. Phys. B **626**, 395-409.
72. D. H. Lyth, D. Wands (2002): Generating the curvature perturbation without an inflaton. Phys. Lett. B **524**, 5-14.
73. T. Moroi, T. Takahashi (2001): Effects of cosmological moduli fields on cosmic microwave background. Phys. Lett. B **522**, 215-221.

74. V. Bozza, M. Gasperini, M. Giovannini, G. Veneziano (2002): Assisting pre-big bang phenomenology through short lived axions. Phys. Lett. B **543**, 14-22.
75. V. Bozza, M. Gasperini, M. Giovannini, G. Veneziano (2003): Constraints on pre big bang parameter space from CMBR anisotropies. Phys. Rev. D **67**, 063514.
76. E. J. Copeland, R. Easther, D. Wands (1997): Vacuum fluctuations in axion-dilaton cosmologies. Phys. Rev. **D56**, 874-888.
77. E. J. Copeland, J. E. Lidsey, D. Wands (1997): S-duality invariant perturbations in string cosmology. Nucl. Phys. B **506**, 407-420.
78. R. Durrer, M. Gasperini, M. Sakellariadou, G. Veneziano (1999): Seeds of large scale anisotropy in string cosmology. Phys. Rev. D **59**, 043511.
79. M. Gasperini, M. Giovannini, G. Veneziano (1995): Primordial magnetic fields from string cosmology. Phys. Rev. Lett. **75**, 3796-3799.
80. D. Lemoine, M. Lemoine (1995): Primordial magnetic fields in string cosmology. Phys. Rev. D **52**, 1995-1962.
81. M. Gasperini, M. Giovannini, G. Veneziano (1995): Electromagnetic origin of the Cosmic Microwave Backgrounds anisotropy in string cosmology. Phys. Rev. D **52** 6651-6655.
82. M. Gasperini, S, Nicotri (2006): Observable (?) cosmological signatures of superstring models. Phys. Lett. B **633**, 155-160.
83. P. J. Steinhardt, N. Turok (2002): Cosmic evolution in a cyclic universe. Phys. Rev. D **65**, 126003.
84. P. J. Steinhardt, N. Turok (2002): A cyclic model of the universe. Science **296**, 1436-1439.



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BIRTH OF THE UNIVERSE AS QUANTUM SCATTERING IN STRING COSMOLOGY

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Abstract

In a Wheeler-De Witt approach to quantum string cosmology, the present state of the Universe arises from the scattering and reflection of the wave function representing the initial string vacuum in superspace. This scenario is described and compared with the more conventional quantum cosmology picture, in which the birth of the Universe is represented as a process of tunnelling "from nothing" in superspace.

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In the standard cosmological model, the birth of our Universe is assumed to coincide with the initial big-bang singularity, characterizing the classical solutions of the Einstein cosmological equations. Near the singularity, however, the Universe approaches the Planck curvature scale and the quantum gravity regime, where a classical description of the space-time manifold is no longer appropriate. By adopting, in that regime, a quantum cosmology approach, one can describe the birth of the Universe as a "tunnelling from nothing" [1]-[3], where the process of tunnelling refers to the Wheeler-De-Witt (WDW) wave function [4] in superspace.

In string cosmology models [5], the Universe starts evolving from the string perturbative vacuum, a state with flat metric and vanishing coupling constant. The curvature and the coupling grow during an initial "pre-big-bang" phase, and this growth, according to the low-energy effective action, leads classically to a singular state which marks the beginning of the standard, post-big-bang cosmological era. By applying a quantum cosmology approach, the transition through the singular big-bang regime can be described as a scattering of the initial pre-big-bang state into a final post-big-bang configuration, in particular as a reflection of the WDW wave function in superspace. The purpose of this paper is to illustrate this effect and to stress analogies and differences with the more conventional tunnelling scenario for the birth of the Universe.

For an easier comparison of the two pictures we shall work with the simplest example of non-trivial WDW equation, in which the effective potential is the one induced by a positive cosmological constant Λ, due to an over-critical number of dimensions. We start with the tree-level, low-energy string effective action [6]

$$S = -\frac{1}{2\lambda_s^2}\int d^4x\sqrt{-g}e^{-\phi}\left(R + \partial_\mu\phi\partial^\mu\phi + \Lambda\right), \tag{1}$$

where ϕ is the dilaton field, λ_s is the fundamental string-length parameter governing the derivative expansion of the effective action, and the extra dimensions have been taken to be completely inert. For a homogeneous, isotropic and spatially flat metric background, with scale factor a and spatial sections of finite volume, we define:

$$\overline{\phi} = \phi - \ln\int(d^3x/\lambda_s^3) - \sqrt{3}\beta, \qquad \beta = \sqrt{3}\ln a. \tag{2}$$

In the cosmic-time gauge, $g_{00} = 1$, the action (1) becomes:

$$S = -\frac{\lambda_s}{2}\int dt e^{-\overline{\phi}}\left(\dot{\overline{\phi}}^2 - \dot{\beta}^2 + \Lambda\right). \tag{3}$$

By using the convenient time reparameterization $dt = d\tau e^{-\overline{\phi}}$ we are finally led to the Hamiltonian

$$H = \frac{1}{2\lambda_s}\left(\Pi_\beta^2 - \Pi_{\overline{\phi}}^2 + \lambda_s^2 \Lambda e^{-2\overline{\phi}} \right), \tag{4}$$

where Π_β, $\Pi_{\overline{\phi}}$ are the canonical momenta

$$\Pi_\beta = \frac{\delta S}{\delta \beta'} = \lambda_s \beta', \qquad \Pi_{\overline{\phi}} = \frac{\delta S}{\delta \overline{\phi}'} = -\lambda_s \overline{\phi}', \tag{5}$$

and a prime denotes differentiation with respect to τ.

This Hamiltonian implies momentum conservation along the β axis,

$$[\Pi_\beta, H] = 0, \qquad \Pi_\beta = \lambda_s \dot{\beta} e^{-\overline{\phi}} = k = \text{const.} \tag{6}$$

The general solution of the classical equations of motion is well known [7], [8] and contains two distinct branches,

$$a(t) = a_0 \left(\tanh |\sqrt{\Lambda} t/2| \right)^{\pm 1/\sqrt{3}}, \qquad \overline{\phi} = \phi_0 - \ln \sinh |\sqrt{\Lambda} t|, \qquad k = \pm \lambda_s \sqrt{\Lambda} e^{-\phi_0} \tag{7}$$

where a_0 and ϕ_0 are integration constants. The two branches are defined over disconnected ranges of time, $t < 0$ and $t > 0$, separated by a singularity of the curvature invariants and of the effective string coupling $e^{\overline{\phi}}$ at $t = 0$ (for simplicity, we have chosen the integration constants so as to make the singular ends of both time ranges coincide at $t = 0$).

We are interested, in particular, in the branch describing a classical approach to the singularity in a state of accelerated expansion, growing curvature, typical of the pre-big-bang regime [5]

$$\begin{aligned} t < 0, \quad a &= a_0 \left[\tanh\left(-\sqrt{\Lambda} t/2 \right) \right]^{-1/\sqrt{3}}, \quad \overline{\phi} - \phi_0 = -\ln \sinh\left(-\sqrt{\Lambda} t \right) \\ \dot{a} &> 0, \quad \ddot{a} > 0, \quad \dot{H} > 0, \qquad k = \lambda_s \sqrt{\Lambda} e^{-\phi_0} > 0, \end{aligned} \tag{8}$$

and in the branch emerging from the singularity in a state of decelerated expansion, decreasing curvature,

$$\begin{aligned} t > 0, \quad a &= a_0 \left[\tanh\left(\sqrt{\Lambda} t/2 \right) \right]^{1/\sqrt{3}}, \quad \overline{\phi} - \phi_0 = -\ln \sinh\left(\sqrt{\Lambda} t \right) \\ \dot{a} &> 0, \quad \ddot{a} < 0, \quad \dot{H} < 0, \qquad k = \lambda_s \sqrt{\Lambda} e^{-\phi_0} > 0. \end{aligned} \tag{9}$$

Both branches have positive canonical momentum $\Pi_\beta = k > 0$, and are related by a duality transformation including time-reflection [5, 8], $a(t) \to a^{-1}(-t)$, $\overline{\phi}(t) \to \overline{\phi}(-t)$. Let us call,

respectively, (+) and (−) the pre- and post-big-bang branches (8) and (9). In the high-curvature, strong coupling regime $\overline{\phi} \to +\infty$ (i.e. near the singularity) they are characterized by a constant and opposite value of the canonical momentum along $\overline{\phi}$, namely

$$\lim_{\overline{\phi}\to+\infty} \Pi_{\overline{\phi}}^{(\pm)} = \lim_{\overline{\phi}\to+\infty} \left(-\lambda_s \dot{\overline{\phi}} e^{-\overline{\phi}}\right)_{\pm} = \mp k. \tag{10}$$

In the low-energy limit, $\overline{\phi} \to -\infty$, the two branches still have opposite canonical momentum $\Pi_{\overline{\phi}}$. The momentum is no longer constant, however, but controlled by $\overline{\phi}$,

$$\lim_{\overline{\phi}\to-\infty} \Pi_{\overline{\phi}}^{(\pm)} \sim \mp\lambda_s \sqrt{\Lambda} e^{-\overline{\phi}} \tag{11}$$

We shall now apply the WDW equation [4], $H\Psi = 0$, to compute the (classically forbidden) probability of transition from one branch to another, assuming in particular as the initial state the pre-big-bang configuration represented classically by the solution (8). For the Hamiltonian (4), the WDW equation is a simple differential equation in the two-dimensional minisuperspace parameterized by $\overline{\phi}$ and β,

$$\left(\partial_{\overline{\phi}}^2 - \partial_\beta^2 + \lambda_s^2 \Lambda e^{-2\overline{\phi}} \right) \Psi(\overline{\phi}, \beta) = 0 \tag{12}$$

(there is no problem of operator ordering, as the order is uniquely fixed by the duality symmetries of the string effective action [9]). By exploiting the conservation property (6) we impose

$$\Pi_\beta \Psi_k = i\partial_\beta \Psi_k = k\Psi_k, \tag{13}$$

(note the role of the time-like coordinate assigned to β, which is monotonically increasing from $-\infty$ to $+\infty$), and eq. (12) can be separated by setting

$$\Psi_k(\overline{\phi}, \beta) = \psi_k(\overline{\phi}) e^{-ik\beta}, \qquad \left(\partial_{\overline{\phi}}^2 + k^2 + \lambda_s^2 \Lambda e^{-2\overline{\phi}} \right) \psi_k(\overline{\phi}) = 0 \tag{14}$$

The general solution of eq. (14) is a linear combination of Bessel functions [10], $J_\nu(z)$ and $J_{-\nu}(z)$, of index $\nu = ik$ and argument $z = \lambda_s \sqrt{\Lambda} e^{-\overline{\phi}}$. In order to fix the boundary conditions we observe that, in the strong coupling regime $\overline{\phi} \to +\infty$, the effective potential of eq. (14) becomes negligible, and the WDW solution can be written in the plane wave form as

$$\Psi_{+\infty}^{(\pm)}(\overline{\phi}, \beta) = \lim_{z\to 0} J_{\pm ik}(z) e^{-ik\beta} \sim e^{-ik(\beta \mp \overline{\phi})}. \tag{15}$$

In this limit, right- and left-moving waves along $\overline{\phi}$ correspond, respectively, to the pre- and post-big-bang classical configurations (8) and (9). Indeed,

$$\Pi_{\overline{\phi}}\Psi^{(\pm)}_{+\infty} = i\partial_{\overline{\phi}}\Psi^{(\pm)}_{+\infty} = \mp k\Psi^{(\pm)}_{+\infty} \tag{16}$$

which is the quantum analogue of the classical relation (10) (the opposite sign with respect to standard conventions, in the differential representation of $\Pi_{\overline{\phi}}$, is due to the negative sign appearing in the definition (5)). Consistently with the chosen pre-big-bang initial conditions, we thus impose that there are only right-moving waves approaching the singularity at $\overline{\phi} \to +\infty$. This is the same as imposing tunnelling boundary conditions [3], which select only outgoing modes at the (singular) boundary of superspace, and uniquely fixes the WDW wave function as (N_k is a normalization factor)

$$\Psi_k(\overline{\phi},\beta) = N_k J_{-ik}(\lambda_s\sqrt{\Lambda}e^{-\overline{\phi}}\ )\ e^{-ik\beta}. \tag{17}$$

In the low-energy limit $\overline{\phi} \to -\infty$, $z \to \infty$, the effective potential induced by Λ becomes dominant, but the solution can still be separated into a left- and a right-moving part according to the asymptotic behaviour of $J_{-\nu}(z)$, namely [10]

$$\begin{aligned} \lim_{\overline{\phi}\to-\infty}\Psi_k(\overline{\phi},\beta) &\sim \Psi^{(+)}_{-\infty} + \Psi^{(-)}_{-\infty}, \\ \Psi^{(\pm)}_{-\infty}(z,\beta) &= N_k\left(\frac{1}{2\pi z}\right)^{1/2}\exp\left[-i(k\beta \pm z) \pm \frac{i\pi}{4} \pm \frac{k\pi}{2}\right], \quad z = \lambda_s\sqrt{\Lambda}e^{-\overline{\phi}} \end{aligned} \tag{18}$$

By applying the momentum operator we find

$$\lim_{\overline{\phi}\to-\infty}\Pi_{\overline{\phi}}\Psi^{(\pm)}_{-\infty}(\overline{\phi},\beta) = \mp z\Psi^{(\pm)}_{-\infty}(\overline{\phi},\beta) \tag{19}$$

so that $\Psi^{(+)}_{-\infty}(\overline{\phi},\beta)$ and $\Psi^{(-)}_{-\infty}(\overline{\phi},\beta)$ correspond, respectively, to the pre- and post-big-bang branches of the low-energy classical solution, according to eq. (11). Starting from a pre-big-bang initial state, we thus obtain a finite probability of transition to the "dual" low-energy state. The transition is represented as a reflection of the wave function in minisuperspace [9], and the probability is measured by the reflection coefficient R_k as

$$R_k = \frac{|\Psi^{(-)}_{-\infty}|^2}{|\Psi^{(+)}_{-\infty}|^2} = e^{-2\pi k} \tag{20}$$

By recalling the definition of k and $\overline{\phi}$, the transition probability for a three-dimensional portion of space of initial proper volume Ω_i at $t \to -\infty$ can be written as

$$R(\Lambda, \Omega_i, g_s) = \exp\left\{-\frac{\sqrt{12}\pi}{g_s^2}\frac{\Omega_i}{\lambda_s^3}\left[\frac{\sqrt{3}}{\lambda_s\sqrt{\Lambda}} + \left(1 + \frac{3}{\lambda_s^2\Lambda}\right)^{1/2}\right]^{\sqrt{3}}\right\}, \tag{21}$$

where $g_s = e^{\phi_s/2}$ is the value of the string coupling at the scale $H_s = \lambda_s^{-1}$. Quite interestingly, this probability is independent of Λ for $\Lambda >> \lambda_s^{-2}$, and it is peaked in the strong coupling regime with a typical instanton-like dependence on the coupling constant, $R \sim \exp(-g_s^{-2})$. This probability is also invariant under the T-duality transformation [11] $(\Omega_i/\lambda_s^3) \to (\lambda_s^3/\Omega_i)$, $g_s^2 \to g_s^2(\Omega_i/\lambda_s^3)^{-2}$.

By contrast, the probability of birth of the Universe from quantum tunnelling may be computed from the Einstein-de Sitter action

$$S = -\frac{1}{2\lambda_p^2}\int d^4x\sqrt{-g}(R + \Lambda) \tag{22}$$

where λ_p is the Planck length. Solving the corresponding WDW equation, with appropriate boundary conditions, the tunnelling probability can be estimated as [1, 2, 3]

$$P \sim \exp\left\{-\frac{4}{\lambda_p^2\Lambda}\right\} \tag{23}$$

The main difference between the above string scenario and the standard one is that, in the latter, the Universe emerges from the quantum era in a classical inflationary regime, and the tunnelling process is completely controlled by the value of the cosmological constant. By contrast, in our case, the quantum era is approached at the end of a classical inflationary epoch. In both cases, however, the appropriate boundary conditions imposed at the big-bang singularity play a crucial role, and the probability of the birth of our Universe is given by the ratio of the (squared modulus of the) final wave function representing a standard decelerated expansion and the initial wave function. Also, in both cases the quantum process seems to favour large values (in Planck or string units) of the cosmological constant.

It is perhaps worth recalling, at this point, the words of Vilenkin while presenting his "tunnelling from nothing" scenario [1]-[3]:

"...Here "nothing" means the vacuum of some more fundamental theory...."

Our work suggests that "nothing" can be just the perturbative vacuum of string theory. The different description of the birth process (a wave reflection, rather than a tunnelling) simply

originates from the deep differences between string and Einstein gravity, which in the string case allow for a long, classical (and inflationary!) pregnancy.

References

[1] A. Vilenkin, Phys. Lett. B117, 25 (1982).

[2] A. Vilenkin, Phys. Rev. D30, 509 (1984).

[3] A. Vilenkin, Phys. Rev. D33, 3650 (1986);

A. Vilenkin, Phys. Rev. D37, 888 (1988).

[4] B. S. De Witt, Phys. Rev. 160, 1113 (1967);

J. A. Wheeler, in *Battelle Rencontres*, ed. by C. De Witt and J. A. Wheeler (Benjamin, New York, 1968).

[5] M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993) ;

M. Gasperini and G. Veneziano, Mod. Phys. Lett. A8, 3701 (1993);

M. Gasperini and G. Veneziano, Phys. Rev. D50, 2519 (1994);

R. Brustein and G. Veneziano, Phys. Lett. B329, 429 (1994).

[6] C. Lovelace, Phys. Lett. B135, 75 (1984);

E. S. Fradkin and A. A Tseytlin, Nucl. Phys. B261, 1 (1985);

C. G. Callan et al., Nucl. Phys. B262, 593 (1985).

[7] M. Muller, Nucl. Phys. B337, 37 (1990).

[8] G. Veneziano, Phys. Lett. B265, 387 (1991).

[9] M. Gasperini, J. Maharana and G. Veneziano, *Graceful exit in quantum string cosmology*, Preprint CERN-TH/96-32 (hep-th/9602078).

[10] M. Abramowicz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1972).

[11] For a review see A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244, 77 (1994).

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relic dilatonic dark matter

Dilaton Production in String Cosmology

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Abstract

We consider the coupled evolution of density, (scalar) metric and dilaton perturbations in the transition from a "stringy" phase of growing curvature and gravitational coupling to the standard radiation-dominated cosmology. We show that dilaton production, with a spectrum tilted towards large frequencies, emerges as a general property of this scenario. We discuss the frame-independence of the dilaton spectrum and of the inflationary properties of the metric background by using, as model of source, a pressureless gas of weakly interacting strings, which is shown to provide an approximate but consistent solution to the full system of background equations and string equations of motion. We combine various cosmological bounds on a growing dilaton spectrum with the bound on the dilaton mass obtained from tests of the equivalence principle, and we find allowed windows compatible with a universe presently dominated by a relic background of dilatonic dark matter.

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but how?

Grishchuk

1. Introduction and motivations.

It is well known that fluctuations of the metric background are amplified in the context of inflationary cosmologies, and that the amplification of their transverse, traceless (spin 2) component can be interpreted as graviton production [1]. Models of the early universe based on the low-energy string effective action (what we shall refer to, for short, as "string cosmology"), are characterized by the additional presence of a dilaton background, $\phi(t)$. It is natural to expect an amplification of the fluctuations $\delta\phi \equiv \chi$ of the dilaton background – with corresponding dilaton production – to accompany that of the metric for a suitable time-evolution of the cosmological fields.

In this paper we discuss such a dilaton production in the context of the so-called "pre-big-bang' scenario [2], characterized by an accelerated evolution from a flat, cold and weakly coupled initial regime to a final hot, highly curved, strong coupling regime, marking the beginning of the standard "post-big-bang" decelerated FRW cosmology. With this background, the spectrum of the produced dilatons tends to grow with frequency, just like in the (previously discussed [2,3]) case of graviton production. As we shall see in Sec.3, the high frequency part of the spectral distribution, Ω_χ, of the dilaton energy density can be parameterized (in units of critical density $\rho_c = H^2/G = M_p^2 H^2$) as

$$\Omega_\chi(\omega,t) = \frac{\omega}{\rho_c}\frac{d\rho_\chi}{d\omega} \simeq GH_1^2\left(\frac{\omega}{\omega_1}\right)^\delta\left(\frac{H_1}{H}\right)^2\left(\frac{a_1}{a}\right)^4, \qquad \delta>0, \qquad \omega<\omega_1 \tag{1.1}$$

where $M_p \simeq 10^{19}$ GeV is the Planck mass. Here H_1 is the curvature scale evaluated at the time t_1 marking the end of the inflationary epoch (which we assume to coincide with the beginning of radiation dominance); $\omega_1 = H_1 a_1/a$ is the maximum amplified proper frequency; $H = \dot{a}/a$, where a, as usual, is the scale factor of the background metric. The integral over ω is thus dominated by the highest frequency ω_1,

$$\Omega_\chi(t) = \int^{\omega_1}\frac{d\omega}{\omega}\Omega_\chi(\omega,t) \simeq GH_1^2\left(\frac{H_1}{H}\right)^2\left(\frac{a_1}{a}\right)^4 \tag{1.2}$$

and the condition $\Omega_\chi < 1$, required to avoid that the dilatons overclose the universe in the radiation-dominated era ($a \sim H^{-1/2}$), yields the constraint $H_1 \lesssim M_p$, already known [2,3] from the graviton spectrum.

H1

omega

a

The produced dilatons, however, cannot be massless. Large distance dilaton couplings can be estimated [4] and turn out to be at least of gravitational strength. This violates the universality of gravity at low energy and, in particular, induces corrections to the effective Newton potential (in the static, weak-field limit), which

mass of dilaton!!!!!!

m = 0 dilaton violates Eq principle

may be reconciled with the present tests of the equivalence principle [5] only for a dilaton mass satisfying [4,6]:

$$m \gtrsim m_0 \equiv 10^{-4}\, eV \tag{1.3}$$

The expression (1.2) is thus valid only until the energy density stays dominated by the relativistic modes, with $\omega(t) > m$. But, at the present time t_0 (with $H_0 \sim 10^{-61}\, M_p$), the maximum frequency ω_1 is

$$\omega_1(t_0) = H_1 \frac{a_1}{a_0} \simeq 10^{-4} \left(\frac{H_1}{M_p}\right)^{1/2} eV \tag{1.4}$$

As $H_1 < M_p$, even the highest mode must then become non-relativistic before the present epoch, because of the constraint (1.3). At the scale $H(t_m) = m$ the modes with $\omega(t_m) \lesssim m$ begin to oscillate coherently, with frequency m, and when they are dominant the dilaton energy density becomes non-relativistic, with

$$\Omega_\chi(t) \simeq GmH_1 \left(\frac{H_1}{H}\right)^2 \left(\frac{m}{H_1}\right)^{\frac{\delta-1}{2}} \left(\frac{a_1}{a}\right)^3, \qquad 0 \le \delta \le 1$$

$$\Omega_\chi(t) \simeq GmH_1 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3, \qquad \delta \ge 1 \tag{1.5}$$

(the dependence on the spectral index δ disappears for fast enough growth of the spectrum, as shown in Sec.5). Such a fraction of critical density grows in time during the radiation era, while in the matter era ($a \sim H^{-2/3}$) it becomes fixed at the maximum constant value

$$\Omega_\chi \simeq Gm^2 \left(\frac{H_1}{H_2}\right)^{1/2} \left(\frac{m}{H_1}\right)^{\frac{\delta-3}{2}}, \qquad 0 \ge \delta \ge 1$$

$$\Omega_\chi \simeq GmH_1 \left(\frac{H_1}{H_2}\right)^{1/2}, \qquad \delta \ge 1 \tag{1.6}$$

where $H_2 \sim 10^6\, H_0$ is the curvature scale at the matter-radiation transition. The condition $\Omega_\chi \le 1$ provides then, for any given inflation scale H_1, an upper limit for the dilaton mass,

$$m \lesssim \left(H_2 M_p^4 H_1^{\delta-4}\right)^{1/(\delta+1)}, \qquad 0 \le \delta \le 1$$

$$m \lesssim H_2^{1/2} M_p^2 H_1^{-3/2}, \qquad \delta \ge 1 \tag{1.7}$$

valid for $m < H_1$ under the assumption that its life-time is sufficiently long to reach the matter-dominated era (if $m > H_1$ the dilaton must decay before becoming dominant with respect to the radiation, as we shall see in Sec.5, and the critical density bound cannot be applied).

In addition to the constraint (1.7), which is an unavoidable consequence of the quantum fluctuations of the dilaton background, one should also consider, in general, the constraints following from possible classical oscillations of the background around the minimum of the potential [7]. The initial amplitude of such oscillations is, however, to a large extent model-dependent and, as discussed in Sec.5, we shall work under the assumption that classical oscillations are negligible with respect to the quantum fluctuations amplified by the cosmological evolution. This assumption will give us the *maximum* allowed region in parameter space.

In the absence of classical oscillations, the upper limit on m obtained from eq.(1.7), combined with the lower limit (1.3), define for each value of δ and H_1 an allowed window for the dilaton mass such that, near the upper end of the window, the produced dilatons can close the universe. Such a dilaton dominance, however, can last only until their energy density is dissipated into radiation, which occurs at a decay scale

$$H_d \simeq \Gamma_d \simeq \frac{m^3}{M_p^2} \tag{1.8}$$

We are thus led to the first interesting result of this analysis. If $\delta \gtrsim 0.1$, the upper limit (1.7) turns out to be larger than the lower bound (1.3) even for inflation scales $H_1 \geq 10^{-5}\ M_p$. Moreover, always for $H_1 \geq 10^{-5}\ M_p$, the largest value of m allowed by eq.(1.7) is $m \simeq 100\ MeV$ (obtained for $\delta \geq 1$), and it implies $H_d \lesssim H_0$. This means that, for fast enough growing spectra, and "realistic" (at least in a string cosmology context) inflation scales $H_1 \geq 10^{-5}\ M_p$, we can be left today with a background of relic dilatons possibly representing a significant fraction of the dark matter background [8]. The allowed ranges of m corresponding to this interesting possibility lie around the upper limits given in (1.7) and thus depend on δ and H_1 in a complicated way. For $0.1 \lesssim \delta \lesssim 0.72$ the range of H_1 for which this possibility can be realized is given by

$$H_1 \lesssim 10^{-(23-32\delta)/(4-\delta)} M_p \tag{1.9}$$

while for $\delta > 0.72$ values of H_1/M_p up to 1 are possible (the case $\delta \geq 1$ is illustrated in **Fig.1**). The lower bound on δ is imposed by the simultaneous requirements $H_1 \geq 10^{-5}\ M_p$ and $m \geq m_0$, together with eq.(1.7). As H_1/M_p is varied between

so that m HAS a mass

10^{-5} and 1 the corresponding dilaton mass vary over the whole domain from 10^{-4} eV to 100 MeV. For lower spectral slopes ($\delta < 0.1$), the present existence of a dominant dilaton background becomes possible only for (unrealistically) low inflation scales, as first pointed out in Ref.[9] for the case of scalar perturbations with a flat ($\delta = 0$) spectrum. Note that, according to eq.(1.9), a final inflation scale H_1 exactly coinciding with the string scale itself, $\sim 10^{-1}\ M_p$, would be compatible with a present light dilaton dominance only for $\delta \gtrsim 0.61$.

The inflation scale H_1 determines also the amplitude of scalar perturbations ψ of the metric itself, and it is thus constrained by the scalar contribution to the CMBR anisotropies (the contribution of tensor perturbations turns out to be negligible in our context, as their spectrum grows very fast with frequency [2,3]). The behavior of the scalar perturbation spectrum, as we shall see, depends in general on the adopted model of matter sources and background evolution, and it is fair to say that our present knowledge of the details of the stringy pre-big-bang phase is to poor to make stringent predictions on the exact value of the spectral index δ. On the other hand, fortunately enough, the properties of a massive dilaton background are only weakly dependent on the value of the spectral index for $\delta > 0$, and rapidly become spectrum-independent as soon as $\delta \geq 1$. The particular example chosen in this paper (see Sec.4) to discuss dilaton production, namely a three-dimensional, isotropic, dilaton-dominated background with negligible matter sources, gives the same spectrum (very fast growing, $\delta = 3$), for scalar (ψ), dilaton (χ) and tensor ($h_{\mu\nu}$) perturbations. However, such an example is chosen here for simplicity reasons only, in order to develop a first qualitative sketch of the scenario associated with dilaton production, and it should not be taken as particularly indicative for what regards the spectral properties of the metric perturbations. For a phenomenological discussion it is better to leave open the possibility of different spectra for ψ and χ (possibility which is in general allowed in this context, as we shall see in Sec.3), and to parameterize the scalar (metric) energy density as

$$\Omega_\psi(\omega, t) \simeq G H_1^2 \left(\frac{\omega}{\omega_1}\right)^{n-1} \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \tag{1.10}$$

with $n - 1$ in general different from δ.

The interesting question to ask at this point is whether, in the same range of H_1 which we believe to be realistic, it is possible to produce enough dilatons to close the present universe and, at the same time, to generate scalar perturbations with a spectrum consistent with the anisotropy observed by COBE [10]. This

amounts to require

$$10^{-5} \lesssim \frac{H_1}{M_p} \lesssim 1, \qquad \Omega_\chi(t_0) \simeq 1, \qquad \Omega_\psi(\omega_0, t_2) \simeq G H_1^2 \left(\frac{\omega_0}{\omega_1}\right)^{n-1} \simeq 10^{-10} \quad (1.11)$$

where $\omega_0 = H_0$ is the minimum amplified frequency corresponding to a wave crossing today the Hubble radius, and t_2 is the time of matter-radiation equilibrium, nearly coincident with the time of recombination.

The answer, perhaps surprisingly, is yes: the last requirement of eq.(1.11) is compatible with eq.(1.9), for the same range of H_1, provided the scalar spectrum is also growing, with

$$1 \lesssim n \lesssim 1.34 \quad (1.12)$$

This allowed range of n is well contained in the range of the spectral index originally determined by COBE [10], $n = 1 \pm 0.5$, and is also consistent with the new recent fit [11] which gives $n = 1.5 \pm 0.5$. It may be interesting to recall, in this context, that growing scalar perturbations, with $n \simeq 1.25$, are also required for a simultaneous fit of the COBE anisotropies and of the observed bulk motion and large voids structures on a $50\, Mpc$ scale [12]. Growing scalar spectra can be obtained in the "hybrid inflation" model proposed by Linde [13] and recently generalized to the class of "false vacuum inflation" [14] (see also [15]). Note also that the condition (1.12) would not be incompatible with the lower bound on δ required for a present dominant dilaton background (according to eq.(1.9)), even in the case of equal scalar and dilaton spectrum, $\delta = n - 1$.

Concluding this qualitative analysis, we can say that the possibility of producing a dilaton background which saturates the closure density, together with scalar perturbations which provide the observed cosmic anisotropies, seems to be naturally associated with a growing dilaton spectrum, $\delta > 0$. The fact that such a spectrum is typical of string-based pre-big-bang models represents, in our opinion, an interesting aspect of such models, and motivates the study of dilaton production in the string cosmology scenario. A requirement analogous to eq.(1.11), formulated however in the context of extended inflation models where the fluctuation spectrum of the Brans-Dicke scalar is not growing, may be satisfied [16] only for a reheating temperature $T_r < 10^{13}\, GeV$, namely for very low scales $H_1 \simeq T_r^2/M_p \leq 10^{-12}\, M_p$. We note, finally, that the possibility of inflationary production of massless scalar particles, associated with excitations of the Brans-Dicke field, was also pointed out in Ref.[17], and previously discussed in Ref.[18] for the massive case (with $m < H_1$), but always in the context of exponential inflation, which is not the

natural inflationary background corresponding to the low energy string effective action.

The paper is organized as follows.

In Sec.2 we present the general exact solutions (for space-independent fields) of the system of background field equations, including classical string sources, following from the tree-level string effective action at lowest order in α'. The explicit form of the solution is displayed, in particular, for a perfect fluid model of sources, in $D = d + 1$ dimensions, for any given equation of state. The low curvature and large curvature limit of such solutions are given both in the Brans-Dicke and in the conformally related Einstein frame. In Sec.3 we derive the coupled system of scalar (metric plus dilaton) perturbation equations, including the perturbations of the matter sources in the perfect fluid form. Such equations are applied to compute the scalar perturbation spectrum for a specific case of background evolution motivated by a model of sources (presented in Sec.4) in which the dominant form of matter is a sufficiently diluted, non-interacting gas of large macroscopic strings. The background describes a phase of growing curvature and accelerated kinematic (of the pre-big-bang type), which is expected to evolve towards the standard, radiation-dominated cosmology. The frame-independence of the inflationary properties of such background is also discussed in Sec.4. The corresponding spectrum of the produced dilatons is discussed in Sec.5, where it is shown that, because of its fast growth with frequency, the phenomenological constraints leave open a window compatible with the possible dominance of not yet decayed dilatons (in the hypothesis of negligible classical oscillations of the dilaton background). The main results of this paper are finally summarized and briefly discussed in Sec.6.

2. General solution of the background field equations.

We will assume the evolution of the Universe to be described at curvatures below the string/Planck scale by the equations

$$R_\mu{}^\nu + \nabla_\mu \nabla^\nu \phi - \frac{1}{2}\delta_\mu{}^\nu \frac{\partial V}{\partial \phi} - \frac{1}{4} H_{\mu\alpha\beta} H^{\nu\alpha\beta} = 8\pi G_D e^{\phi} T_\mu^\nu \tag{2.1}$$

$$R - (\nabla_\mu \phi)^2 + 2\,\nabla_\mu \nabla^\mu \phi + V - \frac{\partial V}{\partial \phi} - \frac{1}{12} H_{\mu\nu\alpha} H^{\mu\nu\alpha} = 0 \tag{2.2}$$

$$\partial_\nu (\sqrt{|g|} e^{-\phi} H^{\mu\alpha\beta}) = 0 \tag{2.3}$$

Such system of equations follows from the low-energy (D-dimensional) effective action of closed (super)string theory [19],

$$S = -\frac{1}{16\pi G_D}\int d^D x\sqrt{|g|}e^{-\phi}\left[R + \partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_{\mu\nu\alpha}H^{\mu\nu\alpha} + V(\phi)\right] + S_M \quad (2.4)$$

Here ϕ is the dilaton field, and $H_{\mu\nu\alpha}$ the field strength of the two-index antisymmetric (torsion) tensor $B_{\mu\nu} = -B_{\nu\mu}$. We have included a possible dilaton potential, $V(\phi)$, and also a possible phenomenological contribution of the matter sources represented by the action S_M, whose metric variation produces the stress tensor $T_{\mu\nu}$.

We shall consider, in this paper, homogeneous backgrounds which are independent of all spacelike coordinates (Bianchi I type, with d Abelian isometries), and for which a synchronous frame exists where $g_{00} = 1$, $g_{0i} = 0 = B_{0i}$ (conventions: $\mu, \nu = 0, 1, ...D = d + 1$; $i, j = 1, 2, ...d$). We shall assume, moreover, that the action S_M describes "bulk" string matter, satisfying the classical string equations of motion in the given background. At tree level V is a constant. In terms of the "shifted dilaton"

$$\overline{\phi} = \phi - \frac{1}{2}\ln|det(g_{\mu\nu})| \quad (2.5)$$

the field equations (2.1)-(2.3) can be written in matrix form as [20]

$$\dot{\overline{\phi}}^2 - 2\ddot{\overline{\phi}} - \frac{1}{8}Tr\,(\dot{M}\eta)^2 - V = 0 \quad (2.6)$$

$$\dot{\overline{\phi}}^2 + \frac{1}{8}Tr\,(\dot{M}\eta)^2 - V = \overline{\rho}e^{\overline{\phi}} \quad (2.7)$$

$$\frac{d}{dt}(e^{-\overline{\phi}}M\eta\dot{M}) = \overline{T} \quad (2.8)$$

(a dot denotes differentiation with respect to the cosmic time t, and we have used units in which $8\pi G_D = 1$, so that $\overline{\rho}$ and $\overline{T}$ have both dimensions L^{-2}). Here M is a $2d \times 2d$ matrix,

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \quad (2.9)$$

where G and B are matrix representation of the $d \times d$ spatial part of the metric (g_{ij}) and of the antisymmetric tensor (B_{ij}), in the basis in which the $O(d, d)$ metric η is in off-diagonal form,

$$\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (2.10)$$

(I is the unit $d \times d$ matrix). $\overline{T}$ is another $2d \times 2d$ matrix representing the spatial part of the string stress tensor [20] (including the possible contribution of an antisymmetric current density, source of torsion). Finally $\overline{\rho}$ is related to the energy density $\rho = T_0{}^0$ by

$$\overline{\rho} = \rho \sqrt{|det(g_{\mu\nu})|} \tag{2.11}$$

The three equations (2.6)-(2.8) correspond, respectively, to to the dilaton equation (2.2) and to the time and space part of eqs.(2.1), (2.3) for the homogeneous background that we have considered. Their combination provides the covariant conservation equation for the source energy density, which can be written in compact form as [20]

$$\dot{\overline{\rho}} + \frac{1}{4} Tr\,(\overline{T}\eta M \eta \dot{M} \eta) = 0 \tag{2.12}$$

The set of equations (2.6)-(2.8), (2.12) is explicitly covariant under the global $O(d,d)$ transformation [21,20]

$$\overline{\phi} \to \overline{\phi}, \quad \overline{\rho} \to \overline{\rho}, \quad M \to \Lambda^T M \Lambda, \quad \overline{T} \to \Lambda^T \overline{T} \Lambda \tag{2.13}$$

where Λ is an $O(d,d)$ constant matrix satisfying

$$\Lambda^T \eta \Lambda = \eta \tag{2.14}$$

For a suitable class of dilaton potentials such a system can be solved by quadratures, following the method presented in Ref.[2].

Here we shall concentrate, in particular, on the case $V = 0$, corresponding to strings in critical space-time dimensions (which does not exclude, however, a description of $d = 3$ evolution in case we add the right number of spectator dimensions in order to compensate the central charge deficit). We introduce a suitable (dimensionless) time coordinate x, such that

$$\overline{\rho} = \frac{1}{L} \frac{dx}{dt} \tag{2.15}$$

(L is a constant with dimensions of length), and we define

$$\Gamma = \int^x \frac{\overline{T}}{\overline{\rho}} dx' \tag{2.16}$$

The equations (2.6)-(2.8) can then be integrated a first time, with the help of eq.(2.16) and of the $O(d,d)$ identity

$$(M\eta\dot{M}\eta)^2 = -(\dot{M}\eta)^2, \tag{2.17}$$

to give [2]

$$\overline{\rho} = \frac{e^{\overline{\phi}}}{4L^2} D \tag{2.18}$$

$$\overline{\phi}' = -\frac{2}{D}(x + x_0) \tag{2.19}$$

$$M\eta M' = \frac{4\Gamma}{D} \tag{2.20}$$

where

$$D = (x + x_0)^2 - \frac{1}{2}\, Tr\, (\Gamma\eta)^2 \tag{2.21}$$

(a prime denotes differentiation with respect to x, and x_0 is an integration constant).

By exploiting the fact that M is a symmetric $O(d,d)$ matrix, $M\eta M = \eta$, and that, because of the definition of $\overline{T}$ (see Ref.[20])

$$M\eta\Gamma = -\Gamma\eta M, \tag{2.22}$$

eqs.(2.19), (2.20) can be integrated a second time to give

$$\overline{\phi}(x) = \phi_0 - 2\int \frac{dx}{D}(x + x_0) \tag{2.22}$$

$$M(x) = P_x \exp(-4\int \frac{dx}{D}\Gamma\eta) M_0 \tag{2.24}$$

where ϕ_0 and M_0 are integration constants (M_0 is a symmetric $O(d,d)$ matrix), and P_x denotes x- ordering of the exponential. For any given "equation of state", providing a relation $\overline{T} = \overline{T}(\overline{\rho})$ between the spatial part of the stress tensor of the sources and their energy density, eqs.(2.23) and (2.24), together with (2.18), represent the general exact solution of the system of string cosmology equations, for space-independent background fields and vanishing dilaton potential.

Such a solution presents in general singularities, for the curvature and the effective coupling constant e^{ϕ}, occurring in correspondence of the zero of $D(x)$. It is important to stress that, near the singularity, the contribution of the matter sources becomes negligible with respect to the curvature terms in the field equations (just like in general relativity, in the case of Kasner's anisotropic solution).

The relative importance of the source term is measured indeed by the ratio (see for instance eq.(2.7))

$$\Omega(x) = -\frac{8\overline{\rho}e^{\overline{\phi}}}{(d-1)\, Tr\, (\dot{M}\eta)^2} \tag{2.25}$$

(we have normalized Ω in such a way that it reduces to the usual expression for the effective energy density in critical units, $\Omega = \rho/\rho_c$, when the dilaton is constant and the metric isotropic). According to eqs.(2.18), (2.20)

$$M\eta\dot{M} = \frac{e^{\overline{\phi}}}{L}\Gamma \tag{2.26}$$

so that, by exploiting the $O(d,d)$ properties of M,

$$Tr\,(\dot{M}\eta)^2 = -\frac{e^{2\overline{\phi}}}{L^2}\,Tr\,(\Gamma\eta)^2 \tag{2.27}$$

Therefore

$$\Omega = \frac{2}{d-1}\frac{D}{Tr\,(\Gamma\eta)^2} \tag{2.28}$$

goes to zero at the singularity ($D \to 0$).

In this limit, the matter contribution becomes negligible and the general solution presented here reduces to the $V = 0$ case of the general vacuum solution of the string cosmology equations [22]. Denoting indeed by t_c a singular point such that $D(t_c) = 0$, $\Gamma(t_c) \neq 0$, from eq.(2.26) we have, near this point,

$$M\eta\dot{M}e^{-\overline{\phi}} = A \tag{2.29}$$

where the constant matrix A satisfies

$$A = \frac{\Gamma(t_c)}{L} = -A^T, \qquad M\eta A + A\eta M = 0 \tag{2.30}$$

because of the property (2.22) of Γ. Moreover, from (2.18) and (2.19)

$$\dot{\overline{\phi}}^2 = \frac{e^{2\overline{\phi}}}{4L^2}(x + x_0)^2 \tag{2.31}$$

so that, by using (2.27),

$$\dot{\overline{\phi}}^2 + \frac{1}{8}\,Tr\,(\dot{M}\eta)^2 = \frac{e^{2\overline{\phi}}}{4L^2}D(t_c) = 0 \tag{2.32}$$

Eqs.(2.29) and (2.32) correspond exactly to the equations defining the general vacuum solution of Ref.[22], for the case of vanishing dilaton potential.

Consider now the particular case in which $B_{\mu\nu} = 0$, and we are in a diagonal, but not necessarily isotropic, Bianchi-I type metric background,

$$g_{00} = 1, \qquad g_{ij} = -a_i^2(t)\delta_{ij} \tag{2.33}$$

(this is the background that will we used here to discuss dilaton production). The matter sources can be represented in the perfect fluid form, but with anisotropic pressure,

$$T_0{}^0 = \rho, \quad T_i{}^j = -p_i \delta_i{}^j, \quad p_i/\rho = \gamma_i = const \tag{2.34}$$

In this case we obtain, from the previous definitions,

$$M\eta M' = 2\begin{pmatrix} 0 & \frac{a_i'}{a_i}\delta_{ij} \\ -\frac{a_i'}{a_i}\delta_{ij} & 0 \end{pmatrix}, \quad \overline{T} = \begin{pmatrix} 0 & \overline{p}_i\delta_{ij} \\ -\overline{p}_i\delta_{ij} & 0 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 0 & \Gamma_i\delta_{ij} \\ -\Gamma_i\delta_{ij} & 0 \end{pmatrix}, \quad \Gamma_i = \gamma_i x + x_i$$

$$D = (x+x_0)^2 - \sum_i (\gamma_i x + x_i)^2 = \alpha(x-x_+)(x-x_-) \tag{2.35}$$

where

$$\overline{p}_i = p_i\sqrt{|g|}, \qquad \alpha = 1 - \sum_i \gamma_i^2$$

$$x_\pm = \frac{1}{\alpha}\left\{ \sum_i \gamma_i x_i - x_0 \pm \left[(\sum_i \gamma_i x_i - x_0)^2 + \alpha(\sum_i x_i^2 - x_0^2) \right]^{1/2} \right\} \tag{2.36}$$

and x_i, x_0 are integration constants. The general solution (2.18), (2.23), (2.24) becomes explicitly [23]

$$a_i = a_{0i}|(x-x_+)(x-x_-)|^{\gamma_i/\alpha}|\frac{x-x_+}{x-x_-}|^{\alpha_i} \tag{2.37}$$

$$e^{\overline{\phi}} = e^{\phi_0}|(x-x_+)(x-x_-)|^{-1/\alpha}|\frac{x-x_+}{x-x_-}|^{-\sigma} \tag{2.38}$$

$$\overline{\rho} = \frac{\alpha}{4L^2}e^{\phi_0}|(x-x_+)(x-x_-)|^{(\alpha-1)/\alpha}|\frac{x-x_+}{x-x_-}|^{-\sigma} \tag{2.39}$$

where

$$\sigma = \sum_i \alpha_i\gamma_i, \quad \alpha_i = \frac{\alpha x_i + \gamma_i(\sum_i \gamma_i x_i - x_0)}{\alpha\left[(\sum_i \gamma_i x_i - x_0)^2 + \alpha(\sum_i x_i^2 - x_0^2)\right]^{1/2}} \tag{2.40}$$

and a_{i0}, ϕ_0 are additional integration constants.

This solution has two curvature singularities at $x = x_\pm$. Near the singularity, the presence of matter becomes negligible,

$$\Omega(x) = \frac{\alpha(x-x_+)(x-x_-)}{(d-1)\sum_i(\gamma_i x + x_i)^2} \to 0 \tag{2.41}$$

and one recovers the anisotropic vacuum solution of string cosmology in critical dimensions [24,25]. Indeed, for $x \to x_\pm$, one has $|x| \sim |t|^{\alpha/(1\pm\alpha\sum\alpha_i\gamma_i)}$, and the solution behaves like

$$a_i(t) \sim |t - t_\pm|^{\beta_i^\pm} \ , \qquad \overline{\phi} \sim -\ln|t - t_\pm| \tag{2.42}$$

where

$$\beta_i^\pm = \frac{x_i \pm \gamma_i x_\pm}{x_0 + x_\pm} \qquad , \qquad \sum_i (\beta_i^\pm)^2 = 1 \ . \tag{2.43}$$

In the large $|x|$ (small curvature) limit, on the contrary, the relation between x and cosmic time is $|x| \simeq |t|^{\alpha/(2-\alpha)}$, and the solution (2.37)-(2.39) behaves like (for $|x| \to \pm\infty$)

$$a_i(t) \sim |t|^{2\gamma_i/(1+\sum\gamma_i^2)}, \quad \overline{\phi} \sim -\frac{2}{1+\sum\gamma_i^2}\ln|t|$$

$$\phi \sim 2\frac{\sum\gamma_i - 1}{1+\sum\gamma_i^2}\ln|t|, \quad \overline{\rho} \sim |t|^{-2\sum\gamma_i^2/(1+\sum\gamma_i^2)} \tag{2.44}$$

The critical density parameter, in this limit, goes to a constant

$$\Omega_\infty = \frac{1 - \sum_i \gamma_i^2}{(d-1)\sum_i \gamma_i^2} \tag{2.45}$$

which is obviously $\Omega_\infty = 1$ for an isotropic, radiation-dominated background with $\gamma_i = 1/d$.

It is interesting to point out that, for any solution a_i, ϕ corresponding to a given set of equations of state, $p_i = \gamma_i\rho$, there are the corresponding "dual" solutions obtained through the reflection $\gamma_i \to -\gamma_i$, which leads to $a_i(\gamma_i) \to a_i(-\gamma_i) = a_i^{-1}(\gamma_i)$, preserving however the values of $\overline{\phi}$ and $\overline{\rho}$ (scale-factor duality [25,26]). Such a duality transformation, combined with the time- inversion $t \to -t$, transforms any given metric describing (for $\gamma_i > 0$) decelerated expansion with decreasing curvature, $\ddot{a}_i < 0$, $H_i > 0$, $\dot{H}_i < 0$, into a new solution describing (for $\gamma_i < 0$) a superinflationary expansion with increasing curvature, $\ddot{a}_i > 0$, $H_i > 0$, $\dot{H}_i > 0$ (see also [2]).

It is convenient, for later use, to write down explicitly the isotropic version of the asymptotic backgrounds (2.42) and (2.44), as a function of the cosmic time t and conformal time η such that $dt = ad\eta$. In the $(d+1)$-dimensional isotropic case, the small curvature limit (2.44) becomes, in cosmic time,

$$a(t) \sim |t|^{2\gamma/(1+d\gamma^2)}, \quad \phi \sim \frac{d\gamma - 1}{\gamma}\ln a, \quad \rho \sim a^{-d(\gamma+1)} \tag{2.46}$$

while in terms of η we have

$$a(\eta) \sim |\eta|^{2\gamma/(1-2\gamma+d\gamma^2)} \tag{2.47}$$

The vacuum, dilaton-dominated limit (2.42) becomes, in the isotropic case,

$$a_\mp(t) \sim |t|^{\mp 1/\sqrt{d}}, \qquad \phi_\mp \sim \sqrt{d}(\sqrt{d} \pm 1) \ln a \tag{2.48}$$

and, in conformal time,

$$a_\mp(\eta) \sim |\eta|^{\mp 1/(\sqrt{d}\pm 1)} \tag{2.49}$$

Note that for $\gamma = 1/d$, and $t \to +\infty$, eq.(2.46) describes the standard, radiation-dominated cosmology with $\phi = const$; the dual case, $\gamma = -1/d$ and $t \to 0$, with $t < 0$, describes instead a typical pre-big-bang configuration [2], with a superinflationary expansion driven by a perfect gas of stretched strings [27]. The dual solution obtained through a more general $O(d,d)$ transformation, applied to the radiation case, corresponds to a non-diagonal metric and an effective viscosity in the source stress tensor, and has been discussed in Ref.[20].

We note, finally, that the solution presented in this section is given explicitly in the Brans-Dicke (BD) frame, whose metric coincides with the σ-model metric to which strings are directly coupled. The passage to the Einstein (E) frame, defined as the frame in which the graviton and dilaton kinetic terms are diagonalized, and the action takes the canonical form

$$S_E = \frac{1}{16\pi G_D} \int d^D x \sqrt{|g_E|} \left[-R(g^E_{\mu\nu}) + \frac{1}{2} g_E^{\mu\nu} \partial_\mu \phi_E \partial_\nu \phi_E \right] \tag{2.50}$$

is obtained through the conformal rescaling

$$g^E_{\mu\nu} = g_{\mu\nu} e^{-2\phi/(d-1)}, \qquad \phi_E = \sqrt{\frac{2}{d-1}}\phi \tag{2.51}$$

The E-transformed scale-factor, a_E, and the cosmic time coordinate, t_E, are thus related to the original BD ones by

$$a_E = a e^{-\phi/(d-1)}, \qquad dt_E = dt e^{-\phi/(d-1)} \tag{2.52}$$

The asymptotic limit (2.46) of the previous general solution thus becomes, in the E frame,

$$a_E(t_E) \sim |t_E|^\beta, \qquad \phi_E \sim \sqrt{\frac{2}{d-1}} \frac{(d-1)(1-d\gamma)}{\gamma - 1} \ln a_E$$

$$\rho_E \sim a_E^{-2/\beta}, \qquad \beta = \frac{2(1-\gamma)}{(d-1)(1+d\gamma^2) - 2(d\gamma - 1)} \tag{2.53}$$

where ρ_E is conformally related to the BD energy density ρ by

$$\rho_E = \rho \frac{\sqrt{|g|}}{\sqrt{|g_E|}} = \rho e^{\phi(d+1)/(d-1)} \tag{2.54}$$

In conformal time,

$$a_E(\eta) \sim |\eta|^{-2(\gamma-1)/(d-1)(1-2\gamma+d\gamma^2)} \tag{2.55}$$

(note that the conformal time coordinate is the same in the E and BD frame,

$$d\eta_E = \frac{dt_E}{a_E(t_E)} = \frac{dt}{a(t)} = d\eta \tag{2.56}$$

because of eq.(2.52)). The high curvature limit (2.48) becomes, in the E frame,

$$a_\mp^E(t_E) \sim |t_E|^{1/d}, \qquad \phi_\mp^E \sim \mp\sqrt{2d(d-1)}\ln a_E \tag{2.57}$$

and, in conformal time,

$$a_\mp^E(\eta_E) \sim |\eta|^{1/(d-1)} \tag{2.58}$$

It should be stressed that the radiation-dominated solution, with $\gamma = 1/d$, $\phi = const$, is obviously the same in both frames, see eqs.(2.53) and (2.48). We note also that in the vacuum, dilaton-dominated case, the duality transformation which is represented in BD frame by the inversion of the scale factor, with related dilaton shift,

$$a_\pm \to a_\mp = a_\pm^{-1}, \qquad \phi_\pm \to \phi_\mp = \phi_\pm - 2d\ln a_\pm \tag{2.59}$$

in the E frame becomes a transformation between the weak coupling and the strong coupling regime,

$$\phi_\pm^E \to \phi_\mp^E = -\phi_\pm^E, \tag{2.60}$$

without changing the metric background.

Concluding this section, we want to stress that the solutions discussed so far describe the situation in which the dilaton potential can be neglected, namely the background evolution at early enough times when the effective coupling e^ϕ is small enough. Indeed, because of non- renormalization theorems, the potential is expected to appear at the non-perturbative level only, and has to be extremely small ($V(\phi) \sim exp[-exp(\phi)]$) in the weak coupling regime. At later times, and large couplings, the main effect of the dilaton potential will be taken into account

in the form of a dilaton mass term (see Sections 3,5), which freezes the Newton constant at its present value.

3. Scalar perturbations with dilaton and perfect fluid sources.

In order to obtain the equations governing the classical evolution of scalar perturbations, we choose to work in the Einstein frame, where the explicit form of the equations is simpler. This is a legitimate choice since, as we shall see at the end of this Section, the scalar perturbation spectrum, just like the graviton spectrum [23], is the same in the Einstein and Brans-Dicke frames.

In the E frame, the background field equations (with $B_{\mu\nu} = 0$ but with a non-vanishing dilaton potential V) take the form

$$2R_\mu{}^\nu - \delta_\mu^\nu R = \partial_\mu\phi\partial^\nu\phi + \delta_\mu^\nu\left[V - \frac{1}{2}(\bigtriangledown\phi)^2\right] + T_\mu{}^\nu \tag{3.1}$$

$$\bigtriangledown_\mu \bigtriangledown^\mu \phi + \frac{\partial V}{\partial\phi} + cT = 0 \tag{3.2}$$

where $c = \sqrt{2/(d-1)}$ (the Einstein frame index, E, will be omitted throughout this Section). The coupling of the dilaton to the stress tensor of the matter sources is fixed by the conformal rescaling (2.51), (2.54). We start, for simplicity, with a $(d+1)$-dimensional isotropic background, with perfect fluid sources,

$$g_{\mu\nu} = diag(1, -a^2\delta_{ij}), \qquad \phi = \phi(t)$$

$$T_\mu{}^\nu = (\rho + p)u_\mu u^\nu - p\delta_\mu^\nu, \qquad u^\mu = \delta_0^\mu \tag{3.3}$$

and we consider the pure scalar part of the metric perturbations, $\delta g_{\mu\nu} \equiv h_{\mu\nu}$, together with the perturbations of the dilaton background, $\delta\phi \equiv \chi$, and of the matter sources, $\delta\rho, \delta p, \delta u^\mu$ (in the linear approximation scalar, vector and tensor perturbations are decoupled, and evolve independently). We use here for the metric the Bardeen variables Φ, ψ, that are invariant under those infinitesimal coordinate transformations which preserve the scalar nature of the fluctuations [28-30]. In the longitudinal gauge we have thus the first-order expressions [30]

$$h_{00} = 2\Phi = h^{00}, \qquad h_{0i} = 0$$

$$h_{ij} = 2a^2\psi\delta_{ij}, \qquad h^{ij} = \frac{2}{a^2}\psi\delta^{ij}$$

$$\delta T_0{}^0 = \delta\rho, \qquad \delta T_i{}^j = -\delta_i^j \delta p, \qquad \delta T_i{}^0 = \frac{\rho + p}{a}\delta u_i \tag{3.4}$$

These are to be inserted into the first order perturbation of the Einstein equations (3.1),

$$\begin{aligned} -2h^{\nu\alpha}R_{\mu\alpha} + 2g^{\nu\alpha}\delta R_{\mu\alpha} - \delta^\nu_\mu(g^{\alpha\beta}\delta R_{\alpha\beta} - h^{\alpha\beta}R_{\alpha\beta}) = \\ = -h^{\nu\alpha}\partial_\mu\phi\partial_\alpha\phi + \partial_\mu\chi\partial^\nu\phi + \partial_\mu\phi\partial^\nu\chi + \delta^\nu_\mu\frac{\partial V}{\partial\phi}\chi - \\ -\frac{1}{2}\delta^\nu_\mu(2g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\chi - h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) + \delta T_\mu{}^\nu \end{aligned} \tag{3.5}$$

and of the dilaton equation (3.2)

$$-h^{\alpha\beta}\bigtriangledown_\alpha\bigtriangledown_\beta\phi + \bigtriangledown_\mu\bigtriangledown^\mu\chi - g^{\alpha\beta}(\delta\Gamma_{\alpha\beta}{}^\mu)\partial_\mu\phi + \frac{\partial^2 V}{\partial\phi^2}\chi + \frac{c}{2}(\delta\rho - d\delta p) = 0 \tag{3.6}$$

Here the covariant derivatives are to be performed with respect to the background metric $g_{\mu\nu}$, and $\delta R_{\mu\nu}, \delta\Gamma$ are to be computed to first order in $h_{\mu\nu}$.

By using the background field equations, the (i,j) component of eq.(3.5), with $i \neq j$, gives

$$\Phi = (d-2)\psi \tag{3.7}$$

which allows us to eliminate everywhere one of the two Bardeen's variables. The $(i,0)$ component gives a constraint which can be written, in terms of the conformal time η, as

$$\partial_i\left\{2(d-1)\left[\frac{a'}{a}(d-2)\psi + \psi'\right] - \chi\phi'\right\} = (\rho + p)a\delta u_i \tag{3.8}$$

(a prime denotes differentiation with respect to η). The $(0,0)$ component provides an expression for the density perturbation $\delta\rho$ in terms of the scalar variables ψ and χ,

$$\begin{aligned} \bigtriangledown^2\psi - d\frac{a'}{a}\psi' - \left[d(d-2)(\frac{a'}{a})^2 - \frac{d-2}{2(d-1)}\phi'^2\right]\psi = \\ = \frac{1}{2(d-1)}(\phi'\chi' + \frac{\partial V}{\partial\phi}a^2\chi + a^2\delta\rho) \end{aligned} \tag{3.9}$$

Finally, the (i,i) component of eq.(3.5) and the perturbed dilaton equation (3.6) give, respectively,

$$\begin{aligned} \psi'' + (2d-3)\frac{a'}{a}\psi' + \left\{(d-2)\left[2\frac{a''}{a} + (d-4)(\frac{a'}{a})^2\right] + \frac{d-2}{2(d-1)}\phi'^2\right\}\psi = \\ = \frac{1}{2(d-1)}(\phi'\chi' - \frac{\partial V}{\partial\phi}a^2\chi + a^2\delta p) \end{aligned} \tag{3.10}$$

$$\chi'' + (d-1)\frac{a'}{a}\chi' + (a^2\frac{\partial^2 V}{\partial\phi^2} - \bigtriangledown^2)\chi =$$

$$= 2(d-1)\phi'\psi' - 2(d-2)\left[\frac{c}{2}(\rho - dp) + \frac{\partial V}{\partial\phi}\right]a^2\psi - \frac{c}{2}a^2(\delta\rho - d\delta p) \qquad (3.11)$$

The linear system formed by the four coupled equations (3.8)-(3.11) determines the classical evolution of four independent perturbation variables $\psi, \chi, \delta\rho$ and δu (an additional relation between δp and $\delta\rho$ is to be provided by the detailed model of matter sources). In the absence of dilaton background ($\phi = 0 = \chi$) one recovers the usual system of equations for hydrodynamical perturbations [30], while in the absence of fluid sources ($T^\nu_\mu = 0 = \delta T^\nu_\mu$) one has the usual perturbation system for a scalar field minimally coupled to the metric [30].

When ϕ and $T_{\mu\nu}$ are both non-vanishing, and δp can be parametrized in terms of $\delta\rho$ as $\delta p = \epsilon(t)\delta\rho$, we may eliminate $\delta\rho$ by means of eq.(3.9), and the system reduces to a pair of second-order differential equations (3.10), (3.11) for the coupled variables ψ and χ. By introducing the bi-dimensional vector

$$Z = \begin{pmatrix} \psi \\ \chi \end{pmatrix} \qquad (3.12)$$

and by the dilaton background as

$$\phi = \beta \ln a, \qquad\qquad \beta = const \qquad (3.13)$$

the abovementioned pair of equations can be represented in compact form as

$$Z_k'' + 2\frac{a'}{a}AZ_k' + (k^2B + C)Z_k = 0 \qquad (3.14)$$

where $Z_k = (\psi_k, \chi_k)$ represents the Fourier component of the perturbation variables, $\bigtriangledown^2 Z_k = -k^2 Z_k$, and

$$A = \begin{pmatrix} \frac{1}{2}(2d - 3 + d\epsilon) & -\frac{1-\epsilon}{4(d-1)}\beta \\ -(d-1)[\frac{cd}{2}(1 - d\epsilon) + \beta] & \frac{1}{2}(d-1) - \frac{c}{4}(1 - d\epsilon)\beta \end{pmatrix}$$

$$B = \begin{pmatrix} \epsilon & 0 \\ -c(1 - d\epsilon)(d-1) & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 2(d-2)\frac{a''}{a} + (d-2)[d - 4 + d\epsilon + \frac{1-\epsilon}{2(d-1)}\beta^2](\frac{a'}{a})^2 & 0 \\ -cd(d-2)(\gamma - \epsilon)[d(d-1) - \frac{\beta^2}{2}](\frac{a'}{a})^2 & 0 \end{pmatrix} \qquad (3.15)$$

(we have neglected here the possible contribution of the dilaton potential, by putting $\partial V/\partial\phi = 0 = \partial^2 V/\partial\phi^2$). We note that a system of coupled scalar perturbation equations similar to (3.14) was previously considered also in Refs.[31,32]

where, however, a scalar field model of source ("inflaton" matter) was used, instead of the fluid model adopted in this paper.

Without further approximations, ψ and χ are thus in general non-trivially mixed, with time-dependent mixing coefficients determined by the explicit model of sources, $\gamma = p/\rho$, $\epsilon = \delta p/\delta\rho$, and by the background kinematics, $a(t)$, $\phi(t)$, according to eq.(3.14). The solution of (3.14) provides in turn, for any given background configuration, a unique determination of the density contrast $\delta\rho/\rho$ through eq.(3.9), and of the velocity perturbation δu_i through eq.(3.8).

Eqs.(3.8)-(3.11) are linear in the perturbations and just describe their classical evolution without specifying their absolute magnitude. As clearly stressed in Ref.[30] (see also Refs.[33,34]), in order to determine the absolute magnitude of the vacuum fluctuations and their spectral distribution, one must express the perturbations in terms of the correctly normalized variables satisfying canonical commutation relations. These can be determined by expanding the action to second order in the fluctuations.

For the pure metric-scalar field system ($T^\nu_\mu = 0$) such a canonical variable is known to be fixed by the following linear combination of χ and ψ [35-37]

$$v = a\chi + z\psi, \qquad\qquad z = \frac{a^2\phi'}{a'}. \tag{3.16}$$

For a pure fluid source ($\phi = 0$), with constant ϵ, the canonical variable is instead [38,39]

$$w = \frac{1}{6}(f - \xi\psi), \qquad\qquad \xi = \frac{a^3}{a'}\sqrt{\frac{\rho + p}{\epsilon}} \tag{3.17}$$

where f is the velocity potential determining the fluid perturbations as

$$\begin{aligned} a\sqrt{\rho + p}\ \delta u_i &= -\sqrt{\epsilon}\ \partial_i f \\ a^2\delta\rho &= -\frac{a^2}{\epsilon}(\rho + p)\psi - \frac{1}{a^2}\left(\frac{a^2 f\sqrt{\rho + p}}{\sqrt{\epsilon}}\right)' \end{aligned} \tag{3.18}$$

(we have assumed $d = 3$ in the previous three equations). The variables v, w play the role of "normal coordinates" decoupling the system of perturbation equations, and reducing the action to the free scalar field form [30]. Only when ψ is expressed in terms of such variables one gets a canonical normalization of the Fourier modes ψ_k, and then the correlation function for the metric fluctuations

$$\langle\psi(x)\psi(x')\rangle = \int \frac{dk}{k}\frac{\sin kr}{kr}|\delta_\psi(k)|^2 \tag{3.19}$$

provides the correct spectral distribution for the metric

$$|\delta_\psi(k)|^2 = k^3|\psi_k|^2 \tag{3.20}$$

and for the dilaton, $\delta_\chi(k)$, through eq.(3.16).

If T_μ^ν and ϕ are both non-vanishing, one could try a perturbative approach to the spectrum (like in Refs.[32,40]), by keeping fixed the definitions of v and w as a zeroth-order approximation. In such case, the constraint (3.8) gives (in $d = 3$)

$$\psi' + \frac{a'}{a}\psi = \frac{1}{4}\phi'\chi - \frac{1}{4}\sqrt{\epsilon(\rho + p)}f \tag{3.21}$$

By eliminating f and χ in terms of v and w through eqs.(3.16), (3.17), by using the constraint (3.21) and the background field equations, one can then express the Fourier mode ψ_k, from eq.(3.9), as

$$\psi_k = \psi_k(v, v', w, w', k) \tag{3.22}$$

Moreover, the system of equations formed by eq.(3.11) and by the combination of eqs.(3.9), (3.10) obtained by eliminating $\delta\rho$, can be written as a system of two second-order differential equations for the coupled modes v_k and w_k. Its solutions, when inserted into eq.(3.22), provide a first approximation to the scalar perturbation spectrum (3.20). From eq.(3.17) one has then the corresponding dilaton spectrum, $|\delta_\chi| = k^{3/2}|\chi_k|$, and from eqs.(3.18), (3.17) the density perturbation spectrum $|\delta_\rho| = k^{3/2}|(\delta\rho/\rho)_k|$.

In general, dilaton and metric fluctuations will have different spectral distributions, $|\delta_\psi| \neq |\delta_\chi|$. The coupled system of equations is rather complicated, but it seems possible, in principle, to obtain a large variety of spectra as the equation of state and the ratio $\delta p/\delta\rho$ are appropriately varied [41].

In this paper we shall consider a model (see Sec.4 for its motivations) in which the universe evolves from a three-dimensional, dilaton-dominated phase of the pre-big-bang type (with negligible fluid sources, $T_\mu^\nu = 0 = \delta T_\mu^\nu$), to the standard radiation-dominated phase ($p = \rho/3$), adiabatic ($\epsilon = 1/3$), and with frozen Newton constant ($\phi = const$). More complicated scenarios will be analyzed in future works [41]. The phase of pre-big-bang inflation is assumed to extend in time from $-\infty$ up to the time $\eta = -\eta_1 < 0$, which marks a sudden transition to the phase of radiation dominance. For $\eta < -\eta_1$ the constraint (3.8) thus becomes

$$\psi_k' + \frac{a'}{a}\psi_k' = \frac{1}{4}\phi'\chi_k \tag{3.23}$$

where, according to eq.(3.16),

$$\chi_k = \frac{v_k}{a} - \frac{z}{a}\psi_k \tag{3.24}$$

When the constraint is inserted into eq.(3.9), and eq.(3.24) is used in order to eliminate χ and χ', we are led to a relation of the form (3.22), namely

$$\psi_k = -\frac{1}{4k^2}\phi'\frac{z}{a}\left(\frac{v_k}{z}\right)' \tag{3.25}$$

In the absence of matter sources, eq.(3.11) becomes equivalent to the combination of eqs.(3.9) and (3.10). By expressing χ_k in terms of v_k according to eq.(3.24), and by eliminating ψ', ψ'' through eq.(3.23), we finally get the canonical perturbation equation [30], valid for $\eta < -\eta_1$,

$$v_k'' + (k^2 - \frac{a''}{a})v_k = 0 \tag{3.25}$$

In the second, radiation-dominated phase ($\eta > -\eta_1$), we assume that the dilaton acquires a mass m, and it stays frozen at the minimum of the potential (with possible small oscillations around it), so that

$$V = 0 = \frac{\partial V}{\partial \phi}, \qquad\qquad \frac{\partial^2 V}{\partial \phi^2} = m^2 \tag{3.26}$$

In this case χ decouples from the metric fluctuations (see eqs.(3.8)-(3.11)), that are coupled now to the fluid perturbations only; the canonical variable for their quantization is thus given by eq.(3.17). As $a''/a = 0$ in the radiation phase, it turns out however that for $\eta > -\eta_1$ both w and $a\chi$ satisfy the free oscillator equation, $w''/w = const$ (apart from the dilaton mass term, assumed to be negligible at early enough times, see Sec.5). As a consequence, ψ and χ will have the same spectrum (identical, in this case, to the tensor perturbation spectrum), which can be computed by adopting a second quantization approach, regarding the amplification of the perturbations as a process of particle production from the vacuum, under the action of the cosmological background fields [30].

The Bogoliubov coefficients $c_\pm$ for such a process are obtained by matching the solution of eq.(3.25) to a general solution of the plane-wave type,

$$v_k = \frac{1}{\sqrt{k}}(c_+ e^{-ik\eta} + c_- e^{ik\eta}) \tag{3.27}$$

valid for $\eta > -\eta_1$. By assuming, for $\eta < -\eta_1$, that

$$a \sim (-\eta)^{-\alpha}, \qquad\qquad \phi = \beta \ln a \tag{3.28}$$

we have

$$\frac{z''}{z} = \frac{a''}{a} = \frac{\alpha(\alpha+1)}{\eta^2} \tag{3.29}$$

The solution of eq.(3.25) describing oscillations with positive frequency at $\eta = -\infty$, and defining the initial vacuum state, is thus given in terms of the second kind Hankel function, $H^{(2)}$, as

$$v_k = \eta^{1/2} H_\nu^{(2)}(k\eta), \qquad\qquad \nu = \alpha + \frac{1}{2} \tag{3.30}$$

The continuity of v_k and v_k' at the transition time $\eta = -\eta_1$ fixes the Bogoliubov coefficient $c_-(k)$, and the corresponding expectation number of particles produced in the mode k. For $k \lesssim 1$ we obtain

$$\langle n(k) \rangle = |c_-(k)|^2 \simeq (k\eta_1)^{-2|\nu|-1} \tag{3.31}$$

(higher mode production turns out to be exponentially suppressed [3, 42], and can be neglected for the purpose of this paper). In terms of the proper frequency $\omega = k/a$, the energy density ρ_χ of the produced dilatons is thus characterized by a spectral distribution $\omega d\rho_\chi/d\omega \simeq \omega^4 |c_-|^2$ which may be written, in units of critical density $\rho_c = H^2/G$,

$$\Omega_\chi(\omega, t) = \frac{\omega}{\rho_c}\frac{d\rho_\chi}{d\omega} \simeq \frac{G\omega^4}{H^2}\left(\frac{\omega}{\omega_1}\right)^{-2-2\alpha} = GH_1^2\left(\frac{\omega}{\omega_1}\right)^{2-2\alpha}\left(\frac{H_1}{H}\right)^2\left(\frac{a_1}{a}\right)^4 \tag{3.32}$$

where $\omega_1(t) = H_1 a_1/a(t)$ is the proper frequency of the highest excited mode (here we have supposed $\alpha \geq -1/2$). This is the same spectrum as that obtained in the graviton case [43], with an intensity normalized to the final inflation scale H_1. It is growing for a phase of superinflationary pre-big-bang expansion ($\alpha < 1$), flat for de Sitter ($\alpha = 1$), and decreasing for power-law inflation ($\alpha > 1$).

It should be stressed that this second quantization approach is convenient to discuss the squeezing properties of the produced radiation [3,34,44-47] but, for what concerns the perturbation spectrum, it is completely equivalent to the more traditional approach in which one computes the parametric amplification of the perturbation amplitude. In this second approach one has indeed, according to the "effective potential" z''/z of eq.(3.25), a mode amplitude which is constant, $|v_k| \simeq 1/\sqrt{k}$, in the initial region $\eta \to -\infty$ where $k^2 >> |z''/z| \simeq \eta^{-2}$, and which grows with power-like behaviour in η in the non-oscillatory region defined by $k^2 << |z''/z|$ (in the subsequent radiation era the solution for v is again oscillating, with frozen amplitude). In the non-oscillatory region

$$v_k = c_1 z + c_2 z \int^\eta \frac{d\eta'}{z^2(\eta')} - k^2 z \int^\eta \frac{d\eta'}{z^2(\eta')} \int^{\eta'} d\eta'' z^2(\eta'') + O(k^4) \tag{3.33}$$

(c_1, c_2 are integration constants) is the general solution of eq.(3.25) to first order in k^2 (the first sub-leading term has been included to have non-trivial derivative of v/z). This gives, for the background (3.28) (with obvious redefinition of c_1, c_2, and introducing a further numerical constant c_3),

$$v_k = c_1|\eta|^{-\alpha} + c_2|\eta|^{1+\alpha} - c_3 k^2 |\eta|^{2-\alpha} \tag{3.34}$$

For $\alpha > 0$ (inflationary expansion) the first term is the dominant one in the the $|\eta| \to 0$ limit, and the wave amplification achieved in this limit can thus be estimated as [30]

$$v_k \simeq \frac{v_k(\eta)}{\sqrt{k}} \left(\frac{1}{v_k}\right)_{k\simeq aH} \simeq \frac{z}{\sqrt{k}} \left(\frac{1}{v_k}\right)_{k\simeq aH} [1 - \frac{k^2\eta^2}{2(1-2\alpha)}] \tag{3.35}$$

The variable at the denominator is to be evaluated at the time $\eta \simeq k^{-1}$, where the mode k "hits" the effective potential barrier $z''/z \simeq \eta^{-2}$ (otherwise stated: at the time of first horizon crossing, when $H = \omega$). By inserting this value of v into eqs.(3.25) and (3.20), and recalling the definition of z, we are led to

$$|\delta_\psi(k)|^2 \simeq k^3 |\psi_k|^2 \simeq \left(\frac{k}{z}\right)^2_{k\simeq aH} \simeq \left(\frac{H^2}{\dot\phi}\right)^2_{k\simeq aH} \tag{3.36}$$

which is the standard expression for the scalar perturbation spectrum [35,48] associated with the inflation-radiation transition (see Sec.4 for a proof of the fact that the same result is recovered in the case of contracting backgrounds). The same spectrum is obtained for the dilaton perturbations as we have, from eq.(3.24),

$$|\delta_\chi|^2 = k^3 |\frac{v_k}{a} - \frac{z}{z}\psi_k|^2 \simeq \left(\frac{k}{z}\right)^2_{k\simeq aH} \simeq |\delta_\psi|^2 \tag{3.37}$$

It is important to stress that this expression, when multiplied by G, exactly coincides with the spectral energy density (3.32) (modulo numerical factors of order unity), evaluated in the radiation era. Indeed, multiplying and dividing eq.(3.37) by $H_1^2 \simeq (a_1\eta_1)^{-2}$ we have

$$\begin{aligned} G|\delta_\chi|^2 &\simeq G\left(\frac{H^2}{\dot\phi}\right)^2_{k\simeq aH} \simeq \left(\frac{G}{a\eta}\right)^2_{k\simeq aH} \simeq GH_1^2 \left(\frac{a_1\eta_1}{a\eta}\right)^2_{k\simeq aH} \\ &\simeq GH_1^2 (k\eta_1)^{2-2\alpha} = GH_1^2 \left(\frac{\omega}{\omega_1}\right)^{2-2\alpha} \simeq \left(\frac{\omega}{\rho_c}\frac{d\rho_\chi}{d\omega}\right)_{rad} \end{aligned} \tag{3.38}$$

in agreement with eq.(3.32) for $a \sim t^{1/2} \sim H^{-1/2}$.

As already stressed in Ref.[14] for the tensor perturbation case, we want to remark finally that the scalar perturbation spectrum is the same in the E and BD frame, as a consequence of the equality of the two conformal time coordinates (see Sec.2). Indeed, quite independently of the computational method (first or second quantization) the spectral behaviour of the energy density is fixed by the Bessel index ν of the solution of eq.(3.25), which depends, in turn, on the slope of the effective potential z''/z. For a generic $d = 3$ background in the E frame we have (recall eq.(2.55))

$$\alpha(\gamma) = \frac{\gamma - 1}{1 - 2\gamma + 3\gamma^2} \tag{3.39}$$

so that, according to eq.(3.29)

$$\left(\frac{z''}{z}\right)_E = \left(\frac{a''}{a}\right)_E = \frac{\gamma}{\eta_E^2} \frac{(\gamma - 1)(3\gamma - 1)}{(1 - 2\gamma + 3\gamma^2)^2} \tag{3.40}$$

Eq.(3.25) for v_k is not conformally invariant, and in the BD frame the effective potential becomes

$$\left(\frac{z''}{z}\right)_{BD} = \left(\frac{z''}{z}\right)_E (a_{BD}, \phi_{BD}) = \left(\frac{a''}{a}\right)_{BD} - \phi'_{BD} \left(\frac{a'}{a}\right)_{BD} - \frac{1}{2}\phi''_{BD} + \frac{1}{4}\phi'^2_{BD} \tag{3.41}$$

In this frame, however, the spectrum is determined by the conformally transformed backgrounds, namely by the solutions (2.46),(2.47) of the BD field equations. By inserting their explicit expressions for $d = 3$ we get

$$\left(\frac{z''}{z}\right)_{BD} = \frac{\gamma}{\eta_{BD}^2} \frac{(\gamma - 1)(3\gamma - 1)}{(1 - 2\gamma + 3\gamma^2)^2} \tag{3.42}$$

which coincides with the effective E-frame potential (3.40) because of the equality $\eta_{BD} = \eta_E$. The same results holds for a dilaton-driven evolution, described by the solution (2.57) and their BD-transformed expressions.

4. Pre-big-bang scenario in the Brans-Dicke and Einstein frame.

As seen in the previous Section, the spectral distribution of the perturbations is uniquely fixed by the explicit form of the background solution. The time-evolution of the background fields is determined, in turn, by the particular model of matter sources. As in our previous work [20,27], our model of sources consists of

a sufficiently diluted gas of classical fundamental strings whose mutual interactions are described, in a mean-field approximation sense, as the interaction of each single string with the background generated by all the others according to the tree-level effective action (2.4). The source stress-tensor appearing in eq.(2.1) is thus given by a sum over all strings (labelled by i) of the stress-tensor of each individual string $T_i^{\mu\nu}$, where

$$T_i^{\mu\nu}(x) = \frac{1}{\pi\alpha'\sqrt{|g|}} \int d\sigma d\tau \left(\frac{dX_i^\mu}{d\tau}\frac{dX_i^\nu}{d\tau} - \frac{dX_i^\mu}{d\sigma}\frac{dX_i^\nu}{d\sigma}\right)\delta^D(X-x) \qquad (4.1)$$

and, for each i, the coordinates X^μ satisfy the string equations of motion in the given background,

$$\frac{d^2X^\mu}{d\tau^2} - \frac{d^2X^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}\left(\frac{dX^\alpha}{d\tau} + \frac{dX^\alpha}{d\sigma}\right)\left(\frac{dX^\beta}{d\tau} - \frac{dX^\beta}{d\sigma}\right) = 0$$

$$g_{\mu\nu}\left(\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} + \frac{dX^\mu}{d\sigma}\frac{dX^\nu}{d\sigma}\right) = 0, \quad g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\sigma} = 0 \qquad (4.2)$$

Here $(2\pi\alpha')^{-1}$ is the string tension, $\Gamma^\alpha_{\mu\nu}$ the Christoffel symbol for the background metric $g_{\mu\nu}$, τ and σ the usual world-sheet time and space variables (we are using the gauge in which the world-sheet metric is conformally flat).

The general exact solution of the system of equations (2.1)-(2.3), (4.1), (4.2) is hard to find and certainly impossible to express in closed form. In some appropriate asymptotic regime, however, the solution of the string equations of motion, when inserted into the energy-momentum tensor (4.1), provides an effective equation of state which allows to describe the string sources in the perfect fluid approximation [20,27], and to recover then the general background solutions of Sec.2. The cosmological solution we are looking for is characterized in particular by having, as initial configuration, the string perturbative vacuum, namely flat spacetime with vanishing torsion and coupling constant, $H_{\mu\nu\alpha} = 0$, $\phi = -\infty$. In this regime strings move freely, do not decay and behave as a pressureless gas with an energy density ρ. We shall assume ρ to be small enough initially so that, as we shall see, it will represent a negligible source of curvature. On the other hand a finite ρ is certainly sufficient to make the dilaton evolve away from the perturbative minimum.

Indeed, the negative branch ($x \leq x_-$) of the general background solution with perfect fluid sources, eqs.(2.37)-(2.31), may be written in the case of vanishing pressure ($\gamma_i = 0$) as

$$a_i(t) = a_{i0}\left|\frac{t-2T}{t}\right|^{\alpha_i}, \quad e^{\overline{\phi}} = \frac{16L^2e^{-\phi_0}}{|(t(t-2T)|}, \quad \overline{\rho} = \frac{1}{L}\frac{dx}{dt} = \frac{e^{\phi_0}}{4L^2} = const$$

$$\alpha_i = \frac{t_i}{T}, \quad T = \left(\sum_i t_i^2\right)^{1/2}, \quad t \leq 0 \tag{4.3}$$

(t_i are integration constants, and we have performed a time translation to shift the singularity from $x = x_-$ to the origin, by choosing $x_0 = -T(e^{\phi_0}/4L)$). This background is certainly consistent with the solution of the string equations of motion (4.2) in the $t \to -\infty$ limit. Indeed, in this limit, the metric is flat,

$$a_i = const, \quad \phi \sim -2\ln(-t), \quad \rho = const \tag{4.4}$$

and the solutions of eqs.(4.2) are characterized by $\sum_i (dx^i/d\tau)^2 = \sum_i (dx^i/d\sigma)^2$. Eq.(4.1) gives then $T_0^0 = const$, $T_i^i = 0$, namely a stress tensor describing dust-like matter in the perfect fluid approximation. For $t \sim -T$, however, the curvature scale begins to increase, the string sources progressively enter a non-oscillating unstable regime [27], and one must then take into account the fact that the ratios $\gamma_i = p_i/\rho$ begin to evolve in time.

In connection with this last point we note that the solution (4.3), which, for $t \leq 0$, gives

$$H_i = \frac{2t_i}{t(t-2T)}, \quad \dot{\overline{\phi}} = -\frac{2(t-T)}{t(t-2T)}, \quad \overline{\rho} e^{\overline{\phi}} = \frac{4}{t(t-2T)} = \rho e^{\phi}\ , \tag{4.5}$$

is characterized by two scales. One is the curvature scale ($|H_T| \sim T^{-1}$ at $t \sim -T$) at which the transition from flat to curved spacetime regime occurs and inflation begins. At $t \sim -T$, the curvature H^2 is of the same order as $\dot{\overline{\phi}}^2$ or ρe^{ϕ} while, much earlier, it was negligible. By contrast, much later than $t = -T$, it is ρe^{ϕ} which becomes negligible and one recovers the vacuum solutions. T is a free phenomenological parameter of the solution. The other scale is the maximal scale $|H_1| \sim t_1^{-1}$, at the time $t \sim -t_1$, after which the solution is no longer valid, because higher orders in α' have to be added to the low energy effective action (2.4). This final scale t_1 is thus determined by the string tension as $t_1 \simeq \sqrt{\alpha'} = \lambda_s$, where λ_s is the fundamental (minimal) length parameter of string theory [49], which may be assumed to coincide roughly with the present value of the Planck length, $l_p = M_p^{-1}$. The important point to stress is that, in any realistic inflationary scenario, T and λ_s cannot be of the same order as we will now show.

When $|t| < T$, the solution describes an accelerated evolution given asymptotically by

$$a_i(t) \sim (-t)^{-\alpha_i}, \quad |\alpha_i| < 1, \quad \sum_i \alpha_i^2 = 1 \tag{4.6}$$

and which is of the type given in eq.(2.42) (we call " accelerated" a configuration in which $\dot{a}$, $\ddot{a}$ and $\dot{H}$ have the same sign, positive for expansion, negative for contraction [2,23,27]). In this metric, the particle horizon along any given spatial direction,

$$d_p^i(t) = a_i(t) \int_{-\infty}^{t} dt' a_i^{-1}(t') \tag{4.7}$$

evolves for $|t| << T$ like the scale factor, $d_p^i \sim a_i$, while the event horizon

$$d_e^i(t) = a_i(t) \int_{t}^{0} dt' a_i^{-1}(t') \tag{4.8}$$

shrinks linearly in time, $d_e^i \sim (-t)$, for $t \to 0$. The ratio of the two proper sizes $r^i(t) = d_p^i/d_e^i$ thus grows in time, for $|t| << T$, as $(-t)^{-\alpha_i - 1} \sim (-\eta)^{-1}$. On the other hand, the horizon problem of the standard cosmological model [50] is solved if, for every spatial direction, the growth of the ratio $r^i(t)$ when $|t|$ is ranging from T to t_1, is large enough to compensate the decreasing of the ratio in the subsequent decelerated phase down to the present time t_0. This implies, in the hypothesis that the pre-big-bang era is followed by the standard radiation-dominated (until $t = t_2$) and matter-dominated evolution,

$$\left(\frac{t_1}{T}\right)^{-\alpha_i - 1} \simeq \left(\frac{H_T}{H_1}\right)^{-\alpha_i - 1} \gtrsim \left(\frac{a_1}{a_0}\right)\left(\frac{t_0}{t_1}\right) =$$

$$= \left(\frac{t_1}{t_2}\right)^{-1/2}\left(\frac{t_2}{t_0}\right)^{-1/3} = \left(\frac{H_1}{H_2}\right)^{1/2}\left(\frac{H_2}{H_0}\right)^{1/3} \simeq 10^{30}\sqrt{\frac{H_1}{M_p}} \tag{4.9}$$

(the same condition is required to solve the flatness problem, see below). For an expanding d-dimensional isotropic background $\alpha_i = 1/\sqrt{d}$ (see eq.(2.48)), and the previous condition gives in particular, for $t_1 \simeq \lambda_s \simeq M_p^{-1}$,

$$T \gtrsim 10^{30\sqrt{d}/(\sqrt{d}+1)}\lambda_s, \quad H_T \lesssim 10^{-30\sqrt{d}/(\sqrt{d}+1)} M_p \tag{4.10}$$

We shall thus assume that the scale T appearing in the solution (4.3) is much larger than the string scale $\lambda_s \simeq M_p^{-1}$.

This fact has an important consequence. In this case the background (4.3) becomes in fact a good zeroth-order approximation to the general solution of the full system of equations, consistent with the string equations of motion not only in the asymptotic limit $t \to -\infty$.

By adopting an iterative approach, let us assume indeed the solution (4.3) to be a zeroth-order approximation, and let us compute the first order corrections by

inserting that solution into the string equations of motion, in order to obtain the corresponding value of $\gamma_i(t)$. To this aim we observe that the given background is characterized, asymptotically, by an accelerated metric with shrinking event horizons (see eq.(4.6)). We recall that, in such a background, the string equations of motion admit oscillating solutions, corresponding to strings with constant proper size L_s, provided L_s is smaller than the size of the event horizon $\sim H^{-1}(t)$ (stable strings), while the solutions describe non oscillating strings with $L_s(t) \sim a(t)$ if $L_s > H^{-1}$ (unstable strings) [27].

The evolution of a network of strings with some initial distribution in backgrounds of the type discussed above can be investigated [51]. One can show that the number $n(L_s, t)$ of strings (per unit length) of given size L_s, at time t, must satisfy in the given background the approximate evolution equation [51]

$$\frac{\partial n}{\partial t} = -H \frac{\partial}{\partial L_s}[n L_s \theta(L_s - H^{-1})] \tag{4.11}$$

where θ is the Heaviside step function. Its general solution can be written in implicit form as [51]

$$n(L_s, a(H)) = n_0(L_s)\theta(H^{-1} - L_s) + f\left(\frac{L_s}{a}\right)\theta(L_s - H^{-1}),$$

$$f\left(\frac{H^{-1}}{a}\right) = n_0(H^{-1})\left(1 + \frac{\partial \ln a}{\partial \ln H}\right)^{-1} \tag{4.12}$$

where n_0 is the initial string distribution.

The energy associated at a time t with stable ($\overline{\rho}_S$) and unstable ($\overline{\rho}_U$) strings can be estimated as

$$\overline{\rho}_S \sim \int_{\lambda_s}^{H^{-1}} L_s n(L_s, t) dL_s, \quad \overline{\rho}_U \sim \int_{H^{-1}}^{\infty} L_s n(L_s, t) dL_s \tag{4.13}$$

However, for a perfect gas of stable strings $p_S = 0$, while, for unstable strings, $p_U = \pm\rho_U/d$, with the sign fixed by the exponent α_i of eq.(4.6), $sign\{p_U\} = -sign\{\alpha_i\}$, as discussed in [27]. Therefore, the ratio $\gamma = p/\rho$ as a function of time, for a perfect gas of strings in an accelerated metric background, can be approximated as

$$\gamma(t) = \pm\frac{1}{d}\frac{\overline{\rho}_U}{\overline{\rho}_U + \overline{\rho}_S} \tag{4.14}$$

By inserting into eqs.(4.13), (4.14) the solution (4.12) expressed for our particular metric (4.6), with an initial string distribution $n_0(L_s) \sim L_s^{-3}$, one then finds for each spatial direction [51]

$$\gamma_i(t) = -\frac{1}{d}\lambda_s H_i(t), \tag{4.15}$$

where H_i is given by eq.(4.5). The above result is valid for $|H_i| < \lambda_s^{-1} \simeq M_p$ and is not very sensitive to the initial string distribution.

We insert now this expression into the right-hand side of the field equations (2.18), (2.19), by recalling that, for the pressureless background (4.3) one has, to zeroth-order,

$$\Gamma_i^{(0)} = x_i = \frac{e^{\phi_0}}{4L} t_i \tag{4.16}$$

Then, to next order,

$$\Gamma_i = x_i + \int_{-\infty}^{x} \gamma_i(x')dx' = \frac{e^{\phi_0}}{4L}\left[t_i - \frac{\lambda_s}{d}\int_{-\infty}^{t}\frac{da_i}{a_i}\right] =$$

$$= \Gamma_i^{(0)}\left[1 - \frac{\lambda_s}{dT}\ln\left(\frac{t-2T}{t}\right)\right] \tag{4.17}$$

According to our iterative approach, the integration of eqs.(2.18), (2.19) with $D(x)$ determined by this new expression for Γ_i provides a first order approximation to the background fields $a(t)$, $\phi(t)$. The corrections to the solution (4.3) due to a non-vanishing effective pressure of the string gas are certainly negligible for $|t| >> T$, in the regime in which the background (4.3) satisfies $H^2 << \rho e^{\phi} \sim \dot{\overline{\phi}}^2$. However, as clearly shown by eq.(4.17), if $T >> \lambda_s$ then the first order corrections keep small also in the $t \sim -T$ regime, in which $H^2 \sim \rho e^{\phi} \sim \dot{\overline{\phi}}^2$, and even in the limit $t \to -t_1 \simeq \lambda_s$, in which $\rho e^{\phi} << \dot{\overline{\phi}}^2 \sim H^2$. Within the assumption that T is very large in string units, the solution (4.3) becomes then a good approximation to the exact solution of the system of background equations and string equations of motion, for the whole range $-\infty \leq t \leq -t_1 \simeq \lambda_s$.

We stress that, in this scenario, when $|t| << T$ the source term ρe^{ϕ} becomes negligible with respect to H^2 and $\dot{\overline{\phi}}^2$ (see eq.(4.5)); quite independently of the exact value of the pressure and of the particular type of equation of state at the scale T, the background rapidly converges, for $|t| << T$, to a phase of vacuum, dilaton-driven accelerated evolution (as discussed in Sec.2), described by the metric (4.6). We are left, therefore, with two phenomenological possibilities.

The first is the case in which T, and then the temporal extension of the regime (4.6), is much larger than the minimal value fixed by eq.(4.9) to secure a phenomenologically sufficient amount of inflation. This means, in conformal time,

$$|\eta_T| >> 10^{30}\left(\frac{H_1}{M_p}\right)^{1/2}|\eta_1| \simeq |\eta_0| \tag{4.18}$$

where η_0 is the time when the largest scale, corresponding to the minimum frequency mode $\omega_0 = H_0$, was pushed out of the event horizon during the pre-big-bang phase. In this case, all today's observable scales crossed the horizon in the dilaton-driven regime (4.6), so that the presently observed perturbation spectrum is wholly determined by the metric behavior of that regime, quite independently of possible earlier matter corrections to the background.

The second possibility is the case of nearly "minimal" inflation, corresponding to the equality in the condition (4.9), which implies then $|\eta_T| \sim |\eta_0|$. In this case the largest scales crossed the horizon when the contribution of the string sources to the metric background was of the same order as the dilaton contribution. As a consequence, the low frequency part of the scalar perturbation spectrum may be affected by the matter corrections, and may be sensitive to the particular type of equation of state. The spectrum is thus to be computed by including the non-trivial mixing induced by the source terms $T_{\mu\nu}$ and their perturbations, $\delta T_{\mu\nu}$, as discussed in Sec.3.

As anticipated in Sec.3, in this paper we will discuss the first possibility only. We shall assume, in particular, that the phase of accelerated evolution responsible for the solution of the standard kinematic problems, and for the amplification of the perturbations (at all presently accessible scales), is described by a three-dimensional, isotropic, dilaton-dominated background with

$$a(t) \sim (-t)^{-1/\sqrt{3}}, \quad a(\eta) \sim (-\eta)^{-1/(\sqrt{3}+1)}, \quad \phi \sim (3+\sqrt{3}) \ln a,$$

$$t \leq -t_1 < 0 \qquad \eta \leq -\eta_1 < 0 \tag{4.19}$$

(according to eqs.(2.48), (2.49)). More complicated scenarios, in particular with higher-dimensional, anisotropic, sourceless backgrounds will be analyzed elsewhere [41]).

The metric (4.19) describes superinflationary expansion [52]. In order to obtain the dilaton spectrum, by applying eq.(3.32), we must transform however the solution (4.19) into the E frame, where it takes the form (see eqs.(2.57), (2.58))

$$a_E(t_E) \sim (-t_E)^{1/3}, \quad a_E(\eta) \sim (-\eta)^{1/2}, \quad \phi_E \sim -\sqrt{12} \ln a_E \tag{4.20}$$

This metric describes, for $t \to 0_-$, a contracting background. Potentially, this represents a difficulty of the whole scenario: indeed, the approximation of a diluted string gas might be no longer valid in a contracting background, as well as the approximated expression (3.36) for the perturbation spectrum, obtained in the

case of inflationary expansion. Most important, it might seem impossible, in a contracting background, to achieve a solution of the standard kinematic problems [50], thus rendering "frame-dependent" the inflationary virtues of the pre-big-bang scenario.

Surprisingly enough, however, this is not the case, as a consequence of the fact that the contraction of the metric (4.20) is of the accelerated type, with $\dot{a} < 0$, $\ddot{a} < 0$, $\dot{H} < 0$ (one can show, in general, that all the BD solutions describing superinflationary expansion, with or without matter sources, are transformed through the Weyl rescaling (2.52) into E backgrounds whose metric describes accelerated contraction [23]). Let us show, first of all, that a phase of accelerated contraction is equally good to solve the kinematic problems of the standard model as a phase of superinflationary expansion, characterized by $\dot{a} > 0$, $\ddot{a} > 0$, $\dot{H} > 0$. Consider indeed the so-called flatness problem: the spatial curvature term becomes negligible with respect to the other terms of the cosmological equations if the ratio

$$r_1(t) = \frac{k}{a^2 H^2} = \frac{k}{\dot{a}^2} \tag{4.21}$$

goes to zero during inflation. Such a condition is certainly satisfied by a metric which behaves, asymptotically, as

$$a(t) \sim (-t)^{\alpha}, \qquad t < 0, \qquad \alpha < 1 \tag{4.22}$$

for $t \to 0_-$. For $\alpha < 0$ this metric parametrizes the known case of pole-inflation (superinflationary expansion [52]). For $0 < \alpha < 1$ one has instead accelerated contraction. In both cases the curvature scale is growing, and $H, \dot{H}$ diverge as $t \to 0_-$.

Accelerated contraction can also provide a solution to the horizon problem. Indeed, by recalling the previous definition of particle (eq.(4.7)) and event (eq.(4.8)) horizon, one finds that the ratio of their proper sizes in the background (4.22),

$$r_2(t) = \frac{d_p(t)}{d_e(t)} \sim (-t)^{\alpha - 1}, \tag{4.23}$$

diverges for $t \to 0_-$. This means that causally connected regions will always cross the horizon, asymptotically, not only in the case of superinflationary expansion ($\alpha < 0$), but also in the case of accelerated contraction ($0 < \alpha < 1$).

It is important to stress that the condition for a successful resolution of the horizon and flatness problems, when expressed in conformal time, is exactly the same for superinflation and accelerated contraction. Quite independently of α, in

fact, the ratio r_2 scales as η^{-1}, while r_1 scales as η^2. The horizon problem is solved if r_2, evaluated at the end of inflation ($\eta = \eta_f$), is larger than a present value of r_2 of order unity, rescaled up to η_f. Namely

$$\frac{|\eta_i|}{|\eta_f|} \gtrsim \frac{|\eta_0|}{|\eta_f|} \tag{4.24}$$

where η_i denotes the beginning of the (contracting or expanding) accelerated evolution (see also eq.(4.9)). The flatness problem is solved if r_1, at $\eta = \eta_f$, is tuned to a small enough value, so that the subsequent decelerated evolution leads to a present value of the ratio $r_1(\eta_0) \lesssim 1$. This implies

$$\left(\frac{\eta_f}{\eta_i}\right)^2 \lesssim \left(\frac{\eta_f}{\eta_0}\right)^2 \tag{4.25}$$

which is clearly equivalent to the previous condition, as expected.

Therefore, if the accelerated phase of pre-big-bang is long enough to solve the kinematic problems in the BD frame, where the metric describes superinflationary expansion, then the solution holds also in the E frame where the metric describes accelerated contraction, because the conditions are the same in conformal time, and the conformal time is the same in the two frames [23]. (We note, incidentally, that the kinematic problems can thus be solved also if one chooses negative integration constants, $t_i < 0$, $\alpha_i < 0$, for the background solution (4.6), corresponding to a metric describing accelerated contraction already in the BD frame). For the solution of the entropy problem, of course, a non-adiabatic phase associated to the inflation-radiation transition is necessarily required, in addition to the accelerated kinematic, as recently stressed also in [53].

For what concerns the dilution of the string gas, we recall that in the BD frame a model of source as a weakly interacting string network is a very good approximation. In that frame, indeed, by starting at some initial time t_i with a packing factor = (average distance/average size) of order unity, one ends up, at any subsequent time t_f, with a number of strings per unit of string volume which is diluted as $n_f/n_i = (a_i/a_f)^d$ (< 1 since the metric is expanding). In the E frame the metric is contracting, but the string proper size $L_s^E(t)$ shrinks with time as $L_s^E(t) = (a_E/a)\lambda_s$, where a is the BD scale factor. As a consequence, the number of strings per unit of string volume scales as $n(t) = (L_s^E/\lambda_s)^d a_E^{-d} = a^{-d}$, and it is again diluted as time goes up, exactly by the same amount as in the BD frame. In other words, one finds that, at the end of inflation, a region of space of initial linear dimensions $O(\lambda_s)$ has become exponentially large in Planck units irrespectively of

the frame one is using. With similar arguments one can show [23] that the heating up of the string gas with respect to the radiation, which is easy to understand in the BD frame where the metric is expanding and the radiation is red-shifted, also occurs in the E frame in spite of the fact that the radiation is blue-shifted because of the contraction.

We want to show, finally, that the result (3.36) for the scalar perturbation spectrum is also valid if the transformed metric, in the E frame, is a contracting one. Consider in fact eqs.(3.33), (3.34), for the mode v_k in the non-oscillating regime. Since the variable $z = a\dot{\phi}/H$ goes like a for $\phi \sim \beta \ln a$, it might seem that for fast enough contraction the second term of the expansion could dominate the first one, asymptotically, thus changing the perturbation spectrum. We must recall, however, that in the scenario that we are considering the universe evolves from an initial phase of pre-big-bang to the standard, decelerated, radiation-dominated expansion. In the BD frame the universe is always expanding, $H_{BD} > 0$, while in the E frame an initial contraction ($H_E < 0$) turns into a final expansion ($H_E > 0$), with a necessary turning point of H_E at some time $\eta^\star$ near the transition time $-\eta_1$. On the other hand, the conformal transformation (2.52) gives (in $d = 3$) $H_E = (H_{BD} - \dot{\phi}/2)e^{\phi/2}$: it follows that $\dot{\phi} \neq 0$ where $H_E = 0$, and that $z \to \infty$ for $\eta \to \eta^\star$, so that the first term of the expansion (3.33) is still the dominant one even in the E frame.

By putting, in this frame, $a'/a \simeq (a'/a)'_\star(\eta - \eta^\star)$ for $\eta \to \eta^\star$ ($z \to \infty$), the amplification of ψ_k in this limit can thus be estimated as (using eq.(3.25))

$$|\psi_k| \simeq |\frac{\phi' z}{k^2 a}\left(\frac{v_k}{z}\right)'| \simeq |\frac{(\phi'_\star)^2}{\sqrt{k}(a'/a)'_\star}\left(\frac{1}{z}\right)_{k\simeq aH}| \simeq \frac{1}{\sqrt{k}}\left(\frac{1}{z}\right)_{k\simeq aH} \tag{4.26}$$

where $\phi'_\star \equiv \phi'(\eta_\star)$, and we have used the background equation $\phi'^2 = -6(a'/a)'$. We thus recover for $|\delta_\psi|^2 = k^3|\psi_k|^2$ in the E frame the standard result (3.36), in spite of the contracting character of the transformed pre-big-bang metric.

5. Phenomenological constraints on the dilaton spectrum.

In the simplified model of pre-big-bang motivated and discussed in the previous Section, and described by the background (4.20), the dilaton perturbations are amplified with a growing spectral distribution: one has, from eq.(3.32),

$$\Omega_\chi(\omega, t) \simeq G H_1^2 \left(\frac{\omega}{\omega_1}\right)^3 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \tag{5.1}$$

The total dilaton energy density $\rho_\chi(t)$ is thus dominated by the highest frequency mode ω_1,

$$\Omega_\chi(t) = \frac{\rho_\chi(t)}{\rho_c(t)} = \int^{\omega_1} \frac{d\omega}{\omega} \Omega_\chi(\omega, t) \simeq G H_1^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \tag{5.2}$$

and since $\rho_\chi(t)$ decreases in time like the radiation density ($\sim a^{-4}$), its value in critical units, $\Omega_\chi(t)$, remains constant during the radiation-dominated era ($H \sim a^{-2}$). The requirements that the produced dilatons do not overclose the universe ($\Omega_\chi < 1$) in the radiation era thus imposes the condition

$$H_1 \lesssim M_p \tag{5.3}$$

which is also needed from a similar constraint on gravitons (this constraint is valid not only for the particular case (5.1) but also, more generally, for all growing dilaton spectra, whose integration leads to an $\Omega_\chi(t)$ similar to that of eq.(5.2)).

If the dilaton would be massless, this would be the end of the story. However, in spite of some recent attempt [54] trying to motivate the possibility of a massless dilaton in a string theory context, present conventional wisdom seems to favor a non-vanishing dilaton mass, with a mass value closely related, in particular, to the phenomenology of supersymmetry breaking (see [55] for a recent discussion). But, even independently from possible supersymmetry motivations, a non-vanishing mass seems to be a compulsory consequence of the fact that dilatons couple non-universally to macroscopic matter, with coupling strength larger than (or at least equal to) the gravitational one [4]. We shall reproduce below, for completeness, the argument given in ref. [4] making it actually slightly more general.

The large distance behavior of dilaton couplings is determined indeed by the string effective action Γ, whose general form, including possible loop corrections, can be written as

$$\Gamma = -\int d^D x \sqrt{|G|}\, [Z_R(\phi) R(G) + Z_\phi(\phi) G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)] + \Gamma_M,$$

$$\Gamma_M = \sum_i \int d^D x \sqrt{|G|} \left[\frac{1}{2} Z_k^i(\phi) G^{\mu\nu} \partial_\mu \psi_i \partial_\nu \psi_i + Z_m^i(\phi) \psi_i^2 + interaction\ \ terms\right] \tag{5.4}$$

Here Z_R, Z_ϕ, Z_k, Z_m are complicated (known in principle, but unknown in practice) coupling functions, $G_{\mu\nu}$ is the (dimensional) sigma-model metric, and we have represented the matter part of the action as a set of (dimensionless) scalar fields ψ_i (fundamental fermions can be added without difficulty [4]). In order to evaluate

the effective dilaton couplings we must restore, first of all, the canonical form of the kinetic energy terms, by rescaling field and masses. The dilaton coupling to matter fields is then obtained from the effective interaction Lagrangian, expressed in terms of the rescaled variables.

To this aim we note that the graviton kinetic term of eq.(5.4) reduces to the canonical Einstein Lagrangian by putting

$$G_{\mu\nu} = g_{\mu\nu} M_p^2 Z_R^{-2/(d-1)} \tag{5.5}$$

where $g_{\mu\nu}$ is the dimensionless Einstein metric. By defining a new scalar field (with canonical dimensions) σ, such that

$$\frac{d\sigma}{d\phi} = M_p^{(d-1)/2} \left[\frac{2d}{d-1} \left(\frac{d \ln Z_R}{d\phi} \right)^2 - 2 \frac{Z_\phi}{Z_R} \right]^{1/2} \tag{5.6}$$

and new rescaled matter fields

$$\hat{\psi}_i = M_p^{(d-1)/2} \sqrt{\frac{Z_k^i(\sigma)}{Z_R(\sigma)}}\ \psi_i \tag{5.7}$$

the action (5.4) can be written in canonical form as

$$\begin{aligned} \Gamma = -\int d^D x \sqrt{|g|} [& -\frac{R(g)}{16\pi G_D} + \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - W(\sigma) + \\ & + \sum_i J \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \hat{\psi}_i \partial_\nu \hat{\psi}_i + \frac{1}{2} \mu_i^2(\sigma) \hat{\psi}_i^2 \right)] \end{aligned} \tag{5.8}$$

where $16\pi G_D = M_p^{1-d}$, $W = V M_p^{d+1} Z_R^{-2/(d-1)}$ and

$$\mu_i^2(\sigma) = \frac{2 M_p^2 Z_m^i(\sigma)}{Z_k^i(\sigma) Z_R^{2/(d-1)}(\sigma)} \tag{5.9}$$

We expand now the effective matter-dilaton interaction Lagrangian, $\mu_i^2 \hat{\psi}_i^2$, around the value of σ which extremizes the dilaton potential (and which can always be assumed to coincide with $\sigma = 0$, after a trivial shift). Defining

$$\frac{1}{2} \mu_i^2(\sigma) \hat{\psi}_i^2 = \frac{1}{2} m_i^2 \hat{\psi}_i^2 + g_i \sigma \hat{\psi}_i^2 + O(\sigma^2)\ , \tag{5.10}$$

we can express the rescaled mass of the matter fields as

$$m_i^2 = \left[\mu_i^2(\sigma) \right]_{\sigma=0} = 2 M_p^2 \left(\frac{Z_m^i}{Z_k^i Z_R^{2/(d-1)}} \right)_{\sigma=0} \tag{5.11}$$

and the effective matter-dilaton couplings (including loop corrections) as

$$g_i = \frac{1}{2}\left(\frac{d\mu_i^2}{d\sigma}\right)_{\sigma=0} = \frac{1}{2}\left(\frac{d\phi}{d\sigma}\frac{d\mu_i^2}{d\phi}\right)_{\sigma=0} =$$

$$= \frac{m_i^2}{2M_p^{(d-1)/2}}\left[\frac{2d}{d-1}\left(\frac{d\ln Z_R}{d\phi}\right)^2 - 2\frac{Z_\phi}{Z_R}\right]^{-1/2}_{\phi=0}\left[\frac{d}{d\phi}\ln\left(\frac{\mu_i^2(\phi)}{M_p^2}\right)\right]_{\phi=0} \tag{5.12}$$

In the weak coupling regime $Z_R = Z_\phi = e^{-\phi} = g^{-2}$, where g is the gauge coupling constant of the superstring unification group, and one finds that the effective dilaton coupling strength g_i/m_i deviates from the standard "gravitational charge" $\sqrt{4\pi G_D}m_i$ by the factor

$$k_i = \frac{g_i}{m_i^2\sqrt{4\pi G_D}} \simeq \left(\frac{d-1}{2}\right)^{1/2}\left[g^2\frac{\partial}{\partial g^2}\ln\left(\frac{\mu_i}{M_p}\right)^2\right]_{\phi=0} \simeq$$

$$\simeq 1 + \left[\frac{\partial}{\partial\phi}\ln\left(\frac{Z_m^i}{Z_k^i}\right)\right]_{\phi=0} \tag{5.13}$$

where the last equality refers to $d = 3$. As already stressed in Ref.[4], eqs.(5.12) and (5.13) clearly show two important phenomenological effects. The first is that string loop contributions violate the universality of the effective dilaton couplings, as the factors k_i being different for particles whose mass has different origins; the second is that the dilaton coupling is even stronger than the graviton coupling, as $k_i \geq 1$ for all known cases [4] and, in particular, for the confinement-generated component of hadronic masses. Such a conclusion has been recently challenged [54] on the basis of a possible new mechanism forcing eq.(5.13) to give a vanishing (or very small) result. It is difficult for us to understand how such a cancellation can occur for any realistic present value of the dilaton.

The existence of a non-universal scalar force of gravitational strength may be reconciled with the Eotvos-Dicke-Braginski experiments only if its range is finite [56]. By considering the present results obtained from tests of the equivalence principle [5], it follows in particular that the dilaton corrections to low energy Newtonian gravity are only allowed if their range is smaller than about $1\,cm$, namely for a dilaton mass

$$m \gtrsim m_0 = 10^{-4}\,eV \tag{5.14}$$

We now turn to the discussion of bounds following from the energy stored in the dilaton perturbations (or, if we prefer, in actual scalar particles associated with that field). Our discussion follows closely that of ref. [8]. We recall now

that the expression (5.2) for the dilaton energy density was obtained neglecting the contribution of the rest energy to the proper oscillation frequency $E(t) = \sqrt{(k/a)^2 + m^2}$ (see eq.(3.27)). Therefore, for a massive dilaton, eq.(5.2) is only valid at early enough times such that $\omega_1(t) = k_1/a > m$ (and thus it is certainly not valid today, in view of the previous bound and of eq.(1.4)). At later times eq.(5.2) is to be corrected to take into account the mass contribution. Let us suppose, first of all, that $m < H_1 = \omega_1(t_1)$, so that eq.(5.2) holds initially, at the beginning of the radiation era, and that Ω_χ stays dominated by the highest mode ω_1 also in the non-relativistic regime. The proper frequency ω_1 is red-shifted as the curvature scale decreases in time, and the dominant mode becomes non-relativistic at a scale $H_{nr} \equiv H(t_{nr})$ such that

$$\omega_1(t_{nr}) = H_1 \frac{a_1}{a_{nr}} = m \tag{5.15}$$

For $H \lesssim H_{nr}$ the oscillation frequency $E_1(t) = \sqrt{\omega_1^2(t) + m^2}$ is dominated by the mass contribution, and the corrected dilaton energy density can be obtained from eq.(5.2) through the rescaling

$$\Omega_\chi(t) \to \frac{m}{\omega_1}\Omega_\chi(t) \simeq GmH_1 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3 \equiv G\frac{m^4}{H^2}\left(\frac{a_{nr}}{a}\right)^3 \tag{5.16}$$

(see also [8]). The cosmological bounds on Ω_χ become then bounds on the dilaton mass, and provide us with a phenomenologically allowed region in the (m, H_1) parameter space.

Consider, first of all, the critical density bound $\Omega_\chi < 1$. Once that the condition (5.3) is satisfied, the dilaton density ρ_χ is certainly subcritical (and constant with respect to the radiation density ρ_γ) for all scales $H > H_{nr}$. When the scale drops below H_{nr}, however, the dilaton density becomes non-relativistic according to eq.(5.16), the ratio ρ_χ/ρ_γ begins to grow in time like the scale factor, and the equality $\rho_\chi = \rho_\gamma$ may be reached at some initial scale $H_i \equiv H(t_i)$. We have now various phenomenological possibilities, depending on the value of H_i and H_{nr}.

Suppose first that the transition to the non-relativistic regime occurs when the universe is already matter-dominated, namely for $H_{nr} \lesssim H_2$ (where $H_2 \sim 10^{-27} eV$ is the scale of radiation-matter transition) which means, according to the definition (5.15)

$$m \lesssim \sqrt{H_1 H_2} \tag{5.17}$$

In the matter-dominated era, the dilaton energy density (in critical units) (5.16) stays fixed at the value

$$\Omega_\chi \simeq G H_1^2 \frac{m}{\sqrt{H_1 H_2}} \tag{5.18}$$

The requirement $\Omega_\chi < 1$ is thus automatically satisfied, in this case, because of eqs.(5.9), (5.17). The same happens if $H_{nr} > H_2$, but the scale H_i of dilaton-radiation equilibrium belongs to the epoch of matter domination, $H_i < H_2$. In this case, indeed, $H_{nr} = m^2/H_1$, and H_i is fixed (according to eq.(5.16)) by

$$\frac{\rho_\chi(t_i)}{\rho_\gamma(t_i)} = \frac{\rho_\chi(t_{nr})}{\rho_\gamma(t_{nr})}\frac{a_i}{a_{nr}} \simeq G H_1^2 \frac{m}{\sqrt{H_1 H_2}} \left(\frac{H_2}{H_i}\right)^{2/3} = 1 \tag{5.19}$$

The condition $H_i < H_2$, reading now $m \lesssim \sqrt{H_1 H_2}(G H_1^2)^{-1}$, again implies $\Omega_\chi < 1$, where Ω_χ is the constant dilaton density (5.18).

Dilatons are always subdominant even if $H_{nr} > H_i > H_2$, but the decay scale H_d (1.7) is larger than H_i, so that dilatons are forced to dissipate their coherent energy density, converting it into radiation before becoming dominant. In this case we have, from eq.(5.16),

$$H_i = H_{nr}\left[\frac{\rho_\chi(t_{nr})}{\rho_\gamma(t_{nr})}\right]^2 = \frac{m^2 H_1^3}{M_p^4} < H_{nr} \tag{5.20}$$

and the condition $H_d > H_i$ reads

$$m > H_1 \left(\frac{H_1}{M_p}\right)^2 \tag{5.21}$$

If, on the contrary, $H_{nr} > H_i > H_2$ and $H_i > H_d$, then in order to avoid contradiction with the presently observed density of non-relativistic matter it is necessary, first of all, to impose that dilatons already decayed, $H_d > H_0$. This gives $m^3 > M_p^2 H_0$ (numerically, $m > 100\ MeV$). This is not sufficient though and one is left, in this case, with two possible alternatives.

The first is the one in which the reheating temperature T_r associated with dilaton decay

$$T_r \simeq \sqrt{M_p H_d} \simeq \left(\frac{m^3}{M_p}\right)^{1/2} \tag{5.22}$$

is smaller than the temperature scale $T_N \sim 1 MeV$ required by nucleosynthesis. This provides

$$m \lesssim 10^4\, GeV \tag{5.23}$$

In such case we must impose that nucleosynthesis occurred before dilaton dominance, $H_N \simeq (1\,MeV)^2/M_p > H_i$, and that the entropy ΔS associated with dilaton decay is small enough, in order not to destroy all light nuclei already formed. The temperature T_d of the radiation gas already present at the scale H_d is in fact, from eq.(5.20),

$$T_d = T_i \left(\frac{a_i}{a_d}\right) \simeq (M_p H_i)^{1/2} \left(\frac{H_d}{H_i}\right)^{2/3} \simeq \left(\frac{m^{10}}{M_p H_1^3}\right)^{1/6} \tag{5.24}$$

The reheating of the radiation gas from T_d to T_r thus produces an entropy increase

$$\Delta S \simeq \left(\frac{T_r}{T_d}\right)^3 \simeq \left(\frac{H_1^3}{m M_p^2}\right)^{1/2} \tag{5.25}$$

By imposing $\Delta S \lesssim 10$ in order to preserve nucleosynthesis [57] one obtains the bound

$$m \gtrsim 10^{-2} \frac{H_1^3}{M_p^2} \tag{5.26}$$

The second alternative corresponds to $T_r \gtrsim T_N$, i.e. $m \gtrsim 10^4\,GeV$, and allows then a nucleosynthesis phase subsequent to dilaton decay. In such case, the only phenomenological constraint is possibly imposed by primordial baryogenesis. The maximum tolerable amount of entropy, in order not to wash-out any pre-existing baryon-antibaryon asymmetry, is somewhat model-dependent, but in general $\Delta S \lesssim 10^5$ seems to be acceptable [57,58]. This implies the bound

$$m \gtrsim 10^{-10} \frac{H_1^3}{M_p^2} \tag{5.27}$$

Note, however, that this last condition may be evaded in the case of low energy baryogenesis and, in particular, in the case of baryogenesis associated with the dilaton decay itself [55,58], occurring at scales not much distant from nucleosynthesis.

The bounds so far considered refer to the case in which the dilaton energy density stays always dominated by the contribution of the highest frequency mode ω_1, even in the non-relativistic regime. This is certainly true for the distribution (5.1), with spectral index $\delta = 3$, but for a more complete phenomenological discussion let us consider also * the coherent oscillations, with frequency m, of the

* The necessity of considering lower modes and of distinguishing two intervals in δ was pointed out to us by A. Starobinsky.

lower modes with $\omega(t_m) \lesssim m$, which begin at the scale $H_m \equiv H(t_m) = m$, when the mode ω_1 is still relativistic ($m > H_{nr}$). For $H \lesssim m$ such modes provide a non-relativistic contribution to the dilaton energy density which can be written (for a general spectrum with $\delta > 0$) as

$$\Omega_\chi(t) \simeq Gm^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{m}{H_1}\right)^{\frac{\delta}{2}} \left(\frac{a_m}{a}\right)^3 = Gm^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{m}{H_1}\right)^{\frac{\delta-3}{2}} \left(\frac{a_1}{a}\right)^3 \quad (5.28)$$

This contribution is initially negligible with respect to the relativistic part of the dilaton energy density, $\sim GH_1^2$, dominated by ω_1. However, during the radiation era it grows in time with respect to GH_1^2, and it may dominate the total dilaton energy if the equality

$$Gm^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{m}{H_1}\right)^{\frac{\delta}{2}} \left(\frac{H}{m}\right)^{3/2} = GH_1^2 \quad (5.29)$$

occurs at a scale $H > H_{nr} = m^2/H_1$, namely for $(m/H_1)^{\delta-1} > 1$, which implies $\delta < 1$.

The previous bounds, obtained in the hypothesis of ω_1-dominance, are thus valid for all growing spectra with $\delta \geq 1$. For lower spectral slopes, $0 < \delta < 1$, the dominant contribution to Ω_χ (the one to be bounded) becomes that of eq.(5.28) and, as a consequence, the bounds acquire a dependence on δ. Indeed, let H_1 be the scale marking the equality $\rho_\chi = \rho_\gamma$. Then the condition $\Omega_\chi < 1$, with Ω_χ given by eq.(5.28) and $0 < \delta < 1$, is always satisfied for $H_i < H_2$, which means

$$m < \left(H_2 M_p^4 H_1^{\delta-4}\right)^{1/(\delta+1)} \quad (5.30)$$

It is also satisfied if $H_i > H_2$, but $H_d > H_i$, which means

$$m > \left(H_1^{4-\delta} M_p^{-2}\right)^{1/(2-\delta)} \quad (5.31)$$

If, however, $H_i > H_2$ and $H_i > H_d$, then the dilatons must have already decayed, $H_d > H_0$. Their decay generates an entropy

$$\Delta S = \left(\frac{T_r}{T_d}\right)^3 = \left(\frac{H_1^{4-\delta} m^{\delta-2}}{M_p^2}\right)^{1/2} \quad (5.32)$$

If $m < 10^4\ GeV$ the reheating temperature is too low to allow nucleosynthesis: we must impose that nucleosynthesis already occurred, $H_i < H_N$, and that [57] $\Delta S \lesssim 10$, which means

$$m \lesssim \left(10^{-2} M_p^{-2} H_1^{4-\delta}\right)^{1/(2-\delta)} \quad (5.33)$$

If, on the contrary, $m > 10^4\ GeV$, the nucleosynthesis scale is subsequent to dilaton decay, and the only possible constraint [57,58] is $\Delta S \lesssim 10^{-5}$, namely

$$m \lesssim \left(10^{-10} M_p^{-2} H_1^{4-\delta}\right)^{1/(2-\delta)} \tag{5.34}$$

This complete the compilation of the phenomenological bounds for a growing dilaton spectra, $\delta > 0$, with $m < H_1$. Let us consider also, for the sake of completeness, the "heavy dilaton" case, $m > H_1$, although this case is very unnatural in a string theory context where H_1 is expected to be close to the Planck scale.

As discussed in Sec.3, in the radiation phase ($H < H_1, a \sim \eta$) the dilaton modes χ_k decouple from the other scalar fluctuations, and satisfy the free equation (which includes in general the mass contribution)

$$\overline{\chi}_k'' + (k^2 + m^2 a^2)\overline{\chi}_k = 0, \qquad \overline{\chi}_k = a\chi_k \tag{5.35}$$

For all modes that are relativistic at the beginning of the radiation era ($\eta = -\eta_1$), namely for $k > ma_1$, one then recover the dilaton spectrum (3.32) by matching the pre-big-bang solution (3.30) with the plane wave (3.27), which is solution of eq.(5.35) in the case of negligible mass. If, however,

$$m > H_1 = \omega_1(t_1) = \frac{k_1}{a_1} \tag{5.36}$$

then the mass term cannot be neglected even in the case of the highest mode k_1. All modes are non-relativistic already at $\eta = -\eta_1$, and satisfy the approximate equation

$$\overline{\chi}_k'' + m^2 a^2 \overline{\chi}_k = 0 \tag{5.37}$$

By using the identity $m^2 a^2 = m^2 H_1^2 a_1^4 \eta^2$, valid in the radiation era, the general solution of (5.37), for $mH_1 a_1^2 \eta^2 \simeq (m/H_1)(\eta/\eta_1)^2 > 1$ can be expressed as

$$\overline{\chi} = \frac{1}{\sqrt{mH_1 a_1^2 \eta}}(c_+ e^{-imH_1 a_1^2 \eta^2} + c_- e^{imH_1 a_1^2 \eta^2}) \tag{5.38}$$

The matching of this solution with the Bessel solution (3.30) gives then the Bogoliubov coefficient for the case $m > H_1$

$$|c_-(k)|^2 \simeq \frac{m}{H_1}(k\eta_1)^{-2|\nu|} \tag{5.39}$$

and the corresponding non-relativistic dilaton spectrum (in critical units)

$$\Omega_\chi(\omega, t) \simeq \frac{G\omega}{H^2}\frac{d\rho_\chi}{d\omega} \simeq G\frac{m\omega^3}{H^2}|c_-|^2 \simeq$$

$$\simeq Gm^2 \left(\frac{\omega}{\omega_1}\right)^{2-2\alpha} \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3 \tag{5.40}$$

(we recall that $2-2\alpha = 3$ for the particular model of pre-big-bang that we have considered).

For a relic background of massive dilatons with $m > H_1$, eq.(5.2) is thus to be replaced by the non-relativistic energy density

$$\Omega_\chi(t) = \int^{\omega_1} \frac{d\omega}{\omega} \Omega_\chi(\omega, t) \simeq Gm^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3 \tag{5.41}$$

One must thus impose the bound

$$m \lesssim M_p \tag{5.42}$$

to avoid an over-critical density of massive dilatons at the beginning of the radiation era. In this case there are no further bounds on m since, as a consequence of eq.(5.42), the dilaton energy is dissipated before a possible dominance. Indeed, the initial scale H_i corresponding to $\Omega_\chi = 1$ is, from eq.(5.41),

$$H_i = H_1 \left(\frac{m}{M_p}\right)^4. \tag{5.43}$$

Therefore, $H_i/H_d = H_1 m/M_p^2 < 1$, just because of eq.(5.42) (unless $H_1 > M_p$, but this is to be excluded to avoid over-critical dominance of other massless particles, such as gravitons, produced by the same background transition).

By collecting all previous phenomenological constraints we obtain a final allowed region in the (m, H_1) plane, which depends on δ and which is illustrated in **Fig.2** for the three cases $\delta = 0$, $\delta = 0.5$ and $\delta = 1$. We recall that for large enough spectral slope, $\delta > 1$, the bounds become slope-independent and coincide with those of the $\delta = 1$ case, which defines then the maximum allowed region for a growing dilaton spectrum (as stressed also in [8]). As clearly shown by the pictures, one of the main effects of a positive spectral index is that light, not yet decayed dilatons become compatible with higher and higher inflation scales as δ ranges from 0 to 1. If we take, for instance, $H_1 \simeq 10^{-5}\ M_p$ as a typical reference value, we find allowed mass windows for a presently dominating background of relic dilatons for all spectra with $\delta \gtrsim 0.1$. For $\delta \gtrsim 1$, moreover, even the TeV mass range, which is supported by some supersymmetry breaking arguments [55], but which lies in the most unfavorable region for the various cosmological constraints, may become compatible with $H_1 \simeq 10^{-5}\ M_p$.

We note, finally, that our allowed regions refer to the cosmological amplification of the quantum fluctuations of the dilaton background. The classical dilaton background is assumed here to sit at the minimum of the dilaton potential, with negligible (with respect to quantum fluctuations) oscillations around it. The classical oscillation amplitude, however, could be too large to be negligible. In that case one should add, to the bounds discussed here, the bounds on m obtained by taking into account the contribution of the classical oscillations to the total cosmological energy density [7]. The initial amplitude of such possible classical oscillations depends, however, on the details of the transition from the accelerated to the decelerated regime. Having neglected such possible additional bounds, the allowed region determined here is to be regarded, for each value of δ, as the *maximal* allowed region in parameter space.

6. Conclusions.

In this paper we have presented the general solution of the equations obtained from the low energy string effective action, for the case of space-independent background fields, vanishing dilaton potential and classical strings as possible matter sources. In the perfect fluid approximation, the solution is uniquely fixed by the choice of the equation of state. We have shown that a model of initial sources such as a pressureless gas of weakly interacting strings provides an approximate, but consistent solution to the full system of background equations and string equations of motion.

This model supports a scenario in which an initial flat perturbative string vacuum evolves towards a high curvature, strong coupling regime through a phase of accelerated expansion or contraction. This is the so-called pre-big-bang epoch, originally motivated in [2,25,27] by the duality symmetries of the string effective action. Accelerated contraction, typical of an Einstein frame representation of the pre-big-bang scenario, works as well as the more conventional inflationary kinematic (accelerated expansion) in order to solve the standard cosmological problems.

We have derived, for an isotropic background, the general coupled system of scalar (metric plus dilaton) perturbation equations, and shown that the transition from the pre-big-bang phase to the standard, radiation-dominated cosmology, is

associated with a copious production of cosmic dilatons, whose spectral distribution grows with frequency. We have discussed the consequent phenomenological bounds on the dilaton mass and on the inflation scale, by combining it with other bounds obtained from tests of the equivalence principle. As a result, we have found allowed windows for the dilaton mass compatible with a possible "dilatonic" solution of the dark matter problem.

The particular model of pre-big-bang considered in this paper leads, however, to scalar metric perturbations which cannot be taken as the origin of the observed CMBR anisotropy, since their spectrum grows too fast with frequency. This can be a welcomed result to people believing in a different source of anisotropy (e.g. in cosmic strings, which could naturally arise in this context as remnants of the violent, non-adiabatic transition from the growing to the decreasing curvature regime). On the other hand, such a result cannot be really taken as typical of our scenario, since we have neglected various effects which may decouple metric and dilaton perturbations leading to a different (possibly flatter) spectrum of metric perturbations. We have in mind, for instance, the contribution of matter sources to the background solutions during the phase of parametric amplification of the vacuum fluctuations and/or a non-trivial evolution of the internal dimensions, whose dynamical compactification introduces at least one additional scalar variables in the perturbations equations.

The most serious omission of this paper is, however, the lack of a detailed description of the transition between the pre-big-bang regime (growing dilaton and curvature scale) and the post-big-bang regime (constant dilaton and decreasing curvature scale). We avoided to face this problem in this paper, as we believe that the low energy effective action adopted here cannot represent any longer an adequate approximation for that purpose. A recent investigation [59] strongly suggests that higher curvature corrections should play a fundamental role in solving the "gracious exit" problem in string cosmology. Furthermore a non-perturbative dilaton potential must be added in the post-big-bang era in order to pin down the dilaton to its present value and to give it a mass.

We do not conceal that the high-curvature transition from inflation to standard cosmology is, at present, the least understood aspect of the whole string cosmology scenario and that it certainly deserves future detailed investigations. We stress however that, once the mechanism which stops the growth of the dilaton and of the curvature and converts non-adiabatically their kinetic energy into radiation is clarified, the phenomenology developed in this paper should remain

valid without further modifications and quite independently of the details of the transition process.

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References

1. L. P. Grishchuk, Sov. Phys. JEPT 40, 409 (1975);
   A. A. Starobinski, JEPT Lett. 30, 682 (1979);
   V. A. Rubakov, M. Sazhin and A. Veryaskin, Phys. Lett. B115, 189 (1982);
   R. Fabbri and M. Pollock, Phys. Lett. B125, 445 (1983);
   L. Abbott and M. Wise, Nucl. Phys. B244, 541 (1984)
2. M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993)
3. M. Gasperini and M. Giovannini, Phys. Rev. D47, 1529 (1993)
4. T. R. Taylor and G. Veneziano, Phys. Lett. B213, 459 (1988);
5. See for instance E. Fischbach and C. Talmadge, Nature 356, 207 (1992)
6. J. Ellis et al., Phys. Lett. B228, 264 (1989).
7. G. D. Coughlan et al., Phys. Lett. B131, 59 (1983);
   J. Ellis, D. V. Nanopoulos and M. Quiros, Phys. Lett. B174,176 (1986);
   J. Ellis, N. C. Tsamish and M. Voloshin, Phys. Lett. B184, 291 (1986);
   B. De Carlos et al., Phys. Lett. B318, 447 (1993)
8. M. Gasperini, Relaxed bound on the dilaton mass in a string cosmology scenario, DFTT-03/94 (January 1994), gr-qc/9401026
9. A. S. Goncharov, A. D. Linde and M I. Vysotsky, Phys. Lett. B147, 279 (1984)
10. G. Smoot et al., Astrophys. J. 396, L1 (1992)
11. C. L. Bennet et al., Cosmic temperature fluctuations from two
    years of COBE DMR observations, astro-ph/9401012 (January 1994)
12. T. Piran et al., Limits on the primordial fluctuation spectrum, void sizes and CMBR anisotropy, astro-ph/9305019
13. A. D. Linde, Phys. Lett. B259, 38 (1991);
    Phys. Rev. D49 (1994) (in press)
14. E. J. Copeland et al., False vacuum inflation with Einstein gravity, SUSSEX-AST 94/1-1 (January 1994), astro-ph/9401011
15. S. Mollerach, S. Matarrese and F. Lucchin, Blue perturbation spectra from inflation, astro-ph/9309054
16. J. Mc Donald, Phys. Rev. D48, 2468 (1993)
17. E. W. Kolb, D. Salopek and M. S. Turner, Phys. Rev. D42, 3925 (1990)
18. M. S. Turner and L. M. Widrow, Phys. Rev. D37, 3428 (1988)
19. C. Lovelace, Phys. Lett. B135, 75 (1984);
    E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. B261, 1 (1985);

C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, Nucl. Phys. B262, 593 (1985)
20. M. Gasperini and G. Veneziano, Phys. Lett. B277, 256 (1992)
21. K. A. Meissner and G. Veneziano, Phys. Lett. B267, 33 (1991)
22. K. A. Meissner and G. Veneziano, Mod. Phys. Lett. A6, 3397 (1991)
23. M. Gasperini and G. Veneziano, Mod. Phys. Lett. A8, 3701 (1993)
24. M.Mueller, Nucl. Phys. B337, 37 (1990)
25. G. Veneziano, Phys. Lett. B265, 287 (1991)
26. A. A. Tseytlin, Mod. Phys. Lett. A6, 1721 (1991)
27. M. Gasperini, N. Sanchez and G. Veneziano, Nucl. Phys. B364, 365 (1991); Int. J. Mod. Phys. A6, 3853 (1991)
28. J. Bardeen, Phys. Rev. D22, 1822 (1980)
29. M. Sasaki, Prog. Theor. Phys. 70, 394 (1983)
30. V. Mukhanov, H. A. Feldman and R. Brandenberger, Phys. Rep. 215, 203 (1992)
31. S. Mollerach and S. Matarrese, Phys. Rev. D45, 1961 (1992)
32. N. Deruelle, C. Gundlach and D. Langlois, Phys. Rev. D46, 5337 (1992)
33. N. Deruelle, C. Gundlach and D. Polarski, Class. Quantum Grav. 9, 137 (1992)
34. L. P. Grishchuk, in Proc. of the 6th Marcel Grossmann Meeting (Kyoto, 1991), ed. by H. Sato (World Scientific, Singapore, 1992)
35. M. Sasaki, Prog. Theor. Phys. 76, 1036 (1986)
36. V. F. Mukhanov, Sov. Phys. JEPT 67, 1297 (1988)
37. E. D. Stewart and D. H. Lyth, Phys. Lett. B302, 171 (1993)
38. V. N. Lukash, Sov. Phys. JEPT 52, 807 (1980)
39. G. V. Chibisov and V. N. Mukhanov, Mon. Not. R. Astron. Soc. 200, 535 (1982)
40. N. Deruelle, C. Gundlach and D. Langlois, Phys. Rev. D45, R3301 (1992)
41. M. Gasperini and G. Veneziano, in preparation
42. N. D. Birrel and P. C. W. Davies, Quantum fields in curved space (Cambridge University Press, Cambridge, England, 1982)
43. F. Lucchin and S. Matarrese, Phys. Lett. B164, 282 (1985);
B. Allen, Phys. Rev. D37, 2078 (1988);
V. Sahni, Phys. Rev. D42, 435 (1990);
L. P. Grishchuk and M. Solokhin, Phys. Rev. D43, 2566 (1991);
M. Gasperini and M. Giovannini, Phys. Lett. B282, 36 (1992)

44. L. P. Grishchuk and Y. V. Sidorov, Phys. Rev. D42, 3413 (1990);
L. P. Grishchuk, Phys. Rev. Lett. 70, 2371 (1993)
45. M. Gasperini and M. Giovannini, Phys. Lett. B301, 334 (1993)
46. M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. D48, R439 (1993)
47. R. Brandenberger, V. Mukhanov and T. Prokopec, Phys. Rev. Lett. 69, 3606 (1992);
Phys. Rev. D48, 2443 (1993)
48. S. W. Hawking, Phys. Lett. B115, 295 (1982);
A. Linde, Phys. Lett. B116, 335 (1982);
A. A. Starobinski, Phys. Lett. B117, 175 (1982);
A. Guth and S. Y. Pi, Phys. Rev. Lett. 49, 1110 (1982);
J. M. Bardeen, P. S. Steinhardt and M. S. Turner, Phys. Rev. D28, 679 (1983)
49. G. Veneziano, Europhys. Lett. 2, 133 (1986)
50. A. Guth, Phys. Rev. D23, 347 (1981)
51. M. Gasperini, M. Giovannini, K. A. Meissner and G. Veneziano, in preparation
52. F. Lucchin and S. Matarrese, Ref. 43
53. Y. Hu, M. S. Turner and E. J. Weinberg, Inflation as the unique solution to the horizon and flatness problem, Fermilab-Pub-92/363-A, astro-ph/9302002
54. T. Damour and A. M. Polyakov, The string dilaton and a least coupling principle, IHES/P/94/1 (January 1994), hep-th/9401069
55. T. Banks, D. V. Kaplan and A. Nelson, Cosmological implications of dynamical supersymmetry breaking, UCSD/PTH 93-26, RU-37, hep-ph/9308292
56. M. Gasperini, Gen. Rel. Grav. 16, 1031 (1984)
57. J. Ellis, D. V. Nanopoulos and M. Quiros, Ref. 7
58. B. De Carlos et al., Ref.7
59. R. Brustein and G. Veneziano, The graceful exit problem in string cosmology, CERN-TH.7179/94 (February 1994)

Figure captions

Fig.1 The dashed area defines the allowed windows for the dilaton mass m (given in units of $m_0 = 10^{-4}\ eV$) and the final inflation scale H_1 which are compatible with a present large contribution of non-relativistic dilatons to Ω, under the assumption that they are produced with a fast enough growing spectrum, $\delta \geq 1$. For lower spectral slopes the allowed window is shifted towards lower values of mass, according to eq.(1.9). Masses higher than $100\, MeV$ are excluded by dilaton decay, masses lower than $10^{-4}\, eV$ by tests of the equivalence principle. Inflation scales higher than M_p are excluded in order to avoid over-critical density in the primordial relativistic particle production.

Fig.2 Maximum allowed region (inside the full lines) relative to the cosmological production of dilatons with growing spectrum, illustrated for three different spectral slopes $\delta = 0$, $\delta = 0.5$ and $\delta = 1$ (the last case applies to all $\delta \geq 1$). The dilaton mass is given in units of $m_0 = 10^{-4} eV$. The lines marked by a, b, c, d, e, f, g represent the most significant bounds quoted in the text, and correspond respectively to: a) $m = m_0$, lower bound on m from the equivalence principle; b) $H_1 = M_p$, upper bound on H_1 from the closure density; c) $T_r = 1\, MeV$, lower bound on the reheating temperature for nucleosynthesis; d) $m = M_p$, upper bound on $m > H_1$ from the closure density; e) $m = (H_2 M_p^4 H_1^{\delta-4})^{1/(\delta+1)}$, upper bound on m from the present matter-to-radiation energy density ratio; f) $m = (10^{-10} M_p^{-2} H_1^{4-\delta})^{1/(2-\delta)}$, upper limit on entropy production in dilaton decay from primordial baryogenesis; g) $m = (10^{-2} M_p^{-2} H_1^{4-\delta})^{1/(2-\delta)}$, upper limit on entropy production from nucleosynthesis.

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Quantum Squeezing and Cosmological Entropy Production[†]

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Abstract

The entropy growth in a cosmological process of pair production is completely determined by the associated squeezing parameter, and is insensitive to the number of particles in the initial state. The total produced entropy may represent a significant fraction of the entropy stored today in the cosmic black-body radiation, provided pair production originates from a change in the background metric at a curvature scale of the Planck order.

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One of the greatest challenges of modern cosmic physics is that of explaining the large level of entropy observed on a cosmological scale. While it seems clear that inflationary kinematics has to play a fundamental role in providing such an explanation, the nature of the (micro)physical mechanism that may have acted as a source of the cosmic entropy is, on the contrary, still unclear.

A natural candidate for such a mechanism, well known even before the advent of the inflationary models, is the production of particles by the changing background metric. The pair production from the vacuum (otherwise stated, in a wave-mechanics language, the parametric amplification of the background fluctuations) provides indeed a natural cosmological arrow [1], which is not inverted even if the expansion turns into a contraction, and which may thus be used to define also an appropriate arrow of time [2]. The problem is, however, that of quantifying in an unambiguous way, and in agreement with the usual notion of entropy, the information loss associated with pair production.

A possible solution to this problem comes from the observation that the production of particles by an external gravitational field can be conveniently represented in the squeezed state formalism [3]. Within this context, one can define indeed two canonically conjugate quantum variables (the so-called "quadrature operators"[4]): one operator has a variance that is "squeezed" with respect to the vacuum, while the variance of the other (the "superfluctuant" one) is correspondingly expanded.

Using this approach, we have recently proposed [5] a way to measure the loss of information associated to the cosmological particle production, in terms of the increased dispersion of the superfluctuant operator. According to the standard (information-theoretic) definition of non-equilibrium entropy, such a coarse graining approach then provides an expression in which the entropy growth ΔS_k is

completely determined, for each mode k, by the squeezing parameter r_k [5] :

$$\Delta S_k = 2r_k. \tag{1}$$

Our definition of entropy is consistent with analogous results, recently obtained in the large squeezing limit, for the entropy of the cosmological perturbations [6]. In our case, however, eq. (1) is an exact result and can be applied also in the small squeezing regime.

By using eq. (1) we can estimate the entropy growth for a realistic case in which particles (e.g. gravitons) are produced by the transition between the inflationary era and the radiation-dominated era (occurring at a given scale H_1). We can assume, in particular, that their mode-number density has a power law behaviour, characterized as usual by the spectral index n (such that $k(d\rho/dk) \simeq k^4 \sinh^2 r_k \sim k^{n-1}$). In such a case the total produced entropy inside a co-moving horizon, obtained by summing the entropy density (1) over all squeezed modes, and measured in units of the usual black-body entropy S_γ associated to the electromagnetic CMB radiation, turns out to be [5]

$$\frac{\Delta S}{S_\gamma} \simeq \mid 5 - n \mid \left(\frac{H_1}{M_P}\right)^{3/2}. \tag{2}$$

This result is particularly interesting for a recently developed string cosmology scenario [7], in which the spectral density of the cosmic graviton background increases with frequency, in such a way that the usual low-frequency bounds on the inflation scale are evaded, and the maximum allowed value of H_1 can be as large as the Planck mass M_P [8]. In such a context $H_1 \sim M_P$ is indeed the natural transition scale, so that the entropy stored in the relic graviton background could be comparable to the observed large scale entropy of the black-body radiation.

The expression (1) for the mode-entropy density, however, was obtained in Ref. [5] for the "minimum uncertainty" squeezed vacuum state, i.e. under the assumption that the initial state, whose fluctuation are amplified by the external

gravitational field, is the vacuum, or a "displaced" vacuum (namely a coherent state, in general). In this paper we shall show that eq. (1) holds exactly, even if the initial state is any eigenstate of the number operator or, more generally, any statistical mixture of number states.

Consider indeed a pair production process described by a Bogoliubov transformation connecting, for each mode k, the particle $(b, b^\dagger)$ and antiparticle $(\tilde{b}, \tilde{b}^\dagger)$ annihilation and creation $|in\rangle$ operators to the $|out\rangle$ ones $(a,\ a^\dagger,\ \tilde{a},\ \tilde{a}^\dagger)$:

$$a_k = \Sigma_k b_k \Sigma^\dagger{}_k \qquad , \qquad \tilde{a}^\dagger_{-k} = \Sigma_k \tilde{b}^\dagger_{-k} \Sigma^\dagger_k . \tag{3}$$

Here Σ_k is the two-mode squeezing operator [4],

$$\Sigma_k = \exp(z_k^* b_k \tilde{b}_{-k} - z_k b_k^\dagger \tilde{b}^\dagger_{-k}) \qquad , \qquad z_k = r_k e^{2i\vartheta_k}, \tag{4}$$

parametrized by the complex number z_k, which depends on the dynamics of the external field leading to the process of pair creation. In terms of the variables x and $\tilde{x}$, whose variance in the final squeezed state is amplified with respect to their initial value, a and $\tilde{a}$ have the differential representation [5]

$$a_k = \frac{i}{2} e^{i\vartheta_k} \left[(\cosh r_k - \sinh r_k)(x - i\tilde{x}) + (\cosh r_k + \sinh r_k)(\partial_x - i\partial_{\tilde{x}}) \right]$$

$$\tilde{a}_{-k} = \frac{i}{2} e^{i\vartheta_k} \left[(\cosh r_k - \sinh r_k)(x + i\tilde{x}) + (\cosh r_k + \sinh r_k)(\partial_x + i\partial_{\tilde{x}}) \right] \tag{5}$$

(the relative phase has been chosen with respect to ϑ_k, in such a way as to identify the x and $\tilde{x}$ operators with the superfluctuant ones).

In the $(x, \tilde{x})$-space representation, the normalized squeezed vacuum wave function $\psi_{o,z_k}(x, \tilde{x})$ (such that $a\psi_o = 0 = \tilde{a}\psi_o$) is thus given by [5]

$$\psi_{o,z_k}(x, \tilde{x}) = \langle x\tilde{x}|\Sigma_k|00\rangle = \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\sigma_k}{2}(x^2+\tilde{x}^2)} \qquad , \qquad \sigma_k = e^{-2r_k}, \tag{6}$$

where $|0\rangle$ is the $|in\rangle$ vacuum state, $b|0\rangle = 0 = \tilde{b}|0\rangle$. When n pairs are already present in the mode k ($b_k^\dagger b_k |n_k n_{-k}\rangle = n|n_k n_{-k}\rangle$), however, the pair production

leads to a two-mode squeezed number state, whose wave function in this space has the explicit representation

$$\psi_{n,z_k}(x,\tilde{x}) = \langle x\tilde{x}|\Sigma_k|n_k n_{-k}\rangle = \langle x\tilde{x}|\frac{(a_k^\dagger \tilde{a}_{-k}^\dagger)^n}{n!}\Sigma_k|0\rangle$$

$$= e^{-2in\vartheta_k}\left(\frac{\sigma_k}{\pi}\right)^{\frac{1}{2}} L_n\left(\sigma_k(x^2+\tilde{x}^2)\right) e^{-\frac{\sigma_k}{2}(x^2+\tilde{x}^2)}$$

$$= e^{in(\pi-2\vartheta_k)}\left(\frac{\sigma_k}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\sigma_k}{2}(x^2+\tilde{x}^2)} \sum_{m=0}^{n} \frac{1}{m!(n-m)!} H_{2m}(\sqrt{\sigma}_k x) H_{2n-2m}(\sqrt{\sigma}_k \tilde{x}) \tag{7}$$

where L_n and H_n are the Laguerre and Hermite polynomials, respectively. It should be noted that, because of pair correlations, this wave function cannot be simply factorized in terms of two decoupled squeezed oscillators in an excited state, which are known to provide the usual representation for the one-mode squeezed number wave function [9]. One can easily verify, however, that x and $\tilde{x}$ are super-fluctuant operators for the state (7), i.e.

$$\frac{(\Delta x)^2_{n,z_k}}{(\Delta x)^2_n} = \frac{(\Delta \tilde{x})^2_{n,z_k}}{(\Delta \tilde{x})^2_n} = e^{2r_k} > 1, \tag{8}$$

where

$$(\Delta x)^2_{n,z_k} = \langle n_k n_{-k}|\Sigma_k^\dagger x^2 \Sigma_k|n_k n_{-k}\rangle$$

$$(\Delta x)^2_n = \lim_{z_k \to 0} (\Delta x)^2_{n,z_k} \tag{9}$$

and the same for $\tilde{x}$.

Suppose now that the initial state is a generic mixture of number states,

$$\overline{\rho}_k = \sum_n P_n(k)|n_k n_{-k}\rangle\langle n_k n_{-k}| \quad , \quad \sum_n P_n = 1. \tag{10}$$

The final state will correspond to the squeezed mixture:

$$\overline{\rho}_{sk} = \sum_n P_n(k)\Sigma_k|n_k n_{-k}\rangle\langle n_k n_{-k}|\Sigma_k^\dagger. \tag{11}$$

Apart from the possible initial entropy, the pair production process is accompanied by a loss of information corresponding to a larger dispersion of x and $\tilde{x}$ around their mean values. According to [5], the reduced density operator associated to such an increased uncertainty, for the initial mixture (10), can be written

$$\rho_{sk} = \sum_n P_n(k) \int dx d\tilde{x} \mid \psi_{n,z_k}(x,\tilde{x}) \mid^2 |x\tilde{x}\rangle\langle x\tilde{x}|. \tag{12}$$

By using the explicit form of the wave function (7), the net entropy growth due to the pair production is thus

$$\Delta S_k = -Tr(\rho_{sk} \ln \rho_{sk}) + [Tr(\rho_{sk} \ln \rho_{sk})]_{\sigma_k=1} = -\sum_n P_n(k) \ln \sigma_k = 2r_k \tag{13}$$

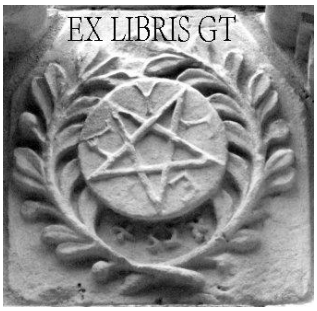
i.e. exactly the same as that obtained starting from the vacuum (eq. (1)). This supports the internal consistency of our definition of entropy [5], and the whole approach to the cosmological entropy based on the squeezed-state formalism [5,6].

We conclude by noting that the result (13) has an important application to the case of an initial thermal mixture.

The inflationary models based on a temperature-dependent phase transition require indeed, as initial condition, a thermal bath at a finite temperature higher than the inflation scale. Such a non-zero temperature affects the spectral energy density of the amplified perturbations, which may deviate from the flat Harrison-Zeldovich form even in the case of pure exponential inflation [10]. The result (13) shows instead that the produced entropy is not affected by finite temperature corrections to the initial state. The entropy depends only on the squeezing parameter (i.e. on the coefficients of the Bogoliubov transformation, determined by the changing geometry), and the expression (2) for the total entropy of a given cosmic background of relic particles is always valid, irrespective of the initial field configuration.

References

1. Hu B L and Pavon D, 1986 Phys.Lett.B180 329;
   Hu B L and Kandrup H E, 1987 Phys.Rev.D35 1776;
   Kandrup H E, 1988 Phys.Rev.D37 3505
2. Hawking S W, Laflamme R and Lyons G W, 1993 Phys.Rev.D47 5342
3. Grishchuk L P and Sidorov Y V, 1989 Class. Quantum Grav.6 L161;
   1990 Phys.Rev.D42 3413;
4. Schumaker B L, 1986 Phys.Rep.135 317;
   Grochmalicki J and Lewenstein M, 1991 Phys.Rep.208 189
5. Gasperini M and Giovannini M, 1993 Phys.Lett.B301 334
6. Brandenberger R, Mukhanov V and Prokopec T, 1992 Phys.Rev.Lett.69 3606; "The entropy of the gravitational field", Preprint Brown-HET-849 (August 1992)
7. Gasperini M, Sanchez N and Veneziano G, 1991 Nucl.Phys.B364 365;
   Veneziano G, 1991 Phys.Lett.B265 287;
   Gasperini M and Veneziano G, 1992 Phys.Lett.B277 256;
   Gasperini M and Veneziano G, 1992 "Pre-big-bang in string cosmology", Astroparticle Phys.1 (in press), CERN-TH.6572/92
8. Gasperini M and Giovannini M, 1992 Phys.Lett.B282 36;
   1993 Phys.Rev.D47 1529
9. Kim M S, de Oliveira F.A.M. and Knight P L, 1990 "New frontiers in quantum electrodynamics and quantum optics", ed. by A.O.Barut (Plenum Press, New York) p.231
10. Gasperini M, Giovannini M and Veneziano G, 1993 "Squeezed thermal vacuum and the maximum scale for inflation", Phys.Rev.D48 (15 July 1993)



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Looking back in time beyond the big bang

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Abstract

String theory can (in principle) describe gravity at all curvature scales, and can be applied to cosmology to look back in time beyond the Planck epoch. The duality symmetries of string theory suggest a cosmological picture in which the imprint of a primordial, pre-big bang phase could still be accessible to present observations. The predictive power of such a scenario relies, however, on our ability to connect in a smooth way the pre-big bang to the present cosmological regime. Classical radiation back reaction seems to play a key role to this purpose, by isotropizing and turning into a final expansion any state of anisotropic contraction possibly emerging from the pre-big bang at the string scale.

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The standard cosmological model is, rightfully, one of the most celebrated scientific conquests of the present century. Such a model, however, cannot be extrapolated back in time beyond an initial regime approaching a state of infinite density and curvature – the so-called "big bang". The history of the Universe from the big bang down to the present time is more or less well known, and its various aspects are under active study since more than forty years [1]. But what happened before the big bang?

This question has not been raised until very recently, mainly because of the lack of a systematic application to cosmology of the powerful instruments of modern theoretical physics, able (in principle) to look back in time beyond the Planck scale. As a consequence, the big bang has represented so far a sort of "Hercule's Pillars " of cosmology. In the ancient times, when nobody knew the world beyond the Straits of Gibraltar, because nobody sailed the sea beyond that point, it was common opinion that Gibraltar would represent the end of the world itself. In the same way, today, the big bang is often popularly referred as the beginning of the Universe, the beginning of spacetime, the beginning of "Everything", just in view of the lack of information about earlier time scales. There are also respectable scientific attempt, in a quantum cosmology context, to explain the origin of the Universe and of the spacetime itself as a process of tunnelling "from nothing", i.e. from some unspecified vacuum [2]. They are affected, however, by problems of boundary conditions [3], arising just because of the ignorance, intrinsic to standard cosmology, about the state of the Universe before it emerged at the Planck scale.

The standard cosmological scenario has been complemented and improved, in many aspects, by the inflationary scenario [4]. Concerning however the very beginning of the Universe, i.e. the state and the evolution of the Universe before the Planck epoch, the situation in conventional inflation is not so much different from that of the standard model, because a phase of conventional inflationary expansion, at constant curvature, cannot be extended back in time for ever [5]. Quoting Alan Guth's recent survey of inflationary cosmology [6]:

"... Nevertheless, since inflation appears to be eternal only into the future, but not to the past, an important question remains open. How did all start? Although eternal inflation pushes this question far into the past, and well beyond the range of observational tests, the question does not disappear."

String theory seems to suggest an answer to this question and, most important, seems to suggest that the beginning of the Universe, namely its evolution at times earlier than

Planckian, might be not completely beyond the range of present observational tests, in contrast to the sentence quoted above. The technical instrument used by string theory to look back in time, beyond the Planck scale and the big bang singularity, is (a general version of) the duality symmetry which, together with supersymmetry, is probably one of the most powerful and important tools of modern theoretical physics (at least, because they are both at the grounds of superstring theory [7], which is at present one of the best candidate for a Theory of Everything).

Just like supersymmetry associates to any bosonic state a fermionic partner, and vice-versa, duality associates to any cosmological configuration with decreasing curvature a geometric partner with growing curvature, and vice-versa. Just like supersymmetry cancellations can eliminate the field theory divergences, duality symmetries are expected to regularize the spacetime and curvature singularities. The assumption of (at least approximate) "self-duality" symmetry, which combines duality and time reversal, suggests in particular a complete model of cosmological evolution, defined in cosmic time from minus to plus infinity, in which the Universe expands around a fixed point of maximal (finite) curvature, controlled by the fundamental length scale L_s of string theory [8].

The big bang singularity is replaced in this context by a phase of high (nearly Planckian) curvature, which marks the transition from an initial accelerated growth of the curvature H and of the string coupling g_s (parametrized by the dilaton ϕ as $g_s = e^{\phi/2}$), to a final state of radiation-dominated, decelerated expansion at constant dilaton. It comes natural, in this context, to call "pre-big bang" the initial phase of growing curvature, in contrast to the subsequent, standard "post-big bang" evolution, with decreasing curvature.

The most revolutionary aspect of this scenario is probably the fact that the high-curvature, Planckian regime is reached *at the end*, and not *at the beginning* of inflation. Thus, the state of the Universe at the Planck scale does not represent an initial condition, but is rather the result of a long and classical pre-big bang (i.e. pre-Planckian) evolution, which starts from a state of very low curvature and small coupling ($H \ll L_s^{-1}$, $g_s \ll 1$), and is well controlled by the low-energy string effective action. In other words, the Universe is far from being a "new-born baby" at the time of the big bang transition, being instead almost in the middle of a very long, possibly infinite, life.

From a phenomenological point of view, the important aspect of this scenario is the fact that the cosmological evolution preceding the Planck epoch may become accessible to present

(direct or indirect) observations. I would like to recall, in particular, three possible effects, referring to observations to be performed *i)* in a not so far future, *ii)* in a near future, and *iii)* to observations already (in part) performed. They are, respectively: the presence of a graviton background much stronger than expected in standard inflation [9], the contribution of massless (or massive) axion fluctuations to the CMB anisotropy spectrum [10], and the production of primordial "seeds" for the cosmic magnetic fields [11].

The predictive power of this scenario relies however on the construction of non-singular models, describing a smooth transition from the pre- to the post-big bang regime. Implementing such a transition is in general problematic in the context of the tree-level, gravi-dilaton string effective action; there are "no-go theorems" [12] excluding a regular transition also in the presence of perfect fluid and axionic Kalb-Ramond sources, and suggesting the need for higher order (quantum loops [13] and higher curvature [14]) corrections. Examples of a complete transition through the strong coupling regime have been implemented, up to date, but only with the help of "ad hoc" corrections: a non-local two-loop [15] or four-loop [16] dilaton potential, a higher derivative dilaton kinetic term [17].

The above non-go theorems are all formulated in the context of homogeneous and isotropic backgrounds. It is known, on the other hand, that the singularity can be "boosted away" already at the tree-level [18] (through an appropriate transformation of the global, pseudo-orthogonal duality symmetry group), provided the metric is allowed to be anisotropic. There are examples [18], dating back to the early studies of the pre-big bang scenario, of anisotropic solutions with a non-trivial axion background which satisfy all the conditions [17] necessary for a "graceful exit" from the pre-big bang phase, and which describe indeed a perfectly smooth transition from an initial growing curvature and dilaton phase, to a final decreasing curvature and dilaton phase.

Such examples are usually regarded as unrealistic, mainly because in the final post-big bang regime the metric background may be contracting; if expanding, it is nevertheless highly anisotropic, with only two dynamical (spatial) dimensions (the background is frozen in all the other space directions).

It should be taken into account, however, that the background transition described by the above solutions generates a large amount of radiation: the quantum fluctuations of the initial pre-big bang state are amplified by the accelerated evolution of the background, and re-enter the horizon in the subsequent decelerated phase, contributing eventually to the

back reaction

post-big bang sources as a gas of relativistic particles. In the post-big bang phase, this radiation tends to become dominant with respect to the axion sources [19], and it is well known that the radiation can isotropize an initially anisotropic metric [20]. In a contracting background, we may expect that the radiation energy density become dominant even faster, and may even turn the initial contraction into a final expansion, as suggested by the general radiation-dominated solution of the gravi-dilaton cosmological equations [21].

To confirm this expectation, I will now present the results of a numerical computation which shows the changes induced by the radiation back reaction in the final state of the regular solutions [18]. The aim is twofold: *1)* to stress the possibility of a smooth connection between the pre-big bang Universe and the present isotropic, expanding Universe also to lowest order in the string effective action, without any "ad hoc" higher order correction (the importance of the low energy string effective action has been recently stressed also in the context of M-theory [22]); *2)* to point out the possible relevance of contracting backgrounds for the solution of the graceful exit problem of string cosmology.

I will concentrate on the particular class of regular backgrounds obtained by boosting the dual of the two-dimensional vacuum Milne solution. The dilaton, the non-vanishing components of the metric and of the antisymmetric axion field, $B_{\mu\nu} = -B_{\nu\mu}$, can be written in the synchronous gauge as follows [18]:

$$\phi = \phi_0 - \ln\left(\beta + \alpha b^2 t^2\right), \quad g_{11} = -\frac{\alpha + \beta b^2 t^2}{\beta + \alpha b^2 t^2}, \quad g_{12} = -\frac{\sqrt{\alpha\beta}\,(1 + b^2 t^2)}{\beta + \alpha b^2 t^2},$$
$$g_{00} = -g_{22} = -g_{33} = 1, \quad B_{12} = g_{12}, \quad \alpha = \cosh\gamma + 1, \quad \beta = \cosh\gamma - 1, \tag{1}$$

where ϕ_0, b and γ are real arbitrary parameters. This background represents an exact anisotropic solution of the tree-level string cosmology equations:

$$R_\mu{}^\nu + \nabla_\mu\nabla^\nu\phi - \frac{1}{4}H_{\mu\alpha\beta}H^{\nu\alpha\beta} = 0, \qquad R - (\nabla_\mu\phi)^2 + 2\nabla_\mu\nabla^\mu\phi - \frac{1}{12}H_{\mu\nu\alpha}H^{\mu\nu\alpha} = 0,$$
$$\partial_\nu\left(\sqrt{|g|}e^{-\phi}H^{\nu\alpha\beta}\right) = 0, \qquad H_{\nu\alpha\beta} = 3!\partial_{[\nu}B_{\alpha\beta]}. \tag{2}$$

The time evolution of the dilaton and of H_1, which represents the rate-of-change of the distance along the x_1 direction between two comoving geodesics, is illustrated in Fig. 1. The parameter H_1, which is the analog of the Hubble parameter for anisotropic, off-diagonal metrics, is defined by $H_1 = \theta_{\mu\nu}n^\mu n^\nu$, where n^μ is a unit space-like vector along x_1, and, in the synchronous frame, $\theta_{\mu\nu} = \nabla_{(\mu}u_{\nu)}$ is the so-called expansion tensor for a congruence of

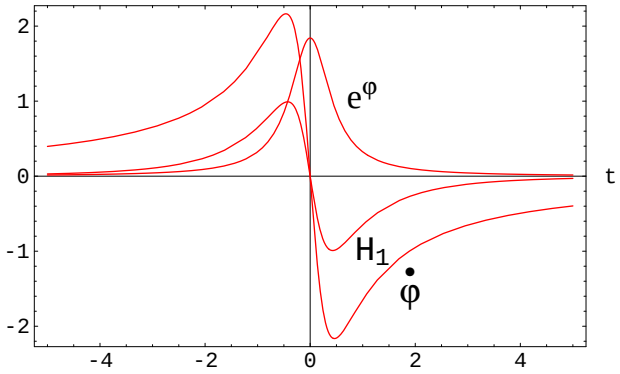

FIG. 1. *Smooth evolution from an expanding pre-big bang configuration to a contracting post-big bang configuration, according to the solution (1), with* $\phi_0 = 0$, *and* $b = \gamma = 1$.

co-moving geodesics u^μ [23]. As clearly shown in Fig. 1, the initial accelerated expansion ($H_1 > 0, \dot{H}_1 > 0$), evolves smoothly into a final decelerated contraction ($H_1 < 0, \dot{H}_1 > 0$). The evolution is non-trivial only in the $\{x_1, x_2\}$ plane, as $H_2 = H_3 = 0$. Notice that, with an appropriate choice of the parameters, it is always possible to bound the peak values to be smaller than one in string units, consistently with the low-energy effective action.

In the above background, the transition to the post-big bang regime amplifies the quantum fluctuations of the initial pre-big bang state. In other words, the final post-big bang state is characterized by a large number of massless particles (gravitons, dilatons, photons...), produced in pairs from the vacuum [19]: their total energy density ρ is bounded by the maximal curvature scale of the background, which, in its turn, is controlled by the string length scale L_s. Their effective averaged stress tensor is traceless [24], and we can thus represent their contribution to the post-big bang background like that of an effective radiation fluid, with $\langle\rho\rangle = \langle 3p\rangle$ and $\langle\rho\rangle \lesssim L_s^{-4}$. This contribution is weighed by the dilaton, in the string frame [8], as $\langle\rho\rangle e^\phi$, and it is initially subdominant at the beginning of the post-big bang phase, but tends to grow in time with respect to the axion.

To take into account this back reaction, I have added to the right hand side of the first of equations (2) the contribution of the effective radiation stress tensor, $e^\phi \langle T_\mu{}^\nu \rangle$ (in units $8\pi G = 1$), and I have numerically integrated the system of equations (2) plus the conservation equation $\nabla_\nu \langle T_\mu{}^\nu \rangle = 0$ (which is still valid in the usual form, in spite of the dilaton [8]). I have imposed the boundary conditions that the background starts initially (at

large and negative times) in the configuration described by the solution (1), and that the radiation keeps negligible until the background is well inside the post-big bang regime. The evolution is thus unchanged in the pre-big bang phase, but the final stage of the post-big bang evolution is qualitatively affected by the radiation back reaction, as illustrated in the three following figures where the results of the numerical integration (plotted as solid curves, with time measured in units of b^{-1}) are compared to the unperturbed solution.

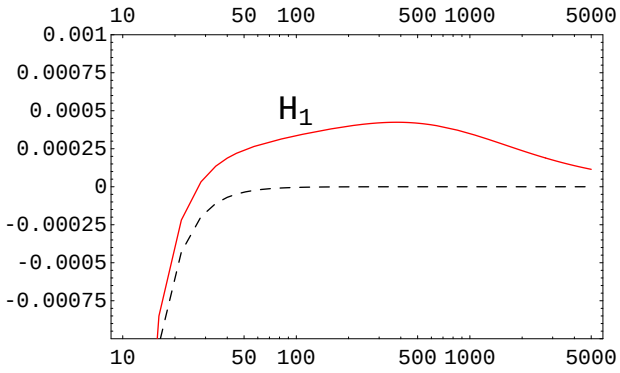

FIG. 2. *Time evolution of H_1. With the inclusion of the radiation back reaction (solid curve) the decelerated contraction of Fig. 1 (dashed curve) becomes decelerated expansion, with H_1 positive and asymptotically decreasing.*

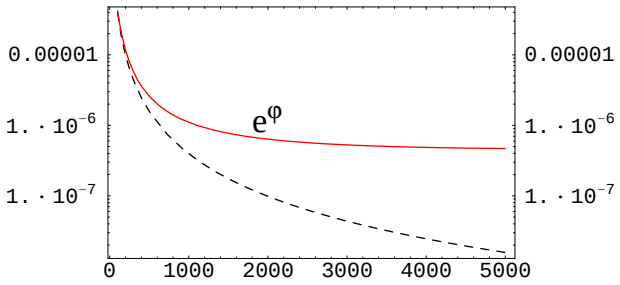

FIG. 3. *Time evolution of the dilaton. The friction of the radiation back reaction tends to stop the dilaton (solid curve), with respect to the axion-dominated solution of Fig. 1 (dashed curve).*

There are three main effects: the contraction turns eventually into a standard decelerated

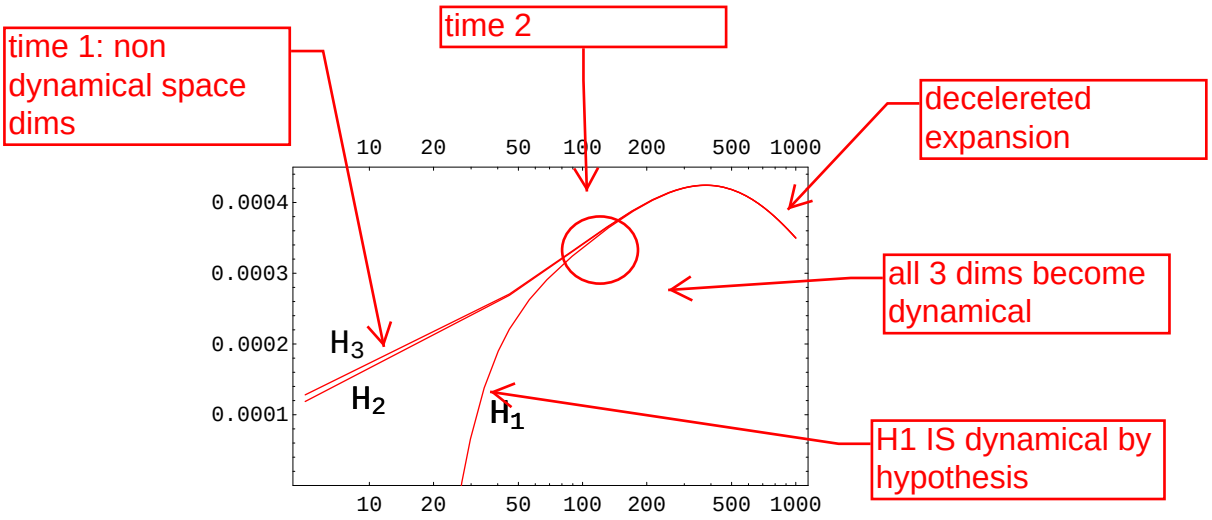

FIG. 4. *Time evolution of H_1, H_2, H_3, with the inclusion of the radiation back reaction. All the spatial dimensions become dynamical, and the background converges to a state with the same rate of decelerated expansion along any direction.*

expansion, with $H_1 > 0$ and $\dot{H}_1 < 0$ (Fig. 2); the dilatons tends to stop (Fig. 3), as the background converges towards the radiation-dominated, frozen-dilaton asymptotic solution; the frozen spatial dimensions start to expand ($H_1, H_2 \neq 0$), and the expansion tends to become isotropic (Fig. 4), asymptotically approaching a state in which $H_1 = H_2 = H_3$, and in which the expansion rates along the three spatial directions are all positive and decreasing.

This example is not completely realistic, for various reasons (for instance, an appropriate non-perturbative dilaton potential is expected to be included, in the post-big bang phase, to give a mass to the dilaton, and to fix the final string coupling to a realistic value [25] $\langle g_s^2 \rangle = \langle e^{\phi} \rangle \sim 10^{-2} - 10^{-4}$). Already from this simple example we can learn, however, that the back reaction of the produced radiation is possibly a key missing ingredient in previous studies of the graceful exit problem. It is a physical effect, not a term added "ad hoc" to the action, which could represent the last step of a a complete transition from the string perturbative vacuum to the present cosmological state. When such a back reaction is included, in particular, it seems possible to have a look at the pre-big bang Universe even following the geodesics of the low-energy string effective action.

MASS of dilaton?????

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REFERENCES

[1] S. Weinberg, *Gravitation and cosmology* (Wiley, New York, 1972); E. W. Kolb and M. S. Turner, *The Early Universe*, (Addison Wesley, Redwood City, Ca, 1990).

[2] A. Vilenkin, Phys. Rev. D **30**, 509 (1984); A. D. Linde, Sov. Phys. JEPT **60**, 211 (1984); Y. Zel'dovich and A. A. Starobinski, Sov. Astron. Lett. **10**, 135 (1984); V. A. Rubakov, Phys. Lett. B **148**, 280 (1984).

[3] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983); S. W. Hawking, Nucl. Phys. B **239**, 257 (1984); S. W. Hawking and D. N. Page, Nucl. Phys. B **264**, 185 (1986).

[4] A. Guth, Phys. Rev. D **23**, 347 (1981).

[5] A. Vilenkin, Phys. Rev. D **46**, 2355 (1992); A. Borde, A. Vilenkin, Phys. Rev. Lett. **72**, 3305 (1994).

[6] A. Guth, *The inflationary Universe*, (Vintage, London, 1998).

[7] M. B. Green, J. Schwartz and E. Witten, *Superstring theory*, (Cambridge U. Press, Cambridge, Ma, 1987).

[8] G. Veneziano, Phys. Lett. B **265**, 287 (1991); M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993); Mod. Phys. Lett. A **8**, 3701 (1993); Phys. Rev. D **50**, 2519 (1994). An updated collections of papers on the pre-big bang scenario is available at `http://www.to.infn.it/~gasperin`.

[9] M. Gasperini and M. Giovannini, Phys. Lett. B **282**, 36 (1992); Phys. Rev. D **47**, 1519 (1993); R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B **361**, 45 (1995).

[10] E. J. Copeland, R. Easther and D. Wands, Phys. Rev. D **56**, 874 (1997); R. Durrer, M . Gasperini, M. Sakellariadou and G. Veneziano, Phys. Lett. B **436**, 66 (1998); Phys. Rev. D **59**, 043511 (1999); M. Gasperini and G. Veneziano, Phys. Rev. D **59**, 043503 (1999).

[11] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75**, 3796 (1995).

[12] R. Brustein and G. Veneziano, Phys. Lett. B **329**, 429 (1994); N. Kaloper, R. Madden

and K. A. Olive, Nucl. Phys. B **452**, 677 (1995); R. Easther, K. Maeda and D. Wands, Phys. Rev. D **53**, 4247 (1996).

[13] I. Antoniadis, J. Rizos and K. Tamvakis, Nucl. Phys. B **415**, 497 (1994); S. J. Rey, Phys. Rev. Lett. **77**, 1929 (1996); M. Gasperini and G. Veneziano, Phys. Lett. B **387**, 715 (1996); S. Foffa, M. Maggiore and R. Sturani, *Loop corrections and graceful exit in string cosmology*, IFUP-TH 8/99 (February 1999).

[14] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B **494**, 315 (1997); M. Maggiore, Nucl. Phys. B **525**, 413 (1998); R. Brandenberger, R. Easther and J. Maia, JHEP **9808**, 007 (1998).

[15] Second paper in Ref. [8].

[16] M. Gasperini, J. Maharana and G. Veneziano, Nucl. Phys. B **472**, 349 (1996).

[17] R. Brustein and R. Madden, Phys. Lett. B. **410**, 110 (1997); Phys. Rev. D **57**, 712 (1998).

[18] M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B **272**, 277 (1991).

[19] M. Gasperini, in *String gravity and physics at the Planck energy scale*, edited by N. Sánchez and A. Zichichi (Kluwer A. P. , Dordrecht, 1996), p. 305.

[20] J. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics*, (Chicago University Press, Chicago, 1983).

[21] Third paper in Ref. [8].

[22] T. Banks, W. Fischler and L. Motl, *Dualities versus singularities*, hep-th/9811194.

[23] R. M. Wald, *General relativity*, (Chicago University Press, Chicago, 1984).

[24] R. L. Abramo, R. H. Brandenberger and V. F. Mukhanov, Phys. Rev. D **56**, 3248 (1997).

[25] V. Kaplunovsky, Phys. Rev. Lett. **55**, 1036 (1985).

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GRACEFUL EXIT IN QUANTUM STRING COSMOLOGY

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Abstract

We write an $O(d,d)$-covariant Wheeler-De Witt equation in the (d^2+1)-dimensional minisuperspace of low-energy cosmological string backgrounds. We discuss explicit examples of transitions between two duality-related cosmological phases, and we find a finite quantum transition probability even when the two phases are classically separated by a curvature singularity. This quantum approach is completely free from operator ordering ambiguities as a consequence of the duality symmetries of the string effective action.

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1 Introduction

The cosmological solutions of the low-energy string effective action, and their duality symmetry properties, have recently motivated the study of an inflationary scenario [1]-[3] in which the universe evolves from the string perturbative vacuum to the high curvature, strong coupling regime. One of the main problems, in this context, is the "graceful" exit from such an accelerated (also called pre-big-bang [1]) phase to the decelerated, decreasing curvature phase typical of the standard (post-big-bang) cosmological evolution. Both phases are present in the exact solutions of the tree-level, lowest order in α', string effective action [2, 3]. However, for vanishing torsion and dilaton potential, these phases correspond to different (duality-related) branches of the solution, defined over disconnected ranges of the time parameter and separated by a singularity of the curvature and of the coupling.

The graceful exit problem would be solved, at the classical level, by an exact cosmological solution connecting smoothly the two branches, and thus describing a continuous evolution from accelerated to decelerated expansion. Unfortunately, confirming a previous conjecture [4], it has been rigorously proved [5] that such a change of branch cannot be simply catalyzed by any (realistic) dilaton potential, if we limit ourselves to lowest order in the α' expansion of the string effective action. Such a no-go theorem has been recently extended, for spatially homogeneous and isotropic manifolds, to the case of non-vanishing torsion background [6] and non-vanishing spatial curvature [7].

A possible way of incorporating branch-changing in the pre-big-bang scenario is thus to resort to the conformal field theory approach [8], where all higher orders in α' are taken into account. On the other hand, in such a "stringy" regime dominated by higher-derivative terms in the effective action, the curvature is expected to approach the Planck scale and thus the quantum gravity regime. This suggests a quantum approach to the graceful exit problem (quantum cosmology methods in a string theory context were previously introduced also in [9, 10]).

By applying the Wheeler-De Witt (WDW) equation to the gravi-dilaton system, we show in this paper that the transition from a pre-big-bang to a post-big-bang classical solution corresponds to a reflection of the wave function in minisuperspace. A classical configuration describing branch-changing gives a reflection coefficient $R = 1$. The reflection probability is in general non-vanishing, however, even if the given classical background forbids branch-

changing. This is the main result of this paper, which may allow a systematic classification of the initial conditions compatible with the present state of our Universe, even ignoring kinematical details during the quantum transition era.

The paper is organized as follows. In Section 2 we derive from the low energy string effective action the WDW equation for homogeneous, spatially flat cosmological backgrounds, including a non-trivial antisymmetric (torsion) tensor. We show, in particular, that the operator ordering problem is trivially solved because of the $O(d,d)$ covariance of the kinetic part of the Hamiltonian, which implies a globally flat minisuperspace metric. In Section 3 we study the free wave equation, and we identify the left and right moving modes in superspace with the two branches of the classical vacuum solution. The reader not interested in technical complications due to the presence of a non-trivial torsion background can move directly to Section 4, where we give two self-contained examples of "quantum" branch changing, in the two-dimensional minisuperspace parameterized by the dilaton field and by the (isotropic) metric scale factor. While the first example has a classic analogue, the second corresponds to a background configuration in which branch changing is classically forbidden. A brief summary, and our concluding remarks, are finally presented in Section 5.

2 O(d,d)-covariant Wheeler-De Witt equation

At low energy, the tree-level, $(d+1)$-dimensional (super)string effective action can be written as [11]

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|} e^{-\phi} \left(R + \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\alpha} H^{\mu\nu\alpha} + V \right). \tag{2.1}$$

Here ϕ is the dilaton field, $H_{\mu\nu\alpha}$ is the field strength of the two-index antisymmetric torsion tensor $B_{\mu\nu} = -B_{\nu\mu}$, and $\lambda_s \equiv (\alpha')^{1/2}$ is the fundamental string length parameter governing the high derivative expansion of the action. Note that we have included a possible dilaton potential V, and we have chosen to work in the String (or Brans-Dicke frame), whose metric coincides with the sigma-model metric to which strings are directly coupled. The more conventional choice of the Einstein frame leads to an equivalent description of the same cosmological scenario [3, 12], but it is less convenient for exploiting the duality symmetries

of the underlying theory.

We shall consider, in this paper, homogeneous backgrounds with d Abelian isometries, for which a synchronous frame exists where $g_{00} = 1$, $g_{0i} = 0 = B_{0i}$, and the fields are independent of all space-like coordinates x^i $(i, j = 1, .., d)$. We also assume spatial sections of finite volume, $(\int d^d x \sqrt{|g|})_{t=\text{const}} < \infty$. For such backgrounds the action can be rewritten as [13]

$$S = -\frac{\lambda_s}{2} \int dt e^{-\overline{\phi}} \left[(\dot{\overline{\phi}})^2 + \frac{1}{8} \text{Tr } \dot{M} (M^{-1})\dot{} + V \right] \tag{2.2}$$

where a dot denotes differentiation with respect to the cosmic time t, and we have chosen to express length and energies in string units through λ_s. Here $\overline{\phi}$ is the " shifted" dilaton field,

$$\overline{\phi} = \phi - \ln |\det \, g_{\mu\nu}|^{1/2} \tag{2.3}$$

(we have absorbed into ϕ the constant shift $-\ln(\lambda_s^{-d} \int d^d x)$ required to secure the scalar behaviour of $\overline{\phi}$ under coordinate reparametrization). Finally, M is the $2d \times 2d$ matrix

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{2.4}$$

where G and B are, respectively, matrix representations of the spatial part of the metric (g_{ij}) and of the antisymmetric tensor (B_{ij}).

For constant V, the whole action (2.2) is invariant under global $O(d, d)$ transformations [13]

$$\overline{\phi} \to \overline{\phi}, \qquad M \to \Omega^T M \Omega \tag{2.5}$$

where

$$\Omega^T \eta \Omega = \eta, \qquad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.6}$$

In addition, M satisfies

$$M \eta M = \eta. \tag{2.7}$$

This $O(d, d)$ symmetry is preserved in the presence of bulk string matter [14] satisfying the string equations of motion, and it reduces to the scale factor duality symmetry [15, 16] (for torsionless, diagonal metric backgrounds) in the particular case in which we restrict Ω to η in eq. (2.5).

By using as time parameter τ, with $dt = e^{-\overline{\phi}} d\tau$, the action (2.2) leads to the Lagrangian (a prime denotes differentiation with respect to τ)

$$L(\tau) = -\frac{\lambda_s}{2} \left[(\overline{\phi}')^2 + \frac{1}{8} \text{Tr } M' (M^{-1})' + e^{-2\overline{\phi}} V \right] \tag{2.8}$$

whose corresponding Hamiltonian is

$$H = -\frac{1}{2\lambda_s}\Pi_{\overline{\phi}}^2 + \frac{4}{\lambda_s}\mathrm{Tr}\ (M\ \Pi_M M\ \Pi_M) + \frac{\lambda_s}{2}Ve^{-2\overline{\phi}} \tag{2.9}$$

where $\Pi_{\overline{\phi}}$ and Π_M are the (dimensionless) canonical momenta

$$\Pi_{\overline{\phi}} = \frac{\delta L}{\delta \overline{\phi}'} = -\lambda_s \overline{\phi}', \qquad \Pi_M = \frac{\delta L}{\delta M'} = \frac{\lambda_s}{8} M^{-1} M' M^{-1}. \tag{2.10}$$

The variation of the action (2.1) with respect to the "lapse" function, $\sqrt{g_{00}}$, provides the canonical constraint $H = 0$. The WDW equation [17], implementing in superspace such a constraint through the differential representation $\Pi_{\overline{\phi}} = \pm i\delta/\delta\overline{\phi}$, $\Pi_M = \pm i\delta/\delta M$, would seem to be affected (as usual) by problems of quantum ordering, as $[M, \Pi_M] \neq 0$. In our context, however, the problems actually disappear because our (d^2+1)-dimensional minisuperspace is globally flat, as a consequence of the $O(d,d)$ symmetry. Indeed, by using the $O(d,d)$ property (2.7), we can always rewrite the M-dependent part of the kinetic operator as

$$\frac{1}{16}\mathrm{Tr}\ M'(M^{-1})' = \frac{1}{16}\mathrm{Tr}\ (M'\eta)^2. \tag{2.11}$$

The corresponding Hamiltonian

$$H = -\frac{1}{2\lambda_s}\Pi_{\overline{\phi}}^2 - \frac{4}{\lambda_s}\mathrm{Tr}\ (\eta\ \Pi_M\ \eta\ \Pi_M) + \frac{\lambda_s}{2}Ve^{-2\overline{\phi}} \tag{2.12}$$

has a flat metric in momentum space, and leads to a WDW equation

$$\left[\frac{\delta^2}{\delta\overline{\phi}^2} + 8\mathrm{Tr}\ \left(\eta\frac{\delta}{\delta M}\eta\frac{\delta}{\delta M}\right) + \lambda_s^2 Ve^{-2\overline{\phi}}\right]\Psi(\overline{\phi}, M) = 0, \tag{2.13}$$

which is manifestly free from problems of quantum ordering.

If we introduce curvilinear coordinates in minisuperspace, adopting for instance the parametrization of eq. (2.9), the ordering imposed by the $O(d,d)$ symmetry is equivalent to the general covariance of the Laplacian operator, as can be easily checked for the simple isotropic case $B = 0$, $G_{ij} = -a^2\delta_{ij}$. In that case the kinetic part of the Hamiltonian (2.12) is represented as

$$\lambda_s H_{\mathrm{Kin}} = -\frac{1}{2}\Pi_{\overline{\phi}}^2 - 4\mathrm{Tr}\ (\eta\Pi_M)^2 \equiv \frac{1}{2}\partial_{\overline{\phi}}^2 - 2d(a\partial_a + a^2\partial_a^2). \tag{2.14}$$

The parametrization of eq. (2.9), on the other hand, corresponds to the metric

$$\gamma_{\mu\nu} = \mathrm{diag}\ \left(-2,\ \frac{a^4}{4d},\ \frac{1}{4da^4}\right), \qquad \mu,\nu = 1,2,3 \quad , \tag{2.15}$$

in the three-dimensional space spanned by the differential operators

$$\Pi_\mu = i\partial_\mu = i\left(\frac{\partial}{\partial\overline{\phi}},\ \frac{\partial}{\partial a^{-2}},\ \frac{\partial}{\partial a^2}\right) \tag{2.16}$$

and the covariant Laplacian gives

$$-\nabla_\mu\nabla^\mu \equiv -\frac{1}{\sqrt{|\gamma|}}\partial_\mu\left(\sqrt{|\gamma|}\gamma^{\mu\nu}\partial_\nu\right) = \frac{1}{2}\partial_{\overline{\phi}}^2 - 2d(a\partial_a + a^2\partial_a^2) \equiv \lambda_s H_{\rm Kin}. \tag{2.17}$$

The ordering fixed by the scale factor duality symmetry of the classical Hamiltonian is thus the same as that imposed by the requirement of general reparametrization invariance in minisuperspace (note that in our case there is no possible contribution to the ordered Hamiltonian from the scalar curvature of superspace [18], as the metric is globally flat).

3 Branch changing as wave reflection

We shall apply, in this paper, the WDW equation (2.13) to study the probability of transition from a given initial background configuration of the pre-big-bang type, to a final configuration typical of the standard cosmological scenario. This amounts to solving eq. (2.13) for a given value of the dilaton potential $V(\phi)$, with appropriate boundary conditions. The systematic study of the transition probability for a "realistic" (supersymmetry breaking) non-perturbative dilaton potential is postponed to a future work. The main goal of this paper is to show that in a quantum cosmology context it is possible to tunnel from one branch to another of the low energy classical solutions, even if the two branches are not smoothly connected, but they are separated by curvature singularities and unphysical regions of finite size.

To this aim we shall work in the simplifying hypothesis of $O(d,d)$ symmetry of the whole (in general non-local) action, assuming $V = V(\overline{\phi})$, because in that case the WDW equation (2.13) can be separated by setting

$$\Psi(\overline{\phi}, M) = \chi_A(M)\psi_A(\overline{\phi}) \tag{3.1}$$

where

$$(M\Pi_M)\chi_A \equiv iM\frac{\delta}{\delta M}\chi_A = -\left(\frac{1}{8}A\eta\right)\chi_A \tag{3.2}$$

and

$$\left[\frac{\delta^2}{\delta\overline{\phi}^2} + \frac{1}{8}\mathrm{Tr}\ (A\eta)^2 + \lambda_s^2 V(\overline{\phi})e^{-2\overline{\phi}}\right]\psi_A(\overline{\phi}) = 0. \tag{3.3}$$

We have used here the momentum conservation $[M\Pi_M, H] = 0$, in agreement with the classical equations of motion obtained from the Lagrangian (2.8), which imply [13, 14]

$$M\Pi_M = -\frac{\lambda_s}{8}M\eta M'\eta = -\frac{1}{8}A\eta \tag{3.4}$$

where A is a constant $2d \times 2d$ matrix satisfying

$$M\eta A + A\eta M = 0. \tag{3.5}$$

If we consider, in particular, the "free" wave equation ($V = 0$), eq. (3.3) is easily solved by a linear superposition of left and right moving waves,

$$\psi_A^{\pm}(\overline{\phi}) = \exp\left\{\pm\frac{i}{2}\overline{\phi}\left[\frac{1}{2}\mathrm{Tr}\ (A\eta)^2\right]^{1/2}\right\}. \tag{3.6}$$

For simplicity, we shall restrict our subsequent discussion to a diagonal, Bianchi I type vacuum background with $G_{ij} = -a_i^2(t)\delta_{ij}$ and $B = 0$, corresponding to [13]

$$A = \begin{pmatrix} 0 & -A_d \\ A_d & 0 \end{pmatrix}, \qquad (A_d)_{ij} = c_i\delta_{ij} \quad , \tag{3.7}$$

where c_i are arbitrary constants. The metric-dependent part of the wave function becomes in this case

$$\chi_A(a_j) = N\exp\left\{-\frac{i}{2}\sum_j c_j \ln a_j\right\}, \tag{3.8}$$

as one can check after realizing that the operator $M\Pi_M$ contains in this case only d independent variables. By defining

$$\alpha_j = c_j\left(\sum_j c_j^2\right)^{-1/2} \equiv c_j\left[\frac{1}{2}\mathrm{Tr}\ (A\eta)^2\right]^{-1/2}, \qquad \sum_j \alpha_j^2 = 1 \tag{3.9}$$

the solutions of the WDW equation can finally be written in the form

$$\Psi_A^{(\pm)}(\overline{\phi}, M) = \chi_A(M)\psi_A^{\pm}(\overline{\phi}) = N_{\pm}\exp\left\{-\frac{i}{2}\left[\mathrm{Tr}\ (A_d)^2\right]^{1/2}\left(\sum_j \alpha_j \ln a_j \mp \overline{\phi}\right)\right\}, \tag{3.10}$$

where $N_{\pm}$ is an overall normalization coefficient.

For fixed α_j, namely for a given eigenstate of $M\Pi_M$, the left and right moving modes $\Psi^{(\pm)}$ correspond to different branches of the exact solution of the vacuum string cosmology equations [2, 3, 13, 15, 16, 19]

$$a_j = a_{j0}|t/\lambda_s|^{\pm\alpha_j}, \qquad \sum_j \alpha_j^2 = 1, \qquad \overline{\phi} - \phi_0 = -\ln|t/\lambda_s| = \mp\sum_j \alpha_j \ln a_j, \tag{3.11}$$

where a_{i0} and ϕ_0 are integration constants. What is important, for our purpose, is that if we apply the momentum operator $\Pi_{\overline{\phi}} = i\partial/\partial\overline{\phi}$ to the right moving wave $\Psi_A^{(+)}$ (the opposite sign with respect to the standard convention is due to the definition of $\Pi_{\overline{\phi}}$, eq. (2.10)), we reproduce the canonical momentum

$$\Pi_{\overline{\phi}} = -\lambda_s \dot{\overline{\phi}} e^{-\overline{\phi}} = -e^{-\phi_0} < 0 \tag{3.12}$$

of a classical configuration corresponding to an accelerated, expanding background, with growing curvature and dilaton coupling:

$$a_i \sim (-t)^{-\alpha_i}, \quad t < 0, \quad \alpha_i > 0, \quad \overline{\phi} - \phi_0 = \sum_j \alpha_j \ln a_j, \quad \dot{\overline{\phi}} > 0. \tag{3.13}$$

By applying $\Pi_{\overline{\phi}}$ to the left mover $\Psi_A^{(-)}$, labelled by the same eigenvalue A, we find instead a configuration with the opposite canonical momentum,

$$\Pi_{\overline{\phi}} = -\lambda_s \dot{\overline{\phi}} e^{-\overline{\phi}} = e^{\phi_0} > 0 \tag{3.14}$$

corresponding again to an expanding branch of the same classical solution, but decelerated and with decreasing curvature:

$$a_i \sim t^{\alpha_i}, \quad t > 0, \quad \alpha_i > 0, \quad \overline{\phi} - \phi_0 = -\sum_j \alpha_j \ln a_j, \quad \dot{\overline{\phi}} < 0. \tag{3.15}$$

A branch transition of the type required to solve the graceful exit problem, involving a scale factor duality transformation and time reversal [1], namely $a(t) \to a^{-1}(-t)$, is thus equivalent in this context to the spatial reflection of the WDW wave function in minisuperspace, $\Psi_A^{(+)} \to \Psi_A^{(-)}$.

4 Two simple examples

As no reflection is possible for free waves, let us introduce an appropriate dilaton potential, considering for simplicity a $d = 3$ isotropic background, and setting

$$B = 0, \qquad a(t) = \exp\left[\beta(t)/\sqrt{3}\right]. \tag{4.1}$$

The lowest order gravi-dilaton effective action,

$$S = -\frac{1}{2\lambda_s}\int d^4x\sqrt{-g}e^{-\phi}\left(R + \partial_\mu\phi\partial^\mu\phi + V\right), \tag{4.2}$$

after integrating by parts, and using as before the convenient time parameterization $dt = d\tau e^{-\overline{\phi}}$, reduces to (in the gauge $g_{00} = 1$):

$$S = -\frac{\lambda_s}{2}\int d\tau\left(\,\overline{\phi}'^2 - \beta'^2 + Ve^{-2\overline{\phi}}\,\right) \tag{4.3}$$

where

$$\overline{\phi} = \phi - \ln\int(d^3x/\lambda_s^3) - \sqrt{3}\beta. \tag{4.4}$$

The corresponding Hamiltonian

$$H = \frac{1}{2\lambda_s}\left(\,\Pi_\beta^2 - \Pi_{\overline{\phi}}^2 + \lambda_s^2 Ve^{-2\overline{\phi}}\,\right), \qquad \Pi_\beta = \lambda_s\beta', \qquad \Pi_{\overline{\phi}} = -\lambda_s\overline{\phi}' \tag{4.5}$$

is a particular case of eq. (2.12), for the torsionless isotropic background considered here. The WDW equation then takes the general form of a two-dimensional Schrödinger-like equation in the plane $(\overline{\phi}, \beta)$:

$$\left[\,\partial_{\overline{\phi}}^2 - \partial_\beta^2 + \lambda_s^2 V(\overline{\phi},\beta)e^{-2\overline{\phi}}\,\right]\Psi(\overline{\phi},\beta) = 0. \tag{4.6}$$

As anticipated in the previous section, we shall assume in this paper $V = V(\overline{\phi})$, in order to separate variables. Let us discuss first the particular case

$$V(\overline{\phi}) = -V_0e^{4\overline{\phi}}, \qquad V_0 = \text{const}, \qquad V_0 > 0 \tag{4.7}$$

as a toy example of classical gravi-dilaton configuration allowing branch changing. A negative non-local dilaton potential, $V(\overline{\phi}) < 0$, although hard to motivate in a realistic superstring theory context, is indeed the only case in which exact analytical solutions are known [1, 20] connecting smoothly the pre- to the post-big-bang regime.

With the above potential, the classical equations of motion following from the Hamiltonian (4.5) imply momentum conservation along the β axis,

$$\Pi_\beta = \lambda_s \beta' = \lambda_s \dot{\beta} e^{-\overline{\phi}} = k = \text{const} \tag{4.8}$$

and are solved exactly by

$$\overline{\phi} = -\frac{1}{2} \ln \left(\frac{\lambda_s^2 V_0}{k^2} + \frac{k^2 t^2}{\lambda_s^2} \right), \qquad a = a_0 \left[\frac{k^2 t}{\lambda_s^2 \sqrt{V_0}} + \left(1 + \frac{k^4 t^2}{\lambda_s^4 V_0} \right)^{1/2} \right]^{1/\sqrt{3}} \tag{4.9}$$

(a_0 is a dimensionless integration constant). This is a regular "self-dual" solution, $a(t)/a_0 = a_0/a(-t)$, characterized by a bell-like shape of the curvature scale and of the coupling $e^{\overline{\phi}}$. It describes a background that evolves from an initial state of accelerated expansion and increasing curvature,

$$\begin{aligned} t \to -\infty, \qquad & a \ \sim (-t)^{-1/\sqrt{3}}, \quad \overline{\phi} \sim -\ln(-t) = \sqrt{3} \ln a = \beta \\ & \dot{a} \ > 0, \quad \ddot{a} > 0, \quad \dot{H} > 0 \end{aligned} \tag{4.10}$$

to a final state of decelerated expansion, decreasing curvature,

$$\begin{aligned} t \to +\infty, \qquad & a \ \sim t^{1/\sqrt{3}}, \quad \overline{\phi} \sim -\ln(-t) = -\sqrt{3} \ln a = -\beta \\ & \dot{a} \ > 0, \quad \ddot{a} < 0, \quad \dot{H} < 0. \end{aligned} \tag{4.11}$$

For the background generated by the potential (4.7), the WDW equation (4.6) can easily be separated by putting $\Psi(\overline{\phi}, \beta) = e^{-ik\beta} \psi_k(\overline{\phi})$, where k belongs to the continuous eigenvalue spectrum of Π_β,

$$\Pi_\beta \Psi_k = i \partial_\beta \Psi_k = k \Psi_k, \qquad [\Pi_\beta, H] = 0 \tag{4.12}$$

and ψ_k satisfies

$$\left(\ \partial_{\overline{\phi}}^2 + k^2 - \lambda_s^2 V_0 e^{2\overline{\phi}} \ \right) \psi_k(\overline{\phi}) = 0 \tag{4.13}$$

(note the role of time-like coordinate assigned to β, monotonically ranging from $-\infty$ to $+\infty$). The general solution for ψ_k is then a linear combination of modified Bessel functions $K_\nu(z)$, $I_\nu(z)$ [21], of complex index $\nu = ik$ and argument $z = \lambda_s \sqrt{V_0} e^{\overline{\phi}}$. We impose the regularity condition [22] $|\Psi_k| < 0$, corresponding to a vanishing wave function in the "impenetrable" region of infinite effective potential, $\psi_k(\overline{\phi}) \to 0$ for $\overline{\phi} \to +\infty$. This condition uniquely selects (modulo a normalization factor) the WDW solution as

$$\Psi_k(\overline{\phi}, \beta) = N K_{ik}(\lambda_s \sqrt{V_0} e^{\overline{\phi}} \ ) \ e^{-ik\beta}. \tag{4.14}$$

For $\overline{\phi} \to -\infty$, i.e. in the low energy regime, this solution contains asymptotically left and right moving waves, as [21]

$$\lim_{\overline{\phi}\to-\infty} \Psi_k(\overline{\phi},\beta) = -\frac{N\pi}{2\sin(ik\pi)}\left[\left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{ik}\frac{e^{-ik(\beta-\overline{\phi})}}{\Gamma(1+ik)} - \left(\frac{\lambda_s\sqrt{V_0}}{2}\right)^{-ik}\frac{e^{-ik(\beta+\overline{\phi})}}{\Gamma(1-ik)}\right] = \\ = \Psi_k^{(+)} + \Psi_k^{(-)} \tag{4.15}$$

As discussed in the previous section, the right movers represent the accelerated, negative time branch (4.10), with $\beta = \overline{\phi}$, the left movers the decelerated, positive time branch (4.11), with $\beta = -\overline{\phi}$. The reflection coefficient, $R_k = |\Psi_k^{(-)}|^2/|\Psi_k^{(+)}|^2$, measures the probability of transition between the two branches of the low energy classical solution. In the low energy limit, according to eq. (4.15), $R_k \to 1$ for all k, as expected because we have considered an example in which the two branches are smoothly connected already at the classical level.

Consider now an example with the opposite potential,

$$V(\overline{\phi}) = V_0 e^{4\overline{\phi}}, \qquad V_0 = \text{const}, \qquad V_0 > 0. \tag{4.16}$$

In this case branch changing is classically forbidden. Indeed, the momentum conservation (4.8) is still valid, but the classical solution becomes

$$\overline{\phi} = -\frac{1}{2}\ln\left(\frac{k^2t^2}{\lambda_s^2} - \frac{\lambda_s^2 V_0}{k^2}\right), \qquad a = a_0\left|\frac{k^2 t}{\lambda_s^2\sqrt{V_0}} + \left(\frac{k^4t^2}{\lambda_s^4 V_0} - 1\right)^{1/2}\right|^{1/\sqrt{3}}. \tag{4.17}$$

The low energy (large time limit) branches (4.10) and (4.11) still exist, but they are now separated by an unphysical region, of extension $|t| < \lambda_s^2\sqrt{V_0}/k^2$, where the dominant energy condition is violated, and the expansion rate (H) and the dilaton coupling ($e^{\overline{\phi}}$) become imaginary. A curvature singularity is present at both ends of such a region, where the branches (4.10) and (4.11) respectively end and start.

Nevertheless, the quantum probability of transition between the two branches is non-vanishing. In fact, the solution of the WDW equation can be factorized as before, with the difference that eq. (4.13) is replaced by

$$\left(\partial_{\overline{\phi}}^2 + k^2 + \lambda_s^2 V_0 e^{2\overline{\phi}}\right)\psi_k(\overline{\phi}) = 0. \tag{4.18}$$

The general solution for ψ_k can now be written as a linear combination of first and second kind Hankel functions [21], $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$. By assuming for the Universe an initial pre-big-bang configuration, we impose that in the high-curvature limit $z \to \infty$ there are only

right moving waves ($\dot{\overline{\phi}} > 0$, $\Pi_{\overline{\phi}} < 0$) approaching the singularity. This condition exactly coincides with the boundary conditions allowing tunnelling through classically forbidden regions of superspace [22] (which select only outgoing waves at the superspace boundary, where classical trajectories can end but not begin), and fixes the wave function as

$$\Psi_k(\overline{\phi}, \beta) = N H^{(1)}_{ik}(\lambda_s \sqrt{V_0} e^{\overline{\phi}}\ )\ e^{-ik\beta}. \tag{4.19}$$

Asymptotically, in the low curvature, perturbative regime $\overline{\phi} \to -\infty$, we then have

$$\begin{aligned} \lim_{\overline{\phi}\to-\infty} \Psi_k(\overline{\phi}, \beta) &= iN \csc(ik\pi) \left[e^{k\pi} \left(\frac{\lambda_s \sqrt{V_0}}{2} \right)^{ik} \frac{e^{-ik(\beta-\overline{\phi})}}{\Gamma(1+ik)} - \left(\frac{\lambda_s \sqrt{V_0}}{2} \right)^{-ik} \frac{e^{-ik(\beta+\overline{\phi})}}{\Gamma(1-ik)} \right] = \\ &= \Psi^{(+)}_k + \Psi^{(-)}_k \end{aligned} \tag{4.20}$$

and the relative amplitude of left and right modes defines the probability

$$R_k = \frac{|\Psi^{(-)}_k|^2}{|\Psi^{(+)}_k|^2} = e^{-2\pi k} \tag{4.21}$$

for transitions from the classical trajectory with $\beta = \overline{\phi}$ to the duality-related one, $\beta = -\overline{\phi}$.

By recalling the definition of k (eq. (4.8)) and of $\overline{\phi}$,

$$k = \frac{\sqrt{3}}{\lambda_s^2} \int d^3x \sqrt{-g} e^{-\phi} H = \text{const} \tag{4.22}$$

we can eventually express the above transition probability as

$$R(g_s, a_s) = \exp\left\{ -\frac{\sqrt{12}\pi}{g_s^2} \frac{\Omega(a_s)}{\lambda_s^3} \right\}. \tag{4.23}$$

Here $\Omega(a_s)$ and $g_s = e^{\phi_s/2}$ are, respectively, the values of the proper spatial volume and of the coupling, at the time $t = t_s$ at which $H = \lambda_s^{-1}$. For values of the coupling $g_s \sim 1$ the probability (4.21) is of order 1 for the formation of "bubbles" of unit proper size (or smaller) in string units at $t = t_s$.

The above example is not "realistic", in the sense that it does not describe the formation of a radiation-dominated or matter-dominated Universe similar to the one we live in today (we postpone the discussion of a more realistic scenario to a forthcoming paper). Nevertheless, it is an example of how the Universe can emerge from the inflationary phase in the right branch corresponding to decelerated expansion, and it is quite interesting that the probability of such a process is peaked in the strong coupling regime, with a typical instanton-like behaviour $R \sim \exp(-g^{-2})$.

We note, finally, that eq. (4.21) is valid for $k > 0$, namely for the transition between two expanding branches with $\mathrm{sign}[H] = \mathrm{sign}[\dot{\beta}] = \mathrm{sign}[k] > 0$, and that it implies $0 < R_k < 1$ ($R_k \to 0$ for $k \to \infty$, as expected when the effective potential in eq. (4.18) becomes negligible). If we consider transitions between contracting branches, $k < 0$, the probability becomes $e^{2\pi k}$ so that, in general, $R_k = e^{-2\pi|k|}$. The appearance of an always negative argument in the exponential is a general consequence of applying tunneling boundary conditions in superspace, as clearly stressed recently also in [23].

5 Conclusion

In string cosmology, a classical description of the background evolution based on the low energy string effective action is allowed both at early and late times (i.e. at large time scales in string units), but it is not allowed in the intermediate epoch, when the background is expected to exit from the inflationary regime. A classical model of transition from the initial string perturbative vacuum to the present standard cosmological regime conflicts both with phenomenological constraints and with formal no-go theorems.

In this paper we have shown that such a transition can be studied quantum mechanically, and can be formulated as a problem of reflection of the Wheeler-De Witt wave function in superspace. By using tunneling boundary conditions, we find that the transition can occur (with finite probability) even in the case of background configurations in which it would be classically forbidden.

This quantum approach to the graceful exit problem is free from ambiguities of operator ordering, because of the underlying $O(d, d)$ symmetry of the kinetic part of the Hamiltonian. It thus seems to provide an appropriate tool for a systematic classification of the initial conditions, irrespective of the (unknown) kinematic details of the high curvature, strong coupling, transition regime.

Various problems, in this approach, are still to be solved, such as the physical interpretation of the wave function and the univocal choice of an appropriate time parameter (these problems affect in general the WDW approach to quantum cosmology, not only our scenario). We believe, however, that the result presented in this paper may improve our understanding

of the birth and of the evolution of the Universe in terms of the basic principles of string theory.

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References

[1] M. Gasperini and G. Veneziano, Astropart. Phys. 1 (1993) 317

[2] M. Gasperini and G. Veneziano, Mod. Phys. Lett. A8 (1993) 3701

[3] M. Gasperini and G. Veneziano, Phys. Rev. D50 (1994) 2519

[4] R. Brustein and G. Veneziano, Phys. Lett. B329 (1994) 429

[5] N. Kaloper, R. Madden and K. A. Olive, Nucl. Phys. B452 (1995) 677

[6] N. Kaloper, R. Madden and K. A. Olive, *Axions and the graceful exit problem in string cosmology*, UMN-TH-1414/95 (hep-th/9510117)

[7] R. Easther, K. Maeda and D. Wands, *Tree-level string cosmology*, SUSSEX-AST-95/9-1 (hep-th/9509074)

[8] E. Kiritsis and C. Kounnas, Phys. Lett. B 331 (1994) 51

[9] M. C. Bento and O. Bertolami, Class. Quantum Grav. 12 (1995) 1919

[10] J. E. Lidsey, Phys. Rev. D52 (1995) R5407

[11] C. Lovelace, Phys. Lett. B135 (1984) 75;

E. S. Fradkin and A. A Tseytlin, Nucl. Phys. B261 (1985) 1;

C. G. Callan et al., Nucl. Phys. B262 (1985) 593.

[12] M. Gasperini, in *Proc. of the 2nd Journée Cosmologie*, ed. by N. Sanchez and H. De Vega (World Scientific, Singapore, 1995) p.429

[13] K. A. Meissner and G. Veneziano, Mod. Phys. Lett. A6 (1991) 3397; Phys. Lett. B 267 (1991) 33.

[14] M. Gasperini and G. Veneziano, Phys. Lett. B 277 (1992) 256

[15] G. Veneziano, Phys. Lett. B265 (1991) 287

[16] A. A. Tseytlin, Mod. Phys. Lett. A6 (1991) 1721

[17] B. S. De Witt, Phys. Rev. 160 (1967) 1113

[18] A. Ashtekar and R. Geroch, Rep. Prog. Phys. 37 (1974) 1211

[19] M. Muller, Nucl. Phys. B337 (1990) 37

[20] C. Angelantonj, L. Amendola, M. Litterio and F. Occhionero, Phys. Rev. D51 (1995) 1607

[21] M. Abramowicz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1972)

[22] A. Vilenkin, Phys. Rev. D33 (1986) 3650; Phys. Rev. D37 (1988) 888.

[23] A. Vilenkin, *Predictions from quantum cosmology* (gr-qc/9507018), to appear in Proc. of the Int. School of Astrophysics "D. Chalonge" (Erice, September 1995), ed. by N. Sanchez (Kluwer Acad. Publ., Dordrecht, 1995)

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Quintessence as a run-away dilaton

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Abstract

We consider a late-time cosmological model based on a recent proposal that the infinite-bare-coupling limit of superstring/M-theory exists and has good phenomenological properties, including a vanishing cosmological constant, and a massless, decoupled dilaton. As it runs away to $+\infty$, the dilaton can play the role of the quintessence field recently advocated to drive the late-time accelerated expansion of the Universe. If, as suggested by some string theory examples, appreciable deviations from General Relativity persist even today in the dark matter sector, the Universe may smoothly evolve from an initial "focusing" stage, lasting untill radiation–matter equality, to a "dragging" regime, which eventually gives rise to an accelerated expansion with frozen Ω(dark energy)/Ω(dark matter).

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I. INTRODUCTION

According to recent astrophysical observations, our Universe, since a red shift of $\mathcal{O}(1)$, appears to have undergone a phase of accelerated expansion [1,2]. This result can be combined with the recent estimates of the average mass density of the Universe [3], $\Omega_m \simeq 0.3 - 0.4$ (in critical units), and with recent measurements of the CMB anisotropy peaks [4], pointing at a nearly critical total energy density, $\Omega_T \simeq 1$. One is then led to the conclusion that the present cosmological evolution, when described in terms of an effective fluid entering Einstein's equations, should be (marginally) dominated by a "dark energy" component ρ_x characterized by a (sufficiently) negative effective pressure, $p_x < -\rho_x/3$.

The simplest candidate for such a missing energy is a positive cosmological constant Λ, of order H_0^2. Such an identification, however, unavoidably raises a series of difficult questions. In particular: a) Why is Λ so small in particle physics units? Explaining a finite but very small value for Λ may turn out to be even harder than finding a reason why it is exactly zero. This is the so-called fine-tuning problem for Λ, see for instance [5]; and b) Why is $\Lambda \sim \rho_{m0}$, where ρ_{m0} is the *present* value (in Planck units) of the (dark) matter energy density? This is the so-called "cosmic coincidence" problem [6].

At present, the most promising scenarios for solving (at least part of) the above problems introduce a single scalar field, dubbed "quintessence" [7], whose potential goes to zero asymptotically (leaving therefore just the usual puzzle of why the "true" cosmological constant vanishes). The scalar field slowly rolls down such a potential reaching infinity (and zero energy) only after an infinite (or very long) time. While doing so, quintessence produces an effective, time-dependent, cosmic energy density ρ_x accompanied by a sufficiently negative pressure, i.e. a sort of effective cosmological constant. By making $\Lambda_{eff} \sim H^2$ time-dependent, this can naturally explain the smallness of the *present* effective vacuum energy density. However, if, as in General Relativity, dust energy and an effective cosmological constant have different time dependence, it can hardly explain why $\Lambda \sim \rho_{m0}$. For a recent review of the relative merits of a cosmological constant and quintessence, see Ref. [8].

As far as identifying quintessence is concerned, the inflaton itself could be a candidate [9]. But also more exotic possibilities have been considered, in particular some motivated by the wish to solve the above-mentioned cosmic coincidence problem [10,11]. In any case, as is the case for the inflaton, the quintessence field does not have, as yet, an obvious place in any fundamental theory of elementary particles. One should also mention, at this point, that, if quintessence may help with the problems typical of the cosmological constant interpretation, it is likely to create a new one of its own: in order to play its role, the quintessence field must be extremely light and can thus mediate a new long–range (of order H_0^{-1}) force, which is

strongly constrained observationally. This is an important constraint to be imposed on any specific scalar field model of quintessence, either minimally or non-minimally [12] coupled to gravity.

At first sight, the search for a quintessence candidate in particle physics looks easier than the one for an inflaton. For instance, fundamental or effective scalar fields with potentials running to zero at infinity are ubiquitous in supersymmetric field theories and/or in string/M-theory. They are usually referred to as moduli fields since, in perturbation theory, they parametrize the space of inequivalent vacua and correspond to exactly flat directions (equivalently, to exactly massless fields). Non-perturbative effects (e.g. gauge-theory instantons) are expected to lift these flat directions, just preserving those that correspond to small or vanishing coupling. Examples are the run-away vacua of supersymmetric gauge theories (see, for instance, [13] for quintessence models based on the latter possibility), or the dilaton modulus ϕ in the limit $\phi \to -\infty$.

However, if we were to take one of these moduli as quintessence, we would immediately run into the problem that the acceleration of the Universe should be accompanied by a drift of interactions towards triviality. For Newton's constant, and even more so for the fine-structure constant, this kind of time variations is very strongly constrained. Furthermore, typical couplings of moduli fields to ordinary matter are of gravitational order, and this creates the already mentioned problem of new, unwanted long-range forces. For all these reasons the conventional attitude towards moduli fields has been (see e.g. [14]) to postulate that they develop non-perturbative potentials, providing them with both a mass and a freezing mechanism (see however [15] for an alternative that is closer, in spirit, to the one advocated here).

Another possible problem with the identification of a string modulus with quintessence is that we would like to freeze the moduli at values that provide the correct values of the coupling constant and unification scale of grand unified theories (GUTs). For instance, the dilaton and compactification volume V_6 should be frozen at values such that [16]

$$\alpha_{GUT}^{-1} \sim (M_P/M_s)^2 \sim e^{-\phi} \ , \qquad M_P/M_{GUT} \sim \alpha_{GUT}^{-2/3}\, g_s^{1/6}, \tag{1.1}$$

where M_P, M_s and M_{GUT} are the Planck, string and GUT scales, $g_s = e^{\phi} V_6 M_s^6$ is the string coupling, and e^{ϕ} is the tree-level effective four-dimensional coupling (thus, in more standard string-theory notation [17], our dilaton is related to the real part of the S modulus by $\mathrm{Re}\{S\} = e^{-\phi}$).

Unfortunately, it looks unlikely that non-perturbative effects will be significant enough in this region to stabilize the moduli. Also, perturbative unification gives too low a value for

M_P/M_{GUT} [16,18]. In this respect, the situation can be drastically improved by considering the M-theory limit, $g_s \to \infty$, while still keeping the four-dimensional effective couplings perturbative ($S \gg 1$) [18]. Even then, the moduli would presumably freeze out in a typical (and cosmologically tiny) particle-physics time, and therefore cannot implement the conventional, slow-roll quintessential scenario. In spite of these difficulties, unconventional models of quintessence based on the stabilization of the dilaton in the perturbative regime are not completely excluded, as recently discussed by one of us [19].

There is, however, another possibility for making the dilaton a candidate for quintessence. As we have already mentioned, the region of large negative ϕ corresponds to the trivial vacuum. The idea that the Universe may have started, long before the big bang, in this region is actually the basis of the so-called pre-big bang scenario in string cosmology (for recent reviews see [20]). Here we are asking instead whether the dilaton can play the role of quintessence at very *late* times (such as today), not by evolving towards $-\infty$ and triviality, but by going towards $+\infty$ and strong coupling. Such a proposal looks absurd at first since, if we do not see a drift towards zero coupling, we do not experience one towards increasing strength either. In order for this idea to make sense we have to assume that the strong-coupling limit of string/M-theory exists, is smooth, and resembles our world. Can this make sense at all?

It has been argued by one of us [21] that the answer to the above question can be affirmative, if we assume a certain structure of the quantum loop corrections to the string effective action suggested by large-N counting arguments. In the strong coupling limit (which could either be the self-dual value $S = 1$ or, if S-duality is broken, $\phi \to +\infty$) gravitational and gauge coupling would be determined entirely by loop corrections (as in the so-called induced-gravity/gauge idea [22]), and would "saturate" at "small" values because of the large number of fields entering the loops (e.g. the large number N of gauge bosons, or the large value of the quadratic Casimir C_A, for gauge groups like E_8). Typically, eq. (1.1) would be replaced (at $\phi \gg 1$) by

$$\alpha_{GUT}^{-1} \sim C_A + \mathcal{O}(e^{-\phi}) \quad , \quad (M_P/M_s)^2 \sim N + \mathcal{O}(e^{-\phi}). \tag{1.2}$$

In this picture there is naturally an asymptotic decoupling mechanism of ordinary matter to the dilaton, whose effective mass goes to zero at late times. The problem remains, of course, of explaining why the cosmological constant vanishes in superstring/M-theory, not only at zero coupling where supersymmetry protects it, but also at infinite (bare) coupling. Possibly, some new, stringy symmetry can explain this. It will simply be assumed to be the case in this paper.

As the dilaton is non-universally coupled to different types of matter fields, its coupling

to ordinary matter can be asymptotically tiny (as to satisfy constraints from gravitational experiments [15]), and much stronger (as first suggested in [23]) to typical dark matter candidates, such as the axion. In that case, the dilaton to dark-matter coupling leads to an initial evolution, which is similar to the "tracking" regime [24] of conventional models of quintessence, but takes place *before* potential energy becomes appreciable. Later on, the interplay of the dark-matter dilatonic charge and of the dilaton potential leads to an accelerated expansion in which the relative fraction of dark energy and dark matter remains fixed (and of order 1), thus offering a possible explanation of the cosmic coincidence, as we will illustrate through explicit examples.

The paper is organized as follows. In Sect. II we present the effective string cosmology equations, in the small curvature –but arbitrary coupling– regime, with generic matter sources non-minimally coupled to the dilaton. In Sect. III we discuss analytically a possibile late-time attractor characterized by a constant positive acceleration and a fixed ratio of dark matter and dark energy. In Sect. IV we provide a semi-quantitative description of the previous phase, during which the dilaton potential can be neglected. This phase is characterized by a "focusing" of the energy densities of the various components of the cosmological fluid (which occurs before the epoch of matter–radiation equilibrium), and by a subsequent "dragging" regime in which the dilaton energy density tends to follow that of non-relativistic (dark) matter. We also discuss here the main phenomenological constraints that have to be imposed on the scenario. In Sect. V we consider a typical example of string cosmology model including radiation, baryonic and cold dark matter, and we present the results of explicit numerical integrations. Our conclusions are summarized in Sect. VI.

II. COSMOLOGICAL EQUATIONS IN THE STRING AND EINSTEIN FRAMES

Our starting point is the string-frame, low-energy, gravi-dilaton effective action [17], to lowest order in the α' expansion, but including dilaton-dependent loop (and non-perturbative) corrections, encoded in a few "form factors" $\psi(\phi)$, $Z(\phi)$, $\alpha(\phi)$, ..., and in an effective dilaton potential $V(\phi)$ (see also [15]). In formulae:

$$S = -\frac{M_s^2}{2}\int d^4x\sqrt{-\widetilde{g}}\,\left[e^{-\psi(\phi)}\widetilde{R} + Z(\phi)\left(\widetilde{\nabla}\phi\right)^2 + \frac{2}{M_s^2}V(\phi)\right] - \frac{1}{16\pi}\int d^4x\frac{\sqrt{-\widetilde{g}}}{\alpha(\phi)}F_{\mu\nu}^2 + \Gamma_m(\phi,\widetilde{g},\text{matter}) \tag{2.1}$$

(conventions: metric signature: $(+,-,-,-)$, $R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu\Gamma_{\nu\alpha}{}^{\beta} - \ldots$, $R_{\mu\nu} = R_{\alpha\mu\nu}{}^{\alpha}$). Here $M_s^{-1} = \lambda_s$ is the fundamental string-length parameter, $\widetilde{g}$ is the sigma-model metric minimally

coupled to fundamental strings, $\widetilde{R}, \widetilde{\nabla}$ are the curvature and the covariant derivative referred to $\widetilde{g}$, and $F_{\mu\nu}$ is the gauge field of some fundamental (GUT) group ($\alpha(\phi)$ is the corresponding gauge coupling). We imagine having already compactified 6 dimensions and having frozen the corresponding moduli at the string scale. Following the basic proposal made in [21], we shall assume that the form factors $\psi(\phi)$, $Z(\phi)$, $\alpha(\phi)$ approach a finite, physically interesting limit as $\phi \to +\infty$ while, in the same limit, $V \to 0$.

The fields appearing in the matter action Γ_m are in general non-minimally and non-universally coupled to the dilaton (also because of the loop corrections [14]). Their gravitational and dilatonic "charge densities", $\widetilde{T}_{\mu\nu}$ and $\widetilde{\sigma}$, are defined as follows:

$$\frac{\delta\Gamma}{\delta\widetilde{g}^{\mu\nu}} = \frac{1}{2}\sqrt{-\widetilde{g}}\;\widetilde{T}_{\mu\nu}, \qquad\qquad \frac{\delta\Gamma}{\delta\phi} = -\frac{1}{2}\sqrt{-\widetilde{g}}\;\widetilde{\sigma}, \tag{2.2}$$

and it is important to stress that, when $\widetilde{\sigma} \neq 0$, the gravi-dilaton effective theory is radically different from a typical, Jordan–Brans–Dicke type model of scalar-tensor gravity [25]. We shall give a prototype form of Γ_m in the following section, after passing to the Einstein frame.

The variation of (2.1) with respect to $\widetilde{g}_{\mu\nu}$ then gives the equations:

$$\begin{aligned}&\widetilde{G}_{\mu\nu} + \psi'\,\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\phi + \left[e^\psi Z - \psi'^{\,2} + \psi''\right]\widetilde{\nabla}_\mu\phi\widetilde{\nabla}_\nu\phi \\ &+\frac{1}{2}\widetilde{g}_{\mu\nu}\left[\left(2\psi'^{\,2} - 2\psi'' - e^\psi Z\right)(\widetilde{\nabla}\phi)^2 - 2\psi'(\widetilde{\nabla}^2\phi) - e^\psi V(\phi)\right] = \lambda_s^2 e^\psi \widetilde{T}_{\mu\nu},\end{aligned} \tag{2.3}$$

where $\widetilde{G}_{\mu\nu}$ is the Einstein tensor, and a prime denotes differentiation with respect to ϕ. The variation with respect to ϕ, using the trace of eq. (2.3) to eliminate $\widetilde{R}$, leads to the equation

$$\begin{aligned}&\left(3\psi'^{\,2} - 2e^\psi Z\right)(\widetilde{\nabla}^2\phi) + \left[e^\psi\left(Z\psi' - Z'\right) + \psi'\left(3\psi'' - 3\psi'^{\,2}\right)\right](\widetilde{\nabla}\phi)^2 \\ &+e^\psi\left(2\psi' V + V'\right) + \lambda_s^2 e^\psi\left(\psi'\widetilde{T} + \widetilde{\sigma}\right) = 0\end{aligned} \tag{2.4}$$

We shall assume an isotropic, spatially flat metric background (appropriate to the present cosmological configuration), and a perfect fluid model of source. In the cosmic-time gauge we thus set

$$\widetilde{g}_{\mu\nu} = \mathrm{diag}\left(1, -\widetilde{a}^2(\widetilde{t})\;\delta_{ij}\right), \qquad \widetilde{T}^\nu_\mu = \mathrm{diag}\left(\widetilde{\rho}, -\widetilde{p}\;\delta^j_i\right), \qquad \phi = \phi(\widetilde{t}), \qquad \widetilde{\sigma} = \widetilde{\sigma}(\widetilde{t}), \tag{2.5}$$

and one can easily check, combining the above equations, that the matter stress tensor is not covariantly conserved (even in this frame), but satisfies the equation

$$\dot{\widetilde{\rho}} + 3\widetilde{H}(\widetilde{\rho} + \widetilde{p}) = \frac{\widetilde{\sigma}}{2}\,\dot{\phi}. \tag{2.6}$$

For the purpose of this paper, and for an easier comparison with previous discussions of the quintessential scenario, it is however convenient to represent the dynamical evolution of the background in the more conventional Einstein frame, characterized by a metric $g_{\mu\nu}$ minimally coupled to the dilaton, and defined by the conformal transformation $\widetilde{g}_{\mu\nu} = c_1^2 g_{\mu\nu} e^{\psi}$. Here c_1^2 parametrizes the asymptotic behaviour of $\psi(\phi)$,

$$c_1^2 = \lim_{\phi \to +\infty} \exp\{-\psi(\phi)\}, \tag{2.7}$$

and thus controls the asymptotic ratio between the string and the Planck scale, $M_P^2 = c_1^2 M_s^2$. In the Einstein frame the action (2.1) becomes:

$$\begin{aligned} S = &-\frac{M_P^2}{2} \int d^4x \sqrt{-g} \left[R - \frac{k(\phi)^2}{2} (\nabla\phi)^2 + \frac{2}{M_P^2} \hat{V}(\phi) \right] \\ &- \frac{1}{16\pi} \int d^4x \frac{\sqrt{-g}}{\alpha(\phi)} F_{\mu\nu}^2 + \Gamma_m(\phi, c_1^2 g_{\mu\nu} e^{\psi}, \text{matter}) \ , \end{aligned} \tag{2.8}$$

where we have defined

$$k^2(\phi) = 3\psi'^2 - 2e^{\psi} Z, \qquad \hat{V} = c_1^4 e^{2\psi} V \ . \tag{2.9}$$

For later use it is also convenient to define a canonical dilaton field by:

$$d\hat{\phi} = \frac{M_P}{\sqrt{2}} k(\phi) d\phi \ , \tag{2.10}$$

although, in solving the equations, it will be easier to work directly with the original field ϕ.

We now choose, also in the Einstein frame, the cosmic-time gauge, according to the rescaling

$$\begin{aligned} &\widetilde{a} = c_1 a e^{\psi/2}, \qquad d\widetilde{t} = c_1 dt e^{\psi/2}, \\ &\rho = c_1^2 e^{2\psi} \widetilde{\rho}, \qquad p = c_1^2 e^{2\psi} \widetilde{p}, \qquad \sigma = c_1^2 e^{2\psi} \widetilde{\sigma} \qquad . \end{aligned} \tag{2.11}$$

From the $(0,0)$ and (i,j) components of eq. (2.3) we obtain, respectively, the Einstein cosmological equations (in units such that $M_P^2 = c_1^2 M_s^2 \equiv (8\pi G)^{-1} = 2$)

$$6H^2 = \rho + \rho_\phi, \tag{2.12}$$

$$4\dot{H} + 6H^2 = -p - p_\phi, \tag{2.13}$$

while from the dilaton equation (2.4) we get

$$k^2(\phi)\left(\ddot{\phi}+3H\dot{\phi}\right)+k(\phi)\,k'(\phi)\,\dot{\phi}^2+\hat{V}'(\phi)+\frac{1}{2}\left[\psi'(\phi)(\rho-3p)+\sigma\right]=0. \tag{2.14}$$

In the above equations $H=\dot{a}/a$, a dot denotes differentiation with respect to the Einstein cosmic time, and we have used the definitions:

$$\rho_\phi=\frac{1}{2}k^2(\phi)\dot{\phi}^2+\hat{V}(\phi),\qquad p_\phi=\frac{1}{2}k^2(\phi)\dot{\phi}^2-\hat{V}(\phi). \tag{2.15}$$

The combination of equations (2.12)–(2.14) leads finally to the coupled conservation equations for the matter and dilaton energy density, respectively:

$$\dot{\rho}+3H(\rho+p)-\frac{1}{2}\dot{\phi}\left[\psi'(\phi)(\rho-3p)+\sigma\right]=0, \tag{2.16}$$

$$\dot{\rho}_\phi+3H(\rho_\phi+p_\phi)+\frac{1}{2}\dot{\phi}\left[\psi'(\phi)(\rho-3p)+\sigma\right]=0. \tag{2.17}$$

For further applications, and for a more transparent numerical integration, it is also convenient to parametrize the time evolution of all variables in terms of the logarithm of the scale factor, $\chi=\ln(a/a_i)$, where a_i corresponds to the initial scale*, and to separate the radiation, baryonic and non-baryonic matter components of the cosmological fluid by setting

$$\rho=\rho_r+\rho_b+\rho_d\equiv\rho_r+\rho_m,\qquad p=\frac{1}{3}\rho_r,\qquad \sigma=\sigma_r+\sigma_b+\sigma_d\equiv\sigma_r+\sigma_m. \tag{2.18}$$

The dilaton equation and the Einstein equation (2.12) can then be written, respectively, as

$$2\,H^2\,k^2\,\frac{d^2\phi}{d\chi^2}\,+\,k^2\left(\frac{1}{2}\rho_m+\frac{1}{3}\rho_r+\hat{V}\right)\frac{d\phi}{d\chi}\,+\,2H^2\,k\,k'\left(\frac{d\phi}{d\chi}\right)^2+2\hat{V}'+\psi'\rho_m\,+\sigma=0, \tag{2.19}$$

$$H^2\left[6-\frac{k^2}{2}\left(\frac{d\phi}{d\chi}\right)^2\right]\;=\;\rho_m+\rho_r+\hat{V}. \tag{2.20}$$

The matter evolution equation (2.16) can be split into the various components as

$$\frac{d\rho_r}{d\chi}+4\rho_r-\frac{\sigma_r}{2}\frac{d\phi}{d\chi}=0, \tag{2.21}$$

$$\frac{d\rho_b}{d\chi}+3\rho_b-\frac{1}{2}\left(\psi'\rho_b+\sigma_b\right)\frac{d\phi}{d\chi}\;=\;0. \tag{2.22}$$

$$\frac{d\rho_d}{d\chi}+3\rho_d-\frac{1}{2}\left(\psi'\rho_d+\sigma_d\right)\frac{d\phi}{d\chi}\;=\;0. \tag{2.23}$$

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*The relation between χ and the redshift z is $\chi=-\ln(1+z)+\ln(a_0/a_i)$, where a_0 is the present value of the scale factor.

Finally, eq. (2.19) is also equivalent to the dilaton conservation equation (2.17), which becomes

$$\frac{d\rho_\phi}{d\chi} + 6\rho_\phi - 6\hat{V}(\phi) + \frac{1}{2}\left(\psi'\rho_m + \sigma\right)\frac{d\phi}{d\chi} \;=\; 0. \tag{2.24}$$

III. ACCELERATED LATE-TIME ATTRACTORS WITH CONSTANT Ω_ϕ

As a first step towards a "dilatonic" interpretation of quintessence we will now discuss the possibility that the above equations, together with a string-theory motivated potential and loop corrections, are asymptotically solved by an accelerated expansion, $\ddot{a} > 0$, with frozen ratio ρ_m/ρ_ϕ of the order of unity. This last property, in particular, is expected to solve (or at least alleviate) the cosmic coincidence problem [10,11].

Under the assumption made in [21] that the form factors appearing in (2.1) have a finite limit as $\phi \to +\infty$, and assuming the validity of an asymptotic Taylor expansion, we write:

$$\begin{aligned} &e^{-\psi(\phi)} \;=\; c_1^2 + b_1 e^{-\phi} + \mathcal{O}(e^{-2\phi}), \qquad\qquad Z(\phi) \;=\; -c_2^2 + b_2 e^{-\phi} + \mathcal{O}(e^{-2\phi}) \;, \\ &\alpha(\phi)^{-1} = \alpha_0^{-1} + b e^{-\phi} + \mathcal{O}(e^{-2\phi}) \;, \end{aligned} \tag{3.1}$$

where c_1^2, c_2^2 are assumed to be of the same order (typically of order 10^2 since, as already noted, $c_1^2 = (M_P/M_s)^2$), and α_0 is to be identified with the unified gauge coupling at the GUT scale, i.e. $\alpha_0 \simeq 1/25$. Unlike the model discussed in [26] our model thus describes, in the strong coupling limit $\phi \to +\infty$, a minimally coupled, canonical scalar field $\hat{\phi} = \sqrt{2}(c_2/c_1)\phi$, see eq. (2.10). In the opposite limit, $\phi \to -\infty$, the gravi-dilaton string effective action reduces, as usual, to an effective Brans–Dicke model with parameter $\omega = -1$. We note that it is not hard to chose $\psi(\phi)$ and $Z(\phi)$ in such a way that the kinetic term of the dilaton keeps the correct sign at all values of ϕ (see the example given in Sect. V).

Similarly, the assumption that V originates from non-perturbative effects, and that $V \to 0$ as $\phi \to \infty$, allows us to write, quite generically:

$$\hat{V}(\phi) \;=\; V_0\, e^{-\phi} + \mathcal{O}(e^{-2\phi}). \tag{3.2}$$

Since the overall normalization of the potential V_0 is non-perturbative, it should be related to the asymptotic value of the gauge coupling α_0 by an expression of the form:

$$V_0 = M_s^4 \,\exp\left(-\frac{4}{\beta\alpha_0}\right) = M_*^4, \tag{3.3}$$

with some model-dependent (one-loop) β-function coefficient β. For a comparison with earlier studies of an exponential potential [27,28] we also note that, when referred to the canonically normalized dilaton field $\hat{\phi}$ defined in (2.10), the Einstein frame potential (3.2) asymptotically exibits an exponential behaviour $\hat{V} \sim \exp(-\lambda\hat{\phi}/M_P)$, with $\lambda = c_1/c_2 = \sqrt{2}/k$ at $\phi \to +\infty$.

It is important to discuss the size of the potential needed for the viability of our scenario. Since the acceleration of the Universe appears to be a relatively recent phenomenon (even, possibly, an *extremely* recent one, as recently argued in [29]), the potential V must enter the game very late, i.e. at an energy scale of the order of $\rho^{1/4} \sim 10^{-3}$ eV. Unless we want to play with an unnaturally large present value of ϕ, this also fixes the scale of the potential in (3.2) as $V_0 \sim (10^{-3}\text{eV})^4$. As far as we know, this is feature is common to all quintessence scenarios: the problem of an outstandingly small cosmological constant is traded for the introduction of another unnaturally small mass scale M_*.

In our context, we easily find that, in order to have a properly normalized potential, we need the constant β appearing in the exponent of (3.3) to be somewhat smaller than the coefficient β_3 of the QCD beta function (see also the discussion after eq. (3.5)), say $\beta \sim 0.6\beta_3$. Given our ignorance of the origin of the dilaton potential, this looks perfectly acceptable, a priori. However, this apparent resolution of the fine-tuning problem should not hide the fact that the potential has to be adjusted very precisely if one wants to start the acceleration of the Universe not earlier that at red-shift $z \sim \mathcal{O}(1)$, and not later than today. To the best of our knowledge there is no obvious explanation, at present, of this aspect of the coincidence problem.

Let us now come to the matter sector of the action (2.8). As a typical example of Γ_m we take:

$$\Gamma_m(\phi, g, \text{matter}) = \int d^4x\sqrt{-g}\ \bar{N}\left[i\partial\!\!\!/ + m_N(\phi)\right] N + \frac{1}{2}\int d^4x\sqrt{-g}\ \left[e^{\zeta(\phi)}(\partial_\mu D)^2 - e^{\eta(\phi)}\mu^2 D^2\right] \tag{3.4}$$

the first term representing baryonic matter, the second (scalar) cold dark matter, while the gauge term appearing explicitly in (2.8) can already represent the radiation component of the cosmic fluid.

The non-observation of appreciable cosmological variations of the coupling constants [30], as well as the precision tests of Newtonian gravity [31] in the context of long-range dilatonic interactions, force us to assume that ordinary matter and radiation have nearly metric couplings to $\widetilde{g}_{\mu\nu}$, i.e. that σ_b, $\sigma_r \simeq 0$ as $\phi \to \infty$. It is not hard to see how such a near–vanishing of dilatonic charges can be achieved starting from the actions (2.8), (3.4).

Following ref. [14] we have:

$$\frac{\sigma_b}{\rho_b} \sim \frac{\partial}{\partial\phi}\left(\ln\,\Lambda_{QCD}\right), \qquad \frac{\sigma_r}{\rho_r} \sim \frac{\partial}{\partial\phi}\left(\ln\,\alpha\right). \tag{3.5}$$

Given that $\Lambda_{QCD} \sim M_s \exp\left(-1/\beta_3\alpha\right)$ (with β_3 the coefficient of the QCD β-function), and using (3.1) for α, it is clear that both σ_b and σ_r are exponentially suppressed at large, positive ϕ. This decoupling mechanism is similar in spirit to the one proposed in [15], although it is supposed to occur at infinite bare coupling.

In the dark matter sector, on the contrary, we shall assume more generic quantum corrections. By taking for instance the action in eq. (3.4), one has for the dilatonic charge of dark matter:

$$\sigma_d \;=\; -\,\zeta'(\phi)\,e^{\zeta(\phi)}(\partial_\mu D)^2 + \eta'(\phi)\,e^{\eta(\phi)}\mu^2 D^2. \tag{3.6}$$

Furthermore, the equations of motion for the D field give a relation between the time-averaged quantities, $e^{\zeta(\phi)}\langle\dot{D}^2\rangle = \mu^2 e^{\eta(\phi)}\langle D^2\rangle$ (which is consistent with the interpretation of D as non-relativistic matter, $p_d = 0$, as assumed in the preceding section), and relate σ_d and ρ_d by a (generally ϕ-dependent) proportionality factor

$$\sigma_d/\rho_d \;\equiv\; q(\phi) \;=\; \eta'(\phi) - \zeta'(\phi). \tag{3.7}$$

The lat–time behaviour we will discuss takes place if we assume that, in the strong coupling limit (i.e., $\phi \gg 1$), $q(\phi)$ tends to a positive constant of order unity, and that the dark matter component dominates over baryonic matter and radiation. Thus, the regime we are considering is characterized (according to eqs. (3.1), (3.2)) by

$$k^2(\phi) = 2c_2^2/c_1^2 = 2/\lambda^2, \quad \sigma = \sigma_d, \quad \rho = \rho_d, \quad q(\phi) = q = \mathcal{O}(1), \quad \sigma_d = q\;\rho_d\,. \tag{3.8}$$

It follows that the dilaton coupling to the stress tensor can be asymptotically neglected with respect to the coupling to the dilatonic charge, as $\psi' \simeq e^{-\phi}/c_1^2 \ll 1$. The dilaton and dark matter conservation equations (2.23), (2.24) and the Einstein equations (2.12), (2.13) can then be written, asymptotically, in the form

$$\dot{\rho}_d + 3H\rho_d - \frac{q}{2}\,\rho_m\dot{\phi} = 0, \qquad \dot{\rho}_\phi + 6H\rho_k + \frac{q}{2}\,\rho_m\dot{\phi} = 0, \tag{3.9}$$

$$1 = \Omega_d + \Omega_k + \Omega_V, \qquad 1 + \frac{2\dot{H}}{3H^2} = \Omega_V - \Omega_k, \tag{3.10}$$

where we have defined

$$\rho_d = 6H^2\Omega_d, \qquad \rho_\phi = \rho_k + \rho_V,$$
$$\rho_k = 6H^2\Omega_k = \dot\phi^2/\lambda^2, \qquad \rho_V = 6H^2\Omega_V = \hat V. \tag{3.11}$$

We now look for solutions with asymptotically frozen dark-matter over dark-energy ratio, and frozen "equation of state". From the constraint (3.10) this is equivalent to the requirement that ρ_k, ρ_V and ρ_d scale in the same way, i.e.

$$\frac{d\log\rho_\phi}{d\chi} \;=\; \frac{d\log\rho_d}{d\chi}, \qquad \frac{d\log\rho_V}{d\chi} \;=\; \frac{d\log\rho_d}{d\chi}. \tag{3.12}$$

The first condition and the conservation equations give

$$\frac{d\phi}{d\chi} \;=\; \frac{6}{q}\,(\Omega_V - \Omega_k)\,. \tag{3.13}$$

Expressing $d\phi/d\chi$ through $\Omega_k = (d\phi/d\chi)^2/6\lambda^2$, and inserting it in the second condition (3.12), we obtain, respectively,

$$\lambda\; q = \sqrt{\frac{6}{\Omega_k}}\,(\Omega_V - \Omega_k)\,, \qquad q = 2\,\frac{\Omega_V - \Omega_k}{1 + \Omega_k - \Omega_V}, \tag{3.14}$$

where in the latter the asymptotic form of the potential (3.2) has been used. The last two equations can be solved for Ω_k and Ω_V,

$$\Omega_k \;=\; \frac{6}{\lambda^2(2+q)^2}, \qquad \Omega_V \;=\; \Omega_k + \frac{q}{q+2} \tag{3.15}$$

giving easily

$$\Omega_\phi \;=\; \frac{12 + q(q+2)\lambda^2}{(q+2)^2\lambda^2}, \qquad w_\phi \;=\; -\frac{q(q+2)\lambda^2}{12 + q(q+2)\lambda^2}, \tag{3.16}$$

where the last equation for $w_\phi = (\Omega_k - \Omega_V)/(\Omega_k + \Omega_V)$ provides the dilaton's equation of state.

The above asymptotic solution, first obained in [32], and recently studied in [33,34], generalizes the results discussed in ref. [27] to the interacting dark matter case, and is very similar to the results obtained by including suitable non-minimal couplings in a Brans–Dicke context [28], or by including an effective bulk viscosity in the dark matter stress tensor [10,35]. Our eq. (3.9) corresponds indeed, formally, to a dissipative pressure $\Pi = -q\rho_m(\dot\phi/6H)$ (in

the notation of [35]). See also [33,34] for a discussion of the parameter values compatible with such an asymptotic solution.

Once Ω_k and Ω_V are given, one can easily compute all the relevant kinematic properties of the asymptotic solution as a function of only two parameters, q and $\lambda = c_1/c_2$, which are in principle calculable for a given string theory model. The asymptotic value of the acceleration, in particular, is fixed by eq. (3.10) as

$$\frac{\ddot{a}}{aH^2} = 1 + \frac{\dot{H}}{H^2} = \frac{q-1}{q+2}. \tag{3.17}$$

One can also easily obtain, through a simple integration, the asymptotic evolution of the Hubble factor and of the dominant energy density,

$$H \sim a^{-3/(2+q)}, \qquad \rho \sim a^{-6/(2+q)}. \tag{3.18}$$

In order to illustrate the range of parameters possibly compatible with present phenomenology, we have plotted in the $\{\lambda\ , q\}$ plane various curves at $\Omega_\phi = \Omega_k + \Omega_V = \text{const}$, and $w_\phi = (\Omega_k - \Omega_V)/(\Omega_k + \Omega_V) = \text{const}$. (Fig. 1). Note that the case discussed in Ref. [27] corresponds to staying on the λ axis. In that case, the critical value of λ below which $\Omega_d/\Omega_\phi \to 0$ is $\sqrt{3}$. The addition of q makes parameter space two-dimensional, with the point $\lambda = \sqrt{3}$ replaced by the left-most curve $\Omega_\phi = 1$. Beyond that curve, i.e. for $\lambda^2 < 6/(2+q)$ (as well as for all values of λ if $q < -2$), the ratio Ω_d/Ω_ϕ goes to zero. However, while in the case of [27] acceleration and a finite ratio Ω_d/Ω_ϕ are incompatible, this is perfectly possible in a large region of the $\{\lambda\ , q\}$ plane.

In fact, it is possible to determine the region of our parameter space that survives the various observational constraints (Type 1a supernovae, CMB anisotropies, large-scale structure ...). The present values of Ω_ϕ and w_ϕ have to lie in the range [36,37] $0.6 \lesssim \Omega_\phi \lesssim 0.7$, and $-1 \leq w_\phi \lesssim -0.4$, but the two allowed intervals are not uncorrelated. Assuming that we are already in the asymptotic regime, the allowed region lies roughly between the two curves $\Omega_\phi = 0.6$ and $\Omega_\phi = 0.7$ and above $q = 2$. Other phenomenological (but somewhat more model-dependent) constraints on q and λ can be obtained from the recent measurements of the position of the third anisotropy peak in the CMB distribution [38], which constrains the value of Ω_ϕ today and at last scattering, as well as the time-averaged equation of state $\langle w_\phi \rangle$ [39]. In the final part of this paper we shall present a model of dark matter that seems to be compatible with all the above-mentioned constraints.

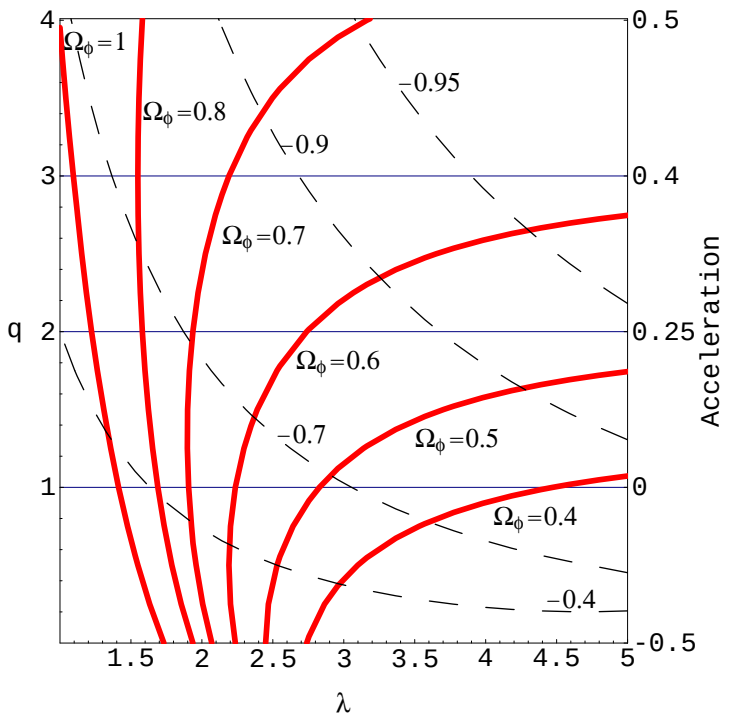

FIG. 1. *The asymptotic configurations in the plane* $\{\lambda\ , q\}$. *The full bold curves correspond to asymptotic solutions with fixed ratios* ρ_ϕ/ρ_d *and with the following values of* Ω_ϕ*:* 1, 0.8, 0.7, 0.6, 0.5, 0.4 . *On the right vertical axis we have reported the corresponding* q*-dependent acceleration parameter,* $\ddot{a}a/\dot{a}^2$. *The thin dashed curves correspond to fixed asymptotic values of the dilatonic equation of state* $w_\phi = p_\phi/\rho_\phi$, *respectively* -0.4, -0.7, -0.9 *and* -0.95.

IV. FOCUSING AND DRAGGING WITH $V = 0$: AN ANALYTIC STUDY OF EARLY-TIME EVOLUTION

Having discussed, in the previous section, the late-time accelerated expansion caused by the interplay of the dilaton potential and the dark-matter dilatonic charge, it looks appropriate to illustrate the earlier evolution, i.e. *before* the dilaton potential starts entering the game. In this section we shall provide a semi-quantitative, analytic analysis of this behaviour as it follows from the string cosmology equations (2.21)–(2.24), by imposing on the non-perturbative normalization (3.3) the constraint $V_0^{1/4} \ll H_{\rm eq}$, where $H_{\rm eq}$ is the curvature scale at the epoch of matter-radiation equality. In such a way the dilaton potential may eventually become important only at late times, in the matter-dominated era. We will show

that this early evolution can be roughly divided in three epochs, providing, altogether, an intermediate attractor that nicely connects to the accelerated behaviour described in Section III.

Let us start by considering an initial, post-big bang and post-inflationary regime of expansion driven by the standard radiation fluid, with negligible dilatonic charge, $\sigma_r = 0$. Possible non-relativistic matter, if present, is highly subdominant with respect to the other components ($\rho_m \ll \rho_\phi,\ \rho_r$) and, consequently, the dilatonic terms in eq. (2.24) can be neglected. The conservation equations can be easily integrated to give

$$\rho_r = \rho_{ri} e^{-4\chi}, \qquad\qquad \rho_\phi = \rho_{\phi i} e^{-6\chi}. \tag{4.1}$$

Therefore, the dilaton (kinetic) energy density, even if initially of the same order as ρ_r, is rapidly diluted like a^{-6}. The dilaton itself, starting from a value $\phi_i \sim \mathcal{O}(1)$ typical of the moderately-strong coupling post-big bang epoch, tends to settle down to a constant value (as already noticed in [40]), that can be easily estimated as follows

$$\rho_k = \frac{k^2}{2} H^2 \left(\frac{d\phi}{d\chi}\right)^2 = \frac{k^2}{12} \left(\frac{d\phi}{d\chi}\right)^2 (\rho_r + \rho_\phi) = \rho_\phi. \tag{4.2}$$

For $\rho_{ri} = \rho_{\phi i}$ we get

$$\frac{d\phi}{d\chi} = \frac{\sqrt{12}}{k} \left(1 + e^{2\chi}\right)^{-1/2}, \tag{4.3}$$

which, for $k = $ const., leads to a solution with asymptotic value $\phi = \phi_1$, related to the initial value $\phi_i = \phi(0)$ by the constant shift

$$\Delta\phi = \phi_1 - \phi_i = \frac{\sqrt{12}}{k} \ln(1 + \sqrt{2}) \simeq \frac{3}{k} \simeq \frac{3}{\sqrt{2}} \frac{c_1}{c_2}, \tag{4.4}$$

independently of ϕ_i and of the initial χ (the last equality holds for ϕ_1 large enough to justify the asymptotic relation $k = \sqrt{2}/\lambda$).

Such an initial regime is effective until the dilaton kinetic energy becomes of the same order as ρ_m. At that point, some oscillations are triggered by the interference term of eq. (2.24), but the dilaton energy density keeps decreasing, on the average, until it enters a "focusing" regime, during which it is diluted at a much slower rate (like a^{-2}), so as to converge, at equality, towards the larger values of ρ_m and ρ_r. Eventually, when dark non-relativistic matter becomes the dominant source ($\rho_d \gtrsim \rho_r$), the dilaton energy density tends to follow the dark matter evolution, as if it were "dragged" by it.

Before turning to a quantitative analysis of these two regimes we note that the time evolution of ρ_ϕ, in the "tracking" quintessence, is determined by the slope of the potential. In the present context, instead, the focusing and dragging effects are not due to the potential, but they are controlled by the non-minimal coupling induced by $(\psi' + q)$ (thus implementing an attractor mechanism already proposed for a class of non-minimal scalar-tensor models of quintessence [41]). Thanks to the focusing effect, which seems to be typical of the string effective action (even if similar, in a sense, to the "self-adjusting" solutions of general relativity with exponential potential [27]), the dilaton energy density at the matter–radiation equality turns out to be fixed independently from its initial value, and only slightly dependent from the initial value of the dilaton, ϕ_i. For large enough values of q, however, even the dependence upon ϕ_i tends to disappear, because the value of the dilaton itself gets focused, as will be discussed in the next section.

For a quantitative analytical study of the "focusing" and "dragging" regimes, we start from eqs. (2.21)–(2.24). Lumping together baryonic and dark matter, neglecting V, and assuming, according to (3.7), $\sigma = \sigma_m = q(\phi)\ \rho_m$, those equations can be easily recast in the form:

$$\rho_r^{-1}\,\frac{d\rho_r}{d\chi} + 4 = 0, \tag{4.5}$$

$$\rho_m^{-1}\,\frac{d\rho_m}{d\chi} + \left[3 \mp \sqrt{3}\,\epsilon\,(\rho_\phi/\rho)^{1/2}\right] = 0, \tag{4.6}$$

$$\frac{d\rho_\phi}{d\chi} + 6\rho_\phi \pm \sqrt{3}\,\epsilon\,\rho_m(\rho_\phi/\rho)^{1/2} = 0, \tag{4.7}$$

where we have introduced the important parameter:

$$\epsilon(\phi)\ \equiv\ \frac{\psi'(\phi) + q(\phi)}{k(\phi)}, \tag{4.8}$$

and the sign ambiguity comes from solving eq. (4.2) for $d\phi/d\chi$ in terms of ρ_ϕ. The focusing solution is then characterized by the relation:

$$\rho_\phi\ =\ \frac{n^2(\phi)\,\rho_m^2}{\rho}, \tag{4.9}$$

i.e. $\Omega_\phi\ =\ n^2(\phi)\,\Omega_m^2$, which holds under the assumption that both ϵ and n are slowly varying. Indeed, we can establish the connection between these two quantities by inserting the ansatz (4.9) into (4.7). This gives:

$$-6 \mp \sqrt{3}\,\frac{\epsilon}{n} \;=\; 2n^{-1}\frac{dn}{d\chi} + 2\rho_m^{-1}\frac{d\rho_m}{d\chi} - \rho^{-1}\frac{d\rho}{d\chi}, \tag{4.10}$$

where on the right–hand side the logarithmic derivative of (4.9) has been taken. By using (4.6) one finally has

$$\rho^{-1}\frac{d\rho}{d\chi} \mp \sqrt{3}\,\epsilon\left[n^{-1} + 2\,\Omega_m n\right] \;=\; 2n^{-1}\frac{dn}{d\chi} \;\simeq\; 0. \tag{4.11}$$

We can now discuss a few cases of interest. During the radiation–dominated phase, and after the kinetic energy of the dilaton is quickly red-shifted away, we can neglect the term with Ω_m in eq. (4.11), we set $d\rho/d\chi = -4\rho$, and obtain:

$$n \;\simeq\; \frac{\sqrt{3}\,\epsilon}{4}, \qquad \frac{d\phi}{d\chi} \;\simeq\; -\frac{3\,\rho_m\,\epsilon}{2\,k\,\rho}, \qquad \rho_\phi \;\simeq\; \frac{3\,\rho_m^2\,\epsilon^2}{16\,\rho}. \tag{4.12}$$

We refer to this behaviour as "focusing" since it implies that ρ_m lies, modulo a factor $(16/3)\epsilon^{-2}$, at the geometric mean between $\rho \sim \rho_r$ and ρ_ϕ. Hence, as we approach radiation-matter equality, ρ_ϕ is effectively focused towards the same common value of the other two components (see eq. (4.15) below). Note that, for a positive ϵ, this happens thanks to a *negative* $d\phi/d\chi$.

In the matter–dominated regime it is no longer safe to neglect the term in Ω_m in eq. (4.11), unless $\epsilon \ll 1$. In that case, the solution is

$$n \;\simeq\; \sqrt{3}\,\epsilon\left(-\rho^{-1}\frac{d\rho}{d\chi}\right)^{-1}, \qquad \Omega_\phi \;\simeq\; 3\,\epsilon^2\Omega_m^2\left(-\rho^{-1}\frac{d\rho}{d\chi}\right)^{-2}. \tag{4.13}$$

During matter domination, using $d\rho/d\chi = -3\rho$, one gets

$$n \;\simeq\; \frac{\epsilon}{\sqrt{3}}, \qquad \frac{d\phi}{d\chi} \;\simeq\; -\frac{2\,\epsilon}{k}, \qquad \rho_\phi \;\simeq\; \frac{\rho_m\epsilon^2}{3}. \tag{4.14}$$

In other words, the focusing regime has been turned into a dragging one: the dilaton energy is dragged along by the (dark) matter energy and keeps a (small) constant ratio to it. Incidentally, at the epoch of exact matter-radiation equality, using $d\rho/d\chi = -3.5\rho$, we easily get (still at small ϵ):

$$\frac{\rho_\phi}{\rho_{eq}} \simeq \frac{3\epsilon^2}{49}, \tag{4.15}$$

which is always smaller than 6% for $\epsilon < 1$.

In order to understand what happens at larger values of ϵ it is useful to find the reason why, for small ϵ, ρ_ϕ/ρ_m stays constant. This comes about because the corrections to the a^{-3} and a^{-6} laws for ρ_m and ρ_ϕ, due to the non-vanishing ϵ, push the two towards each other. It is easy to check that, precisely if $\rho_\phi/(\rho_m + \rho_\phi) = \epsilon^2/3$, both energies scale like $a^{-(3+\epsilon^2)}$. We note, incidentally, that the above ratio of energies nicely fits with the value given in (4.14) when $\epsilon \ll 1$. If $\epsilon < 1$, the decrease of ρ_ϕ is still slower than the a^{-4} of ρ_r, which justifies neglecting the latter. However, if $\epsilon > 1$, this is no longer the case and we have a third kind of behaviour, which can be called "total dragging". In that case, as shown by a simple analysis, all three components of ρ scale like radiation, with the following sharing of the "energy budget" (remember that we are always at $\Omega = 1$):

$$\Omega_\phi = \frac{\Omega_m}{2} = \frac{1}{3\epsilon^2}, \qquad \Omega_r = \frac{\epsilon^2 - 1}{\epsilon^2}. \tag{4.16}$$

In the next section we will see how numerical integration confirms in full detail the analytic behaviour we have discussed. We end this Section by discussing some constraints on our parameters.

As already mentioned, we assume the ordinary components of matter (radiation and baryons) to have a nearly metric coupling to $\widetilde{g}_{\mu\nu}$ (see discussion after eq. (3.4)). To be more specific, let us define the ratios between dilatonic charges and energy densities in a way similar to that used for cold dark matter in eq. (3.7), i.e.

$$q_r(\phi) \equiv \sigma_r/\rho_r, \qquad q_b(\phi) \equiv \sigma_b/\rho_b. \tag{4.17}$$

Since it is precisely the ratio $(\psi' + q_{r,b})/k$, which controls both the effective coupling of the dilaton to ordinary macroscopic matter, as well as a possible time-dependence of the fundamental constants [42,43], we shall assume that both q_b and q_r are at most of order ψ', in agreement with the discussion after eq. (3.5). We then find that there are neither appreciable violations of the equivalence principle in the context of macroscopic gravitational interactions, nor significant contributions to the time-variation of the fundamental constants, both effects being controlled by ψ'/k for $q_{r,b} \to 0$. In the strong coupling regime we have $\psi'/k \sim e^{-\phi}/(c_1 c_2)$. For a non-negative ϕ_i, and c_1^2, c_2^2 of order 10^2, there is no appreciable deviation from the standard cosmological scenario down to the epoch of matter-radiation equality, so that one easily satisfies the early-Universe constraints on dark energy, as reported for instance in [44].

The dilaton charge of dark matter is not restricted by the experimental tests of long–range gravitational interactions: this is the reason why we can play with it in order to produce an acceleration. Still, from the above discussion on the early phases of the universe,

it is clear that high values of the dark-matter parameter ϵ may result in dangerously high values for Ω_ϕ, and thus in radical deviations from the standard cosmological scenario. Until radiation-matter equality the situation is relatively harmless: we can easily estimate the dilaton energy density at the equality and at the nucleosynthesis scale, $H_N \sim 10^{10} H_{\rm eq}$, using the fact that the dilaton, during the focusing regime, is not significantly shifted away from the value $\phi_i + \Delta\phi$, fixed by eq. (4.4). Because of the focusing behaviour we find $\Omega_\phi(\text{nucl}) \sim 10^{-10}\ \Omega_\phi(\text{eq})$, and therefore the most stringent bound comes at equality, where, thanks to (4.15), it is comfortably satisfied for $\epsilon < 1$.

During the dragging phase, however, we must certainly impose $\epsilon < 1$, otherwise, the phenomenon of "total dragging" takes place. This would represent a dramatic deviation from the standard cosmological scenario, since all the components $\rho_\phi, \rho_r, \rho_d$ (except baryonic matter) would redshift in the same way (a^{-4}) from equality until the potential starts to be felt. Even if $\epsilon < 1$, but not sufficiently small, the unusual scaling $\rho_m \propto a^{-3-\epsilon^2}$ tends to change the global temporal picture between now and the epoch of matter-radiation equality and, from eqs. (4.14), values of $\Omega_\phi \sim \epsilon^2/3$ (while in agreement with possible constraints at last scattering [44]) can be dangerously high. In our context, a bound $\Omega_\phi(\text{drag}) < 0.1$, i.e. $\epsilon^2(\text{drag}) < 0.3$, appears to be necessary in order to agree with the observed CMB spectrum and with the standard scenario of structure formation.

On the other hand, due to the smallness of $\psi' \sim e^{-\phi}/c_1^2$ in the dragging regime, an upper bound on ϵ effectively turns into a bound on the value of q/k, i.e. on the dilatonic charge of the dark matter component. The above constraints thus translate into a bound for the combination λq:

$$\lambda\ q(\phi_{\rm drag})\ <\ 0.8\,, \tag{4.18}$$

where we used the already mentioned asymptotic relation $\lambda = \sqrt{2}/k$. It is clear that a constant q cannot satisfy the above bound and, at the same time, provide the present acceleration of the Universe by means of the mechanism described in Section III (see also Fig. 1), that requires $q\lambda \gtrsim 4$.

A time– (or, better, ϕ–) dependent q, however, is allowed. For this reason we have to consider cold dark matter models like the one of eq. (3.4), whose dilatonic charge (3.7) switches on at large enough values of the dilaton. The transition to large values of ϕ is rapidly activated as the potential comes into play, $\rho_V \sim \rho_\phi$. At that point, the dilaton energy density stops decreasing and freezes at a constant value, necessarily crossing, at some later moment, the matter energy density, $\rho_\phi \sim \rho_d$. From then on, the dilaton starts rolling towards $+\infty$, triggering the effect of the dilatonic charge, which rapidly freezes the ratio ρ_ϕ/ρ_m and (for suitable values of q) leads to the accelerated asymptotic regime described

by eqs. (3.16)–(3.17). Explicit numerical examples of such a behaviour will be discussed in Section V.

For a realistic picture, in which the positive acceleration regime starts around the present epoch (and not much earlier) and the standard scenario of structure formation is implemented successfully, we have to require that the contribution of the dilatonic charge (as well as the effect of the dilaton potential) come into play only at a late enough epoch. The importance of this constraint was already discussed in the context of other scalar-tensor models of quintessence [28] where, for instance, the non-minimal coupling of the scalar field to the trace of the dark matter stress tensor was assumed to be ϕ-dependent, to interpolate between a small and a large mixing regime.

V. NUMERICAL EXAMPLES

Finally, after the analytic discussion of the previous section, it seems appropriate to illustrate the "run-away" dilaton scenario with some numerical example, both in order to confirm the validity of some approximations made in deriving the analytic results, and in order to see how the various regimes we discussed can be put together. To this aim, we shall numerically integrate eqs. (2.19)–(2.23), using eq. (2.20) as a constraint on the set of initial data, and assuming an explicit model for the dilatonic charges and the dilaton potential. Also, following the "induced-gravity" ideas [21], we shall specialize the loop form-factors according to eq. (3.1), using the "minimal" choice

$$e^{-\psi(\phi)} = e^{-\phi} + c_1^2\,, \qquad\qquad Z(\phi) = e^{-\phi} - c_2^2\,. \tag{5.1}$$

First of all, for a clear illustration of the "focusing" and "dragging" regimes, let us put $V = 0$, $\sigma_r = 0 = \sigma_b$, and $\sigma_d = q\rho_d$, with $q = \text{const}$. By choosing, in particular, $c_1^2 = 100$, $c_2^2 = 30$, we have integrated eqs. (2.19)–(2.23) for three different values of the charge, $q = 0$, $q = 0.01$, and $q = 0.1$, starting from the initial scale $H_i = 10^{40} H_{\rm eq}$,

$$\left(\frac{a_i}{a_{\rm eq}}\right) = \left(\frac{H_{\rm eq}}{H_i}\right)^{1/2} = \left(\frac{\rho_{mi}}{\rho_{ri}}\right) = 10^{-20}, \tag{5.2}$$

and using $\rho_{\phi i} = \rho_{ri}$, $\phi_i = -2$ as initial conditions. It should be noted that such initial conditions are generic, in the sense that different initial values of ρ_ϕ and ϕ may change the fixed value reached by ϕ during the focusing phase, but do not affect in a significant way the subsequent evolution, as will be discussed at the end of this section.

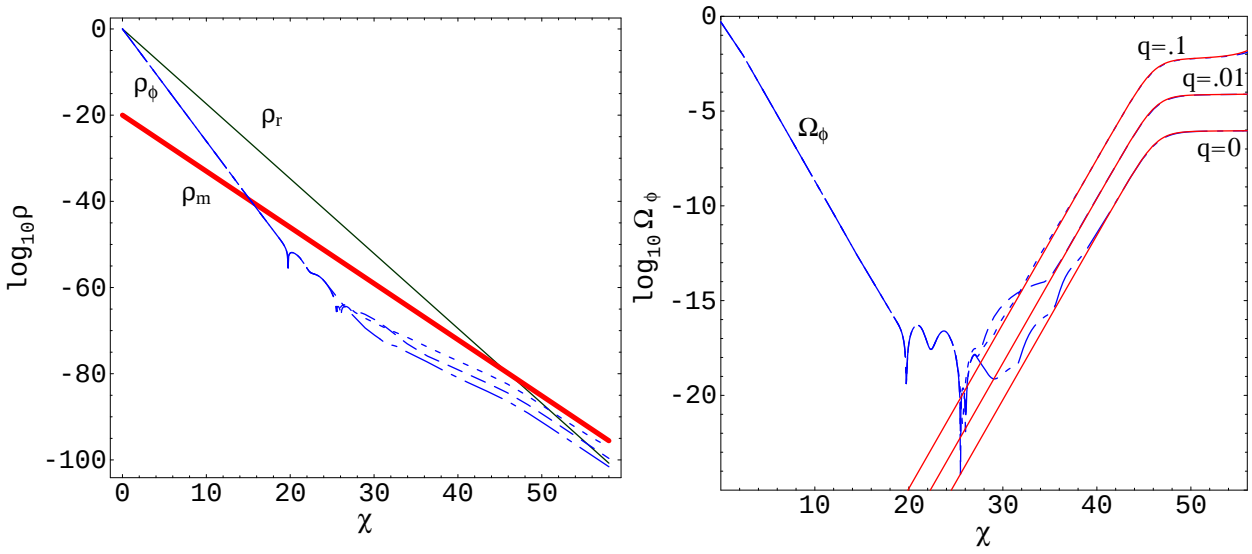

FIG. 2. *Time evolution of ρ_ϕ for $q = 0$ (dash-dotted curve), $q = 0.01$ (dashed curve) and $q = 0.1$ (dotted curve). The initial scale is $a_i = 10^{-20} a_{\rm eq}$, and the epoch of matter-radiation equality corresponds to $\chi \simeq 46$. Left panel: the dilaton energy density is compared with the radiation (thin solid curve) and matter (bold solid curve) energy density. Right panel: the dilaton energy density (in critical units) is compared with the analytical estimates (4.12), (4.13), (4.14) for the focusing and dragging phases.*

The results of this first numerical integration are illustrated in Fig. 2. The left panel clearly displays the initial regime of fast dilaton dilution ($\rho_\phi \sim a^{-6}$), the subsequent focusing regime ($\rho_\phi \sim a^{-2}$, see eq. (4.12)) triggered (after some oscillations) soon after ρ_ϕ falls below ρ_m, and the final dragging regime ($\rho_\phi \sim \rho_m$, see eq. (4.14)) in the epoch of matter domination (the epoch of matter-radiation equality corresponds to $\chi \simeq 46$). Note that the constant values of q have been chosen small enough to avoid the phenomenon of "total dragging", see Section IV. Note also that, in this example, ρ_m always coincides with ρ_d. In the right panel the evolution of Ω_ϕ, obtained through the numerical integration, is compared with the analytic estimates (4.12), (4.13), (4.14), for the three different values of q. In all cases, Ω_ϕ grows like a^2 during the focusing regime (in the radiation era), while the final stabilization $\Omega_\phi = $ const, after the epoch of matter-radiation equality ($\chi \gtrsim 46$), clearly illustrates the effect of the dragging phase during which ρ_ϕ and ρ_m evolve in time with the same behaviour.

For a realistic model of quintessence, however, a constant dilatonic charge cannot drive the Universe towards an asymptotic accelerated regime and, simultaneously, satisfy all the required phenomenological constraints during the earlier epochs (as discussed in the previ-

ous sections). By keeping σ_b, $\sigma_r \simeq 0$ at large coupling (see eq. (3.5) and the discussion thereafter), we shall thus consider the explicit model of scalar dark matter (3.4), with the following simple loop form–factors

$$e^{-\zeta(\phi)} = 1 + e^{q_0\phi}/c^2, \qquad e^{\eta(\phi)} = \text{const} \tag{5.3}$$

(note that, by a field redefinition, one of the two loop factors can always be taken to be trivial: what really matters is the ratio e^{ζ}/e^{η}). Using (3.7) we immediately get

$$q(\phi) = \frac{\sigma_d}{\rho_d} = q_0 \, \frac{e^{q_0\phi}}{c^2 + e^{q_0\phi}} \, , \tag{5.4}$$

which is exponentially suppressed in the perturbative regime, and approaches $q = q_0$ at large coupling (for $q_0 > 1$ it is thus compatible with an asymptotically accelerated cosmological configuration, see Fig. 1). For our numerical example we shall choose $q_0 = 2.5$ and $c^2 = 150$, but the behaviour of the solution is rather stable, at late times, against large variations of the latter parameter (see the discussion at the end of this section).

In addition, we have to specify the form of the dilaton potential. In agreement with its non-perturbative origin, and with the assumtion of exponential suppression at strong coupling (see Section IV), the simplest choice is a difference of terms of the type $e^{-\beta/\alpha(\phi)}$. We shall thus consider the bell-like potential (in units $M_P^2 = 2$)

$$V(\phi) = m_V^2 \left[\exp\left(-e^{-\phi}/\beta_1\right) - \exp\left(-e^{-\phi}/\beta_2\right)\right], \qquad 0 < \beta_2 < \beta_1, \tag{5.5}$$

which leads, asymptotically, to the large-ϕ behaviour of eq. (3.2). The mass scale m_V, related to the mass M_* of eq. (3.3), will be fixed at $m_V = 10^{-3} H_{\rm eq}$, together with $\beta_1 = 10, \beta_2 = 5$, for a realistic scenario that starts accelerating at a phenomenologically acceptable epoch.

With all the parameters fixed, we have numerically integrated the evolution equations (2.19)–(2.23), for our model of charge (5.4) and potential (5.5), using the same initial conditions as in the previous example, but separating the dark and baryonic components inside ρ_m. In particular, we have set, initially, $\rho_{di} = 10^{-20}\rho_{ri}$, $\rho_{bi} = 7 \times 10^{-21}\rho_{ri}$.

The resulting late-time evolution of the various energy densities is shown in the left panel of Fig. 3. Dark matter and baryonic energy densities evolve in the same way, until the potential comes into play, starting at a scale around $\chi \simeq 49$. The potential first tends to stabilize ρ_ϕ to a constant but then (thanks to the contribution of q) the system eventually evolves towards a final regime in which ρ_ϕ and ρ_d are closely tied up, and their asymptotic evolution departs from the trajectory of the standard, decelerated scenario (in particular, they both scale, asymptotically as $a^{-6/(2+q_0)}$, see eq. (3.18). It is amusing to conjecture that

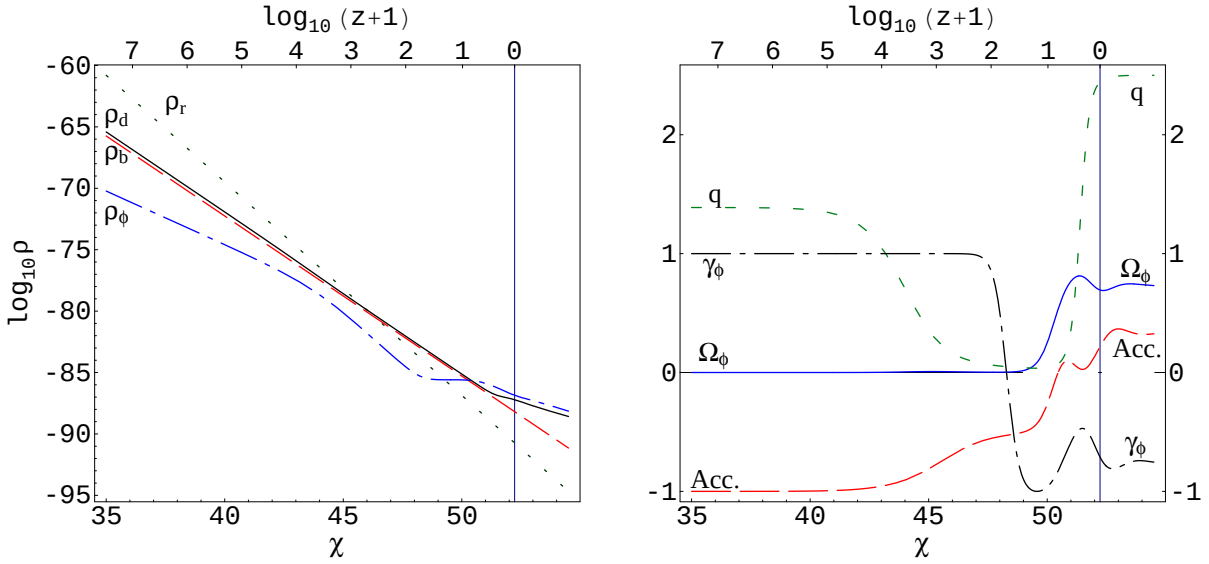

FIG. 3. *Left panel: Late-time evolution of the dark matter (solid curve), barionic matter (dashed curve), radiation (dotted curve) and the dilaton (dash-dotted curve) energy densities, for the string cosmology model specified by eqs. (5.4), (5.5). The upper horizontal axis gives the* $\log_{10}$ *of the redshift parameter. Right panel: for the same model, the late-time evolution of* q *(fine-dashed curve),* w_ϕ *(dash-dotted curve),* Ω_ϕ *(solid curve) and of the acceleration parameter* $\ddot{a}a/\dot{a}^2$ *(dashed curve).*

the different time-dependence of ρ_b and r_d could be responsible for the present small ratio ρ_b/ρ_d.

In the right panel we have plotted the time evolution of the dilatonic charge q, of the energy density Ω_ϕ, of the equation of state w_ϕ, and of the acceleration parameter $\ddot{a}/aH^2$. When the potential energy becomes important, all the above quantities rapidly approach their asymptotic values given in eqs. (3.16)-(3.17). Note that, with our choice of parameters, we have $q_0 = 2.5$ and $\lambda = c_1/c_2 = \sqrt{10/3}$, corresponding to an asymptotic value $\Omega_\phi \simeq 0.733$, slightly exceeding the best fit value suggested by present observations [36,37]. It is important to stress, however, that the asymptotic attractor may be preceded by a (short) oscillating regime, which, as illustrated in the right panel of Fig. 3, can easily allow for values of the cosmological parameters different from the asymptotic ones to be compatible with present observations. Note also that, when switching from the focusing to the dragging phases, the dilaton starts to move back towards decreasing values of q, as will be illustrated also by a subsequent numerical integration. This may slow down the evolution of ρ_ϕ with respect to ρ_m during the dragging, as shown for instance in the left panel of Fig. 3. Because of this effect,

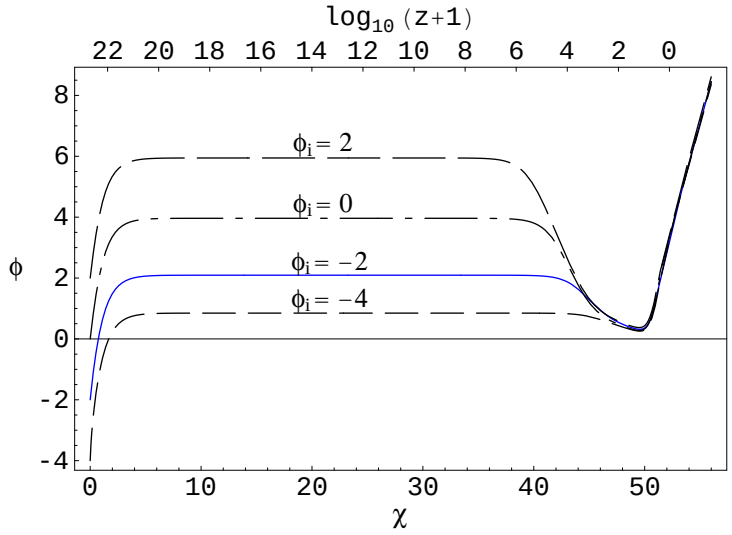

FIG. 4. *Time evolution of the dilaton field, for different initial conditions* $\phi_i = -4, -2, 0, 2$. *All the other parameters are the same as in the example of Fig. 3. After the* plateau *associated with the focusing regime, and for a strong enough dilatonic charge, the solutions tend to converge to a common value of* ϕ. *The subsequent running to* $+\infty$, *driven by the potential, is thus completely independent from the initial value.*

however, the dilaton can easily satisfy, during the dragging phase, the phenomenological bounds discussed in the previous sections. This does not require fine tuning, the validity if the bounds being guaranteed for a large basin of initial conditions by a convergent behaviour of the solutions during dragging.

During the focusing phase, in fact, the dilaton is practically frozen, as can be argued from eq. (4.12), and its effective constant value, as determined by eq. (4.4), depends on ϕ_i. However, if such a value is high enough, the presence of the dilatonic charge may become important, and may contribute to the focalization towards the epoch of matter-radiation equality, as already anticipated. This is illustrated in Fig. 4, which shows the time evolution of the dilaton obtained by numerically integrating the same model as in Fig. 3, for different initial values $\phi_i = -4, -2, 0, 2$. Although we start with different dilaton values at the *plateau* associated with the focusing regime, all the solutions tend to converge as we enter the dragging regime, so as to make the subsequent (potential-dominated) evolution *insensitive* to the initial value of the dilaton[†].

[†]The *preceding* evolution, of course, is not sensitive either, since during focusing the order of

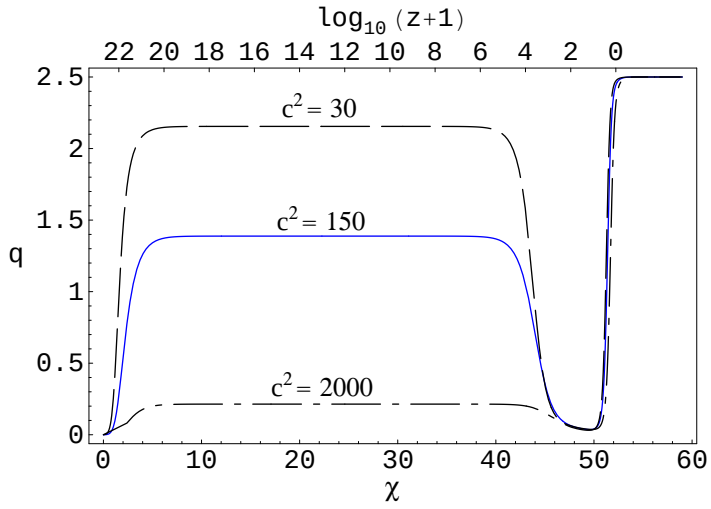

FIG. 5. *Time evolution of $q(\phi)$, from eq. (5.4), for three different values of the parameter c. All the other parameters are the same as in the example of Fig. 3. During the dragging phase the value of q converges to the regime $q \ll 1$.*

This new focusing effect, which is very different from the one of the energy densities during the radiation-dominated phase, can also be understood analytically by writing the solution of eq. (4.14) as:

$$\chi - \chi_{\rm eq} \; = \; - \int_{\phi_{\rm eq}}^{\phi} \frac{k(\bar\phi)}{2\,\epsilon(\bar\phi)}\, d\bar\phi \,. \tag{5.6}$$

Since k is almost constant, a variation $\delta\phi_{\rm eq}$ on the initial value of ϕ changes the solution $\phi(\chi)$ by an amount $\delta\phi(\chi) = [\epsilon(\phi)/\epsilon(\phi_{\rm eq})]\delta\phi_{\rm eq}$, which rapidly decreases (with $q(\phi)$) during the dragging phase. This is why the solution has become independent of the initial value of ϕ by the time the potential becomes an important component.

For the same reason, the model is only weakly affected by variations of the parameter c in eq. (5.4), which roughly gives the transition scale between small and large dilatonic charges: $\phi_s = (2/q_0)\log c$. Indeed, because of the above mechanism the dilaton is pushed back during the dragging phase the dilaton is pushed back, with a velocity as high as needed to reach, in any case, the safe zone $q \ll 1$. This effect is illustrated in Fig. 5, where we have plotted the time evolution of $q(\phi)$, for the same model as Fig. 3, and for three different values of c.

It should be noted, in conclusion, that the above class of models depends in crucial way

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magnitude of Ω_ϕ is given by Ω_m^2 as in (4.12).

on three important parameters: m_V, q_0 and the ratio $\lambda = c_1/c_2$. The first one controls the transition time between the epoch of standard cosmological evolution and the final accelerated regime (as can be easily checked, for instance, by repeating the numerical integration of Fig. 3 with different values of m_V). The other two parameters control the asymptotic properties of the model (acceleration, equation of state, ...), as discussed in Section III. Future precision data, both from supernovae observations and from measurements of the CMB anisotropy, could give us a good determination of these parameters, thus providing important indirect information on the parameters of the string effective action in the strong coupling regime.

VI. CONCLUSION

Let us conclude by summarizing the main points of our work. We have argued that a run-away dilaton can provide an interesting model of quintessence under a well-defined set of assumptions that we list hereafter:

- The limit of supertring theory, as its bare four-dimensional effective coupling goes to infinity (so called induced gravity/gauge or compositeness limit), should exist and make sense phenomenologically, i.e. should yield reasonable values for the unified gauge coupling at the GUT scale and for the ratio M_P/M_{GUT}, thanks to the large number of degrees of freedom at M_{GUT};

- In the visible-matter sector, the couplings to the dilaton, either direct or through the trace of the energy-momentum tensor (i.e. via a conformally rescaled metric), should vanish in the $\phi \to +\infty$ limit;

- In the dark matter sector, there should be a surviving coupling to the dilaton (and thus violations of the strong and/or weak equivalence principles) even in the $\phi \to +\infty$ limit;

- The dilaton potential should be non-perturbative, go to zero asymptotically, and have an absolute scale not too far from the present energy density.

Under these circumstances, it is natural for the dilaton energy in critical units, Ω_ϕ, to be: i) subdominant during radiation domination; ii) a (small) fraction of the total energy at matter-radiation equality; iii) a (small) fraction of Ω_m during the earlier epoch of matter domination; iv) a fraction of dark-matter energy since a red-shift $\mathcal{O}(1)$. This very last phase is characterized by an accelerated expansion.

In other words, this framework seems to be naturally consistent with present astrophysical observations and with known cosmological constraints. From a theoretical point of view the model appears to combine nicely previous ideas [15] on dilaton stabilization and decoupling with those recently advocated (e.g. in [28]) so as to have acceleration while keeping the ratio Ω(dark energy)/Ω(dark matter) constant.

It must be stressed, of course, that the analysis presented in this paper is still preliminary, and that various problems are still open. In particular, a precise computation of the CMB anisotropy spectrum, and of the spacing of acoustic peaks expected in this context, as well as a comparison with currently available measurements [4], could provide significant bounds on the parameters of the string cosmology models we have discussed. Such an investigation is postponed to future work. Nevertheless, we believe that the results of this paper are encouraging, as they suggest that the dilaton, which can hardly play the role of the inflaton in the standard inflationary scenario [45], could play instead a successful role as the quintessential field in post-inflationary, late-time cosmology.

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REFERENCES

[1] A. G. Riess *et al*, Astron. J. **116**, 1009 (1998); S. Perlmutter *et al*, Astrophys. J **517**, 565 (1999).

[2] A. G. Riess *et al*, astro-ph/0104455.

[3] N. A. Bachall, J. P. Ostriker, S. Perlmutter and P. J. Steinhardt, Science **284**, 1481 (1999).

[4] P. de Bernardis *et al*, Nature **404**, 955 (2000).

[5] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).

[6] P. Steinhardt, in *Critical problems in Physics*, edited by by V. L. Fitch and D. R. Marlow (Princeton University Press, Princeton, NJ, 1997).

[7] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. **80**, 582 (1998); J. A. Frieman and I. Vaga, Phys. Rev. D **57**, 4642 (1998); M. S. Turner and M. White, Phys. Rev. D **56**, 4439 (1997).

[8] P. Binètruy, in *Proceedings of the LXXI Les Houches Summer School "The Primordial Universe"*, edited by P. Binètruy, R. Schaeffer, J. Silk and F. David (Springer-Verlag, Berlin, 2000), p. 397.

[9] P. J. E. Peebles and A. Vilenkin, Phys. Rev. D **59**, 063505 (1999); M. Peloso and F. Rosati, JHEP **9912**, 026 (1999).

[10] L. P. Chimento, A. S. Jakubi and D. Pavon, Phys. Rev. D **62**, 063508 (2000); W. Zimdahl, D. J. Schwarz, A. B. Balakin and D. Pavon, astro-ph/0009353; S. Sen and A. A. Sen, Phys. Rev. D **63**, 124006 (2001); A. A. Sen and S. Sen, gr-qc/0103098.

[11] A. Hebecker and C. Wetterich, Phys. Rev. Lett. **85**, 3339 (2000); L. Amendola and D. Tocchini-Valentini, astro-ph/0011243.

[12] J. P. Uzan, Phys. Rev. D **59**, 123510 (1999); T. Chiba, Phys. Rev. D **60**, 083508 (1999); L. Amendola, Phys. Rev. D **60**, 043501 (1999); F. Perrotta, C. Baccigalupi and S. Matarrese, Phys. Rev. D **61**, 023507 (2000); D. J. Holden and D. Wands, Phys. Rev. D **61**, 043506 (2000); N. Bartolo and M. Pietroni, Phys. Rev. D **61**, 023518 (2000); O. Bertolami and P. J. Martins, Phys. Rev. D **61**, 064007 (2000); R. De Ritis , A. A. Marino, C, Rubano and P. Scudellaro, Phys. Rev. D **62**, 043506 (2000); R. Bean and J. Magueijo, astro-ph/0007199; S. Sen and T. R. Seshadri, gr-qc/0007079; N. Banerjee and

D. Pavon, Phys. Rev. D **63**, 043504 (2001); E. Gunzig *et al*, Phys. Rev. D **63**, 067301 (2001).

[13] A. Masiero, M. Pietroni and F. Rosati, Phys. Rev. D **61**, 023504 (2000).

[14] T. R. Taylor and G. Veneziano, Phys. Lett. B **213**, 459 (1988).

[15] T. Damour and K. Nordtvedt, Phys. Rev. D **48**, 3436 (1993); Phys. Rev. Let. **70**, 2217 (1993); T. Damour and A. M. Polyakov, Nucl. Phys. B **423**, 532 (1994); Gen. Rel. Grav. **26**, 1171 (1994).

[16] V. Kaplunovsky, Phys. Rev. Lett. **55**, 1036 (1985); M. Dine and N. Seiberg, Phys. Rev. Lett. **55**, 366 (1985); R. Petronzio and G. Veneziano, Mod. Phys. Lett. A **2**, 707 (1987).

[17] M. B. Green, J. Schwartz and E. Witten, *Superstring theory* (Cambridge University Press, Cambridge, England, 1987).

[18] E. Witten, Nucl. Phys. B **471**, 135 (1996).

[19] M. Gasperini, Phys. Rev. D **64**, 043510 (2001).

[20] M. Gasperini, Class. Quantum Grav. **17**, R1 (2000); G. Veneziano, in *Proceedings of the LXXI Les Houches Summer School "The Primordial Universe"*, edited by P. Binètruy, R. Schaeffer, J. Silk and F. David (Springer-Verlag, Berlin, 2000), p. 581; M. Gasperini, in *Proc. of the Second SIGRAV School on "Gravitational Waves in Astrophysics, Comology and String Theory"* (Centre A. Volta, Como, April 1999), edited by I. Ciufolini et al. (IOP Publishing, Bristol, 2001), p. 280; J. E. Lidsey, D. Wands and E. J. Copeland, Phys. Rep. **337**, 343 (2000). An updated collections of papers on the pre-big bag scenario is also available at `http://www.ba.infn.it/~gasperin`.

[21] G. Veneziano, in *"The Geometric Universe"*, edited by L. J. Mason, K. P. Tod, S. T. Tsou and N. M. J. Woodhouse (Oxford University Press, Oxford, 1998), p. 235; G. Veneziano, *"Large-N bounds –and compositeness limit of– gauge and gravitational interactions"*, to appear.

[22] A. D. Sakharov, Dokl. Akad. Nauk. SSSR **177** 70 (1967); [Sov. Phys. Dokl. **12**, 1040 (1968)]; Ya. B. Zel'dovich, ZhETF Pisma **6**, 922 (1967); [JETP Lett. **6**, 345 (1967)]; D. Amati and G. Veneziano, Phys. Lett. **105**B, 358 (1981); Nucl. Phys. B **204**, 451(1982), and references therein.

[23] T. Damour, G. W. Gibbons and C. Gundlach, Phys. Rev. Lett. **64**, 123 (1990).

[24] B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988); I. Zlatev, L. Wang and P.

J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999); Phys. Rev. D **59**, 123504 (1999).

[25] M. Gasperini, Phys. Lett. B **470**, 67 (1999).

[26] B. Boisseau, G. Esposito-Farese, D. Polarski and A. A. Starobinsky, Phys. Rev. Lett. **85**, 2236 (2000).

[27] P. G. Ferreira and M. Joyce, Phys. Rev. D **58**, 023503 (1998).

[28] L. Amendola and D. Tocchini-Valentini, Phys. Rev. D **64**, 043509 (2001).

[29] M. S. Turner and A. G. Riess, astro-ph/0106051.

[30] See for instance R. W. Hellings, in *Gravitational measurements, fundamental metrology and constants*, (Erice, June 1987), edited by V. N. Melnikov (Kluwer Acad. Pub., Dordrecht, The Netherlands, 1988), Nato ASI Series, Vol. C230, p. 181.

[31] E. Fischbach and C. Talmadge, Nature **356**, 207 (1992).

[32] C. Wetterich, Astron. Astrophys. **301**, 321 (1995).

[33] L. Amendola, Phys. Rev. D **62**, 043511 (2000).

[34] D. J, Holden and D. Wands, Phys. Rev. D **61**, 043506 (2000).

[35] W. Zimdahl and D. Pavón, astro-ph/0105479.

[36] L. Wang, R. R. Caldwell, J. P. Ostriker and P. J. Steinhardt, Astrophys. J. **530**, 17 (2000).

[37] A. Balbi, C. Baccigalupi, S. Matarrese, F. Perrotta and N. Vittorio, Astrophys. J. **547**, L89 (2001).

[38] P. de Bernardis et al., astro-ph/0105296.

[39] M. Doran, M. Lilley and C. Wetterich, astro-ph/0105457.

[40] T. Barreiro, B. de Carlos and E. J. Copeland, Phys. Rev. D **58**, 083513 (1998).

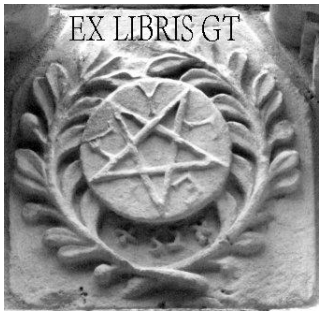
[41] T. Damour and K. Nordtvedt, Ref. [15]; N. Bartolo and M. Pietroni, Phys. Rev. D **61**, 023518 (2000).

[42] T. Damour, in *Proc. of the Les Houches School*, (1992) (gr-qc/9606079).

[43] T. Damour Nucl. Phys. Proc. Suppl. **80**, 41 (2000).

[44] R. Bean, S. H. Hansen and A. Melchiorri, astro-ph/0104162.

[45] R. Brustein and P. J. Steinhardt, Phys. Lett. B **302**, 196 (1993).



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Birth of the Universe in String Cosmology

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Abstract

The decay of the string perturbative vacuum into our present cosmological state is associated to the transition from a phase of growing curvature and growing dilaton, to a phase of decreasing curvature and frozen dilaton. The possible approaches to a classical and quantum description of such a transition are introduced and briefly discussed.

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BIRTH OF THE UNIVERSE IN STRING COSMOLOGY

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1. Introduction

In the context of the pre-big bang cosmological scenario[1], based on the string effective action, the process of "birth of the Universe" corresponds to the transition from the string perturbative vacuum to the standard radiation-dominated regime, passing through a high-curvature and strong coupling phase, in which quantum and "stringy" effects may become important. Such a process may be qualitatively illustrated as in Fig. 1, by plotting the time evolution of the curvature scale from the initial vacuum down to the present cosmological state. The aim of this paper is to introduce, and briefly discuss, the classical and quantum approach to the transition from the pre-big bang to the post-big bang regime.

Before starting the discussion, let me stress that the physical system that we call Universe evolves in time, like all physical systems. For our convenience, its evolution can be divided into various phases: now we are in the matter-dominated phase, but in the past there was certainly a radiation-dominated phase, and an explosive phase of very high curvature and density, that we may call "big bang". I cannot find any convincing reason, however, to believe that before the big bang there was "nothing".

fine!!

To make an analogy with another physical system, consider for instance the β-decay of a neutron. The initial neutron is transformed into a proton, an electron and a neutrino. For these three particles, the decay process is a sort of "big bang" which marks the beginning of their existence. This does not means, however, that these three particles spring out of nothing: initially, the system was in a different quantum state, representing a neutron.

In the same way, coming back to cosmology, the high curvature big bang phase certainly marks the beginning of the present state of the Universe, i.e. of the stan-

dard Friedmann phase that we can observe today. It seems quite reasonable, however, to wonder about the state of the Universe preceeding the big bang explosion. The answer suggested by string theory is illustrated in Fig. 1: our present cosmological phase, decelerated, with decreasing curvature, might have been complemented by a dual phase, accelerated, with growing curvature, which seems natural to call "pre-big bang". In this context, the very early cosmological evolution from the perturbative vacuum, i.e. from a configuration with asymptotically flat metric and vanishing coupling constant, can be consistently described by the lowest-order string effective action, as will be discussed in the following Section.

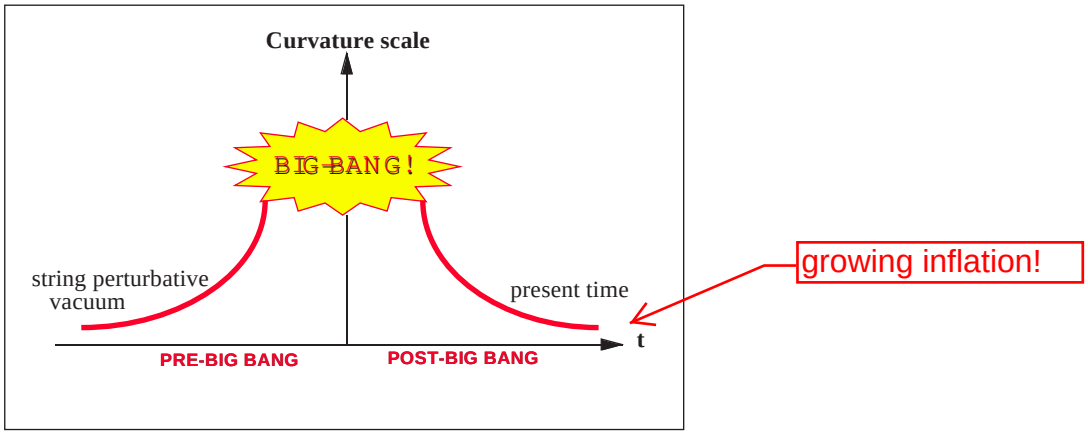

Figure 1: *Curvature scale versus time in the pre-big bang scenario.*

2. Low-energy pre-big bang evolution

In the context of the pre-big bang scenario, the global evolution of the universe can be conveniently represented in the phase space spanned by the Hubble factor and by the dilaton kinetic energy.

Consider in fact the lowest order gravi-dilaton effective action[2], which in d critical spatial dimensions, and in the string frame, takes the form

$$S = -\frac{1}{2\,\lambda_s^{d-1}} \int d^{d+1}x\,\sqrt{|g|}\,e^{-\phi}\,\left(R + \partial_\mu\phi\partial^\mu\phi\right). \tag{2.1}$$

($\lambda_s \equiv (\alpha')^{1/2}$ is the basic string length parameter). For an isotropic and spatially flat background (but we can also consider more general initial conditions, spatially curved[3] and inhomogeneous[4]), the asymptotic solutions approaching the singularity

can be written as[5]:

$$a(t) = (\mp t)^{\mp 1/\sqrt{d}}, \qquad \overline{\phi}(t) = -\ln(\mp t), \tag{2.2}$$

scale factor

where a is the scale factor, and $\overline{\phi}$ is the so-called "shifted" dilaton:

$$\overline{\phi} = \phi - d\ln a - \ln \int \frac{d^d x}{\lambda_s^d} \tag{2.3}$$

(we are assuming spatial sections of finite volume). Such solutions are characterized by four branches, depending on the range of time, and on the power of the scale factor. As illustrated in Fig. 2, these solutions are represented by the bisecting lines in the plane spanned by $\sqrt{d}H$ and $\dot{\overline{\phi}}$ ($H = \dot{a}/a$, and a dot denotes differentiation with respect to the cosmic time t).

The string perturbative vacuum is characterized by $H = 0 = \dot{\overline{\phi}}$, and corresponds to the origin of that plane. In the limits $|H| \to \infty$, $|\dot{\overline{\phi}}| \to \infty$, the background approaches a curvature singularity. The four branches of the solution describe expansion or contraction depending on the sign of H, and represent a pre-big bang configuration (evolving towards the curvature singularity) or a post-big bang configuration (evolving from the singularity towards an asymptotically flat spacetime), depending on the sign of $\dot{\overline{\phi}}$. Notice that, in the isotropic case, only the branch called "expanding pre-big bang" in Fig. 2 describes a true evolution from the perturbative vacuum (i.e. from a state of zero string coupling), because in the contracting branch the true dilaton ϕ is decreasing[1], and the string coupling e^{ϕ} is also decreasing.

The four branches of the solution are connected by T-duality transformations[5]:

$$a \to \tilde{a} = a^{-1}, \qquad \overline{\phi} \to \overline{\phi}, \tag{2.4}$$

and t-reversal transformations:

$$a(t) \to a(-t), \qquad \overline{\phi}(t) \to \overline{\phi}(-t). \tag{2.5}$$

A monotonic (expanding or contracting) transition from pre- to post-big bang thus requires a combination of both T and t transformations, as illustrated in Fig. 2. Smooth self-dual solutions, characterized by $a(t) = a^{-1}(-t)$, and defined over the whole time range $-\infty \leq t \leq +\infty$, would automatically connect the pre- and post-big bang regime, avoiding the curvature singularity. Such solutions are possible, but only at the price of adding to the action an "ad-hoc", non-local potential for the shifted dilaton[1,6].

The simplest potential is the one induced by a cosmological constant, $V(\overline{\phi}) = \Lambda$. In that case the solutions are characterized by the condition[5]

+ lambda!

$$\dot{\overline{\phi}}^{\,2} - (\sqrt{d}H)^2 = \Lambda, \tag{2.6}$$

representing an hyperbola in the plane of Fig. 2, and the initial vacuum is shifted to a state of flat metric and linearly evolving dilaton. The solution is still characterized

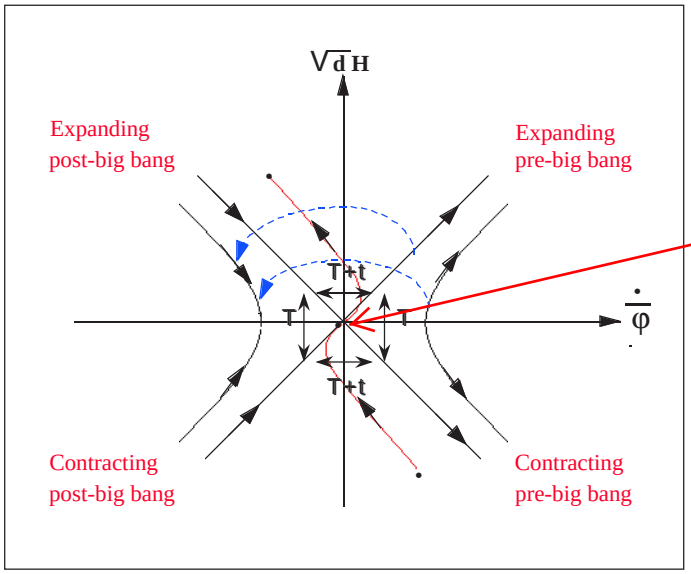

Figure 2: *Global view of the possible cosmological evolution from and towards the string perturbative vacuum.*

by four branches disconnected by a curvature singularity, so that a transition from the pre- to the post-big bang phase remains classically forbidden. It may be allowed, however, at a quantum level.

transition prob WDW

If we compute, with the Wheeler-De Witt (WDW) equation[7], the probability of transition to a post-big bang phase with $\Lambda \neq 0$, we find indeed that such a probability is finite and non-vanishing, in spite of the presence of a curvature singularity disconnecting, classically, the two regimes[6,8] (such transitions are represented by the dashed curves of Fig. 1).

The transition from a state of positive $\dot{\bar{\phi}}$ to a state of negative $\dot{\bar{\phi}}$ is allowed even classically, and even in the absence of a dilaton potential, provided we add to the lowest-order action (2.1) the higher-derivative corrections, arising from an expansion of the string effective action in powers of the curvature (the so-called α' expansion[9]). Already to first order in α', we can find indeed exact solutions which smoothly interpolate between the pre- and post-big bang regime[10], connecting the perturbative vacuum to a final state of constant curvature and linearly evolving dilaton, as illustrated by the solid curve of Fig. 2.

Fig. 2 illustrates the two possibilities (classical, solid line, and quantum, dashed lines) of transition from pre- to post-big bang, and thus provides a schematic summary of my subsequent discussion.

3. Quantum string cosmology

The quantum approach to the transition is based on the WDW equation, obtained

from the low-energy string effective action. All the standard quantum cosmology problems (time parameter, probability interpretation, semiclassical limit) remain, except perhaps the problem of quantum ordering, since the ordering is fixed by the duality symmetry[6]. The aim of this approach is to compare, and possibly contrast, the quantum results obtained in the context of the standard cosmological scenario, with the results obtained, with the same method and the same assumptions, in the context of the pre-big bang scenario.

SCS/=pBBS

Witten on tunnel for instantons(??)

The main difference between the two scenarios is qualitatively illustrated in Fig. 3, in which we compare classical cosmology, quantum cosmology with tunnelling boundary conditions[11,12], and quantum string cosmology for the pre-big bang scenario. While in classical cosmology the initial singularity is unavoidable, in quantum cosmology the Universe may avoid the singularity, emerging from the quantum regime through a tunnelling process. The natural suggestion of string theory is that tunnelling in not "from nothing"[11], but from the preceeding pre-big bang phase. Also, tunnelling in not through a curvature singularity, but through a string phase of high (but finite) curvature.

3.1. *Quantum transition from pre- to post big bang*

For a sort, but self-contained, illustration of how this program could be implemented, we can start with the simplest gravi-dilaton action (2.1), supplemented by a dilaton potential $V(\phi)$, in $d = 3$ isotropic dimensions:

$$S = \frac{\lambda_s}{2} \int dt \, \frac{e^{-\overline{\phi}}}{N} \left(\dot{\beta}^2 - \dot{\overline{\phi}}^2 - N^2 \, V \right). \tag{3.1}$$

N is the usual lapse function, and $\beta = \sqrt{3} \ln a$. The variation of the lapse, in the cosmic time gauge $N = 1$, gives the Hamiltonian constraint

$$\Pi_\beta^2 - \Pi_{\overline{\phi}}^2 + \lambda_s^2 \, V(\beta, \overline{\phi}) \, e^{-2\,\overline{\phi}} = 0 \ , \tag{3.2}$$

and the WDW equation

$$\left[\partial_{\overline{\phi}}^2 - \partial_\beta^2 + \lambda_s^2 \, V(\beta, \overline{\phi}) \, e^{-2\,\overline{\phi}} \right] \psi = 0, \tag{3.3}$$

where we have introduced the canonical momenta:

$$\Pi_\beta = \frac{\delta S}{\delta \dot{\beta}} = \lambda_s \, \dot{\beta} \, e^{-\overline{\phi}}, \qquad \Pi_{\overline{\phi}} = \frac{\delta S}{\delta \dot{\overline{\phi}}} = -\lambda_s \, \dot{\overline{\phi}} \, e^{-\overline{\phi}}. \tag{3.4}$$

In the absence of a dilaton potential the WDW equation (3.3) reduces to the massless Klein-Gordon equation, and we recover a plane wave representation of the four branches of the classical solution, $\psi_\beta^{(\pm)} \psi_{\overline{\phi}}^{(\pm)} \sim e^{\pm i k \beta \pm i k \overline{\phi}}$, where

$$\Pi_\beta \, \psi_\beta^{(\pm)} = \pm k \, \psi_\beta^{(\pm)}, \qquad \Pi_{\overline{\phi}} \, \psi_{\overline{\phi}}^{(\pm)} = \pm k \, \psi_{\overline{\phi}}^{(\pm)}. \tag{3.5}$$

They corresponds to expansion ($\Pi_\beta > 0$), contraction ($\Pi_\beta < 0$), growing dilaton ($\Pi_{\overline{\phi}} < 0$), decreasing dilaton ($\Pi_{\overline{\phi}} > 0$), according to eq. (3.4). A transition from a state of pre-big bang expansion to a state of post-big bang expansion thus corresponds, in this representation, to a transition from a state of positive momentum Π_β and negative momentum $\Pi_{\overline{\phi}}$, to a state with positive Π_β and $\Pi_{\overline{\phi}}$. In other words, the transition corresponds to a monotonic evolution along the β direction, and to a reflection along the $\overline{\phi}$ direction.

This suggests to look at the transition as at a scattering process[6,8] in the minisuperspace spanned by β and $\overline{\phi}$, in which β plays the role of the time-like coordinate, and $\overline{\phi}$ plays the role of the space-like coordinate, as illustrated in Fig. 4. The initial pre-big bang state, $\psi \sim e^{+ik\overline{\phi}-ik\beta}$, is partially transmitted towards the singularity, and partially reflected back to a post-big bang configuration, $\psi \sim e^{-ik\overline{\phi}-ik\beta}$. The transition probability is controlled by the reflection coefficient, $R = |\psi^{(+)}_{\overline{\phi}}|^2/|\psi^{(-)}_{\overline{\phi}}|^2$ (see Ref. [13] for a rigorous definition of scalar products in the appropriate Hilbert space).

The boundary conditions, in this context, are unambiguously fixed by the choice of the perturbative vacuum as the initial state of the Universe. It may be noted that, with this choice, there are only outgoing waves at the singular spacetime boundary ($\overline{\phi} \to +\infty$), just like in the case of tunnelling boundary conditions[11]. We may thus look at the process as at a sort of "tunnelling from the string perturbative vacuum", even if, in this minisuperspace representation, the WDW wave function is actually reflected.

When the reflection is induced by a cosmological constant, $V = \Lambda$, we have checked[8] that the transition probability is indeed very similar to the tunnelling probability. Both are exponentially suppressed, unless the cosmological constant is very large and the proper size of the transition volume is very small in string units. The only basic difference is that, in a string cosmology context, the coupling constant is controlled by the dilaton and it is thus running in Planckian units. As a consequence, the Universe tends to emerge from the pre-big bang phase in the strong coupling regime, since the transition probability has a typical instanton-like behaviour $\sim \exp(-g_s^{-2})$, where $g_s = \exp(\phi/2)$ is the string coupling parameter.

It is important to stress, finally, that the WDW eq. (3.3) is free from problems of operator ordering. Indeed, thanks to the duality symmetry of the effective action, the WDW Hamiltonian is associated to a globally flat minisuperspace.

For a more general discussion of this point, we may add to the effective action (3.1) a non-trivial antisymmetric tensor (also called torsion) background, $B_{\mu\nu} \neq 0$. The kinetic part of the action may then be written in compact form as

+Bmunu

$$S = -\frac{\lambda_s}{2}\int dt e^{-\overline{\phi}}\left[(\dot{\overline{\phi}})^2 + \frac{1}{8}\mathrm{Tr}\ \dot{M}(M^{-1})\dot{}\right], \tag{3.6}$$

torsion,
Kalb-Ramond

where M is a symmetric $2d \times 2d$ matrix, including the spatial part of the metric,

$G \equiv g_{ij}$, and of the torsion, $B \equiv B_{ij}$:

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}. \tag{3.7}$$

This action is invariant under global $O(d,d)$ transformations[14] which leave invariant the shifted dilaton,

$$\overline{\phi} \to \overline{\phi}, \qquad M \to \Omega^T M \Omega, \tag{3.8}$$

where

$$\Omega^T \eta \Omega = \eta, \qquad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3.9}$$

The Hamiltonian associated to torsion-graviton background,

$$H \sim \mathrm{Tr}\ (M\ \Pi_M M\ \Pi_M), \qquad \Pi_M = \delta S/\delta M, \tag{3.10}$$

would seem to have ordering problems, because $[M, \Pi_M] \neq 0$. However, thanks to the $O(d,d)$ properties of M,

$$M\eta M = \eta, \tag{3.11}$$

we can always rewrite the kinetic part of the action in terms of the flat $O(d,d)$ metric η:

$$\mathrm{Tr}\ \dot{M}(M^{-1})\dot{} = \mathrm{Tr}\ (\dot{M}\eta)^2. \tag{3.12}$$

The corresponding Hamiltonian

$$H \sim \mathrm{Tr}\ (\eta\ \Pi_M\ \eta\ \Pi_M) \tag{3.13}$$

has a flat metric in momentum space, with no ordering problems for the corresponding WDW equation.

It may be noted that, for a general curvilinear parametrization of superspace, the ordering fixed by the duality symmetry is equivalent to the ordering fixed by the requirement of reparametrization invariance[6]. It is crucial, for this equivalence, the fact that there are no contributions to the ordered Hamiltonian from the scalar curvature of minisuperspace, because minisuperspace is globally flat.

3.2. *Quantum transition from expansion to contraction*

The transition discussed in the previous Section was characterized by a monotonic evolution of the scale factor. In the context of the pre-big bang scenario, however, it is also important to consider the possibility of transitions from expansion to contraction. Indeed, if we start with a higher-dimensional and anisotropic perturbative vacuum, the initial growth of the dilaton requires a large enough number of expanding dimensions[1], in general larger than 3. On the other hand, a model of dynamical dimensional reduction requires only three expanding dimensions, with all the other dimensions contracting down to a final compactification scale. The late-time transition of an appropriate subsection of the spatial manifold, from the

expanding pre-big bang phase to the contracting pre-big bang phase, may thus be useful to implement a realistic cosmological scenario.

If the dilaton potential depends only on $\overline{\phi}$, the action (3.1) is duality-invariant, the momentum along β is conserved, $[\Pi_\beta, H] = 0$, the evolution of the scale factor is monotonic, and the transition from expansion to contraction is classically forbidden. Such a transition is allowed at the quantum level, however, where it can be represented (in a third quantization formalism) as a process of pair production (pairs of universes, in this case), out of the string perturbative vacuum[15]. This process is described by the same action, same Hamiltonian, same minisuperspace as before, but with different boundary conditions, and with a 90 degree rotation of the time axis (see Ref. [16] for a classification of the different types of scattering of the perturbative vacuum in the $(\beta, \overline{\phi})$ minisuperspace).

Consider in fact the configuration illustrated in Fig. 5, where we have one incoming wave in the weak coupling regime $\overline{\phi} \to -\infty$, and two waves, one incoming, the other outgoing, in the strong coupling regime $\overline{\phi} \to +\infty$. All the asymptotic states appearing in Fig. 5 are eigenstates of Π_β, with positive momentum $k > 0$. However, if $\overline{\phi}$ is the time-like coordinate, the final state is a mixture of positive and negative frequency modes, $\psi_{\overline{\phi}}^{(\pm)}$. In a quantum field theory context, this configuration describes a Bogoliubov process of pair creation out of the vacuum, in which the negative frequency mode is re-interpreted as an anti-particle of positive frequency (i.e. positive energy) and opposite spatial momentum.

BOGO!!!

(Feynman)

In our case the spatial momentum is Π_β, and a negative value of Π_β corresponds to a contraction. The final state of the process illustrated in Fig. 5 thus describes, in a third quantization formalism, the production from the vacuum of a universe–anti-universe pair, one expanding, the other contracting.

With such a boundary conditions, the process cannot be interpreted as a tunnelling. It can be interpreted, instead, as an "anti-tunnelling from the string perturbative vacuum", i.e as a process in which the WDW wave function is parametrically amplified in superspace[15]. The probability of the process is indeed controlled by the Bogoliubov coefficient which weights the anti-universe content of the final state, and which becomes, in the parametric amplification regime, the inverse of the quantum mechanical transmission coefficient associated to the scattering process.

Tricky!!!!!!!

a solution to the antimatter pb

4. Classical evolution from pre- to post-big bang

A classical description of the transition, in terms of an exact solution connecting in a smooth way the pre- and post-big bang regime, is known to be excluded in the context of the lowest-order string effective action, for any local (and realistic) dilaton potential[17]. It may be allowed, however, when higher order corrections to

GV

the action are taken into account.

In string theory there are two types of higher order corrections: higher-derivative terms, appearing in the so-called α'-expansion[9], and and higher loops in the string coupling parameter $g_s = e^{\phi/2}$. The first is a truly "stringy" effect, due to the presence of the minimal, fundamental length $\lambda_s = (\alpha')^{1/2}$; the second is a quantum effect, controlled by the expectation value of the dilaton field.

Both types of corrections are probably required for a complete and successful description of the transition. For pedagogical reasons, however, it is better to discuss their effects separately, starting with the α' corrections that are probably the first to come into play when the background evolves from the perturbative vacuum[10].

4.1. *Higher-derivative corrections*

1 alpha' corrections

To first order in α', i.e. including all terms required by the gravi-dilaton sector of string theory up to four derivatives in the tree-level action, the action can be written as[9]:

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|} e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{\alpha'}{4} \left(R^2_{GB} - (\nabla\phi)^4 \right) \right], \qquad (4.1)$$

where $R^2_{GB} \equiv R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the usual Gauss-Bonnet invariant. Note that there are no free parameters, except a possible number of order unity in front of the α' corrections, depending on the particular (super)string theory (bosonic, heterotic, ...) adopted. We have chosen here a convenient field-redefinition that eliminates higher-than-second derivatives from the equations of motion, but at the price of introducing dilaton-dependent α' corrections.

A numerical integration of the equations of motion (in any number of dimensions) shows that, already to this order in α', the effect of the higher-derivative terms is to contrast the growth of the curvature and of the dilaton, leading the background to a final regime in which the curvature is constant, and the dilaton is linearly growing (in cosmic time). This is clearly illustrated in Fig. 6, which shows the result of a numerical integration of the equations of motion, with the perturbative vacuum ($H = 0 = \dot{\phi}$) as initial conditions at $t \to -\infty$. The plot refers to the case $d = 3$, but the result is qualitatively the same[10] for any d.

It is important to stress that such a state of frozen curvature and linear dilaton may represent a solution to all orders in α' of the tree-level action. Indeed, the complete set of the string σ-model β-function equations, for a Bianchi-type I gravi-dilaton background, has been shown to reduce, to all orders, to an algebraic set of $d + 1$ equations in $d + 1$ unknowns, representing the (anisotropic) linear rate of growth of the log of the metric and of the string coupling[10]. The existence of real solutions for this system is a sufficient condition for the existence of an exact (in the sense of conformal field theory, i.e. to all orders in α') solution with constant curvature and linearly evolving dilaton.

CFT condition

It may be noted that a solution with $H = \text{const}$, $\dot{\phi} = \text{const}$, is in general allowed in many higher-derivative models of gravity. Such a fixed point of the cosmological

equations, however, is in general disconnected from the trivial fixed point $H = 0 = \dot{\phi}$ (the perturbative vacuum, in this case) by a singularity, or by an unphysical region in which the curvature becomes imaginary. For the action (4.1), on the contrary, the constant fixed point is a late time time attractor for all isotropic backgrounds emerging from the string perturbative vacuum[10].

This attraction property, unfortunately, is not invariant under field redefinition, as we have checked, until the action is truncated at a given finite order in α'. Also, and most important, the growth of the curvature is stopped, but the transition is not completely performed. Indeed, the final fixed point is in the post-big bang regime $\dot{\bar{\phi}} < 0$, as illustrated in Fig. 2, but the transition cannot proceed further towards the lower curvature regime, without including additional corrections or an appropriate dilaton potential in the effective action.

The reason for this incompleteness is that the action (4.1) is invariant under time reflections, but is not duality-invariant. As a consequence, what is regularized is the expanding pre-big bang branch of the lowest order solution, and its time-symmetric counterpart, the *contracting post-big bang* branch. This is illustrated in Fig. 7, where we plot the evolution in time of the curvature scale, according to a numerical integration of the equations of motion. The dashed curves are the four branches of the corresponding lowest-order, singular solution. The expanding post-big bang branch remains singular, and cannot be smoothly connected to the regularized pre-big bang branch. More corrections are to be added to the effective action.

4.2. *Loop corrections*

2 One loop corrections

One-loop corrections to the string cosmology equations have been discussed by various authors, in $D = 2$ and $D = 4$ spacetime dimensions. In two dimensions[18], the one-loop corrections to the so-called CGHS model[19] lead to the action

$$S = S_{\rm tree} + \frac{k}{2}\int d^2x\sqrt{-g}\left(R\Box^{-2}R + \epsilon\phi R\right), \tag{4.2}$$

where $S_{\rm tree}$ is the tree-level action, and k is a dimensionless parameter that depends on the number of conformal scalar field present in the model. Notice that the trace-anomaly term $R\Box^{-2}R$ has been supplemented by the local covariant counterterm $\epsilon\phi R$, that one is free to add to preserve classical symmetries[20].

In four dimensions, the dimensionally reduced one-loop action can be written as[21]

$$S = S_{\rm tree} - \alpha'\delta\int d^4x\sqrt{-g}\xi(\sigma)F\left[R^2, (\partial_\mu\phi)^4, R(\partial_\mu\phi)^2, ...\right] \tag{4.3}$$

where δ is a model-dependent number of the order of unity, $\xi(\sigma) \approx -(2\pi/3)\cosh\sigma$ is a function of the modulus field σ parametrizing the size of the internal compactified manifold, and F is a complicated four-derivative function of the metric and of the dilaton, that in the Einstein frame factorizes into the Gauss-Bonnet form times a dilaton-dependent function. The tree-level action, in eq. (4.3), includes the modulus field and the first order α' corrections.

a bounce in curvature!!!!!!!!!!!

Both the analytical and numerical solutions of the actions (4.2) and (4.3) show that the effect of loop corrections is to induce a bounce in the curvature[18,21]: the curvature grows, reaches a maximum and then decreases, with a typical "bell-like" shape which mimics the effects of a transition from pre- to post-big bang.

The dilaton, however, remains growing from $-\infty$ to $+\infty$. It is true that a monotonic evolution of the dilaton from weak to strong coupling is equivalent, via S-duality transformations, to a smooth interpolation between two different weak coupling regimes. The final value of the string coupling should go to a finite constant, however, and not to zero. To this aim the dilaton has to be stopped, and this probably requires the addition of a realistic, non-perturbative potential.

The problem with the cosmological solutions of the one-loop corrected action, at least for the cases that we have analyzed, is that the final background configuration is still characterized by $\dot{\overline{\phi}} > 0$. This means, in other words, that there is no branch changing: the background is still in the pre-big bang regime (in spite of the fact that the curvature is decreasing), and the dilaton is not ready to be attracted to any stable minimum of the potential[17].

remaining pb

We may thus conclude, to the best of our present understanding of string cosmology, that α' and loop corrections can separately implement different aspects of the transition from pre- to post-big bang. They are both required, however, together with an appropriate non-perturbative potential, for the description of a complete transition.

5. Conclusion

String theory can provide (at least in principle) a consistent description of the Universe at all curvature scale. The duality symmetries of string theory, in particular, suggest a cosmological scenario in which the Universe evolves from the string perturbative vacuum, through a pre-big bang phase characterized by an accelerated growth of the curvature and of the string coupling (controlled by the dilaton).

I do not conceal that what I have presented here is no more than an approximate sketch of a complete and possibly realistic cosmological scenario. A lot of work is still needed to clarify all the aspects of this scenario, and in particular the dynamical details of the transition to the standard cosmological regime.

It seems to me, however, that such a work is worth to be done, because the study of this pre-big bang scenario, and of the associated non-standard phenomenology[1,22], may provide an efficient way to test string theory as well as alternative models of Planck scale physics.

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1. M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993); Mod. Phys. Lett. A8, 3701 (1993); Phys. Rev. D50, 2519 (1994). An updated collection of papers on the pre-big bang scenario is available at `http://www.to.infn.it/teorici/gasperini/`.
2. C. Lovelace, Phys. Lett. B135, 75 (1984); E. S. Fradkin and A. A Tseytlin, Nucl. Phys. B261, 1 (1985); C. G. Callan et al., Nucl. Phys. B262, 593 (1985); A. Sen, Phys. Rev. Lett. 55, 1846 (1985).
3. E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. D50, 4868 (1994); R. Easther, K. Maeda and D. Wands, Phys. Rev. 53, 4247 (1996); M. S. Turner and E. J. Weinberg, *Pre-big bang inflation requires fine tuning*, hep-th/9705035.
4. G. Veneziano, *Inhomogeneous pre-big bang string cosmology*, hep-th/9703150.
5. G. Veneziano, Phys. Lett. B265, 287 (1991).
6. M. Gasperini, J. Maharana and G. Veneziano, Nucl. Phys. B472, 349 (1996).
7. B. S. De Witt, Phys. Rev. 160, 1113 (1967); J. A. Wheeler, in *Battelle Rencontres*, ed. by C. De Witt and J. A. Wheeler (Benjamin, New York, 1968).
8. M. Gasperini and G. Veneziano, Gen. Rel. Grav. 28, 1301 (1996).
9. See for instance R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B293, 385 (1987).
10. M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B494, 315 (1997).
11. A. Vilenkin, Phys. Rev. D30, 509 (1984); D33, 3560 (1986); D37, 888 (1988).
12. A. D. Linde, Sov. Phys. JETP 60, 211 (1984); Lett. Nuovo Cimento 39, 401 (1984); Y. B. Zel'dovich and A. A. Starobinski, Sov. Astron. Lett. 10, 135 (1984); V. A. Rubakov, Phys. Lett. B148, 280 (1984).
13. M. Cavagliá and V. De Alfaro, Gen. Rel. Grav. (1997), in press (gr-qc/9605020).
14. K. A. Meissner and G. Veneziano, Mod. Phys. Lett. A6, 3397 (1991); Phys. Lett. B267, 33 (1991); M. Gasperini and G. Veneziano, Phys. Lett. B277, 256 (1992).
15. A. Buonanno, M. Gasperini, M. Maggiore and C. Ungarelli, Class. Quantum Grav. 14, L97 (1997).
16. M. Gasperini, in *Proc. of the 2th Int. Workshop on "Fundamental physics at the birth of the Universe"*, Rome, May 1997 (hep-th/9706049).
17. R. Brustein and G. Veneziano, Phys. Lett. B329, 429 (1994); N. Kaloper, R. Madden and K. Olive, Nucl. Phys. B452, 677 (1995); Phys. Lett. B371, 34 (1996); R. Easther, K. Maeda and D. Wands, Ref. 3.
18. S. J. Rey, Phys. Rev. Lett. 77, 1929 (1996); M. Gasperini and G. Veneziano, Phys. Lett. B387, 715 (1996).
19. C. Callan, S. Giddings, J. Harvey and A. Strominger, Phys. Rev. D45, R1005 (1992).
20. A. Bilal and C. Callan, Nucl. Phys. B394, 73 (1993); S.de Alwis, Phys. Lett. B289, 278 (1992); B300, 330 (1993); J. G. Russo, L. Susskind and L. Thorlacius, Phys. Rev. D46, 3444 (1992); D47, 533 (1993).
21. I. Antoniadis, J. Rizos and K. Tamvakis, Nucl. Phys. B415, 497 (1994); J. Rizos and K. Tamvakis, Phys. Lett. B326, 57 (1994); R. Easther and K. Maeda, Phys. Rev. D54, 7252 (1996).
22. See for instance M. Gasperini, *Relic gravitons from the pre-big bang: what we know and what we do not know*, hep-th/9607146; and *Relic dilatons in string cosmology*, gr-qc/9611059.

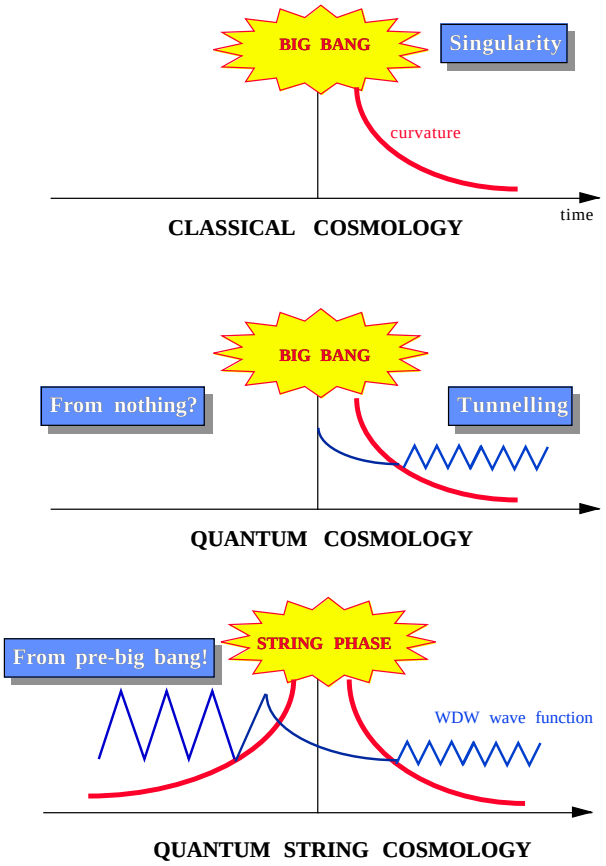

Figure 3: *A qualitative comparison of classical cosmology, quantum cosmology with tunnelling boundary conditions, and quantum string cosmology for the pre-big bang scenario.*

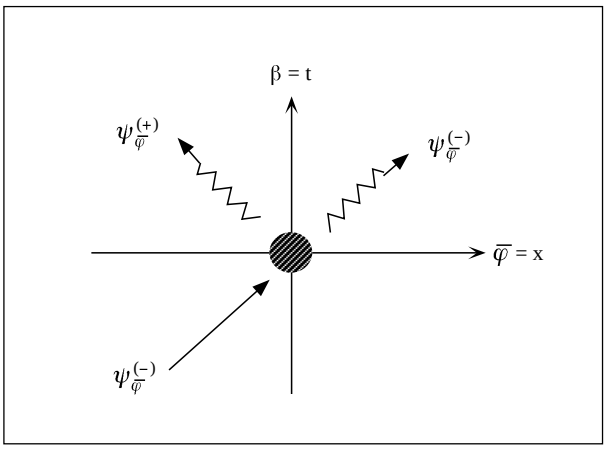

Figure 4: *Transition from pre- to post-big bang represented as a spacelike reflection of the wave function, in the minisuperspace spanned by β and $\overline{\phi}$.*

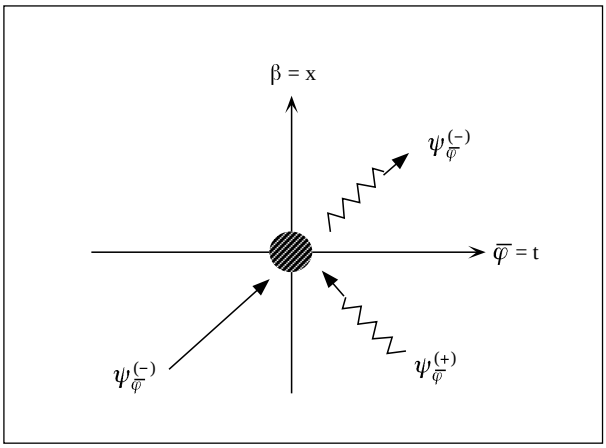

Figure 5: *Production from the vacuum of a universe–anti-universe pair, one expanding, the other contracting.*

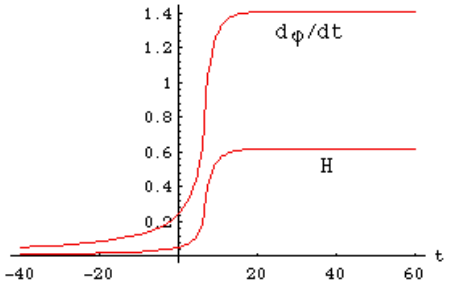

Figure 6: *Numerical integration of the equations of motion for the action (4.1), in $d = 3$, and with pre-big bang initial conditions.*

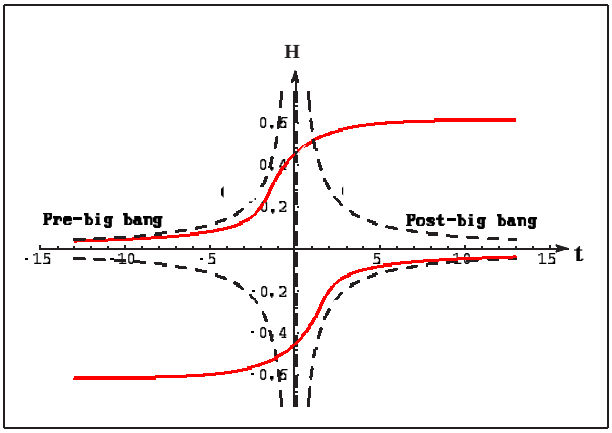

Figure 7: *Time-reversal symmetry of the regularized solution (the dashed curves represent the four branches of the lowest order, singular solution).*

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Primordial Magnetic Fields from String Cosmology

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Abstract

Sufficiently large seeds for generating the observed (inter)galactic magnetic fields emerge naturally in string cosmology from the amplification of electromagnetic vacuum fluctuations due to a dynamical dilaton background. The success of the mechanism depends crucially on two features of the so-called pre-big-bang scenario, an early epoch of dilaton-driven inflation at very small coupling, and a sufficiently long intermediate stringy era preceding the standard radiation-dominated evolution.

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It is widely believed that the observed galactic (and intergalactic) magnetic fields, of microgauss strength, are generated and maintained by the action of a cosmic dynamo [1]. Any dynamo model, however, requires a primordial seed field; in spite of many attempts [2]-[5], it is fair to say that no compelling mechanism able to generate the required seed field (coherent over the Mpc scale, and with an energy density to radiation density ratio $\rho_B/\rho_\gamma \gtrsim 10^{-34}$) has yet been suggested.

A priori, an appealing mechanism for the origin of the seed field is the cosmological amplification of the vacuum quantum fluctuations of the electromagnetic field, the same kind of mechanism which is believed to generate primordial metric and energy density perturbations [6]. The minimal coupling of photons to the metric background is, however, conformally invariant (in $d = 3$ spatial dimensions). As a consequence, a cosmological evolution involving a conformally flat metric (as it is effectively the case in inflation) cannot amplify magnetic fluctuations, unless conformal invariance is broken. Possible attempts to generate large enough seeds thus include considering exotic higher-dimensional scenarios, or coupling non-minimally the electromagnetic field to the background curvature [2] with some "ad hoc" prescription, or breaking conformal invariance at the quantum level through the so-called trace anomaly [4].

In critical superstring theory the electromagnetic field $F_{\mu\nu}$ is coupled not only to the metric ($g_{\mu\nu}$), but also to the dilaton background (ϕ). In the low energy limit such interaction is represented by the string effective action [7], which reads, after reduction from ten to four external dimensions,

$$S = -\int d^4x\sqrt{-g}e^{-\phi}(R + \partial_\mu\phi\partial^\mu\phi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}) \tag{1}$$

were $\phi = \Phi - \ln V_6 \equiv \ln(g^2)$ controls the tree-level four-dimensional gauge coupling (Φ being the ten-dimensional dilaton field, and V_6 the volume of the six-dimensional compact internal space).

In the inflationary models based on the above effective action [8, 9] the dilaton background is not at all constant, but undergoes an accelerated evolution from the string perturbative vacuum ($\phi = -\infty$) towards the strong coupling regime, where it is expected to remain frozen at its present value. In this context, the quantum fluctuations of the electromagnetic field

can thus be amplified *directly* through their coupling to the dilaton, according to eq.(1). In the following we will discuss the conditions under which such mechanism is able to produce large enough primordial magnetic fields to seed the galactic dynamo (a scalar-vector coupling similar to that of eq.(1) was previously discussed in [3], but ϕ was there identified with the conventional inflaton undergoing a dynamical evolution much different from the dilaton evolution considered here).

Let us first define a few important parameters of the inflationary scenario (also called "pre-big-bang" scenario) discussed in [9]. The phase of growing curvature and dilaton coupling ($\dot{H} > 0$, $\dot{\phi} > 0$), driven by the kinetic energy of the dilaton field, is correctly described in terms of the lowest order string effective action only up to the conformal time $\eta = \eta_s$ at which the curvature reaches the string scale $H_s = \lambda_s^{-1}$ ($\lambda_s \equiv \sqrt{\alpha'}$ is the fundamental length of string theory). A first important parameter of this cosmological model is thus the value ϕ_s attained by the dilaton at $\eta = \eta_s$. Provided such value is sufficiently negative (i.e. provided the coupling $g = e^{\phi/2}$ is sufficiently small to be still in the perturbative region at $\eta = \eta_s$), it is also arbitrary, since there is no perturbative potential to break invariance under shifts of ϕ. For $\eta > \eta_s$ high-derivatives terms (higher orders in α') become important in the string effective action, and the background enters a genuinely "stringy" phase of unknown duration. It was shown in [10] that it is impossible to have a graceful exit to standard cosmology without such an intermediate stringy phase. An assumption of string cosmology is that the stringy phase eventually ends at some conformal time η_1 in the strong coupling regime. At this time the dilaton, feeling a non-trivial potential, freezes to its present constant value $\phi = \phi_1$ and the standard radiation-dominated era starts. The total duration η_1/η_s, or the total red-shift z_s encountered during the stringy epoch (i.e. between η_s and η_1), will be the second crucial parameter besides ϕ_s entering our discussion. For the purpose of this paper, two parameters are enough to specify completely our model of background, if we accept that during the string phase the curvature stays controlled by the string scale, that is $H \simeq g M_p = \lambda_s^{-1}$ (M_p is the Planck mass) for $\eta_s < \eta < \eta_1$.

We will work all the time in the String (also called Brans-Dicke) frame, in which test strings move along geodesic surfaces. In this frame the string scale λ_s is constant, while

the Planck scale $\lambda_p = e^{\frac{\phi}{2}}\lambda_s$ grows from zero (at the initial vacuum) to its present value, reached at the end of the string phase. We have explicitly checked that all our results also follow in the more commonly used (but less natural in a string context) Einstein frame, in which the gravi-dilaton action is diagonalized in standard canonical form. Moreover, our parameterization in terms of ϕ_s and η_1/η_s is completely frame-independent, since the dilaton field and the conformal time coordinate are the same in the String and Einstein frame [9].

We shall now consider, in the above background, the amplification of the quantum fluctuations of the electromagnetic field, assuming that, at the very beginning, it was in its vacuum state. In a four-dimensional, conformally flat background, the Fourier modes A_k^μ of the (correctly normalized) variable corresponding to the standard electromagnetic field, and obeying canonical commutation relations, satisfy the equation

$$A_k'' + [k^2 - V(\eta)]A_k = 0 \quad , \quad V(\eta) = g(g^{-1})'' \quad , \quad g(\eta) \equiv e^{\frac{\phi}{2}} \tag{2}$$

This equation is valid for each polarization component, and is obtained from the action (1) with the gauge condition $\partial_\nu[e^{-\phi}\partial^\mu(e^{\frac{\phi}{2}}A^\nu)] = 0$ (a prime denotes differentiation with respect to η). Note the analogy with the equation for the tensor part of metric perturbations [6]. The latter have the same form as (2) with the inverse of the string coupling, g^{-1}, replaced simply by the Einstein-frame scale factor $a_E = ae^{-\phi/2}$, expressed in conformal time. Equation (2) clearly shows the absence of parametric amplification whenever ϕ is constant.

The effective potential $V(\eta)$ grows from zero like η^{-2}, for $\eta \to 0_-$, in the phase of dilaton-driven inflation (where [8, 9] $\phi \sim -\sqrt{3}\ln|\eta|$), is expected to reach some maximum value during the string phase, and then goes rapidly to zero at the beginning of the radiation-dominated era (where $\phi = $ const). The approximate solution of eq. (2), for a mode k "hitting" the effective potential barrier at $\eta = \eta_{ex}$, and with initial conditions corresponding to vacuum fluctuations, is given by:

$$\begin{aligned}
A_k &= \frac{e^{-ik\eta}}{\sqrt{k}} \quad , \qquad\qquad \eta < \eta_{ex} \\
A_k &= g^{-1}(\eta)[C_k + D_k \int^\eta d\eta' \;\; g^2(\eta')] \quad , \quad \eta_{ex} < \eta < \eta_{re} \\
A_k &= \frac{1}{\sqrt{k}}[c_+(k)e^{-ik\eta} + c_-(k)e^{ik\eta}] \quad , \quad \eta > \eta_{re}
\end{aligned} \tag{3}$$

where η_{ex} and η_{re} are the times of exit and reentry of the comoving scale associated with k, defined by the conditions $k^2 = |V(\eta_{ex})| = |V(\eta_{re})|$ ($C, D, c_\pm$ are integrations constants). We are following here the usual convention for which a mode in the underbarrier region is referred to, somewhat improperly, as being "outside the horizon". Hence the names "exit" and "reentry". Moreover, we are considering a background in which the potential $V(\eta)$ keeps growing in the string phase until the final time η_1, so that a mode crossing the horizon during dilaton-driven inflation remains outside the horizon during the whole string phase, i.e. $\eta_{re} \geq \eta_1$.

The Bogoliubov coefficients $c_\pm(k)$, determining the parametric amplification of a mode $k < |V(\eta_1)|$, are easily determined by matching these various solutions. One finds:

$$\begin{aligned} 2ike^{-ik(\eta_{ex}\mp\eta_{re})}c_\pm = & \mp \frac{g_{ex}}{g_{re}}(-\frac{g_{re}{}'}{g_{re}} \mp ik) \pm \frac{g_{re}}{g_{ex}}(-\frac{g_{ex}{}'}{g_{ex}} + ik) \pm \\ & \pm \frac{1}{g_{ex}g_{re}}(-\frac{g_{ex}{}'}{g_{ex}} + ik)(-\frac{g_{re}{}'}{g_{re}} \mp ik)\int_{\eta_{ex}}^{\eta_{re}} g^2 d\eta \end{aligned} \tag{4}$$

If we remember that reentry occurs during the radiation epoch in which the dilaton freezes to a constant value ($g'_{re} \simeq 0$), it is easy to estimate the complicated looking expression (4) and to obtain, for the leading contribution, the amazingly simple and intuitive result:

$$|c_-| \simeq \frac{g_{re}}{g_{ex}} \equiv e^{-\frac{1}{2}(\phi_{ex}-\phi_{re})} \tag{5}$$

expressing the fact that the amplification of the electromagnetic field depends just (to leading order) on the ratio of the gauge couplings at reentry and at exit.

The coefficient (5) defines the energy density distribution ($\rho_B(\omega)$) over the amplified fluctuation spectrum, $d\rho_B/d\ln\omega \simeq \omega^4|c_-(\omega)|^2$, where $\omega = k/a$ is the red-shifted, present value of the amplified proper frequency. We are interested in the ratio

$$r(\omega) = \frac{\omega}{\rho_\gamma}\frac{d\rho_B}{d\omega} \simeq \frac{\omega^4}{\rho_\gamma}|c_-(\omega)|^2 \simeq \frac{\omega^4}{\rho_\gamma}\left(\frac{g_{re}}{g_{ex}}\right)^2 \tag{6}$$

measuring the fraction of electromagnetic energy stored in the mode ω (in particular, for the intergalactic scale, $\omega_G \simeq (1Mpc)^{-1} \simeq 10^{-14}$Hertz), relative to the background radiation energy ρ_γ.

The ratio $r(\omega)$ stays constant during the phase of matter-dominated as well as radiation-dominated evolution, in which the universe behaves like a good electromagnetic conductor [2]. In terms of $r(\omega)$ the condition for a large enough magnetic field to seed the galactic dynamo is [2]

$$r(\omega_G) \gtrsim 10^{-34} \tag{7}$$

Using the known value of ρ_γ and $e^{\phi_{re}}$ we thus find, from eqs.(6, 7):

$$g_{ex}(\omega_G) < 10^{-33} \tag{8}$$

i.e. a very tiny coupling at the time of exit of the (inter)galactic scale ω_G (we may note, incidentally, that it looks very unlikely that such a large ratio of g_{re}/g_{ex} can be obtained through the trace anomaly mechanism of Ref. [4]).

In order see whether or not the previous condition can be fulfilled we go back to our two-parameter cosmological model. We recall that, in the string phase, the curvature stays controlled by the string scale, so that $H \simeq \lambda_s^{-1} =$ constant in the String frame, and $a_1/a_s = \eta_s/\eta_1 = z_s$. The discussion is greatly helped by looking at **Fig.1** where we plot, on a double-logarithmic scale against the scale factor a, the evolution of the coupling strength (i.e. of $e^{\phi/2}$) and that of the "horizon" size (defined here by $a|V|^{-1/2}$, whose behavior coincides with that of the Hubble radius H^{-1} during the dilaton driven epoch). The horizon curve has an inverted trapezoidal shape, corresponding to the fact that $V = 0$ during the radiation era, that $\dot{\phi}$ and H are approximately constant during the string era, and that, during the dilaton-driven era [8, 9],

$$a = (-t)^\alpha, \qquad \alpha = -\frac{1}{\sqrt{3}}\sqrt{1-\Sigma}, \qquad a|V|^{-\frac{1}{2}} \simeq a(t)\int_t^0 dt' a^{-1}(t') \simeq a^{\frac{1}{\alpha}} \tag{9}$$

Here $\Sigma \equiv \sum_i \beta_i^2$ represents the possible effect of internal dimensions whose radii b_i shrink like $(-t)^{\beta_i}$ for $t \to 0_-$ (for the sake of definiteness we show in the figure the case $\Sigma = 0$). The shape of the coupling curve corresponds to the fact that the dilaton is constant during the radiation era, that $\dot{\phi}$ is approximately constant during the string era, and that it evolves like

$$g(\eta) = a^\lambda, \qquad \lambda = \frac{1}{2}(3 + \frac{\sqrt{3}}{\sqrt{1-\Sigma}}) \tag{10}$$

during the dilaton-driven era [8, 9] ($\Sigma = 0$ is the case shown in the picture).

We can now easily see when a sufficient amplification is achieved. The galactic scale of length ω_G^{-1} was about 10^{25} in string (or Planck) units at the beginning of the radiation era. By definition, at earlier times it evolves as a straight line with a positive slope on our plot and thus inevitably hits the horizon curve sometimes during the string or the dilaton-driven era. At that time, the value of g should have been smaller than 10^{-33}. One can easily convince her/himself that this is all but impossible provided: i) $z_s = a_1/a_s$ is a sufficiently large, and ii) the dilaton evolution during the string era is sufficiently fast. For the first condition a red-shift z_s of 10^{10} is necessary, while for the average ratio $\dot{\phi}/H$ during the string era a value below but not too far from the one just before η_s is sufficient. The combination of i) and ii) also implies that the coupling at the onset of the string era has to be smaller that 10^{-20} or so, which thus supports the scenario advocated in [10] for the gracious exit problem.

We can express our results more quantitatively by showing the allowed region in the $g_s - z_s$ plane in order to have sufficiently large seeds. Considering the possibility of galactic scale exit during the string or the dilaton-driven phase, we find from eq.(6) that $r(\omega_G)$ can be expressed, in the two cases, respectively as:

$$r(\omega_G) \simeq \left(\frac{\omega_G}{\omega_1}\right)^4 e^{-\phi_{ex}(\omega_G)} \simeq \left(\frac{\omega_G}{\omega_1}\right)^{4+\frac{\phi_s}{\ln z_s}} \quad , \quad \omega_s < \omega_G < \omega_1 \tag{11}$$

and

$$r(\omega_G) \simeq \left(\frac{\omega_G}{\omega_1}\right)^{4-2\gamma} z_s^{-2\gamma} e^{-\phi_s} \quad , \quad \omega_G < \omega_s \tag{12}$$

where $\gamma = \lambda\alpha/(\alpha - 1)$, $\omega_1 = H_1 a_1/a \simeq 10^{11}$Hertz is the maximal amplified frequency, and $\omega_s = \omega_1/z_s$. In the previous formulae (11), (12) we used the fact that, according to our model of background, the transition scale H_1 has to be of the order of the Planck mass M_p, so that $\rho_\gamma(t) \simeq H_1^4[a_1/a(t)]^4 = {\omega_1}^4$.

The resulting limits obtained by imposing eq.(7) are plotted in **Fig.2**, where they provide the right-side border of the allowed region (the shaded area). According to the previous spectrum, however, the amplitude of the electromagnetic perturbations is not constant outside the horizon, but tends to grows, asymptotically, with a power-like behavior (similar effects are known to occur also for metric perturbations in string cosmology backgrounds [11]). A

further bound on the allowed region thus emerges in the context of this model, as we must require for consistency that our perturbative treatment of the vacuum fluctuations remains valid at all the times, or, in other words, that the energy density of the produced photons is smaller than critical also in the radiation era, i.e.

$$r(\omega) < 1 \tag{13}$$

at all ω. For electromagnetic perturbations crossing the horizon during the dilaton-driven period, in particular, this condition is most stringent at $\omega = \omega_s$.

Using eqs. (11), (12) and (13) we find that the two different branches of the photon spectrum give the same bounds, $g_s < 1$ and $\log_{10} g_s > -2 \log_{10} z_s$, the latter determining the left border of the allowed region. Actually, a slightly tighter bound ($r \lesssim 0.5$) could also be imposed in order to avoid perturbations of the standard nucleosynthesis scenario [12], but this does not affect in significant way the result presented in **Fig.2**, which should be regarded, in any case, as an order of magnitude estimate of the allowed region.

We want also to mention that in the described scenario is not only possible to seed the galactic dynamo (according to eq.(7)), but also to produce directly fields of the same strength as that of the galactic magnetic field, by requiring [2] $r(\omega_G) \simeq 10^{-8}$. In our case this is achieved if $z_s > 10^{21}$ and if the coupling is sufficiently tiny at the end of the dilaton driven era, $\log_{10} g_s < -43$. The possible impact of such a small value on the problem of freezing out the classical oscillations of the dilaton background, in the presence of a "realistic" supersymmetry breaking, non-perturbative potential, is left as an interesting subject for future research.

We want to recall, finally, that our results were obtained in the framework of the tree-level, string effective lagrangian. We know that we could have corrections coming either from higher loops (expansion in e^{ϕ}) or from higher curvature terms (α' corrections). Since we work in a range of parameters where the dilaton is deeply in his perturbative regime we expect our results to be stable against loop corrections, at least for scales leaving the horizon during the dilaton driven phase. As to the α' corrections, they are instead invoked in the basic assumption that the dilaton-driven era ends when the curvature reaches the string scale λ_s^{-2}, and leads to a quasi-de Sitter epoch. It should be clear, nevertheless, that

the basic ideas of our mechanism and its implementation do not depend too strongly on the detailed features of the string epoch.

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Note added

While this paper was being written, we received a paper by D. Lemoine and M. Lemoine, "Primordial magnetic fields in string cosmology", whose content overlap with ours for what concerns the effects of dilaton-driven inflation on the amplification of electromagnetic perturbations. Their model of background does not include, however, a sufficiently long, intermediate stringy era whose presence is instead crucial to produce the large amplification discussed here.

References

[1] E. N. Parker, *Cosmical Magnetic fields* (Clarendon, Oxford, England, 1979);

Y. B. Zeldovich, A. A.Ruzmaikin and D. D. Sokoloff, *Magnetic fields in astrophysics* (Gordon and Breach, New York, 1983)

[2] M. S. Turner and L. M. Widrow, Phys. Rev. D37, 2743 (1988).

[3] B. Rathra, Astrophys. J. Lett. 391, L1 (1992)

[4] A. D. Dolgov, Phys. Rev. D48, 2499, (1993).

[5] E. R. Harrison, Phys. Rev. Lett. 30, 188 (1973);

T. Vachaspati, Phys. Lett. B265, 258 (1991);

W. D. Garretson, G. B. Field and S. M. Carrol, Phys. Rev. D46, 5346 (1992);

B. Cheng, and A. V. Olinto, Phys. Rev. D50, 2421 (1994).

[6] L. P. Grishchuk, Sov. Phys. JEPT 40, 409 (1975);

A. A. Starobinski, JEPT Lett. 30, 682 (1979)

[7] C. Lovelace, Phys. Lett. B135, 75 (1984);

E. S. Fradkin and A. A Tseytlin, Nucl. Phys. B261, 1 (1985);

C. G. Callan et al., Nucl. Phys. B262, 593 (1985)

[8] G. Veneziano, Phys. Lett. B265, 287 (1991);

M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993);

M. Gasperini and G. Veneziano, Mod. Phys. Lett. A8, 3701 (1993)

[9] M. Gasperini and G. Veneziano, Phys. Rev. D50, 2519 (1994)

[10] R. Brustein and G. Veneziano, Phys. Lett. B329, 429 (1994)

[11] R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov and G. Veneziano, Phys. Rev. D (1995) (in press), CERN-TH. 7544/94 (December 1994)

[12] B. Cheng, D. N. Schramm and J. W. Truran, Phys. Rev. D49, 5006 (1994)

Figure captions

Fig. 1

Plot showing the evolution of the horizon scale H^{-1} (thick lines) and of the coupling scale $g = e^{\phi/2}$ (dashed lines) while the galactic scale ω_G^{-1} (thin line) was beyond the horizon. Two case with a different z_s are compared, showing that, for a sufficiently fast variation of the dilaton during the string era, a larger z_s helps achieving the bound $g_{ex}(\omega_G) < 10^{-33}$.

Fig. 2

The shaded area represents the allowed region determined by the conditions (7) and (13), and defines the values of z_s, g_s compatible with an amplification of the electromagnetic vacuum fluctuations large enough to seed the galactic magnetic field.

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Electromagnetic Origin of the CMB Anisotropy in String Cosmology

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Abstract

In the inflationary scenarios suggested by string theory, the vacuum fluctuations of the electromagnetic field can be amplified by the time-evolution of the dilaton background, and can grow large enough to explain both the origin of the cosmic magnetic fields and of the observed CMB anisotropy. The normalization of the perturbation spectrum is fixed, and implies a relation between the perturbation amplitude at the COBE scale and the spectral index n. Working within a generic two-parameter family of backgrounds, a large scale anisotropy $\Delta T/T \simeq 10^{-5}$ is found to correspond to a spectral index in the range $n \simeq 1.11 - 1.17$.

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In the standard inflationary scenario, the anisotropy of the Cosmic Microwave Background (CMB) recently detected by COBE [1, 2] is usually attributed to the cosmological amplification of the quantum fluctuations of the metric. These consist of both tensor (gravitational waves) and scalar perturbations, the latter being coupled to the energy density fluctuations. The observed inhomogeneities of the CMB radiation could also emerge from the vacuum fluctuations of the electromagnetic radiation itself, through their contribution to $\delta\rho/\rho$. However, the minimal coupling of photons to the metric is conformally invariant in $d = 3$ spatial dimensions, and it is difficult, in general, to obtain a significant amplification of the electromagnetic fluctuations in the context of the standard inflationary scenario [3].

In the inflationary models based on the low energy limit of critical superstring theory [4, 5, 6], the electromagnetic field $F_{\mu\nu}$ is coupled not only to the metric ($g_{\mu\nu}$) but also to the dilaton (ϕ) background, according to the dimensionally reduced, effective action [7]

$$S = -\int d^4x |det(g_{\mu\nu})|^{1/2} e^{-\phi}(R + \partial_\mu\phi\partial^\mu\phi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}) \qquad (1)$$

In this context ϕ, which controls the tree-level four-dimensional gauge coupling $g^2 = e^\phi$, is rapidly changing in time and can amplify directly the electromagnetic vacuum fluctuations [8, 9]. On the other hand, tensor metric perturbations, as well as the scalar perturbations induced by dilatonic fluctuations, are characterized in this context by "blue" spectra strongly tilted towards large frequencies [6, 10, 11], with an amplitude on large angular scales which is far too low to match COBE's observations. It becomes thus crucial, for the purpose of testing string theory through its astrophysical consequences [12], to decide whether or not the CMB anisotropy could originate from the vacuum quantum fluctuations of the electromagnetic field itself, after they have been amplified by the time-evolution of the dilaton background.

The purpose of this paper is to show that, in an appropriate range for two arbitrary parameters characterizing a generic string cosmology scenario, such an electromagnetic origin of the anisotropy is possible, is consistent with the linearization of perturbations around a nearly homogeneous background, and is also consistent with various phenomenological constraints (following from pulsar timing data and nucleosynthesis). In that range of parameters, moreover, the same mechanism that amplifies the electromagnetic vacuum fluctuations can also be responsible for the production of the observed galactic (and extragalactic) magnetic

fields [8], and can thus explain why the average energy density of the cosmic magnetic fields and of the CMB radiation are of the same order.

Let us consider the evolution of the quantum fluctuations of the electromagnetic field, according to the action (1). In a four-dimensional, conformally flat background, the Fourier modes A_k^μ of the (canonically normalized) electromagnetic variable satisfy the equation

$$A_k'' + [k^2 - V(\eta)]A_k = 0 \quad , \quad V(\eta) = g(g^{-1})'' \quad , \quad g(\eta) \equiv e^{\phi/2} \tag{2}$$

This equation is valid for each polarization component, and is obtained from the action (1) with the gauge condition $\partial_\nu[e^{-\phi}\partial^\mu(e^{\frac{\phi}{2}}A^\nu)] = 0$ (a prime denotes differentiation with respect to the conformal time η). Note the analogy with the tensor part of the metric perturbation equations [13], which has the same form as (2) with the inverse of the coupling, g^{-1}, replaced simply by the Einstein-frame scale factor $a_E = g^{-1}a$.

In our context $V(\eta)$ represents an effective potential barrier, approaching zero as $\eta \to \pm\infty$. A mode of comoving frequency k, "hitting" the barrier at the time $\eta = \eta_{ex}(k)$, is thus parametrically amplified just like in the case of tensor perturbations. The modulus of the Bogoliubov coefficient $|c(k)|$ describing this amplification turns out to be given, to leading order, by the ratio of the gauge coupling at reentry and at exit [8]

$$|c(k)| \simeq \frac{g_{re}}{g_{ex}} \equiv \exp\{-\frac{1}{2}[\phi(\eta_{ex}) - \phi(\eta_{re})]\} \tag{3}$$

where $\eta_{ex}(k)$ and $\eta_{re}(k)$ are defined by $k^2 = |V(\eta_{ex})| = |V(\eta_{re})|$. The Bogoliubov coefficient $c(\omega)$ defines the energy distribution $\rho(\omega)$ of the amplified fluctuation spectrum, through the relation $d\rho/d\ln\omega \simeq (\omega^4/16\pi^2)|c(\omega)|^2$, where $\omega(t) = k/a(t)$ is the red-shifted, present value of the amplified proper frequency. We are interested, in particular, in the ratio

$$r(\omega) = \frac{\omega}{\rho_{cmb}}\frac{d\rho}{d\omega} \simeq \frac{\omega^4}{16\pi^2\rho_{cmb}}\left[\frac{g_{re}(\omega)}{g_{ex}(\omega)}\right]^2 \tag{4}$$

measuring the fraction of electromagnetic energy stored in the mode ω, relative to the CMB energy density ρ_{cmb}.

In order to compute this ratio, we must use the explicit time evolution of the dilaton background, as predicted by the inflationary models based on the string effective action [5, 6].

In such models the dilaton undergoes an accelerated evolution from the string perturbative vacuum ($g = 0$, $\phi = -\infty$) towards the strong coupling regime, where it is expected to remain frozen at its present value ($g = g_1 = e^{\phi_1/2} = \text{const}$). The initial phase of growing curvature and dilaton coupling (also called "pre-big-bang" scenario [5, 6]) is driven by the kinetic energy of the dilaton field (with negligible contributions from the dilaton potential), and can be described in terms of the lowest order string effective action only up to the time $\eta = \eta_s$ at which the curvature reaches the string scale $H_s = \lambda_s^{-1} \equiv (\alpha')^{-1/2}$ [determined by the string tension $(\alpha')^{-1}$]. The value ϕ_s of the dilaton at $\eta = \eta_s$ is the first important parameter of our scenario. Provided such value is sufficiently negative it is also arbitrary, since we are still in the perturbative regime at $\eta = \eta_s$, and there is no perturbative potential to break invariance under shifts of ϕ.

For $\eta > \eta_s$, however, higher orders in α' become important in the string effective action, and the background enters a genuinely "stringy" phase of unknown duration, assumed to end at $\eta = \eta_1$ with a smooth transition to the standard radiation-dominated regime (where $\phi = \phi_1$ =const). As shown in [14], it is impossible to have a graceful exit to standard cosmology without such an intermediate stringy phase, after which the dilaton, feeling a non-trivial potential, is attracted to its present constant value. The total red-shift $z_s = a_1/a_s$ encountered during the stringy epoch, will be the second crucial parameter (besides ϕ_s) entering our discussion. For our purpose, two parameters are enough to specify completely our model of background, if we accept that during the string phase the curvature stays controlled by the string scale, that is $H \simeq \lambda_s^{-1}$ for $\eta_s < \eta < \eta_1$.

We will work in the so-called String frame, in which the string scale λ_s is constant, while the Planck scale $\lambda_p = \lambda_s e^{\phi/2}$ grows from zero (at the initial vacuum) to its present value $\lambda_p \simeq 10^{-19}(GeV)^{-1}$ reached at the end of the string phase. In the low energy phase driven by the dilaton, the dilaton evolution is exactly known [4, 5, 6] and is given, in the String frame, by

$$\phi = (3 + \sqrt{3}) \ln a + const = -\sqrt{3} \ln |\eta| + const, \qquad \eta < \eta_s \tag{5}$$

Internal dimensions affects slightly the above numerical constants without affecting the results of this paper. During the string phase the curvature stays constant ($H \simeq \lambda_s^{-1}$) so that,

in this frame, the string epoch is characterized by a de Sitter-like evolution of the metric background, with $\eta_s/\eta_1 = a_1/a_s \equiv z_s$. In addition, the rate of growth of the coupling during the string phase should also be bounded by the string scale (like the space-time curvature). Defining $\dot{\phi} \simeq 2\beta H \simeq 2\beta\lambda_s^{-1}$, where β is some constant of order of unity [for instance, eq.(5) gives $\beta \simeq 2.37$ in the dilaton driven epoch], the "average" time behaviour of the dilaton between η_s and η_1 can be parameterized as

$$\phi = -2\beta \ln|\eta| + const, \qquad \beta = -\frac{(\phi_s - \phi_1)}{2\ln z_s}, \qquad \eta_s < \eta < \eta_1 \tag{6}$$

For this model of background, the effective potential $V(\eta)$ of eq.(2) grows like η^{-2} for $\eta \to 0_-$ in the dilaton-driven phase, keeps growing during the string phase , where it reaches a maximal value $\sim \eta_1^{-2}$ around the final time η_1, and then goes rapidly to zero at the beginning of the radiation dominated era, where $g(\eta) = g_1$ =const. A mode hitting the barrier (or "crossing the horizon") during the dilaton phase thus remains under the barrier during the whole string phase. As a consequence, $\eta_{re} > \eta_1$ and $\phi_{re}(\omega) = \phi_1$ =const for all ω.

The spectral distribution $r(\omega)$ is now completely fixed in terms of our two parameters ϕ_s and z_s. Using $\rho_{cmb}(\eta_1) \simeq M_p^2 H_1^2 \simeq g_1^{-2} H_1^4$, ($M_p$ being the present value of the Plank Mass), eq.(4) leads simply to

$$r(\omega) \simeq \frac{g_1^2}{16\pi^2}\left(\frac{\omega}{\omega_1}\right)^4 e^{-\Delta\phi_{ex}(\omega)} \tag{7}$$

where $\Delta\phi_{ex}(\omega) = \phi_{ex}(\omega) - \phi_1$, and $\omega_1 = H_1 a_1/a \simeq (g_1/4\pi)^{1/2} 10^{11}$Hz is the maximal amplified frequency (the amplitude of modes $\omega > \omega_1$ is exponentially suppressed, and will be neglected throughout this paper). For modes $\omega > \omega_s \equiv \omega_1/z_s$, crossing the horizon during the string phase, we thus obtain the spectrum

$$r(\omega) \simeq \frac{g_1^2}{16\pi^2}\left(\frac{\omega}{\omega_1}\right)^{4-2\beta} \qquad , \quad \omega_s < \omega < \omega_1 \tag{8}$$

For modes crossing the horizon in the dilaton phase ($\omega < \omega_s$) we have instead, from eqs.(5,6,7),

$$r(\omega) \simeq \frac{g_1^2}{16\pi^2}\left(\frac{\omega}{\omega_1}\right)^{4-\sqrt{3}} z_s^{-\sqrt{3}} e^{-\phi_s} \qquad , \quad \omega < \omega_s \tag{9}$$

The above electromagnetic spectrum has been obtained by using a homogeneous and isotropic model of metric (and dilaton) background. It is thus valid provided the amplified

fluctuations remain, at all times, small perturbations of a nearly homogeneous configuration, with a negligible back-reaction on the metric. This requires $r(\omega)\lesssim 1$, at all ω. This bound is satisfied, for $(g_1/4\pi)\lesssim 1$, provided

$$(g_s/g_1)\gtrsim z_s^{-2} \tag{10}$$

which restricts the allowed region in the two-dimensional parameter space (z_s, g_s) of our background model.

Consider now a length scale ω^{-1} reentering the horizon in the radiation era. The associated electromagnetic perturbation represents, at the time of reentry, a coherent field over the horizon scale which, consistently with the bound (10), could be strong enough to seed the galactic dynamo mechanism, or the galactic magnetic field itself [8]. Soon after reentry, however, the perturbation may be expected to thermalize and homogenize rapidly since, unlike metric (scalar and tensor) perturbations, photons are not decoupled from matter in the radiation era, and the shape of their spectrum remains frozen only outside the causal horizon.

For all scales reentering the horizon after the decoupling epoch, however, the electromagnetic perturbations can contribute to the inhomogeneity of the CMB radiation, with a spectral distribution $\rho = r(\omega)$ determined by eqs.(8,9). Such a spectrum grows with frequency, with the position of its peak fixed in the plane (ω, r) (i.e. $r \simeq g_1^2/16\pi^2$ at $\omega \simeq \omega_1$). The perturbation amplitude $r(\omega)$ at a given scale ω can thus be uniquely determined as a function of the unknown duration and slope of the "stringy" branch (8) of the spectrum, namely in terms of the two parameters z_s, g_s.

A large enough perturbation to match COBE's observations [1, 2], $\Delta T/T \simeq 10^{-5}$ at the present horizon scale ω_0 would require a spectral energy density such that, in critical units, $\Omega(\omega_0) \equiv \rho_c^{-1}[d\rho(\omega)/d\ln\omega]_{\omega=\omega_0} \simeq 10^{-10}$. In terms of our variable $r(\omega)$ this implies

$$r(\omega_0) \simeq 10^{-6}, \qquad\qquad \omega_0 \simeq 10^{-18} Hz \tag{11}$$

If the scale ω_0 crossed the horizon during the dilaton phase (i.e. if $z_s \lesssim 10^{29}$), this condition is compatible with eq.(10) only in a very small region of parameter space. In such case an electromagnetic origin of the CMB anisotropy is possible, but requires a certain degree of

fine-tuning. If, on the contrary, the horizon crossing of ω_0 occurred during the string phase ($z_s \gtrsim 10^{29}$), and we define as usual the spectral index n for eq.(8) as $n-1 = 4-2\beta$, then the present electromagnetic contribution to the anisotropy at the scale ω_0 can be written as

$$\log_{10} r(\omega_0) \simeq -29(n-1) + \frac{1}{2}(5-n)\log_{10}(\frac{g_1}{4\pi}), \qquad z_s \gtrsim 10^{29} \tag{12}$$

This equation (which is the main result of this paper) relates the perturbation amplitude at the scale ω_0 to the spectral index n, and provides a condition on the parameter space (z_s, ϕ_s) which is always compatible with eq.(10), as $(g_1/4\pi) \leq 1$. The requirement (11), in particular, is satisfied for

$$n \simeq \frac{35 + \frac{5}{2}\log_{10}(g_1/4\pi)}{29 + \frac{1}{2}\log_{10}(g_1/4\pi)} \tag{13}$$

Typical values of $(g_1/4\pi)^2$ range from 10^{-1} to 10^{-3} [15]. The COBE observations are thus accounted for, in this context, for values of the spectral index that are typically in the range $n \simeq 1.11 - 1.17$. Such a spectral index is certainly flat enough to be well consistent with the analysis of the first two years of the COBE DMR data [2]. It may be worth recalling that slightly growing ($n > 1$, also called "blue") spectra, like this, have been invoked [16] to explain the claimed bulk flow and large voids in the galaxy distribution, on scales of order 10^2Mpc (see also [17]). Our range of values, moreover, is consistent with the upper bound $n < 1.5$ recently obtained by using the COBE FIRAS limits on the CMB spectral distortions [18]. Note that, according to eq.(13), n depends very weakly on the precise value of g_1, so that our estimate is quite stable, in spite of the rather large theoretical uncertainties about g_1. Note also that the above value of n corresponds to an average value of $\dot{\phi}/2H \simeq \beta$ of about 1.9, which is not far from the value 2.37 characteristic of the dilaton-driven era.

In order to give some concrete estimate of the phenomenological bounds relevant for the problem we will set $g_1/4\pi \simeq 1$ in the following [if the value of $(g_1/4\pi)^2$ would be smaller the constraints which we will discuss will be satisfied even better, thanks to the flatter ensuing spectrum]. In such case, the CMB anisotropy can receive a complete electromagnetic explanation provided the parameters of our background are constrained to lie on (or near) the half-line $\log_{10} g_s = -1.90 \log_{10} z_s$, $\log_{10} z_s > 29$. The resulting perturbation spectrum

$$r(\omega) \simeq \left(\frac{\omega}{10^{11} Hz}\right)^{6/29}, \qquad \omega < 10^{11} Hz \tag{14}$$

gives then $r \simeq 10^{-5}$ at the intergalactic scale $\omega_G = (1Mpc)^{-1}$, which is large enough to seed not only the galactic dynamo, but also the cosmic magnetic field directly [8]. This results holds in general for any realistic value of g_1, and leads to consider a scenario in which the CMB anisotropy and the primordial magnetic fields have a common origin. The peak value of order unity of the electromagnetic spectrum can then easily explain the (otherwise mysterious, to the best of our knowledge) coincidence that the total energy density of our galactic magnetic field, $\rho_B = \rho_{cmb} \int^{\omega_1} r(\omega) d\omega$, is of the same order of magnitude as the CMB energy density. We also note that, at the scale corresponding to the end of nucleosynthesis ($\omega_N \simeq 10^{-12}$Hz), eq.(14) predicts $r \simeq 10^{-4.7}$. It is thus automatically consistent with the bounds following from the presence of strong magnetic fields at nucleosynthesis time [19], which impose $r(\omega_N) \lesssim 0.05$.

The above discussion refers to the case in which all scales inside our present horizon crossed the horizon, for the first time, during the string phase, i.e. for $z_s \geq 10^{29}$. Such a phase was characterized by a de Sitter-like metric evolution, with a curvature scale of Planckian order. In the standard inflationary context such a background configuration is forbidden, as it would lead to an overproduction of tensor perturbations: a phase of constant Planckian curvature can last only up to a total red-shift $z \leq 10^{19}$ [20], to be compatible with the bounds obtained from pulsar timing data [21]. In a string cosmology context, however, we must recall that the de Sitter-like evolution of the metric refers to the String frame, where tensor metric perturbations are also coupled to the dilaton background [10]. As a consequence, the spectral distribution $r_g(\omega)$ of tensor perturbation is growing with frequency (instead of being flat like in the standard de Sitter scenario), with a peak value which is again of order one around 10^{11}Hz. Moreover, the growth is so fast ($r_g \sim \omega^{n+1}$, where n is the spectral index of the electromagnetic perturbations), that the contribution of r_g is negligible at the scale ω_0, and it is also largely consistent with the pulsar bound [21] which requires $r_g(\omega_P) \lesssim 10^{-2}$ at $\omega_P \simeq 10^{-8}$Hz.

A further remark related to the long duration of the string phase concerns the validity of the spectrum (8), which has been obtained from the tree-level, low energy action (1). It is true that, in the string phase, we may have corrections coming both from higher loops

(expansion in e^{ϕ}) and from higher derivative terms (α' corrections). However, in order to reproduce the large scale anisotropy we have to work in a range of parameters where the dilaton is deeply in his perturbative regime. Eq.(14) holds in fact for $g_s = e^{\phi_s/2} \lesssim 10^{-55}$. We thus expect our results to be stable against loop corrections, at least at all the scales which are relevant for the observed anisotropy and for the generation of primordial magnetic fields.

As to the α' corrections, they are instead crucial in the basic assumption that the dilaton driven era leads to a quasi de Sitter epoch when the curvature reaches the string scale. Concerning the higher derivative corrections of the form $(\alpha' F_{\mu\nu} F^{\mu\nu})^m$, $m \geq 2$, they can modify in principle the equation determining the evolution of electromagnetic fluctuations (eq.(2)). We are expanding, however, our perturbations around the vacuum background $F_{\mu\nu} = 0$. Therefore, no higher curvature correction may provide significant contributions as long as we work in the region of parameter space in which perturbations can be consistently treated linearly, namely in the region in which eq.(10) is satisfied.

We thus believe that the main conclusion of this paper, namely that an electromagnetic origin of the CMB anisotropy is allowed in a realistic string cosmology scenario, is not only compatible with the various phenomenological bounds, but is also quite independent of the (unknown) kinematic details of the high energy "stringy" phase, preceding the phase of standard cosmological evolution.

References

[1] G. F. Smoot et al., Astrophys. J. 396, L1 (1992)

[2] C. L. Bennett et al., Astrophys. J. 430, 423 (1994)

[3] M. S. Turner and L. M. Widrow, Phys. Rev. D37, 2743 (1988);

T. Vachaspati, Phys. Lett. B265, 258 (1991);

B. Ratra, Astrophys. J. Lett. 391, L1 (1992);

W. D. Garretson, G. B. Field and S. M. Carrol, Phys. Rev. D46, 5346 (1992);

A. D. Dolgov, Phys. Rev. D48, 2499, (1993).

[4] G. Veneziano, Phys. Lett. B265, 287 (1991);

[5] M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993);

M. Gasperini and G. Veneziano, Mod. Phys. Lett. A8, 3701 (1993).

[6] M. Gasperini and G. Veneziano, Phys. Rev. D50, 2519 (1994)

[7] C. Lovelace, Phys. Lett. B135, 75 (1984);

E. S. Fradkin and A. A Tseytlin, Nucl. Phys. B261, 1 (1985);

C. G. Callan et al., Nucl. Phys. B262, 593 (1985)

[8] M. Gasperini, M. Giovannini and G. Veneziano, *Primordial magnetic fields from string cosmology*, CERN-TH/95-85, April 1995 (hep-th/9504083)

[9] D. Lemoine and M. Lemoine, *Primordial magnetic fields in string cosmology*, April 1995

[10] M. Gasperini and M. Giovannini, Phys. Rev. D47, 1529 (1993)

[11] R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov and G. Veneziano, Phys. Rev. D (1995) (in press), CERN-TH. 7544/94 (December 1994)

[12] M. Gasperini, in *Proceedings of the 2nd Journée Cosmologie*, Paris 1994, ed. by N. Sanchez and H. de Vega (World Scientific, Singapore, in press) (astro-ph/9406056)

[13] L. P. Grishchuk, Sov. Phys. JEPT 40, 409 (1975);

A. A. Starobinski, JEPT Lett. 30, 682 (1979).

[14] R. Brustein and G. Veneziano, Phys. Lett. B329, 429 (1994)

[15] G. Veneziano, in *Proceedings of the PASCOS '94 Conference*, Syracuse N.Y., May 1994 (CERN-TH 7502/94)

[16] T. Piran et al., MNRAS 265, 681 (1993)

[17] S. Mollerach, S. Matarrese and F. Lucchin, Phys. Rev. D50, 4835 (1994)

[18] W. Hu, D. Scott and J. Silk, Astrophys. J. 430, L97 (1994)

[19] D. Grasso and H. R. Rubinstein, Astropart. Phys. 3, 95 (1995)

[20] M. Gasperini and M. Giovannini, Phys. Lett. B301, 334 (1993)

[21] D. R. Stinebring et al., Phys. Rev. Lett. 65, 285 (1990)

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Inflation and Initial Conditions in the Pre-Big Bang Scenario

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The pre-big bang scenario describes the evolution of the Universe from an initial state approaching the flat, cold, empty, string perturbative vacuum. The choice of such an initial state is suggested by the present state of our Universe if we accept that the cosmological evolution is (at least partially) duality-symmetric. Recently, the initial conditions of the pre-big bang scenario have been criticized as they introduce large dimensionless parameters allowing the Universe to be "exponentially large from the very beginning". We agree that a set of initial parameters (such as the initial homogeneity scale, the initial entropy) larger than those determined by the initial horizon scale, H^{-1}, would be somewhat unnatural to start with. However, in the pre-big bang scenario, the initial parameters are all bounded by the size of the initial horizon. The basic question thus becomes: is a maximal homogeneity scale of order H^{-1} necessarily unnatural if the initial curvature is small and, consequently, H^{-1} is very large in Planck (or string) units? In the impossibility of experimental information one could exclude "a priori", for large horizons, the maximal homogeneity scale H^{-1} as a natural initial condition. In the pre-big bang scenario, however, pre-Planckian initial conditions are not necessarily washed out by inflation and are accessible (in principle) to observational tests, so that their naturalness could be also analyzed with a Bayesan approach, in terms of "a posteriori" probabilities.

Recently, the validity of the pre-big bang scenario as a viable inflationary model has been questioned on the grounds of its initial conditions [1].

The main criticism raised against models in which the Universe evolves from the flat, zero-interactions, string perturbative vacuum is mainly based on two points [1]. The first concerns the homogeneity problem, in particular the largeness of the initial homogeneous region in string (or Planckian) units; the second concerns the flatness problem, and in particular the two large dimensionless parameters (the inverse of the string coupling and of the curvature, in string units) characterizing the Universe at the beginning of inflation. The fact that, as a consequence of these large numbers, "the pre-big bang Universe must be very huge and homogeneous from the very beginning" is quoted as a serious problem, supporting the conclusion that "the current version of the pre-big bang scenario cannot replace usual inflation" [1].

I agree with the remarks concerning the initial size of the Universe (indeed, the need for an initial state with a Universe very large in Planck units was already noted in the first paper on the pre-big bang scenario [2] and, even before, in the context of string-driven superinflation [3]; in particular, the condition on the duration of inflation, reported in [1] as eq. (8), was already derived in [4]). The large initial size of the Universe is only part of the conditions to be imposed at the onset of pre-big bang inflation, and I also agree with the fact that a successful pre-big bang scenario requires an initial state characterized by very small (or very large) dimensionless ratios measuring the initial curvature and coupling constant, and possibly leading to a fine-tuning problem, as first pointed out in [5]. I disagree, however, with the conclusion presented in [1], and I would like to point out some arguments, hoping to clarify a different point of view on a large initial Universe.

I will concentrate, in particular, on the largeness of the initial horizon scale, which can be thought to be at the ground of the various objections discussed in [1]. The large dimensionless ratios of the initial state, when referred to the Einstein frame in which the Planck length is fixed, correspond indeed to a small initial curvature in Planck units, and then to a large horizon (in Planck length units), allowing a large homogeneous domain as initial condition. I do not pretend, of course, to provide a final answer to all problems. The modest aim of this paper is to stress that the problems raised in [1] reduce, in the end, to the question of whether the horizon scale, *irrespective of its size*, may be a natural scale for determining the inflationary initial conditions (in particular, the size of the initial homogeneous region), and to suggest the possibility that the answer is not negative "a priori", at least when the initial conditions are imposed well inside the classical regime, like in the case of the

pre-big bang scenario.

Let me start recalling that the kinematical problems of the standard scenario can be solved by two classes of accelerated backgrounds [6]. Consider, for instance, the flatness problem, requiring a phase in which the ratio $r = k/a^2H^2 \sim \dot{a}^{-2}$ decreases, so as to compensate its growth up to the present value $r \lesssim 1$ during the subsequent phase of standard evolution. By parametrizing the scale factor as $a \sim |t|^\beta$, the decrease of $r \sim |t|^{2(1-\beta)}$ can be arranged either by 1) $\beta > 1$, $t \to +\infty$, or 2) $\beta < 1$, $t \to 0_-$. Both classes of backgrounds are accelerated, as sign $\dot{a} =$ sign $\ddot{a}$. The first class corresponds to power-inflation, and includes de Sitter inflation in the limit $\beta \to \infty$. The second class includes superinflation for $\beta < 0$, and accelerated contraction for $0 < \beta < 1$.

The main kinematic difference between the two classes is the behaviour of the event horizon, whose proper size is defined by

$$d_e(t) = a(t) \int_t^{t_M} dt' a^{-1}(t'). \tag{1}$$

Here t_M is the maximal future extension of the cosmic time coordinate for the inflationary manifold. Therefore, $t_M = +\infty$ for the first class, and $t_M = 0$ for the second class of backgrounds. In both cases we find that the integral converges, and that $d_e(t) \sim |H|^{-1}(t)$, so that the horizon size is constant or growing for class 1), shrinking for class 2), following the inverse behaviour of the curvature scale. The phase of pre-big bang evolution, in particular, is dual to a phase of standard, decelerated evolution: its accelerated kinematics is characterized by a growing curvature scale (i.e. growing $|H|$), and may be represented as superinflation, in the string frame, or accelerated contraction, in the Einstein frame [6].

In order to recall the criticism of [1] we will now compare the kinematics of standard de Sitter inflation and pre-big bang superinflation, for an oversimplified cosmological model in which the standard radiation era begins at the Planck scale, and it is immediately preceeded by a phase of accelerated (inflationary) evolution. Also, for the sake of simplicity, we will identify at the end of inflation the present value of the string length L_s with the Planck length L_p (at tree-level, they are related by $L_p = \langle g \rangle L_s = \langle \exp \phi/2 \rangle L_s$, with a present dilaton expectation value $\langle g \rangle \sim 0.1 - 0.01$).

At the beginning of the radiation era the horizon size is thus controlled by the Planck length $L_p \sim L_s$, while the proper size of the homogeneous and causally connected region inside our present Hubble radius, rescaled down at the Planck epoch according to the standard decelerated evolution of $a(t)$, is unnaturally larger than the horizon by the factor $\sim 10^{30} L_p$. During the inflationary epoch, the ratio

$$\frac{\text{proper size horizon scale}}{\text{proper size homogeneous region}} \sim \frac{H^{-1}(t)}{a(t)} \sim \eta \tag{2}$$

must thus decrease at least by the factor 10^{-30}, so as to push the homogeneous region outside the horizon, of the amount required by the subsequent decelerated evolution. Since the above ratio evolves linearly in conformal time $\eta \sim \int a^{-1} dt$, the condition of sufficient inflation can be written as

$$|\eta_f|/|\eta_i| \lesssim 10^{-30}, \tag{3}$$

where η_i and η_f mark, respectively, the beginning and the end of the inflationary epoch.

Let us now compare de Sitter inflation, $a \sim (-\eta)^{-1}$, with a typical dilaton-dominated superinflation, $a \sim (-t)^{-1/\sqrt{3}} \sim (-\eta)^{-1/(\sqrt{3}+1)}$ (the same discussed in [1]).

In the standard de Sitter case the horizon and the Planck length are constant, $H^{-1} \sim L_s \sim L_p$; as we go back in time, according to eq. (3), $a(t)$ reduces by the factor 10^{-30} so that, at the beginning of inflation, we find a homogeneous region just of size L_p, like the horizon. In the superinflation case, on the contrary, during the conformal time interval (3), $a(t)$ is only reduced by the factor $a_i/a_f = 10^{-30/(1+\sqrt{3})} \sim 10^{-11}$, so that the size of the homogeneous region, at the beginning of inflation, is still large in string units, $\sim 10^{30\sqrt{3}/(1+\sqrt{3})} L_s \sim 10^{19} L_s$. The situation is even worse in Planck units since, at the beginning of inflation, the string coupling $\exp \phi/2$, and thus the Planck length L_p, are reduced with respect to their final values by the factor [4] $L_p/L_s = |\eta_f/\eta_i|^{\sqrt{3}/2} \sim 10^{-15\sqrt{3}}$, so that $10^{19} L_s \sim 10^{45} L_p$. This, by the way, is exactly the initial size of the homogeneous region evaluated in the Einstein frame in which L_p is constant, and the above dilaton-driven evolution is represented as a contraction, with $a \sim (-\eta)^{1/2}$ (see Fig. 1 for a qualitative illustration of the differences between de Sitter inflation and pre-big bang inflation).

According to [1], case (a) of Fig. 1 provides an acceptable example of inflationary scenario, as the initial homogeneity scale is contained within a single domain of Planckian size. Case (b), on the contrary, is not satisfactory because of the initial homogeneity on scales much greater than Planckian, $10^{19} L_s \sim 10^{45} L_p$. Quoting Ref. [1], this situation "is not much better than the situation in the non-inflationary big bang cosmology, where it was necessary to assume that the initial size of the homogeneous part of our Universe was greater than $10^{30} L_p$".

I would like to stress, however, that in case (b) the initial homogeneous region is large in Planck units, *but not larger than the horizon itself*. Indeed, during superinflation, the horizon scale shrinks linearly in cosmic time. As we go backwards in time, for the particular example that we are considering, the horizon increases by the factor $H_i^{-1}/H_f^{-1} = |t_i|/|t_f| = (\eta_i/\eta_f)^{\sqrt{3}/(1+\sqrt{3})}$, so that, at the beginning of inflation, $H^{-1} \sim 10^{30\sqrt{3}/(1+\sqrt{3})} L_s \sim 10^{19} L_s \sim 10^{45} L_p$, i.e. the horizon size is just the same as that of the homogeneous region (as illustrated in Fig.

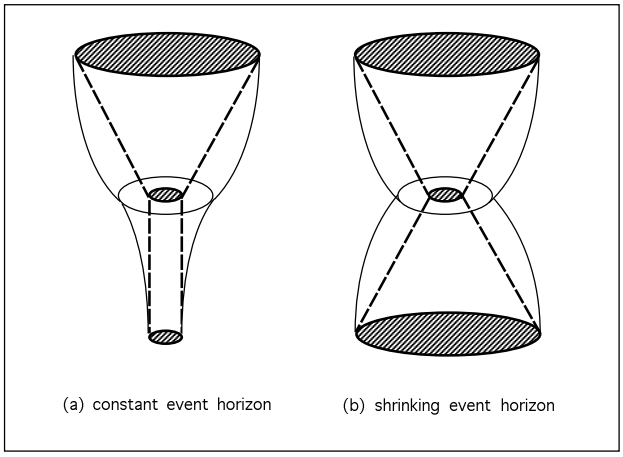

FIG. 1. *Qualitative evolution of the horizon scale and of the proper size of a homogeneous region for (a) standard de Sitter inflation, and (b) pre-big bang superinflation, represented in the Einstein frame as a contraction. The time direction coincides with the vertical axis. The three horizontal spatial sections corresponds, from top to bottom, to the present time, to the end and to the beginning of inflation. The shaded area represents the horizon, and the dashed lines its time evolution. The full curves represent the time evolution of the border of the homogeneous region, controlled by the scale factor.*

1). In this sense, both initial conditions, in cases (a) and (b), seem to be equally natural. The difference is that in case (b) the initial horizon is large in Planck units, while in case (a) it is of order one. This is an obvious consequence of the different curvature scales at the beginning of inflation.

The question about the naturalness of the initial conditions seems thus to concern the unit of length used, in particular, to measure the size of the initial homogeneous domain, and, more generally, to characterize the initial geometric configuration at the onset of inflation: which basic length scale has to be used, the Planck (or string) length, or the radius of the causal horizon?

This, I believe, is the question to be answered. Providing a definite answer may deserve a careful analysis, which is outside the scope of this brief paper. Let me note that, according to [1], it is the Planck (or string) scale that should provide the natural units for the size of the initial homogenous patches and for the initial curvature and coupling scale. This is certainly reasonable when initial conditions are imposed on a cosmological state approaching the high-curvature, quantum gravity regime. In the pre-big bang scenario, however, initial conditions are to be imposed when the Universe is deeply inside the low-curvature, weak coupling, classical regime. In that regime the Universe does not know about the Planck length, and the causal horizon H^{-1} could represent a natural candidate for controlling the set of initial conditions. For what concerns homogeneity, however, I am not suggesting that the horizon (which is the maximal homogeneity scale) should be always *assumed* as the natural scale of homogeneity. I am suggesting that this possibility should be discussed on the ground of some quantitative and objective criterium, as attempted for instance in [7], and not discarded a priori, as in [1] (see also [8] for a discussion of "generic" initial conditions in a string cosmology context).

One might think that, accepting the horizon size as a natural homogeneity scale, there is no need of inflation to explain our present homogeneous Universe [1]. This is not the case, however, because if we go back in time without inflation our Universe should start in the past from a homogeneous region unnaturally larger than the horizon (see Fig. 1). Only with inflation the homogeneous region, going back in time, re-enters inside the horizon. So, only if there is inflation, an initial homogeneity scale of the order of the horizon scale is enough to reproduce our present Universe.

Also, one might think, as noted in [1], that the classical homogeneity of the horizon might be destroyed by quantum fluctuations amplified during the contraction preceeding the onset of the inflationary era, in such a way as to prevent the formation of a large homogeneous domain. This problem has been recently discussed in [9] for the case of a homogeneous string cosmology background with negative spatial curvature: it has been shown that quantum fluctuations die off much faster than classical inhomogeneities as they approach the initial perturbative vacuum, and remain negligible throughout the perturbative pre-big bang phase. For classical perturbations, however, the situation is different, and no general result is presently available. The initial amplitude of the classical inhomogeneities is not normalized to a vacuum fluctuation spectrum, the results of [9] cannot be applied, and inflation can occurr successfully or not depending on the initial distribution of the classical amplitudes.

Finally, one might argue that a large initial horizon, assuming a saturation of the bound imposed by the holographic principle in a cosmological context [10], implies a large initial entropy, $S =$ (horizon area in Planck units), and thus a small probability for the initial configuration. Indeed, if S is large, the probability that such a configuration be obtained through a process of quantum tunnelling (proportional to $\exp[-S]$) is exponentially suppressed, as emphasized in [1]. However, in the pre-big bang scenario, quantum effects such as tunnelling or reflection of the Wheeler-De Witt wave function are expected to be important towards *the end* of inflation [11], and *not the beginning*, as they may be effective *to exit* [12], eventually, from the inflationary regime, *not to enter* it and to explain the origin of the initial state. A large entropy of the initial state, in the weakly coupled, highly classical regime, can only correspond to a large probability of

such configuration, (proportional to $\exp[S]$), as expected for classical and macroscopic configurations.

In conclusion, let me come back on the large dimensionless parameters characterizing the initial state of pre-big bang inflation [1]. The physical meaning of those parameters, i.e. the fact that the initial string coupling and curvature are very small in string (or Planck) units, is to be understood as a consequence of the perturbative initial conditions, suggested by the underlying duality symmetries. On the other hand, whenever inflation starts at curvature scales smaller than Planckian, the initial state is necessarily characterized by a large dimensionless ratio – the inverse of the curvature in Planck units. If one believes that such large numbers should be avoided, then should be prepared to accept the fact that natural initial conditions are only possible in the context of models in which inflation starts at the Planck scale: for instance chaotic inflation, as pointed out in [1].

This is a rather strong conclusion, that rules out, as a satisfactory explanation of our present cosmological state, not only the pre-big bang scenario, but any model in which inflation starts at scales smaller than Planckian (unless we have a scenario with different stages of inflation responsible for solving different problems). Even for a single stage of inflation very close to the Planck scale, however, we are not free of problems, as we are led, eventually, to the following question: can we trust the naturalness of inflation models like chaotic inflation, in which classical general relativity is applied to set up initial conditions at Planckian curvature scales, i.e. deeply inside the non-perturbative, quantum gravity regime?

The Planckian regime is certainly problematic to deal with, both in the string and in the standard inflationary scenario: in string cosmology, in particular, it prevents a simple solution of the "graceful exit" problem [13]. The pre-big bang scenario, however, tries to look back in time beyond the Planck scale by using the powerful tools of superstring theory, in particular its duality symmetries. According to duality, the pre-Planckian Universe approaches initially the state of a low-energy system, and initial conditions are to be set up in a regime well described by the lowest order effective action, in which all quantum and higher-order corrections are small, and under control. It is true, however, that the presence of the Planckian regime can indirectly affect the initial conditions also in a string cosmology context, as it imposes a finite duration of the low-energy dilaton-driven phase: the initial homogeneity scale, as a consequence, has to be large enough to emerge with the required size at the Planck epoch, and to avoid the need for a further period of high-curvature, Planckian inflation [14].

It should be stressed, finally, that the main difference from the standard scenario, in which any tracks of the pre-Planckian cosmological state is washed out by inflation, is probably the fact that the pre-Planckian history may become visible, in the sense that its phenomenological consequences can be tested (at least in principle) even today [15]. So, while in the context of standard inflation the naturalness criterium can be safely applied to select an initial state at the Planck scale, it seems difficult (in my opinion) to apply the same criterium in a string cosmology context, and to discard a model of pre-Planckian evolution only on the grounds of the large parameters characterizing the initial conditions. Such initial conditions have consequences accessible to observational tests, and the analysis of the "a posteriori" probabilities with the Bayesan approach of [7] suggests that a state with a large initial horizon may become "a posteriori" natural, because of the duality symmetries intrinsic to the pre-big bang scenario.

However, much further work is certainly needed before a final conclusion is reached. Irrespective of the final results, such work will certainly improve our present understanding of string theory and of the physics of the early Universe.

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[1] N. Kaloper, A. Linde and R. Bousso, Phys. Rev. D **59**, 043508 (1999).
[2] G. Veneziano, Phys. Lett. B **265**, 287 (1991).
[3] M. Gasperini, N. Sanchez and G. Veneziano, Nucl. Phys. B **364** 365 (1991).
[4] M. Gasperini and G. Veneziano, Phys. Rev. D **50**, 2519 (1994).
[5] M. S. Turner and E. J. Weinberg, Phys. Rev. D **56**, 4604 (1997).
[6] M. Gasperini and G. Veneziano, Mod. Phys. Lett. A **8**, 3701 (1993).
[7] A. Buonanno, T. Damour and G. Veneziano, Nucl. Phys. B **543**, 275 (1999).
[8] D. Clancy, J. E. Lidsey and R. Tavakol, Phys. Rev. D **59**, 063511 (1999); Phys. Rev. D **58**, 44017 (1998); D. Clancy, A. Feinstein, J. E. Lidsey and R. Tavakol, Phys. Lett. B **451**, 303 (1999).
[9] A. Gosh, G. Pollifrone and G. Veneziano, Phys. Lett. B **440**, 20 (1998).
[10] W. Fischler and L. Susskind, *Holography and cosmology*, hep-th/9806039; D. Bak and S. J. Rey, *Holographic principle and string cosmology*, hep-th/9811008; A. K. Biswas, J. Maharana and R. K. Pradhan, *The holography*

hypothesis and pre-big bang cosmology hep-th/9811051; G. Veneziano, Phys. Lett. B **454**, 22 (1999).

[11] M. Gasperini, Int. J. Mod. Phys. A **13**, 4779 (1998).

[12] M. Gasperini, J. Maharana and G. Veneziano, Nucl. Phys. B **472**, 394 (1996); M. Gasperini and G. Veneziano, Gen. Rel. Grav. **28**, 1301 (1996).

[13] R. Brustein and G. Veneziano, Phys. Lett. B **329**, 429 (1994); N. Kaloper, R. Madden and K. A. Olive, Nucl. Phys. B **452**, 677 (1995); R. Easther, K. Maeda and D. Wands, Phys. Rev. D **53**, 4247 (1996); I. Antoniadis, J. Rizos and K. Tamvakis, Nucl. Phys. B **415**, 497 (1994); S. J. Rey, Phys. Rev. Lett. **77**, 1929 (1996); M. Gasperini and G. Veneziano, Phys. Lett. B **387**, 715 (1996); M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B **494**, 315 (1997); M. Maggiore, Nucl. Phys. B **525**, 413 (1998); R. Brustein and R. Madden, Phys. Lett. B. **410**, 110 (1997); Phys. Rev. D **57**, 712 (1998); R. Brandenberger, R. Easther and J. Maia, JHEP **9808**, 007 (1998); M. Maggiore and A. Riotto, Nucl. Phys. B **548**, 427 (1999) ; S. Foffa, M. Maggiore and R. Sturani, Nucl. Phys. B **552**, 395 (1999).

[14] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B **494**, 315 (1997); M. Maggiore and R. Sturani, Phys. Lett. B **415**, 335 (1997).

[15] See for instance R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B **361**, 45 (1995); M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75**, 3796 (1995); R. Durrer, M . Gasperini, M. Sakellariadou and G. Veneziano, Phys. Rev. D **59**, 043511 (1999).

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TOWARDS A NON-SINGULAR PRE-BIG BANG COSMOLOGY

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Abstract

We discuss general features of the β-function equations for spatially flat, $(d+1)$-dimensional cosmological backgrounds at lowest order in the string-loop expansion, but to all orders in α'. In the special case of constant curvature and a linear dilaton these equations reduce to $(d+1)$ algebraic equations in $(d+1)$ unknowns, whose solutions can act as late-time regularizing attractors for the singular lowest-order pre-big bang solutions. We illustrate the phenomenon in a first order example, thus providing an explicit realization of the previously conjectured transition from the dilaton to the string phase in the weak coupling regime of string cosmology. The complementary role of α' corrections and string loops for completing the transition to the standard cosmological scenario is also briefly discussed.

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1 Introduction

Standard cosmology [1] assumes that the primordial Universe was in a hot, dense, and highly curved state, very close indeed to the so-called big-bang singularity. By contrast, the duality symmetries of string theory [2, 3, 4] motivate a class of "pre-big bang" cosmological models [3, 5] in which the Universe starts very near the cold, empty and flat perturbative vacuum. This scenario, although theoretically appealing, can only make sense phenomenologically if such unconventional initial conditions evolve naturally into those of the standard scenario at some later time, smoothing out the big-bang singularity. Does this happen?

The early evolution of a Universe that starts at very low curvature and coupling is well described by the low-energy, tree-level string effective action [6, 7]. The corresponding field equations imply that the string perturbative vacuum, with vanishing coupling constant $g_s = e^{\phi/2} = 0$, is actually unstable towards small homogeneous fluctuations of the metric and the dilaton ϕ, which can easily ignite an accelerated growth of the curvature and of the coupling [3, 5]. However, in the absence of higher-order corrections to the effective action, such a growth is unbounded: as conjectured in [8] (and later proved to a large extent in [9]), a singularity in the curvature and/or the coupling is reached in a finite amount of cosmic time, for any realistic choice of the (local) dilaton potential. Is such a singularity unavoidable?

Higher-order corrections to the effective action of string theory are controlled by two independent expansion parameters. One is the field-dependent (and thus in principle space-time-dependent) coupling g_s, which controls the importance of string-loop corrections. The other parameter, α', controls the importance of finite-string-size corrections, which are small if fields vary little over a string-length distance $\lambda_s = \sqrt{\alpha'}$. Only when this second expansion parameter is small, can the higher-derivative corrections to the action be neglected and does string theory go over to an effective quantum field theory.

As discussed in previous examples [10]-[13], quantum corrections arising in the strong coupling regime can regularize the curvature singularity of the tree-level pre-big bang models, already at the one-loop order. With the exception of the very particular case of two space-time dimensions, the inclusion of loops is necessarily associated with the appearance of higher-derivative terms in the effective action, and thus requires also, for consistency, the inclusion of higher orders in α'. Instead, if the initial value of the string coupling is small enough, it is quite possible that the Universe reaches the high-curvature regime in which

higher-derivative (α') corrections become important, while the coupling is still small enough to neglect loop corrections.

According to the above considerations, we shall discuss hereafter the regularization of the curvature singularity just as a result of "stringy" α' corrections, but at lowest order in g_s. We will consider, in particular, the possibility that a cosmological background evolving from the perturbative vacuum be attracted into a state of constant curvature and linearly running dilaton, i.e. of constant H and $\dot{\phi}$ ($H = \dot{a}/a$ is the Hubble parameter in the so-called string frame, i.e. in the frame most directly related to the σ-model parametrization of the action, and the dot stands for differentiation with respect to cosmic time in that frame).

A high-curvature string phase with frozen string-size values of H and $\dot{\phi}$ was previously conjectured as a crucial ingredient for a successful pre-big bang scenario, and has important phenomenological consequences [14]. Such a state exists in general as a cosmological solution of higher-curvature effective actions, but is in general disconnected from the perturbative vacuum (the trivial solution with $H = \dot{\phi} = 0$) by a singularity, or by an unphysical region in which H becomes imaginary. String theory, on the contrary, provides examples in which the two states are smoothly joined by the evolution of the background already to first order in α', as we will show in this paper. In this sense string theory automatically implements, without a scalar potential and in any number of dimensions, the "limiting curvature" hypothesis previously introduced ad-hoc, with various mechanisms [15, 16], to regularize the curvature of cosmological backgrounds.

A final state with linearly evolving dilaton can also be obtained in the context of one-loop-regularized models. The main difference between the loop case and the one at hand is that our final shifted dilaton $\overline{\phi}$ satisfies, in d isotropic spatial dimensions, $\dot{\overline{\phi}} \equiv \dot{\phi} - dH < 0$. This is a necessary condition for the background to be subsequently attracted by an appropriate potential in a state with expanding metric ($H > 0$) and frozen dilaton ($\dot{\phi} = 0$), the starting point of standard cosmology. In fact, since the dilaton keeps growing after the transition to the string phase, the effects of loops, of a non-perturbative dilaton potential, and of the back-reaction from particle production must eventually become important, and are expected to play an essential role in the second transition from the string phase to the usual hot big bang scenario. Leaving this second transition to further investigation, the main purpose of this paper is to give arguments in favour of the occurrence of a smooth evolution, in the weak

coupling regime, from the dilaton phase to the constant curvature string phase dominated by the α' corrections.

The paper is organized as follows. In Section 2 we discuss the general structure of tree-level, σ-model β-functions for generic Bianchi-type I backgrounds (including a time-dependent dilaton). In Section 3 we specialize the equations to the case of constant curvature and linear dilaton, and show that in this case we obtain a system of $(d+1)$ algebraic equations in $(d+1)$ unknowns (d Hubble constants and $\dot{\phi}$). The equations look like those for the fixed points of a renormalization group (RG) flow in physical cosmic time (as opposed to the RG "time"). In Section 4 we consider an example to first order in α' (i.e. four derivatives), determine the fixed points, and show, by numerical integration, that any isotropic pre-big bang background necessarily evolves smoothly towards the regular fixed points, thus avoiding the singularity. The fixed points have in general an attraction basin of finite area, in the space of initial conditions, also for anisotropic backgrounds. Section 5 contains our conclusions.

2 Bianchi I tree-level β-functions

It is well known [6, 7] that the field equations for the background fields appearing in the σ-model action of string theory correspond to the vanishing of a set of β-function(al)s. We shall call the latter "σ-model β-functions" in order to avoid a possible confusion with another set of effective β-functions to be introduced later. It is also known [6, 7] that the σ-model β-functions vanish on a set of field equations that can be derived by varying a space-time effective action Γ, which is also the generating functional of the string S-matrix.

We shall consider in this paper a σ-model background in $(d+1)$ dimensions, consisting of just a time-dependent dilaton $\phi(t)$ and an anisotropic Bianchi-type I metric, $g_{\mu\nu}$, which can be conveniently parametrized as:

$$g_{\mu\nu} = \mathrm{diag}\left(N^2(t), -a_i^2(t)\delta_{ij}\right); \qquad i,j = 1,...,d. \tag{2.1}$$

Here $N(t)$ is the lapse function and $a_i(t)$ are the scale factors along the d different spatial directions. We will not consider, for the sake of simplicity, an additional antisymmetric-tensor background $B_{\mu\nu}$, which is needed in order to discuss the $O(d,d)$ symmetries [17, 18] associated with the presence of d Abelian isometries, but our considerations can easily be extended to such a case.

Using the general covariance of the string effective action and the fact that, at tree-level in the string-loop expansion, the dependence on a constant dilaton is fixed in the string frame, we can immediately write the exact, tree-level effective action for this background, in terms of the fields

$$\overline{\phi} = \phi - \sum_i \beta_i, \qquad\qquad \beta_i = \ln\ a_i, \tag{2.2}$$

in the form:

$$\Gamma = \int dt N e^{-\overline{\phi}} \mathrm{L}\left(\dot{\beta}_i^{(n)}, \dot{\overline{\phi}}_i^{(n)}\right). \tag{2.3}$$

The effective Lagrangian L is a general function of the (properly covariantized) time derivatives of the fields β_i and $\overline{\phi}$, at all orders in $n \geq 1$:

$$\dot{\beta}_i^{(n)} = \prod_{k=1}^{n} \left(\frac{1}{N}\frac{d}{dt}\right)^k \beta(t), \qquad\qquad \dot{\overline{\phi}}^{(n)} = \prod_{k=1}^{n} \left(\frac{1}{N}\frac{d}{dt}\right)^k \overline{\phi}(t). \tag{2.4}$$

In this paper we shall limit our attention to the case of critical (super)strings in which no cosmological constant appears in L. Thus, even when we discuss the case $d \neq d_{\rm crit}$ we shall assume that other "passive" sectors are present to cancel the central charge deficit $(d - d_{\rm crit})/3\alpha'$.

There are just three kinds of σ-model β-function equations that follow from (2.3). They correspond, respectively, to the field equation for the lapse, for each scale factor, and for the dilaton $\overline{\phi}$. However, these $(d+1)$ equations are not independent. The relation among them follows from the general fact that, in a theory with coordinate reparametrization invariance, the set of "non-dynamical" field equations representing constraints on the initial data are covariantly conserved [1], as a consequence of the Bianchi identities. In our particular case, the residual time reparametrization invariance implies the relation:

$$\dot{\beta}_i \frac{\delta\Gamma}{\delta\beta_i} + \dot{\overline{\phi}}\frac{\delta\Gamma}{\delta\overline{\phi}} = N\frac{d}{dt}\frac{\delta\Gamma}{\delta N}. \tag{2.5}$$

On the equations of motion each one of the three terms in eq. (2.5) vanishes separately, so that only $(d+1)$ equations are independent, to all orders, in agreement with a general property of the σ-model β-functions [19].

The variation of the action (2.3), in the cosmic time gauge $N = 1$, gives rise to the following system of field equations:

$$\frac{\partial \mathrm{L}}{\partial \dot{\beta}_i} - e^{\overline{\phi}}\frac{d}{dt}\left(e^{-\overline{\phi}}\frac{\partial \mathrm{L}}{\partial \ddot{\beta}_i}\right) + \ldots = e^{\bar{\phi}} Q_i, \tag{2.6}$$

$$- \mathrm{L} - \mathrm{e}^{\overline{\phi}} \frac{\mathrm{d}}{\mathrm{dt}} \left(\mathrm{e}^{-\overline{\phi}} \frac{\partial \mathrm{L}}{\partial \dot{\overline{\phi}}} \right) + \mathrm{e}^{\overline{\phi}} \frac{\mathrm{d}^2}{\mathrm{dt}^2} \left(\mathrm{e}^{-\overline{\phi}} \frac{\partial \mathrm{L}}{\partial \ddot{\overline{\phi}}} \right) + \ldots = 0 \,, \tag{2.7}$$

$$\mathrm{L} - \dot{\overline{\phi}} \frac{\partial \mathrm{L}}{\partial \dot{\overline{\phi}}} - \dot{\beta}_i \frac{\partial \mathrm{L}}{\partial \dot{\beta}_i} - 2\ddot{\beta}_i \frac{\partial \mathrm{L}}{\partial \ddot{\beta}_i} - 2\ddot{\overline{\phi}} \frac{\partial \mathrm{L}}{\partial \ddot{\overline{\phi}}} + e^{\bar{\phi}} \frac{d}{dt} \left(e^{-\bar{\phi}} \dot{\beta}_i \frac{\partial \mathrm{L}}{\partial \ddot{\beta}_i} + e^{-\bar{\phi}} \dot{\overline{\phi}} \frac{\partial \mathrm{L}}{\partial \ddot{\overline{\phi}}} \right) + \ldots = 0. \tag{2.8}$$

Equation (2.6), obtained by varying β_i, defines a set of d charges Q_i whose conservation follows from the fact that Γ depends only upon derivatives of the β_i. The second equation is the (shifted) dilaton equation, while the last equation is the so-called Hamiltonian constraint, following from the variation of the lapse. Relation (2.5) can be directly checked at the level of eqs. (2.6)–(2.8).

Equation (2.5) can be exploited in several ways when solving the system (2.6)–(2.8). The first way, which we adopt here, is to solve the first two equations and to impose the constraint only on the initial data. After solving the equations numerically, we can double check that the constraint remains valid at all times. Alternatively, we can just use the constraint and the conservation equations and ignore the dilaton equation. Equation (2.5) then guarantees that the dilaton equation is automatically satisfied provided $\dot{\overline{\phi}}$ is not identically zero.

3 The constant curvature case

Let us now consider a very special class of backgrounds, those with constant curvature and a linear dilaton, $\dot{\beta}_i$ and $\dot{\overline{\phi}}$ constant, where the fields are referred to the so-called string frame defined by the action (2.3). In our coordinate system they correspond to the ansatz:

$$ds^2 = dt^2 - \sum_i e^{2H_i t} dx^i dx^i, \qquad \phi(t) = ct + \phi_0 \tag{3.1}$$

parametrized by the $(d+1)$ constants c and H_i.

The only known (all-order) solutions of this type have a trivial Minkowski metric and a constant or linear dilaton [20] depending on whether $d = d_{\rm crit}$ or $d > d_{\rm crit}$. The existence of other high-curvature solutions of this type is all but excluded [21], however, provided the dilaton is not a constant, $c \neq 0$. Note, incidentally, that it is quite crucial that one looks at constant curvature solutions in the string frame [21]. Because of the dilaton's time-dependence this is <u>not</u> equivalent to a constant curvature in the Einstein frame, for which no-go theorems probably apply [22]. Let us thus discuss the necessary and sufficient

conditions for the existence of solutions of this type, which, in accordance with previously used terminology [14], we will term string-phase solutions.

It is clear, by inspection, that, for this class of backgrounds, eqs. (2.6)–(2.8) reduce to $(d+2)$ algebraic equations in the $(d+1)$ unknowns H_i and c. However, as already discussed, only $(d+1)$ equations are really independent, giving rise to the hope that isolated solutions other than the already mentioned trivial ones might exist. Actually, from eq. (2.6), we easily see that the l.h.s. is constant for a string-phase solution. Thus, in order for the r.h.s. to be constant as well, two possibilities exist: either $\overline{\phi}$ itself is constant with Q_i arbitrary, or $Q_i = 0$ for all i. Let us discuss these two possibilities in turn:

1) $\dot{\overline{\phi}} \equiv 0$.

   This case looks more attractive at first sight. However, with $\dot{\overline{\phi}} \equiv 0$, the remaining two equations are independent (see discussion in the previous Section) and give two constraints among the d conserved charges Q_i. Since these charges are already related by the initial (pre-big bang) conditions, fine-tuned initial conditions would be needed in order to flow into a string phase of this kind. For this reason we will concentrate here on the second alternative.

2) $\dot{\overline{\phi}} \neq 0,\ Q_i = 0$.

   This case looks at first even worse because, instead of having two constraints among the d conserved charges, we now have set all of them to zero while they are certainly non-vanishing on the pre-big bang solution. However, if $\dot{\overline{\phi}} < 0$, both sides of eq.(2.6) will go exponentially to zero at late times, irrespectively of the Q_i's. Therefore, as we shall see explicitly in the following section, string-phase solutions with $\dot{\overline{\phi}} < 0$ can play the role of late-time attractors for solutions coming from pre-big bang initial conditions.

Although $\dot{\overline{\phi}} < 0$ is a necessary condition for the above phenomenon to occur, it turns out not to be always sufficient. This can be most easily understood by using a RG description. Our differential equations (2.6)–(2.8) define a set of RG equations in physical (cosmic) time where $\dot{\overline{\phi}}$, $\dot{\beta}_i$ play the role of running couplings. Time-derivatives of the couplings define some new kind of β-functions whose zeros correspond to the constant-curvature, linear-dilaton fixed points. Trivial Minkowski space (with $\dot{\overline{\phi}} = 0$) corresponds to a trivial quadratic zero

of the β-functions and, consequently, is a late-time (early-time) attractor for post-big bang (pre-big bang) initial conditions.

By contrast, a non-trivial simple zero with $\dot{\tilde{\phi}} < 0$ is a late-time attractor from any initial condition sufficiently close to it. In order that pre-big bang initial conditions flow to this attractor, it is necessary that no other zeros or singularities separate the trivial fixed point from the non-trivial one. If this is the case, the long-conjectured transition from the dilaton to the string phase does indeed take place. In the next section we shall see explicit examples of such an interesting phenomenon.

4 A first-order example

To the first order in α', and in the string frame, the simplest effective action that reproduces the massless bosonic sector of the tree-level string S-matrix can be written in the form [7]:

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|} e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{k\alpha'}{4} R^2_{\mu\nu\alpha\beta} \right] , \tag{4.1}$$

where $k = 1, 1/2$ for the bosonic and heterotic string, respectively (remember that we have assumed the torsion background to be trivial). The corresponding field equations are equivalent, in the sense discussed in [7], to the conditions of σ-model conformal invariance represented by the vanishing of the β-functions.

In order to discuss cosmological solutions, it is convenient to perform a field redefinition (preserving, however, the σ-model parametrization of the action) that eliminates terms with higher than second derivatives from the effective equations. This can be easily done, as is well known, by replacing the square of the Riemann tensor with the Gauss–Bonnet invariant [23] $R^2_{GB} \equiv R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$, at the price of introducing dilaton-dependent α' corrections. The field redefinition

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + 4k\alpha' \left[R_{\mu\nu} - \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}(\nabla\phi)^2 \right] , \quad \tilde{\phi} = \phi + k\alpha' \left[R + (2d-3)(\nabla\phi)^2 \right] , \tag{4.2}$$

truncated to first order in α', leads in particular to the following simple form of the action (dropping the tilde over the redefined fields):

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|} e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{k\alpha'}{4} \left(R^2_{GB} - (\nabla\phi)^4 \right) \right] , \tag{4.3}$$

which we will use throughout this section.

We have explicitly checked that the results of this section are qualitatively reproduced also by using different parametrizations of the string effective action, for instance the one obtained by transforming into the string frame a pure Gauss–Bonnet action in the Einstein frame. Such results are not invariant, however, under field redefinitions that are truncated by keeping first order terms in α' in the effective action. With an additional truncated redefinition we can in fact modify eq.(4.3), without re-indroducing second derivatives, and obtain, in particular, the special action that was shown to be off-shell equivalent to the conditions of σ-model conformal invariance [24] through Zamolodchikov-like equations. The same action has also been recently proposed as the correct one to achieve a higher-order extension of the T-duality symmetry [25]. Such an action does have fixed points; however, these are not smoothly connected to the perturbative vacuum. This field-redefiniton dependence of the background properties is unavoidable as long as the α' expansion is truncated at a given finite order.

We specialize the Bianchi I background, for simplicity, to the case in which the spatial sections are the product of two isotropic, conformally flat ("external" and "internal") manifolds, respectively d- and n-dimensional, described by the metric

$$g_{00} = N^2(t), \quad g_{ij} = -\delta_{ij} e^{\beta(t)}, \quad g_{ab} = -\delta_{ab} e^{\gamma(t)}, \quad i,j = 1,...,d; \quad a,b = d+1,...,d+n \tag{4.4}$$

(note that the total number of spatial dimensions, previously denoted by d, is now redefined to be $d+n$). After integration by parts, the action (4.3) can be written as

$$S \propto \int dt e^{d\beta + n\gamma - \phi} \left[\frac{1}{N} \left(-\dot\phi^2 - d(d-1)\dot\beta^2 - n(n-1)\dot\gamma^2 - 2dn\dot\beta\dot\gamma + 2d\dot\beta\dot\phi + 2n\dot\gamma\dot\phi \right) + \right.$$

$$\left. \frac{k\alpha'}{4N^3} \left(c_1\dot\beta^4 + c_2\dot\gamma^4 + c_3\dot\phi\dot\beta^3 + c_4\dot\phi\dot\gamma^3 + c_5\dot\phi\dot\beta\dot\gamma^2 + c_6\dot\phi\dot\beta^2\dot\gamma + c_7\dot\beta^2\dot\gamma^2 + c_8\dot\beta\dot\gamma^3 + c_9\dot\beta^3\dot\gamma - \dot\phi^4 \right) \right], \tag{4.5}$$

where

$$\begin{aligned}
c_1 &= -\frac{d}{3}(d-1)(d-2)(d-3), \quad c_2 = -\frac{n}{3}(n-1)(n-2)(n-3), \\
c_3 &= \frac{4}{3}d(d-1)(d-2), \quad c_4 = \frac{4}{3}n(n-1)(n-2), \quad c_5 = 4dn(n-1), \quad c_6 = 4dn(d-1), \\
c_7 &= -2dn(d-1)(n-1), \; c_8 = -\frac{4}{3}dn(n-1)(n-2), \; c_9 = -\frac{4}{3}dn(d-1)(d-2).
\end{aligned} \tag{4.6}$$

Here we are working, for convenience, with the original dilaton ϕ, but the action can be easily converted to the general form of section 2 using the definition $\phi \equiv \overline{\phi} + d\beta + n\gamma$.

Let us first look for constant curvature solutions in the isotropic case $n = 0$. By varying the action with respect to ϕ and N, and setting $\dot{\phi} = x = \text{const}$, $\dot{\beta} = y = \text{const}$, in the gauge $N = 1$, we get the two independent algebraic equations

$$\begin{aligned} x^2 &+ d(d-1)y^2 - 2dxy - \frac{k\alpha'}{4}\left(c_1 y^4 + c_3 x y^3 - x^4\right) - \\ &- (dy - x)\left[-2x + 2dy + \frac{k\alpha'}{4}\left(c_3 y^3 - 4x^3\right)\right] = 0, \\ x^2 &+ d(d-1)y^2 - 2dxy - \frac{3}{4}k\alpha'\left(c_1 y^4 + c_3 x y^3 - x^4\right) = 0. \end{aligned} \tag{4.7}$$

We have explicitly checked that they have real solutions for any d from 1 to 9. For $d = 3, 6, 9$, in particular, the coordinates of the fixed point in the plane $(\dot{\phi}, \dot{\beta})$ are given by (in units $k\alpha' = 1$)

$$\begin{aligned} d &= 3 \qquad x = \pm 1.40..., \qquad y = \pm 0.616..., \\ d &= 6 \qquad x = \pm 1.37..., \qquad y = \pm 0.253..., \\ d &= 9 \qquad x = \pm 1.38..., \qquad y = \pm 0.163..., \quad . \end{aligned} \tag{4.8}$$

Exactly the same results follow from the system of equations obtained by varying $\{N, \beta\}$ and $\{\phi, \beta\}$. In the last case one gets an additional fixed point, which corresponds, however, to $\dot{\overline{\phi}} = x - dy = 0$. In that case the third equation is no longer a consequence of the other two: indeed, by imposing the constraint $\delta S/\delta N = 0$, one finds that the additional solution has to be discarded, in agreement with the general discussion of the previous sections. Although we have not made an exhaustive search, it seems that non-isotropic fixed points are excluded in the absence of an exact duality symmetry.

By integrating numerically the field equations for β and ϕ, and imposing the constraint on the initial data, we have verified that, for any given initial condition corresponding to a state of pre-big bang evolution from the vacuum (i.e. $0 < \dot{\beta} < x$, $\dot{\overline{\phi}} = \dot{\phi} - d\dot{\beta} > 0$), the solution is necessarily attracted to the expanding fixed points (4.8). This is illustrated in Fig. 1, for various numbers of dimensions.

Fig. 1. Curvature regularization of a pre-big bang background as a consequence of the first-order α' corrections (in units $k\alpha' = 1$). The dashed curve shows the singular behaviour of the zeroth-order solution in $d = 9$. The solid curves approach asymptotically the constant values of eq. (4.8).

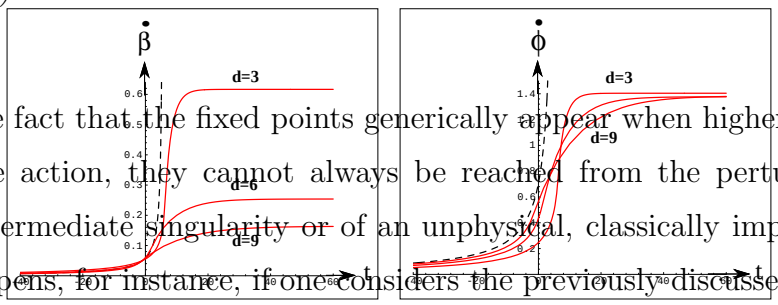

Fig. 1

In spite of the fact that the fixed points generically appear when higher curvature terms are added to the action, they cannot always be reached from the perturbative vacuum, because of an intermediate singularity or of an unphysical, classically impenetrable region. This is what happens, for instance, if one considers the previously discussed action with the Gauss–Bonnet term alone, which does not correspond, in the string frame, to the correct string effective action. The Gauss–Bonnet invariant, by itself, may parametrize the first-order α' corrections only in the Einstein frame, and only in $d = 3$ [7].

For the string effective action (4.3), on the contrary, the fixed points are continuously joined to the perturbative vacuum ($\dot\beta = 0 = \dot\phi$) by the smooth flow of the background in cosmic time, as illustrated by the "β-functions" $\ddot\beta(\dot\beta)$, $\ddot\phi(\dot\phi)$ plotted in Fig. 2 ($\dot\phi$ and $\dot\beta$ are obviously not independent, being related by the constraint equation). They show the turning of the curves $\ddot\beta$, $\ddot\phi$ for the numerical solutions of Fig. 1. The simple action (4.3) thus implements already to first order in α' a smooth transition from the dilaton phase to the string phase of the pre-big bang scenario, in agreement with previous assumptions [14].

Fig. 2. Smooth evolution of the background from the perturbative vacuum, $\dot\beta = 0 = \dot\phi$, to the non-trivial fixed points with $\dot\beta$ and $\dot\phi$ constant reported in eq. (4.8). The dashed curve corresponds to the singular zeroth-order solution.

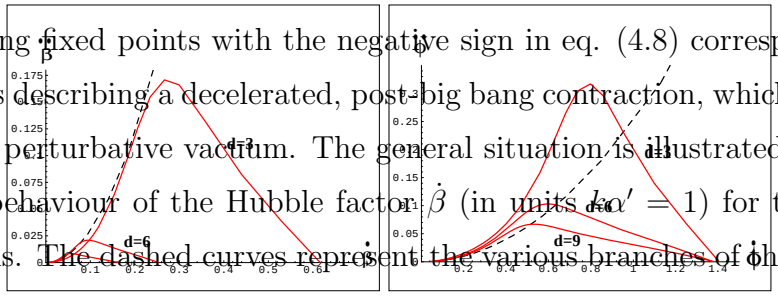

Fig. 2

The contracting fixed points with the negative sign in eq. (4.8) correspond to the time-reversed solutions describing a decelerated, post-big bang contraction, which is also smoothly connected to the perturbative vacuum. The general situation is illustrated in Fig. 3, which shows the time behaviour of the Hubble factor $\dot\beta$ (in units $k\alpha' = 1$) for the case of $d = 3$ spatial dimensions. The dashed curves represent the various branches of the singular zeroth-order solution [5]:

$$\sqrt{d}\dot\beta = \pm|t|^{-1}, \qquad \dot\phi = \pm|t|^{-1}, \tag{4.9}$$

evolving towards ((a) and (c)) or from ((b) and (d)) the singularity, expanding ((a) and (b)) or contracting ((c) and (d)). The position of the singularity has been made to coincide with the origin of the time axis, which thus separates the pre-big bang ($t < 0$) from the post-big bang ($t > 0$) configurations. The four dashed curves are related by T-duality and time-reversal transformation as follows:

$$\begin{aligned} \mathrm{T-duality} \quad &: \qquad (a) \Longleftrightarrow (c), \qquad (b) \Longleftrightarrow (d). \\ \mathrm{t-reversal} \quad &: \qquad (a) \Longleftrightarrow (d), \qquad (b) \Longleftrightarrow (c). \end{aligned} \tag{4.10}$$

The numerical integration shows that, at least according to the model (4.3), only expanding pre-big bang and contracting post-big bang configurations are regularized, to first order in α', as illustrated by the solid curves. In the context of a theory that is exactly duality-invariant, one may expect, however, a more symmetric situation in which the symmetry pattern of the zeroth-order solutions is maintained after regularization.

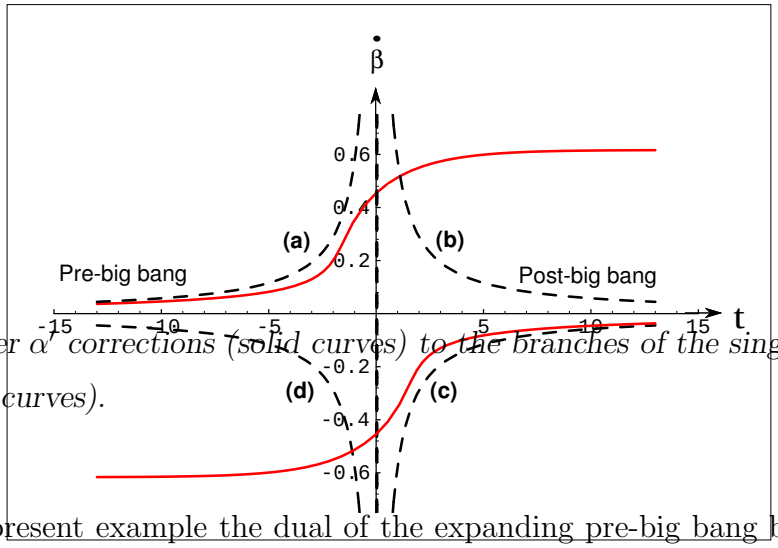

Fig. 3. First-order α' corrections (solid curves) to the branches of the singular zeroth-order solution (dashed curves).

Since in the present example the dual of the expanding pre-big bang branch is not regularized, no smooth monotonic evolution from growing to decreasing curvature is possible, unlike in models where one-loop corrections are included [10]–[13]. In the loop case, however, the final state of the background tends to remain in the pre-big bang sector with $\dot{\overline{\phi}} > 0$, $\dot{\beta} > 0$, because of the final growing rate of the dilaton. The expanding fixed point determined by the α' corrections corresponds instead to a final configuration of the post-big bang type, with $\dot{\overline{\phi}} < 0$, $\dot{\beta} > 0$, as illustrated in Fig. 4 by a numerical integration of the $d = 3$ field equations. This means that, unlike what happens in the one-loop model of [13], it is not impossible for the background to be attracted by an appropriate potential in the expanding, frozen-dilaton state of the standard scenario.

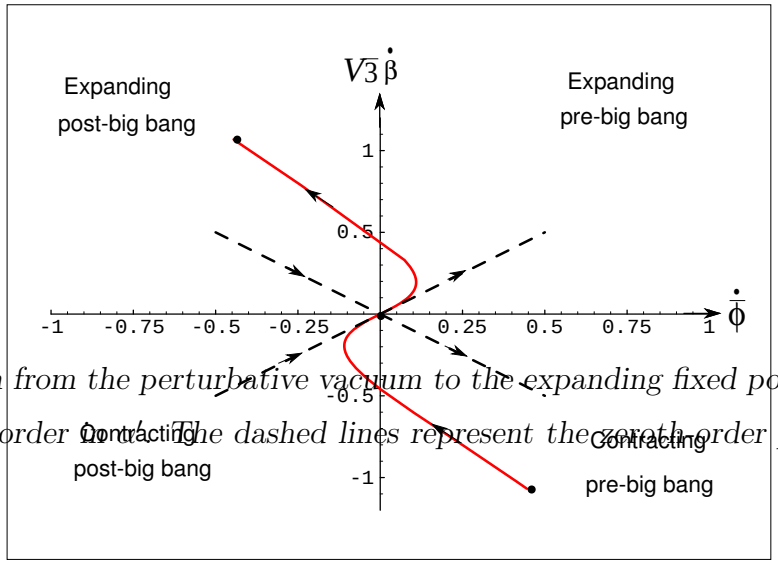

Fig. 4. Evolution from the perturbative vacuum to the expanding fixed point (and its time-reversal) to first order in α'. The dashed lines represent the zeroth-order pre- and post-big bang solutions.

Fig. 4

Let us finally stress, to conclude the discussion of our example, that if we start with sufficiently anisotropic initial conditions, the background in general evolves towards a curvature singularity. However, the isotropic fixed points may also attract anisotropic backgrounds, and the set of anisotropic configurations that are eventually attracted by the isotropic fixed point spans a region of finite size in the space of initial conditions. This property of the first-order action (4.3) is illustrated in Fig. 5, where we have plotted various curves $\dot{\gamma}(t)$ and $\dot{\gamma}(\dot{\bar{\phi}})$ obtained through a numerical integration by varying the initial conditions of $\dot{\gamma}$, at fixed initial conditions for $\dot{\beta}$. The plots refer to the case $d = 2$, $n = 1$ and $\dot{\beta} > 0$, but they are qualitatively the same for any d and n, and can obviously be symmetrically extended to the plane $\dot{\beta} < 0$. When the initial conditions are inside the attraction basin of the fixed points, in this case a segment of the $\dot{\gamma}$ axis, the curvature is stabilized for a particular, fixed initial value of $\dot{\beta}$. In this figure, for the case $d = 2$, $n = 1$, a regular, fixed initial value of $\dot{\beta}$ singularity is not avoided.

Fig. 5. Attraction basin of the fixed points in the space of anisotropic pre-big bang initial conditions, for the case $d = 2$, $n = 1$, and for a particular, fixed initial value of $\dot{\beta}$.

Fig. 5

The size of the attraction basin depends of course on the details of the first-order action. By using, for instance, the $d = 3$ effective action obtained by transforming from the Einstein to the string frame the Gauss–Bonnet invariant, we have found a much larger attraction basin than the one illustrated in Fig. 5, so large as to include part of the region $\dot{\gamma} < 0$, namely contracting initial conditions. The lack of a complete isotropization is not a negative aspect of the model since, in a higher-dimensional background, the cosmological evolution should indeed implement an effective dimensional reduction by separating three expanding dimensions from the contracting "internal" ones.

5 Conclusions

The evolution of a cosmological background from the string perturbative vacuum may lead to a regime in which higher-derivative contributions to the string effective action are large, while string loop effects are still negligible. We have considered, in that regime, the possible existence of a "string-phase", with the background curvatures frozen at a scale controlled by the string length parameter λ_s, corresponding to an exponential evolution of the scale factor and a linear evolution of the dilaton (in cosmic time). We have shown that solutions of this kind may represent an exact solution (to all orders in α') of the tree-level action, and we have discussed an explicit example (to first order in α') in which all expanding isotropic backgrounds, evolving initially from the perturbative vacuum, are necessarily attracted to that constant curvature state.

The main purpose of this paper was to point out the importance of α'corrections for a singularity-free cosmology: by implementing a mechanism of limiting curvature, they can regularize cosmological backgrounds even in the absence of quantum loop effects. The emergence of such a high-curvature string phase, in the weak coupling regime, leads to a cosmological scenario rich of interesting phenomenological consequences.

In a string-theory context, however, the properties of a background obtained to first order in α' (and to any finite order in the α' expansion) are characterized by a certain degree of ambiguity [7], because they are not invariant under field redefinitions. A truly unambiguous cosmological background should correspond to the solution of an exact conformal field theory [26], which automatically includes all orders in α'. In addition, the quantum back-reaction of loops and radiation, as well as an appropriate non-perturbative dilaton potential, are certainly required to complete the transition from the string phase to the radiation-dominated, constant dilaton phase of the standard scenario. In this sense, the results of this paper are still preliminary to the formulation of a complete and realistic string cosmology scenario. Nevertheless, they confirm previous conjectures, clarify the relative role of α' and loop corrections, and motivate a more systematic study of higher-order and exact conformal solutions with constant curvature and linear dilatonic evolution.

Acknowledgements

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References

[1] S. Weinberg, Cosmology and gravitation (Wiley, New York, 1972).

[2] R. Brandenberger and C. Vafa, Nucl. Phys. B316 (1989) 391.

[3] G. Veneziano, Phys. Lett. B265 (1991) 287.

[4] A. A. Tseytlin and C. Vafa, Nucl. Phys. B372 (1992) 443.

[5] M. Gasperini and G. Veneziano, Astropart. Phys. 1 (1993) 317; Mod. Phys. Lett. A8 (1993) 3701; Phys. Rev. D50 (1994) 2519. An updated collection of papers on the pre-big bang scenario is available at `http://www.to.infn.it/teorici/gasperini/`.

[6] C. Lovelace, Phys. Lett. B135 (1984) 75; E. S. Fradkin and A. A Tseytlin, Nucl. Phys. B261 (1985) 1; C. G. Callan et al., Nucl. Phys. B262 (1985) 593; A. Sen, Phys. Rev. Lett. 55 (1985) 1846.

[7] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B293 (1987) 385.

[8] R. Brustein and G. Veneziano, Phys. Lett. B329 (1994) 429.

[9] N. Kaloper, R. Madden and K. A. Olive, Nucl. Phys. B452 (1995) 677; Phys. Lett. B371 (1996) 34; R. Easther, K. Maeda and D. Wands, Phys. Rev. D53 (1996) 4247.

[10] I. Antoniadis, J. Rizos and K. Tamvakis, Nucl. Phys. B415 (1994) 497;
J. Rizos and K. Tamvakis, Phys. Lett. B326 (1994) 57.

[11] R. Easther and K. Maeda, *One-loop superstring cosmology and the non-singular universe*, hep-th/9605173, Phys. Rev. D, in press.

[12] S. J. Rey, Phys. Rev. Lett. 77 (1996) 1929.

[13] M. Gasperini and G. Veneziano, Phys. Lett. B387 (1996) 715.

[14] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B361 (1995) 45; M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. 75 (1995) 3796; M. Gasperini, in "String gravity and physics at the Planck energy scale", ed. by N. Sanchez and A. Zichichi (Kluwer Acad. Pub., Dordrecht, 1996), p. 305; R. Brustein,

M. Gasperini and G. Veneziano, *Peak and end point of the relic gravitaton background in string cosmology*, CERN-TH/96-37 (hep-th/9604084); A. Buonanno, M. Maggiore and C. Ungarelli, *Spectrum of relic gravitational waves in string cosmology*, IFUP-TH 25/96 (gr-qc/9605072).

[15] M. Gasperini, in "Advances in theoretical physics", ed. by E. R. Caianiello (World Scientific, Singapore, 1991), p. 77.

[16] V. Mukhanov and R. Brandenberger, Phys. Rev. Lett. 68 (1992) 1969.

[17] K. A. Meissner and G. Veneziano, Mod. Phys. Lett. A6 (1991)3397; Phys. Lett. B267 (1991) 33; M. Gasperini and G. Veneziano, Phys. Lett. B277 (1992) 256.

[18] A. Sen, Phys. Lett. B271 (1991) 295; S. F. Hassan and A. Sen, Nucl. Phys. B375 (1992) 103.

[19] G. Curci and G. Paffuti, Nucl. Phys. B286 (1987) 399.

[20] C. Myers, Phys. Lett. B199 (1987) 371.

[21] M. Gasperini and M. Giovannini, Phys. Lett. B287 (1992) 56.

[22] D. G. Boulware and S. Deser, Phys. Lett. B175 (1986) 409; S. Kalara, C. Kounnas and K. A. Olive, Phys. Lett. B 215 (1988) 265; S. Kalara and K. A. Olive, Phys. Lett. B218 (1989) 148.

[23] B. Zwiebach, Phys. Lett. B156 (1985) 315.

[24] N. E. Mavromatos and J. L. Miramontes, Phys. Lett. B201 (1988) 473.

[25] K. A. Meissner, *Symmetries of higher-order string gravity actions*, CERN-TH/96-291 (hep-th/9610131).

[26] E. Kiritsis and C. Kounnas, Phys. Lett. B331 (1994) 51.

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CERN-PH-TH/2009-249
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A covariant and gauge invariant formulation of the cosmological "backreaction"

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Abstract

Using our recent proposal for defining gauge invariant averages we give a general-covariant formulation of the so-called cosmological "backreaction". Our effective covariant equations allow to describe in an explicitly gauge invariant form the way classical or quantum inhomogeneities affect the average evolution of our Universe.

1 Introduction

It is well known that the homogeneous and isotropic Friedman-Lemaitre-Robertson-Walker (FLRW) metric describes the geometric properties of our Universe only on sufficiently large scales of distance, and has to be interpreted as the "averaged" cosmological metric: namely, the metric emerging from an appropriate smoothing-out of the local inhomogeneities and anisotropies. The same interpretation of averaged variables has to be assigned to the matter energy and momentum density, sourcing the FLRW metric in the cosmological Einstein equations.

A problem (a rather old one, see for instance [1]) thus appears due to the fact that the Einstein equations for the *averaged geometry* are different, in general, from the *averaged Einstein equations*. This is because the averaging procedure does not commute, in general, with the non-linear differential operators appearing in Einstein's equations. As a consequence, the dynamics of the averaged geometry is affected by so-called "backreaction" terms, originating from the contribution of the inhomogeneities present in the metric and matter sectors.

Interest in these themes has considerably risen after the suggestion that the cosmic acceleration, recently observed on large scales, could be unrelated to phantomatic dark-energy sources, but – perhaps more simply – to the dynamical effects of the backreaction (see e.g. [2]) thus solving the well-known "coincidence problem". This possibility has focused current researches on a new problem: the gauge invariance of the averaging procedure. Indeed, in the absence of gauge invariance, the computed backreaction effects depend not only on the hypersurface chosen to compute the average integrals, but also on the chosen coordinate frame (see e.g. [3]-[6] for recent discussions).

At present, the most commonly used averaging procedure in a cosmological context is based on the foliation of space-time into three-dimensional hypersurfaces comoving with the matter sources, and on the volume integration over such spacelike hypersurfaces [7] (see [8] for a recent review, and [6, 9] for an extension to more general hypersurfaces). This procedure is applied, in particular, to the scalar part of the cosmological Einstein equations, i.e. to the so-called Hamiltonian constraint and Raychaudhuri's equation. By spatially averaging such equations on a comoving domain D (and considering, for simplicity, dust fluid sources) one obtains [7]:

$$\left(\frac{\dot{a}_D}{a_D}\right)^2 = \frac{8\pi G}{3}\langle\rho\rangle_D - \frac{1}{6}\left(\langle Q\rangle_D + \langle\mathcal{R}\rangle_{\mathcal{D}}\right), \tag{1.1}$$

$$-\frac{\ddot{a}_D}{a_D} = \frac{4\pi G}{3}\langle\rho\rangle_D - \frac{1}{3}\langle Q\rangle_D. \tag{1.2}$$

Here the dot denotes derivatives with respect to the cosmic time of the comoving synchronous coordinates, a_D is an effective scale factor, related to the volume V_D of the integration domain (normalized by a reference volume scale V_{D_0}) by $a_D = (V_D/V_{D_0})^{1/3}$, where

$$V_D = \int_D d^3x \sqrt{|\det g_{ij}|}, \tag{1.3}$$

and g_{ij} is the intrinsic metric of the comoving hypersurfaces. Also, the brackets denote spatial average over D, namely

$$\langle \cdots \rangle_D = \frac{1}{V_D} \int_D d^3x \sqrt{|\det g_{ij}|}\,(\cdots)\,, \tag{1.4}$$

ρ is the energy density of the dust sources, and $\mathcal{R}$ is the scalar intrinsic curvature associated with the spatial metric g_{ij}. Finally, $\langle Q \rangle$ is a correction called "kinematical" backreaction, arising in a particular gauge (see section 2) from the averages of two scalar quantities: the trace Θ of the expansion tensor and the scalar shear σ^2:

$$\langle Q \rangle_D = \frac{2}{3}\left(\langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2\right) - 2\langle \sigma^2 \rangle_D. \tag{1.5}$$

The above averaged equations are obtained within a spatial slicing of the space-time manifold induced by the flow lines of the matter sources. In this paper we present a covariant version of the effective equations for the averaged cosmological quantities (called hereafter, for simplicity, averaged cosmological equations), based on a recently proposed gauge invariant averaging prescription [4]: the backreaction effects we obtain depend on the hypersurface chosen to define the physical observer, but *do not depend* on the particular choice of coordinates. In the appropriate class of gauges we recover the recent results given in [6, 9]. In addition, for an observer at rest with respect to the matter sources, we recover the same results as in [7, 8].

The paper is organized as follows. In Sect. 2 we present an explicitly covariant version of the Hamiltonian constraint, Raychaudhuri's equation, and of the projected conservation equation for a generic foliation of spacetime and energy-momentum tensor. In Sect. 3 we briefly summarize the main aspects of our gauge-invariant averaging prescription, and we derive the corresponding general-covariant version of the averaged cosmological equations. Our conclusive remarks are briefly presented in Sect. 4.

2 Covariant ADM equations

In order to introduce spatial averages of physical quantities we consider a general class of foliations of spacetime by spacelike hypersurfaces $\Sigma(A)$ over which a scalar field $A(x)$ takes

constant values (the so-called level-sets of A). Let n_μ be the future-directed unit normal to $\Sigma(A)$, defined by

$$n_\mu = -\frac{\partial_\mu A}{(-\partial_\mu A \partial_\nu A \; g^{\mu\nu})^{1/2}} \quad , \qquad\qquad n_\mu n^\mu = -1 \tag{2.1}$$

(we are using the metric signature $(-,+,+,+)$). Let us also introduce the projector $h_{\mu\nu}$ into the hypersurfaces by:

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \, , \qquad\qquad h_{\mu\rho} h^\rho_\nu = h_{\mu\nu} \, , \qquad\qquad h_{\mu\nu} n^\mu = 0 \, . \tag{2.2}$$

The Einstein equations $G_{\mu\nu} = T_{\mu\nu}$ (we use units in which $8\pi G = 1$) can then be projected along n^μ and $h^{\mu\nu}$, and give rise to three (sets of) equations that can be chosen in the following form:

$$G_{\mu\nu} n^\mu n^\nu = T_{\mu\nu} n^\mu n^\nu \equiv \varepsilon \, , \tag{2.3}$$

$$G_{\mu\nu} n^\mu h^\nu_\rho = T_{\mu\nu} n^\mu h^\nu_\rho \equiv J_\rho \, , \tag{2.4}$$

$$R_{\mu\nu} h^\mu_\rho h^\nu_\sigma = T_{\mu\nu} h^\mu_\rho h^\nu_\sigma - \frac{1}{2} h_{\rho\sigma} T \equiv \mathcal{S}_{\rho\sigma} - \frac{1}{2} h_{\rho\sigma} T \, . \tag{2.5}$$

They correspond to an explicitly covariant version of the so-called Arnowitt-Deser-Misner (ADM) equations.

It is always possible to make contact with the more conventional ADM formalism by choosing a class of gauges in which the scalar field $A(x)$ is homogeneous. We shall call it the ADM gauges. In such a gauge the normal vector n_μ takes the form:

$$n_\mu = N(-1,0,0,0) \, , \qquad\qquad n^\mu = \frac{1}{N}(1, -N^i) \, , \tag{2.6}$$

where N and N^i are, respectively, the so-called lapse function and shift vector. In this gauge:

$$h^\mu_i = \delta^\mu_i \, , \qquad\qquad h^0_0 = 0 \, , \qquad\qquad h_{ij} = g_{ij} \, , \tag{2.7}$$

where g_{ij} is the induced 3–metric (or first fundamental form) on Σ. The spacetime metric of the foliated spacetime is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad , \tag{2.8}$$

and its inverse by:

$$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu = -N^{-2}(\nabla_0 - N^i \nabla_i)^2 + {}^{(3)}g^{ij} \nabla_i \nabla_j \quad . \tag{2.9}$$

where ${}^{(3)}g^{ij}$ is the inverse of the 3×3 metric g_{ij}.

In this class of ADM gauges Eqs. (2.3) and (2.4) reduce to the convential form of the Hamiltonian and momentum constraints, respectively, while Eq. (2.5) generates the

second order evolution equations. In the ADM context such evolution equations are splitted into twice as many equations, those defining the second fundamental form (or extrinsic curvature) K_i^j, and those describing the evolution of K_i^j itself (see, e.g. [8]).

In order to give a covariant and gauge invariant formulation of the cosmological back-reaction we shall make use of the covariant Eqs. (2.3)–(2.5), which lead to an explicitly scalar form of the Hamiltonian constraint and Rachayduri's equation. To this purpose, let us first consider the spacetime flow generated by the timelike vector field n_μ, and define the projected expansion tensor of the flow worldlines as

$$\Theta_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \nabla_\alpha n_\beta = \frac{1}{3} h_{\mu\nu}\Theta + \sigma_{\mu\nu} + \omega_{\mu\nu} \,, \tag{2.10}$$

where

$$\Theta \equiv \nabla_\mu n^\mu, \qquad \sigma_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \left(\nabla_{(\alpha} n_{\beta)} - \frac{1}{3} h_{\alpha\beta} \nabla_\tau n^\tau \right), \quad \omega_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \nabla_{[\alpha} n_{\beta]}, \tag{2.11}$$

are the expansion scalar, the shear tensor and the rotation tensor, respectively. In our case, with n_μ given by Eq. (2.1), we have a zero rotation tensor and we can write

$$\sigma_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu}\Theta \,, \qquad \sigma^2 \equiv \frac{1}{2}\sigma_\nu^\mu \sigma_\mu^\nu = \frac{1}{2}\left(\Theta_\nu^\mu \Theta_\mu^\nu - \frac{1}{3}\Theta^2 \right) \,, \tag{2.12}$$

where σ^2 is the shear scalar.

Consider first the Hamiltonian constraint (2.3), whose left hand side can be decomposed as follows:

$$R_{\mu\nu} n^\mu n^\nu + \frac{1}{2} R = \frac{1}{2}\left(\Theta^2 - \Theta_\nu^\mu \Theta_\mu^\nu \right) + \frac{1}{2}\mathcal{R}_s \,. \tag{2.13}$$

In the ADM gauge the term in brackets on the right hand side can be expressed in terms of the extrinsic curvature as:

$$\Theta^2 - \Theta_\nu^\mu \Theta_\mu^\nu = K^2 - K_j^i K_i^j \,, \tag{2.14}$$

while $\mathcal{R}_s$ is a scalar which, in the ADM gauge, goes over to the intrinsic scalar curvature $\mathcal{R}$ associated with the induced metric g_{ij}. We can thus rewrite the Hamiltonian constraint completely in terms of scalar quantities as:

$$\mathcal{R}_s + \Theta^2 - \Theta_\nu^\mu \Theta_\mu^\nu = \mathcal{R}_s + \frac{2}{3}\Theta^2 - 2\sigma^2 = 2T_{\mu\nu} n^\mu n^\nu = 2\varepsilon \,. \tag{2.15}$$

Coming now to the explicitly scalar form of Rachayduri's equation, we note that it corresponds to the linear combination of Eq. (2.3) and of the trace of Eq. (2.5), leading to:

$$R_{\mu\nu} n^\mu n^\nu = T_{\mu\nu} h^{\mu\nu} - \frac{1}{2} T. \tag{2.16}$$

After some straightforward calculation the above equation takes the form:

$$\begin{aligned} -n^\mu \nabla_\mu \Theta &= 2\sigma^2 + \frac{1}{3}\Theta^2 - \nabla^\nu (n^\mu \nabla_\mu n_\nu) + \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) n^\mu n^\nu \\ &= 2\sigma^2 + \frac{1}{3}\Theta^2 - \nabla^\nu (n^\mu \nabla_\mu n_\nu) + \varepsilon + \frac{1}{2} T \,. \end{aligned} \tag{2.17}$$

So far we have made no assumptions on the form of $T_{\mu\nu}$. Let us now restrict ourselves to the case of a perfect fluid with:

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{2.18}$$

where u_μ is the 4-velocity comoving with the fluid, and ρ and p are, respectively, the (scalar) energy density and pressure in the fluid's rest frame (the generalization to several non-interacting fluids is straightforward). We stress that u_μ and n_μ are in general distinct: the former depends on the properties of the matter sources, the latter depends on the choice of the hypersurfaces on which we want to average, hence should be determined by the particular problem at hand. With this model of sources we can then express the basic quantities entering our equations (2.15) and (2.17) as:

$$\varepsilon = T_{\mu\nu} n^\mu n^\nu = (\rho + p)(u^\mu n_\mu)^2 - p \,, \qquad T = T^\mu_\mu = -\rho + 3p \,. \tag{2.19}$$

For later use, we note that the trace T of the energy momentum tensor is unaffected by the possible "tilt" (misalignment) between u_μ and n_μ, while the sources of the covariant ADM Eqs. (2.3)–(2.5) – namely the objects that we may call the ADM energy density ε, the ADM pressure $\pi = \mathcal{S}^\rho_\rho/3$, and the ADM current J_ν – become:

$$\varepsilon = \rho - (\rho + p)\left(1 - (u^\mu n_\mu)^2\right) \,, \tag{2.20}$$

$$\pi = p - \frac{1}{3}(\rho + p)\left(1 - (u^\mu n_\mu)^2\right) \,, \tag{2.21}$$

$$J_\nu = (\rho + p) u^\beta n_\beta (1 + u^\mu n_\mu) u_\nu . \tag{2.22}$$

On the other hand, a straightforward calculation in the ADM gauge leads to:

$$(u^\mu n_\mu)^2 = 1 + {}^{(3)}g^{ij} u_i u_j \geq 1 \,, \tag{2.23}$$

meaning that this quantity is always larger than 1. We can thus introduce a "tilt angle" α_T such that $\sinh^2 \alpha_T = (u^\mu n_\mu)^2 - 1$, and we can rewrite ϵ and π in the more convenient form

$$\varepsilon = \rho + (\rho + p) \sinh^2 \alpha_T \,, \qquad \pi = p + \frac{1}{3}(\rho + p) \sinh^2 \alpha_T \,. \tag{2.24}$$

Let us finally consider the condition following from the projected energy-momentum conservation law, $n^\mu \nabla^\mu T_{\mu\nu} = 0$. It is known that such condition is not implied by the

two projected scalar Einstein equations considered above, and has to be added to the set of averaged cosmological equations as an additional independent constraint [7, 8]. In the general context we are considering (with $n_\mu \neq u_\mu$), the projected conservation equation can be written explicitly as follows:

$$u^\mu \partial_\mu \left[(\rho + p)\, u^\rho n_\rho \right] + n^\mu \partial_\mu p + (\rho + p) \left[\nabla_\mu u^\mu u^\rho n_\rho - \Theta^{\mu\nu} u_\mu u_\nu \right] = 0. \tag{2.25}$$

3 Covariant averaged equations

Let us begin from the four dimensional integral of a scalar $S(x)$ as defined in [4]:

$$I(S, \Omega) = \int_{\Omega(x)} d^4x \sqrt{-g(x)}\, S(x) \equiv \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)}\, S(x) W_\Omega(x). \tag{3.1}$$

The integration region $\Omega \subseteq \mathcal{M}_4$ is defined in terms of a suitable scalar window function W_Ω, selecting a region with temporal boundaries determined by the space-like hypersurfaces $\Sigma(A)$ (defined in Sect. 2), and with spatial boundary determined by the coordinate condition $B < r_0$, where B is a (positive) function of the coordinates with space-like gradient $\partial_\mu B$, and r_0 is a positive constant. As we are interested in the variation of the volume averages along the flow lines normal to $\Sigma(A)$, we choose in particular the following window function:

$$W_\Omega(x) = n^\mu \nabla_\mu \theta(A(x) - A_0) \theta(r_0 - B(x)), \tag{3.2}$$

where θ is the Heaviside step function, and n_μ is defined in Eq. (2.1).

As discussed in [4], if $B(x)$ is a scalar function the integral (3.1) is not only a scalar under general coordinate transformations but is also gauge invariant (to all orders): namely, it is invariant under the local field reparametrizations induced by any coordinate change when old and new fields are evaluated at the same space-time position. If B is not a scalar, the spatial boundary can be a source of breaking of covariance and gauge invariance. In [10] we will discuss in more detail such a breaking, and confirm that it goes away in the limit of large spatial volumes (with respect to the typical scale of inhomogeneities) [4]. Using the window function (3.2), the integral (3.1) becomes:

$$I(S, A_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta(A(x) - A_0) (-\partial_\mu A\, \partial^\mu A)^{1/2} \theta(r_0 - B(x))\, S(x)\,. \tag{3.3}$$

Let us now consider the derivative of $I(S, A_0)$ with respect to A_0, a quantity that, like I itself, is covariant and gauge invariant (apart from a possible gauge dependence induced by the spatial boundary):

$$\frac{\partial I(S, A_0)}{\partial A_0} = - \int d^4x \sqrt{-g(x)}\, \delta'\left(A(x) - A_0\right) (-\partial_\mu A\, \partial^\mu A)^{1/2} \theta(r_0 - B(x))\, S(x)$$

$$= -\int d^4x\sqrt{-g(x)}\,\partial_0\delta\left(A(x)-A_0\right)\left[\partial_0 A(x)\right]^{-1}\left(-\partial_\mu A\,\partial^\mu A\right)^{1/2}\theta(r_0-B(x))\,S(x). \tag{3.4}$$

Within adapted ADM coordinates, where A is homogeneous, we can always choose those with vanishing shift (i.e. with $g_{00}=-N^2$ and $g_{0i}=0$). In such coordinates we have:

$$\begin{aligned}\frac{\partial I(S,A_0)}{\partial A_0} &= -\int d^4x\sqrt{-g}\;\partial_0\delta\left(A(t)-A_0\right)(-g^{00})^{1/2}\theta(r_0-B(x))\,S(x)\\ &= \int d^4x\,\delta\left(A(t)-A_0\right)\partial_0\left[\sqrt{-g}(-g^{00})^{1/2}\theta(r_0-B(x))\,S(x)\right]\\ &= \int d^4x\,\sqrt{|\gamma|}\delta(A(t)-A_0)\Big[\theta(r_0-B(x))\left(N\Theta\,S+\partial_0 S\right)-\delta(r_0-B(x))S\partial_0 B\Big],\end{aligned} \tag{3.5}$$

where $\gamma=\det g_{ij}$, and we have used that, in these coordinates,

$$\Theta = N^{-1}\partial_0\log\sqrt{\gamma}. \tag{3.6}$$

The above equation can be easily recast into the following covariant and gauge invariant form,

$$\frac{\partial I(S,A_0)}{\partial A_0} = I\left(\frac{\partial_\mu A\partial^\mu S}{\partial_\mu A\partial^\mu A},A_0\right)+I\left(\frac{S\,\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}},A_0\right)-2I\left(\frac{\partial_\mu A\partial^\mu B}{\partial_\mu A\partial^\mu A}S\delta(r_0-B),A_0\right), \tag{3.7}$$

that reduces to Eq. (3.5) in the special coordinates we have been using. Note that the last factor of the above equation is absent if n^μ is orthogonal to the gradient of B ($n^\mu\partial_\mu B=0$), namely if B does not depend on the time coordinate of the gauge used in Eq. (3.5). We shall restrict ourselves to this case hereafter.

Let us define now the covariant averaging prescription for a scalar $S(x)$ on the hypersurfaces of constant A, following [4], as:

$$\langle S\rangle_{A_0} = \frac{I(S,A_0)}{I(1,A_0)}. \tag{3.8}$$

Using Eq. (3.5) we can easily obtain the derivative of the averaged scalar $\langle S\rangle_{A_0}$ as:

$$\frac{\partial\langle S\rangle_{A_0}}{\partial A_0} = \langle\partial_{A_0}S\rangle_{A_0}+\left\langle S\frac{N\Theta}{\partial_0 A}\right\rangle_{A_0}-\langle S\rangle_{A_0}\left\langle\frac{N\Theta}{\partial_0 A}\right\rangle_{A_0}, \tag{3.9}$$

and Eq. (3.7) immediately gives us the generally covariant version of the above equation:

$$\frac{\partial\langle S\rangle_{A_0}}{\partial A_0} = \left\langle\frac{\partial_\mu A\partial^\mu S}{\partial_\mu A\partial^\mu A}\right\rangle_{A_0}+\left\langle\frac{S\,\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0}-\langle S\rangle_{A_0}\left\langle\frac{\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0}. \tag{3.10}$$

This is the covariant and gauge invariant generalization of the Buchert-Ehlers commutation rule [11], and will be the starting point for our generalization of the averaged cosmological

equations. However, let us first illustrate the precise connection to the special version of this rule obtained in [11].

In ADM coordinates, with A homogeneous, Eq. (3.3) reads:

$$I(S, A_0) = \int_{\Sigma_{A_0}} d^3x \sqrt{|\gamma(t_0, \vec{x})|}\, S(t_0, \vec{x}) \theta\left(r_0 - B(t_0, \vec{x})\right) \,, \tag{3.11}$$

where we have called t_0 the time when $A(t)$ takes the constant values A_0, and the averages are referred to a section of the three-dimensional hypersurface Σ_{A_0} where $A(x) = A_0$. This is exactly the type of spatial integrals used in [7] (with a domain D determined by the condition $B(x) < r_0$), and thus leads to the same definition of averages (see Eq. (1.4)). Let us compute, in this case, the corresponding version of Eq. (3.10). Multiplying both sides by $\partial_t A_0$ we obtain the equation

$$\begin{aligned} \partial_t \langle S \rangle_{A_0} &= \left\langle \partial_t S - N^i \partial_i S \right\rangle_{A_0} + \langle SN\Theta \rangle_{A_0} - \langle S \rangle_{A_0} \langle N\Theta \rangle_{A_0} \\ &= \langle N n^\mu \partial_\mu S \rangle_{A_0} + \langle SN\Theta \rangle_{A_0} - \langle S \rangle_{A_0} \langle N\Theta \rangle_{A_0} \,, \end{aligned} \tag{3.12}$$

which generalises the commutation rule given in [11] to the case of non-vanishing shift vector. For $N^i = 0$ the standard result is recovered.

Let us come now to the covariant formulation of the averaged cosmological equations. Starting from the generally covariant volume integral

$$I(1, A_0) = \int d^4x \sqrt{-g}\, \delta(A(x) - A_0) \sqrt{-\partial_\mu A \partial^\mu A}\, \theta(r_0 - B(x)) \tag{3.13}$$

we define, along the lines of the previous approach [7], an effective scale factor $\widetilde{a}$ such that

$$\frac{1}{\widetilde{a}} \frac{\partial \widetilde{a}}{\partial A_0} = \frac{1}{3 I(1, A_0)} \frac{\partial I(1, A_0)}{\partial A_0}. \tag{3.14}$$

We then find, using Eqs. (3.7), (3.8),

$$\frac{1}{\widetilde{a}} \frac{\partial \widetilde{a}}{\partial A_0} = \frac{1}{3} \left\langle \frac{\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}} \right\rangle_{A_0} \,. \tag{3.15}$$

We are now in the position of presenting the covariant generalization of the averaged equation (1.1). Taking the square of the previous equation, using the Hamiltonian constraint in the form of Eq. (2.15), and explicitly reintroducing Newton's constant in the formulae, we easily obtain:

$$\begin{aligned} \left(\frac{1}{\widetilde{a}} \frac{\partial \widetilde{a}}{\partial A_0} \right)^2 &= \frac{8\pi G}{3} \left\langle \frac{\varepsilon}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} - \frac{1}{6} \left\langle \frac{\mathcal{R}_s}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} \\ &\quad - \frac{1}{9} \left[\left\langle \frac{\Theta^2}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} - \left\langle \frac{\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}} \right\rangle_{A_0}^2 \right] + \frac{1}{3} \left\langle \frac{\sigma^2}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} \,. \end{aligned} \tag{3.16}$$

In order to arrive at the covariant generalization of the second Buchert's equation (1.2) we start with the simple relation

$$\frac{1}{\widetilde{a}}\frac{\partial^2\widetilde{a}}{\partial A_0^2}=\frac{\partial}{\partial A_0}\left(\frac{1}{\widetilde{a}}\frac{\partial\widetilde{a}}{\partial A_0}\right)+\left(\frac{1}{\widetilde{a}}\frac{\partial\widetilde{a}}{\partial A_0}\right)^2. \tag{3.17}$$

Using Eq. (3.15), and the general commutation rule (3.10) for the first term on the right hand side, we obtain

$$\begin{aligned}-\frac{1}{\widetilde{a}}\frac{\partial^2\widetilde{a}}{\partial A_0^2} &= \frac{2}{9}\left\langle\frac{\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0}^2+\frac{1}{6}\left\langle\frac{\partial_\mu A\partial^\mu(\partial_\nu A\partial^\nu A)}{(-\partial_\mu A\partial^\mu A)^{5/2}}\Theta\right\rangle_{A_0}\\ &\quad -\frac{1}{3}\left\langle\frac{\Theta^2}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}+\frac{1}{3}\left\langle\frac{\partial_\mu A\partial^\mu\Theta}{(-\partial_\mu A\partial^\mu A)^{3/2}}\right\rangle_{A_0}.\end{aligned} \tag{3.18}$$

Inserting then the covariant Raychaudhuri's equation (2.17) in the last term of the above equation we are lead to the final result,

$$\begin{aligned}-\frac{1}{\widetilde{a}}\frac{\partial^2\widetilde{a}}{\partial A_0^2} &= \frac{4\pi G}{3}\left\langle\frac{\varepsilon+3\pi}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}-\frac{1}{3}\left\langle\frac{\nabla^\nu(n^\mu\nabla_\mu n_\nu)}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}+\frac{1}{6}\left\langle\frac{\partial_\mu A\partial^\mu(\partial_\nu A\partial^\nu A)}{(-\partial_\mu A\partial^\mu A)^{5/2}}\Theta\right\rangle_{A_0}\\ &\quad -\frac{2}{9}\left[\left\langle\frac{\Theta^2}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}-\left\langle\frac{\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0}^2\right]+\frac{2}{3}\left\langle\frac{\sigma^2}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0},\end{aligned} \tag{3.19}$$

where we have used the relation $\rho-3p=\varepsilon-3\pi$ (see Eq. (2.24)). Equations (3.16) and (3.19), together with the averaged conservation equation discussed below, are the main results of this paper.

Let us now observe that Eq. (3.16), written in the ADM gauge, and multiplied by $(\partial_t A_0)^2$, reduces to the equation recently presented in [6, 9]:

$$\begin{aligned}\left(\frac{1}{\widetilde{a}}\frac{\partial\widetilde{a}}{\partial t}\right)^2 &= \frac{8\pi G}{3}\langle N^2\rho\rangle_{A_0}+\frac{8\pi G}{3}\left\langle N^2(\rho+p)\sinh^2\alpha_T\right\rangle_{A_0}-\frac{1}{6}\langle N^2\mathcal{R}\rangle_{A_0}\\ &\quad -\frac{1}{9}\left(\langle N^2\Theta^2\rangle_{A_0}-\langle N\Theta\rangle_{A_0}^2\right)+\frac{1}{3}\langle N^2\sigma^2\rangle_{A_0}.\end{aligned} \tag{3.20}$$

We have used Eq. (2.24) to replace the ADM parameter ε with the fluid proper energy and pressure. When $\sinh^2\alpha_T=0$ (namely when the averaging hypersurfaces coincide with those orthogonal to the fluid velocity, $n_\mu=u_\mu$), we exactly recover the corresponding Buchert's equation (see the second paper of [7]). In the synchronous gauge ($N=1$) and for dust sources ($p=0$) we recover instead Buchert's Eq. (1.1), apart from the additional contribution arising from a nonvaninshing tilt angle α_T.

Consider now the second equation (3.19), and let us first note that

$$\frac{1}{\widetilde{a}}\frac{\partial^2\widetilde{a}}{\partial t^2}=\left(\frac{\partial A_0}{\partial t}\right)^2\frac{1}{\widetilde{a}}\frac{\partial^2\widetilde{a}}{\partial A_0^2}+\frac{1}{\widetilde{a}}\frac{\partial^2 A_0}{\partial t^2}\frac{\partial\widetilde{a}}{\partial A_0}. \tag{3.21}$$

We go then to the ADM coordinates, as before, imposing in addition the convenient gauge choice $N^i = 0$, and we apply Eq. (2.24) to express ε and π in terms of ρ and p. By using Eq. (3.19) to eliminate the first term on the right hand side of Eq. (3.21) we obtain

$$\begin{aligned} -\frac{1}{\widetilde{a}}\frac{\partial^2 \widetilde{a}}{\partial t^2} &= \frac{4\pi G}{3}\left\langle N^2(\rho+3p)\right\rangle_{A_0} + \frac{8\pi G}{3}\left\langle N^2(\rho+p)\sinh^2\alpha_T\right\rangle_{A_0} \\ &- \frac{2}{9}\left(\langle N^2\Theta^2\rangle_{A_0} - \langle N\Theta\rangle^2_{A_0}\right) + \frac{2}{3}\langle N^2\sigma^2\rangle_{A_0} \\ &- \frac{1}{3}\left\langle \Theta\frac{\partial N}{\partial t}\right\rangle_{A_0} - \frac{1}{3}\left\langle N\ {}^{(3)}g^{ij}\nabla_i\nabla_j N\right\rangle_{A_0}, \end{aligned} \tag{3.22}$$

in agreement with [6]. Again, for $\sinh^2\alpha_T = 0$, we also recover the corresponding Buchert's equation (see for instance [8]). We may note that, when $\sinh^2\alpha_T \neq 0$, the "tilt effects" give a negative contribution (assuming $\rho + p > 0$) to the average cosmic acceleration.

Let us conclude this section by noting that, as anticipated in Sect. 2, the set of averaged cosmological equations has to be complemented by the general-covariant average of the conservation equation (2.25). Such an operation can be performed straightforwardly, according to the general procedure outlined above. We shall present here, for simplicity, the covariant version of the averaged conservation equation in the particular case in which the space-time foliation is referred to the fluid comoving frame. In such a case, setting $n_\mu = u_\mu$ in Eq. (2.25), and applying the general-covariant commutation rule (3.10), we obtain

$$\frac{\partial}{\partial A_0}\langle\rho\rangle_{A_0} = -\left\langle\frac{\Theta\, p}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0} - \langle\rho\rangle\left\langle\frac{\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0} \tag{3.23}$$

(note that, in this case, the scalar A corresponds to the velocity potential of the fluid sources). Going, as before, to the ADM coordinates, and multiplying by $\partial_t A_0$, we recover the known result

$$\frac{\partial}{\partial t}\langle\rho\rangle_D + \langle N\Theta\, p\rangle_D - \langle\rho\rangle\langle N\Theta\rangle_D = 0, \tag{3.24}$$

already presented in various papers [7, 8].

4 Conclusion

The main results of this paper are the covariant and gauge invariant formulation of the Buchert-Ehlers commutation rule, Eq. (3.10), and of the effective equations for the averaged evolution of a perfect fluid-dominated Universe, Eqs. (3.16), (3.19),(3.23). The average is performed over a generic class of hypersurfaces, not necessarily orthogonal to the fluid flow lines. We stress that our results allow to compute averaged quantities in a completely arbitrary coordinate system.

The results obtained in this paper can be directly applied to the case of the quantum cosmological backreaction, by using the correspondence between quantum expectation values and classical averages performed over all three-dimensional space, as illustrated in details in [4]. Hence, in particular, can be applied to study the effect of the quantum fluctuations within the canonical formalism of cosmological perturbation theory.

At the same time, the classical averaged equations (possibly extended to light-like hypersurfaces) may provide a covariant starting point for determining whether present inhomogeneities can significantly contribute to the observed cosmic acceleration.

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References

[1] G. F. R. Ellis, in "*10th International Conference on General Relativity and Gravitation*", ed. by B. Bertotti, De Felice and A. Pascolini (Reidel, Dordrecht, 1984), p. 215.

[2] S. Rasanen, JCAP **02**, 003 (2004).

[3] N. Li and D. J. Schwarz, Phys. Rev. **D76**, 083011 (2007).

[4] M. Gasperini, G. Marozzi, and G. Veneziano, JCAP **03**, 011 (2009).

[5] J. Larena, Phys. Rev. **D79**, 084006 (2009).

[6] I. A. Brown, J. Behrend and K. A. Malik, JCAP **11**, 027 (2009).

[7] T. Buchert, Gen. Rel. Grav. **32**, 105 (2000); **33**, 1381 (2001).

[8] T. Buchert, Gen. Rel. Grav. **40**, 467 (2008).

[9] I. A. Brown, G. Robbers and J. Behrend, JCAP **04**, 016 (2009); I. A. Brown, L. Schrempp and K. Ananda, arXiv:0909.1922 [gr-qc].

[10] M. Gasperini, G. Marozzi, and G. Veneziano, in preparation.

[11] T. Buchert and J. Ehlers, Astron. Astrophys. **320**, 1 (1997).

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Dilatonic Interpretation of the Quintessence?

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We discuss the possibility that "quintessential effects", recently displayed by large scale observations, may be consistently described in the context of the low-energy string effective action, and we suggest a possible approach to the problem of the cosmic coincidence based on the link between the strength of the dilaton couplings and the cosmological state of our Universe.

In a string theory context [1], the macroscopic and large scale gravitational interactions are described by a low-energy effective action which necessarily contains a scalar field – the dilaton – and which can be written, in general, as the action of a non-minimally coupled and self-interacting scalar-tensor theory.

The dilaton ϕ controls the strength of the gravitational coupling and – in superstring models of unification – also the strength of all other interactions, since the expectation value of the dilaton should fix the fundamental ratio between string and Planck mass, and the gauge coupling constant of GUT theories [2]. At the tree-level,

$$g_s^2 = \exp\langle\phi\rangle \simeq (M_S/M_P)^2 \simeq \alpha_{\rm GUT}. \qquad (1)$$

It may be possible (and even auspicated, in certain string-inspired cosmological scenarios [3]) that in the very early past of our Universe the dilaton was varying very rapidly in time, and the effective gravitational interactions were very different from those described by the Einstein equations. At present, however, the dilaton has to be frozen, to be consistent with the observed values of the coupling constants. The string effective action should thus contain an appropriate potential to allow the solution $\phi = \phi_0 = $ const, or a regime of slow-enough dilaton variation. In such a regime, in addition, the dilatonic interactions must become weak enough and/or short-range, so that the action may provide a correct description of present macroscopic gravity.

The potential energy of a nearly constant dilaton, on the other hand, introduces into the gravitational equations a cosmological term, which might lead the Universe to a phase of cosmic repulsion and accelerated expansion. It seems thus natural to wonder whether the dilaton could simulate consistently, in a string theory context, the effects of the so-called "quintessence" [4], the phantomatic scalar field introduced to fit recent cosmological observations [5], and to (possibly) alleviate some problems posed by the phenomenological need for a vacuum source with negative pressure and very small energy density, which seems to dominate our present Universe. If the answer would be positive, there would be no need to assume the existence of new exotic scalar fields, and/or to invent *ad hoc* models of scalar interactions.

In this paper we shall assume that the dilaton is frozen, for some mechanism, already in the radiation era, with a potential energy small enough to avoid disturbing the standard cosmological evolution down to the equilibrium epoch. It could be argued that a frozen scalar field with, a very small but non-vanishing potential energy, is a model of quintessence indistinguishable from adding a "pure" cosmological constant to the action. This is certanly true for a minimally coupled field, but is not true, in general, for scalars non-minimally coupled to the geometry and to the other matter fields, like the dilaton.

For non-minimally coupled fields, in fact, a cosmological transition (i.e. a change in the equation of state of the dominant cosmological sources) tends to shift the field away from the initial equilibrium position, and a subsequent transition to the frozen, potential-dominated regime is not at all guaranteed. The dilaton, in particular, is coupled to the trace of the matter stress tensor, and a constant solution is possible in the (traceless) radiation era, as we shall see, but not in the era of matter domination.

Thus, even if the dilaton is "sleeping" in the radiation era, it necessarily "wakes up" and starts rolling down (or up) the potential after the equilibrium time, when the Universe enters the matter-dominated regime. However, if the dilaton mass is not too small, the dilaton may bounce back, and approach again the freezing position, just when its potential energy starts to become critical. It thus becomes a non-trivial consequence of the potential and of the "stringy" dilaton dynamics if the decelerated, matter-dominated era is followed by a phase of potential domination and accelerated expansion.

The main purpose of this paper is to discuss, in the

above dilatonic scenario, the problem of the "cosmic coincidence" [6], which seems to affect the so-called "trackers solutions" [7] arising in models with power-law or exponential [8] potentials, as well as in models of quintessential inflation [9]. Our effort, in particular, is to understand whether or not such a problem may be avoided, or relaxed, in a dilatonic scenario in which the ratio of the matter to scalar energy density goes asymptotically to zero, and not to a constant like in recent attempts to solve the coincidence problem based on bulk viscosity [10] and on non-minimal scalar-tensor couplings [11]. The scenario discussed in this paper is also different from other, non-minimally coupled scalar models of quintessence [12], because the dilaton potential and couplings are not *ad hoc*, but (in principle) prescribed by string theory, and because the present value of the dilaton field is not an arbitrary parameter, but has to be determined so as to fix a realistic set of GUT coupling constants, according to eq. (1) (the tree-level relation between the dilaton and the fundamental constants could be non-trivially modified, however, by loop corrections [13]).

Let us start our discussion with the tree-level, lowest order in α', gravi-dilaton string effective action, minimally coupled to perfect fluid sources:

$$S = -\frac{1}{2\,\lambda_s^2}\int d^4x\,\sqrt{|g|}\,e^{-\phi}\,\left[R + (\nabla_\mu\phi)^2 + V(\phi)\right] + S_m \tag{2}$$

($\lambda_s = M_s^{-1}$ is the fundamental string length parameter). Consider a homogeneous, isotropic and spatially flat background. By varying the action with respect to g_{00}, g_{ij} and ϕ, and using the dilaton equations to simplify the g_{ij} equation, we get, respectively:

$$\dot\phi^2 + 6H^2 - 6H\dot\phi - V = e^\phi\rho, \tag{3}$$

$$\dot H - H\dot\phi + 3H^2 + \frac{V'}{2} = \frac{1}{2}e^\phi p, \tag{4}$$

$$\dot\phi^2 + 12H^2 - 6H\dot\phi - 2\ddot\phi + 6\dot H + V' - V = 0, \tag{5}$$

where $V' = \partial V/\partial\phi$, and we have chosen the cosmic time gauge. Note also that we have chosen units in which $2\lambda_s^2 = 1$, so that e^ϕ represents, in string units, the effective four-dimensional Newton constant $16\pi G$. The combination of the above equations leads to the usual covariant conservation of the energy density:

$$\dot\rho + 3H(\rho + p) = 0. \tag{6}$$

It is useful, at this point, to rewrite the dilaton equation by eliminating H^2 and $\dot H$. This gives the condition

$$\ddot\phi + 3H\dot\phi - \dot\phi^2 + \frac{1}{2}e^\phi(\rho - 3p) + V' + V = 0. \tag{7}$$

A stable solution $\phi = \phi_0 = $ const is thus possible only if

$$3p - \rho = 2e^{-\phi}(V + V') = \text{const}, \tag{8}$$

which, combined with eq. (6), leaves only three possibilities: *i)* vacuum, $\rho = p = 0$, $V + V' = 0$; *ii)* cosmological constant, $\rho = -p = \rho_0 = $ const, $V + V' = -2e^{\phi_0}\rho_0 = $ const; *iii)* radiation, $\rho = 3p$, $V + V' = 0$.

The first two cases corresponds to an accelerated de Sitter solution, $H^2 = $ constant. Only in the third case we can obtain a decelerated background, which coincides with the standard radiation-dominated solution provided $V(\phi_0), V'(\phi_0) \ll e^{\phi_0}\rho$. Let us thus suppose that, for some mechanism (to be discussed elsewhere), the dilaton is attracted to the equilibrium position $V + V' = 0$ early enough in the radiation era, and that the potential energy is always subdominant during the whole radiation epoch, $V(\phi_0) \ll H_{\rm eq}^2$, in such a way as to avoid any conflict with standard big bang nucleosynthesis (unlike other models of extended quintessence, see [14]).

It should be noted that such a dilaton configuration extremizes the effective (canonical) potential when transformed to the Einstein frame. In the (tilded) Einstein frame variables, defined by the conformal transformation

$$\tilde g_{\mu\nu} = g_{\mu\nu}e^{-\phi}, \qquad \tilde\phi = \phi, \tag{9}$$

the dilaton in fact is minimally coupled to the metric, but non-minimally coupled to the fluid sources, and the cosmological equations (3-5) become (in units $16\pi G = 1$):

$$6\tilde H^2 = \tilde\rho + \tilde V + \frac{\dot\phi^2}{2}, \tag{10}$$

$$4\dot{\tilde H} + 6\tilde H^2 = -\tilde p + \tilde V - \frac{\dot\phi^2}{2}, \tag{11}$$

$$\ddot\phi + 3\tilde H\dot\phi + \frac{1}{2}(\tilde\rho - 3\tilde p) + \tilde V' = 0, \tag{12}$$

where $\tilde V = e^\phi V$, $\tilde\rho = e^{2\phi}\rho$, $\tilde p = e^{2\phi}p$, and the dot denotes the derivative with respect to the Einstein cosmic time, $d\tilde t = dte^{-\phi/2}$. Thus, in the radiation era, $\tilde V' = 0$ is the necessary condition for $\phi = $ const.

Let us now approach the problem of the cosmic coincidence considering the Einstein frame equations (10–12), which we rewrite in the matter dominated era $\rho = \rho_m$, $p = 0$ (omitting the tilde, for simplicity), and taking into account the possibility of a more general (loop-induced, see below) matter-dilaton coupling, parametrized by the function $\alpha(\phi)$:

$$6H^2 = \rho_m + V + \frac{\dot\phi^2}{2}, \tag{13}$$

$$4\dot H + 6H^2 = V - \frac{\dot\phi^2}{2}, \tag{14}$$

$$\ddot\phi + 3H\dot\phi + \frac{1}{2}\alpha(\phi)\rho_m + V' = 0 \tag{15}$$

($\alpha = 1$ corresponds to the previous, lowest-order action). Their combination gives

$$\dot\rho_m + 3H\rho_m - \frac{1}{2}\alpha(\phi)\rho_m\dot\phi = 0. \tag{16}$$

Assuming that the matter-dominated era starts at the equilibrium time with $\phi = \phi_0$, $\dot\phi = 0 = V'$, $V_0 \equiv V(\phi_0) \ll 6H_{\rm eq}^2$, $\rho_m \simeq 6H_{\rm eq}^2$, we ask the question: is it possible for the phase of matter domination to evolve into a subsequent phase dominated by the potential energy of the scalar field, and corresponding to an effective negative pressure $p_\phi/\rho_\phi \equiv (\dot\phi^2/2 - V)/(\dot\phi^2/2 + V) < 0$?

In the absence of the non-minimal coupling to matter (i.e $\alpha = 0$, pure general relativity) the answer would be yes, for all values of $V_0 < 6H_{\rm eq}^2$. In that case, indeed, the dilaton wold keep constant at the minimum, and the Universe would enter the potential-dominated phase as soon as $\rho_m \lesssim V_0$, rapidly approaching a final regime with $p_\phi/\rho_\phi = -1$, and $H^2 = V_0/6 = $ const. However, given the (semi-)infinite range of allowed values for V_0, why $V_0 \sim \rho_m(t_0) \sim H_0^2$ (as suggested by osservations), i.e. why the potential energy of the scalar field is just of the same order as the present matter energy density? (cosmic "coincidence").

The dilaton, however, *is* non-minimally coupled to macroscopic matter ($\alpha \neq 0$, in general), and cannot stay frozen at the minimum when the Universe is driven by ρ_m (see eq. (15)). The above question concerning the possible advent of the quintessential regime thus becomes: for which values of the initial potential $V_0 < H_{\rm eq}^2$ the dilaton may come back to the minimum, and the Universe may enter the potential-dominated phase, with total negative pressure? If the answer would point, only and precisely, at the value $V_0 \sim H_0^2$, the coincidence problem would disappear. If the answer would indicate instead a restricted range of values for V_0, the problem would be nevertheless alleviated.

The answer to the above question obviously depends on the shape of the potential, and on the coupling $\alpha(\phi)$. Let us start assuming, first of all, that α is a constant, or that its dilaton dependence (in the range of interest of our problem) is so weak to be consistently neglected. For the dilaton potential, we can ask the assistance of string theory. Although the full and detailed form of $V(\phi)$ is largely unknown (mainly because of non-perturbative effects), we know, however, that in the weak coupling regime ($\phi \to -\infty$, $g_s \ll 1$), the supersymmetry breaking potential [15] of critical superstring theory has to be strongly suppressed in an istantonic way, i.e. $V \sim \exp(-1/g_s^2) \equiv \exp[-\exp(-\phi)]$. In the regime of moderately strong coupling ($-\phi \sim 1$), on the other hand, the potential must exhibit some structure, leading to a minimum ϕ_0 such that [2]

$$g_s^2 = \exp(\phi_0) \simeq 0.1 - 0.01, \qquad (17)$$

(according to eq. (1)). Finally, in the strong coupling regime ($g_s \sim 1$), we may expect an exponential growth of the potential, due to the factor e^ϕ induced by the transformation from the string to the Einstein frame. Such an exponential behaviour could be suppressed, at very strong coupling $g_s \gg 1$, by some non-perturbative effect or by loop corrections, but we are not concerned with this possibility here since the matter coupling is expected to shift the dilaton towards negative values, and then towards the weak coupling regime (see eq. (15)).

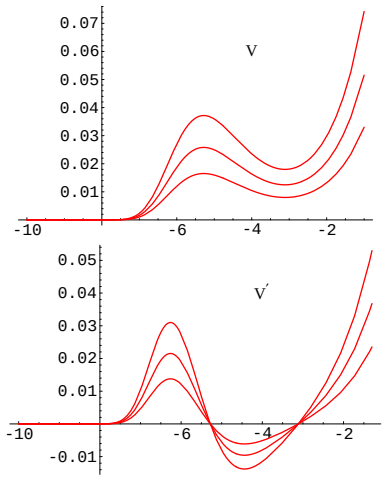

FIG. 1. *Plot of $V(\phi)$ and $V'(\phi)$ from eq. (18), with $k_1 = k_2 = \beta = \gamma = 1$, $\epsilon = 0.1$, $\phi_1 = -3$. The three curves, from top to bottom, correspond respectively to $m = 1/10$, $m = 1/12$, $m = 1/15$, in units of $H_{\rm eq}$.*

What is imporant, in this context, is that the potential barrier separating the minimum ϕ_0 from the perturbaive regime is not infinitely high, and it is possible for the shifted dilaton to escape to $-\infty$, provided its mass is light enough, and/or its coupling to matter (i.e., the initial acceleration) is sufficiently strong. The possible transition to a phase dominated by the potential energy V_0 thus becomes a precise problem of balance between the initial force aiming at moving the dilaton away from the minimum, $-\alpha\rho/2 < 0$, and the restoring force generated by the potential, $-V' > 0$.

In order to illustrate this mechanism, let us consider a "minimal" example of dilaton potential satisfying the above-mentioned string theory requisites, and controlled only by one dimensional parameter m, related to the effective (low-energy) mass of the dilaton. Such a potential can be simply parametrized (in the Einstein frame) as follows:

$$V = m^2 \left[e^{k_1(\phi-\phi_1)} + \beta e^{-k_2(\phi-\phi_1)} \right] e^{-\epsilon \exp[-\gamma(\phi-\phi_1)]} \qquad (18)$$

(see also [16]), where $k_1, k_2, \phi_1, \epsilon, \beta, \gamma$ are dimensionless numbers of order one, whose precise values are not crucial for the purpose of this paper, provided they determine a minimum around a value ϕ_0 consistent with a superstring unification scenario (eq. (1)). For our illustrative purpose we will choose the particular values $k_1 = k_2 = \beta = \gamma = 1$, $\epsilon = 0.1$, $\phi_1 = -3$, in such

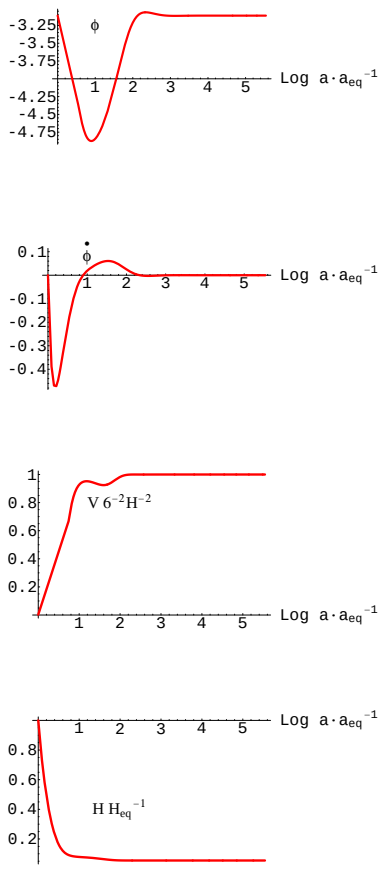

FIG. 2. *Numerical integration of eqs. (13-16) with* $\alpha = 1$, $m = 10^{-1} H_{\rm eq}$, *and initial conditions* $\phi = \phi_0 = -3.112$, $\dot\phi = 0$, $a = a_{\rm eq} = 1$, $H = H_{\rm eq} = 1$, $\rho_m = 6H_{\rm eq}^2 - V_0$.

a way that the minimum is at $\phi_0 = -3.112...$, and $g_s^2 = \exp(\phi_0) \simeq 0.045$, in agreement with eq. (17). (Many other choices are possible, of course. In particular, the qualitative behaviour of the potential remains the same if we increase ϵ, provided the values of k_1 and k_2 are also simultaneously increased, or the value of γ is decreased).

With the above choice of parameters, the dilaton potential (18) and its gradient V' are plotted in Fig. 1, for various values of m. We may note that, as m is decreased, the effective potential barrier is lowered, the restoring force $-V'$ becomes weaker, and it becomes easier for the dilaton to escape from the minimum, as previously anticipated.

With such a potential, we have numerically integrated eqs. (13-16), at fixed α (in particular, $\alpha = 1$), and initial condition at the equilibrium epoch: $\phi = \phi_0 = -3.112$, $\dot\phi = 0$, $H = H_{\rm eq}$, $\rho_m = 6H_{\rm eq}^2 - V_0$, $a = a_{\rm eq}$. We have found that, when the mass parameter $m < H_{\rm eq}$ is not too small (for instance, $m = 10^{-1} H_{\rm eq}$), the cosmological evolution is typically the one illustrated in Fig. 2: the dilaton is shifted from the minimum but is subsequently re-attracted to it, its velocity is damped and goes to zero after some oscillations, the potential energy asymptotically becomes critical, and the Hubble parameter freezes at a constant value determined by V_0 (and corresponding to a de Sitter equation of state $p_\phi/\rho_\phi = -1$).

When the mass parameter m is too small, however, the evolution is qualitatively different, and is illustrated in Fig. 3, where we present the results of a numerical integration with the same initial conditions, but a mass ten times smaller, $m = 10^{-2} H_{\rm eq}$: the velocity $\dot\phi$ is still damped and goes to zero, asymptotically, but it keeps negative for ever, and the dilaton runs monotonically towards the perturbative regime $\phi \to -\infty$, $V \to 0$.

We observe now that, in a realistic model of quintessence, the potential energy of the dilaton should represent an important fraction of the critical energy density (in particular [5], $V_0 \simeq (2/3)\rho_c \simeq 4H^2$) just at the present epoch, i.e. when the radiation energy density ρ_r (which evolves independently from the dilaton) has been reduced by a factor $(\rho_m/\rho_r)_0 = (a_0/a_{\rm eq}) \sim 10^4$ with respect to the matter energy density. This requires $V_0 \sim H_0^2$, where H_0 is the present value of the Hubble radius, and since (from eq. (18)) $V_0 = 2m^2 \cosh(0.112) \exp[-0.1 \exp(0.112)] \simeq 2m^2$, this implies $m \sim H_0 \sim 10^{-6} H_{\rm eq}$. The qustion is now whether or not, for such a value of m, the solution may become asymptotically dominated by the potential.

The answer is strongly dependent on the value of the dilaton coupling $\alpha(\phi)$. If $\alpha = 1$, for instance, the value $m \sim H_0 \sim 10^{-6} H_{\rm eq}$ is certainly to be excluded because, as shown by a numerical integration, the dilaton returs to the minimum of the potential only if $m \gtrsim 0.077 H_{\rm eq}$. In that case, the regime of potential domination occurs too early, as illustrated in Fig. 2 where $V_0 \sim \rho_m$ for $a \sim 10 a_{\rm eq}$, i.e. for $\rho_r \sim 10^{-1} \rho_m$. If $\alpha = 10^{-1}$, instead, we find that all the values of m from $H_{\rm eq}$ down to $10^{-7} H_{\rm eq}$ are compatible with a dilaton trapped around the minimum. The realistic value $m \sim 10^{-6} H_{\rm eq}$ is still to be fixed by hand, but the choice is restricted within a limited range and, in this sense, the coincidence problem seems to be alleviated.

It must be stressed, at this point, that the dilaton potential in the Einstein frame also determines the mass $\tilde m$ of the (canonically normalized) dilaton field, through its expansion around the minimum:

$$\tilde m^2 = V''(\phi_0) = 2m^2 \cosh(\phi_0 - \phi_1) e^{-\frac{1}{10}\exp(\phi_1 - \phi_0)} \times \left[1 - 0.1 e^{-(\phi_0 - \phi_1)} - 10^{-2} e^{-2(\phi_0 - \phi_1)}\right] \simeq 2m^2, \qquad (19)$$

where $(\phi_0 - \phi_1) \simeq -0.112$. A realistic scenario, with $m \sim H_0 \sim 10^{-33}$ eV, thus correspond to a very long (infinite, in practice) range for the dilatonic interactions. It follows that the present value of the dilaton coupling,

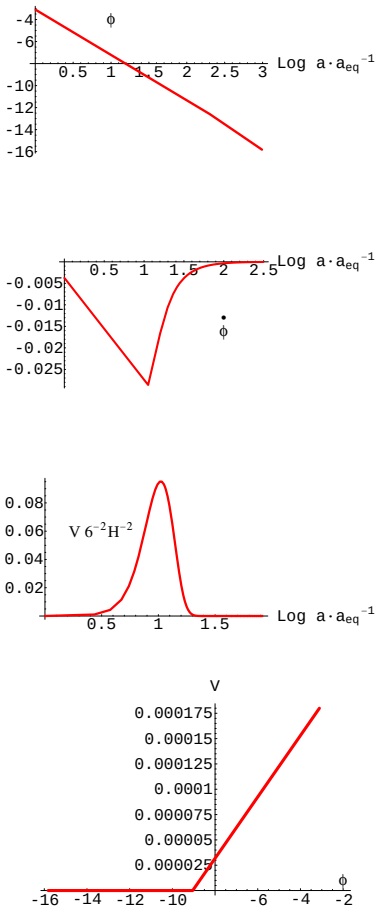

FIG. 3. *Numerical integration of eqs. (13-16) with* $\alpha = 1$, $m = 10^{-2} H_{\rm eq}$, *and the same initial conditions as in Fig. 2.*

$\alpha_0 \equiv \alpha[\phi(t_0)]$, has to be strongly suppressed, to be compatible with the present tests of the gravitational interaction. In particular [17], $\alpha_0 \lesssim 10^{-4}$ for composition-dependent dilatonic couplings, strongly constrained by the precise tests of the equivalence principle; $\alpha_0 \lesssim 10^{-2}$ for universal dilatonic couplings, constrained by tests of post-Newtonian gravity. Such phenomenological constraints can be accounted for, in principle, by including in the string effective action the quantum loop corrections, to all orders [17] (indeed, they are to be included when the string coupling g_s is not negligibly small, like in the case we are considering; higher-curvature α' corrections, on the contrary, can be safely neglected, as the curvature scales we are considering are always small in string units, $\lambda_s^2 H^2 \ll 1$).

The loop corrections modify the dilaton coupling to the matter sources, and can be parametrized, in our case, by three effective dilaton "form factors", $Z_\rho(\phi)$, $Z_p(\phi)$, $Z_\phi(\phi)$, appearing in the string frame action (2), and defined by:

$$\frac{\delta S_m}{\delta g_{00}} = \frac{1}{2} Z_\rho T_{00}, \quad \frac{\delta S_m}{\delta g_{ij}} = \frac{1}{2} Z_p T_{ij}, \quad \frac{\delta S_m}{\delta \phi} = Z_\phi T, \quad (20)$$

(T is the trace of the stress tensor). The transformation to the Einstein frame leads then to cosmological equations in which the canonically rescaled fields $\tilde\rho, \tilde p, \tilde\phi$, are minimally coupled to the metric, according to eqs. (10, 11). The dilaton, however, turns out to be non-minimally coupled to the trace of the fluid stress tensor, as in eqs. (15,16), through the coupling function $\alpha(\phi)$ that depends on Z_ϕ.

In the absence of a closed and explicit expression for the loops corrections one might assume, in the spirit of [17], that the present value of the coupling is small because the dilaton is attracted towards a local extremum ϕ_m of the coupling function. One can then expand (to first order) $\alpha(\phi) = k(\phi - \phi_m)$, where k is a dimensionless number of order one. In that case, if ϕ_m exactly coincides with ϕ_0, and with the initial position of the dilaton, then the dilaton is decoupled from matter already in the radiation era, it cannot be shifted from the minimum, and the coincidence problem remains.

If, however, the initial dilaton $\phi_{\rm eq}$ at equilibrium is slightly shifted from the extremum, because $\phi_m \neq \phi_0$ and/or $\phi_{\rm eq} \neq \phi_0$, a difference $|\phi_{\rm eq} - \phi_m| \lesssim 10^{-1}$ is already enough (as previously mentioned) to enlarge the allowed values of m to the range $H_{\rm eq} \to 10^{-7} H_{\rm eq}$, which includes $m \sim H_0$. In such a case one can easily match present observations, concerning both the cosmic equation of state and the present coupling of matter to scalar, long-range forces, as illustrated in Fig. 4 for an initial dilaton slightly tilted from the minimum. By choosing the appropriate value of V_0, the quintessential regime with critical potential and frozen Hubble radius may start indeed around a red-shift $a \sim 10^4 a_{\rm eq}$, and just when $H \sim H_0 \sim 10^{-6} H_{\rm eq}$. Note also that, in the example of Fig. 4, $\dot\phi/H$ and $\alpha(\phi)$ tend asymptotically to zero, and remain small enough to satisfy the present phenomenological constraints [17].

The main conclusion of this discussion – quite irrespective of the given particular examples, and of their possible relevance for a complete and fully realistic scenario – is that a string cosmology interpretation of quintessence suggests a close relationship between the problem of the cosmic coincidence, and the problem of fixing the present value of the dilaton coupling to matter (see also [18], for previous discussions of the link between quintessence and possible deviations from Newtonian gravity). In particular, a strong value of the coupling $\alpha(\phi) \sim 1$ – which is excluded by the gravitational phenomenology – would be *indirectly forbidden*, in such a context, *also by the present large scale configuration* of our Universe. Viceversa, given $\alpha(\phi)$ in agreement with phenomenology, the allowed range of the dilaton potential turns out to be fixed, and the coincidence problem possibly alleviated, once one assumes $V_0 < H_{\rm eq}^2$.

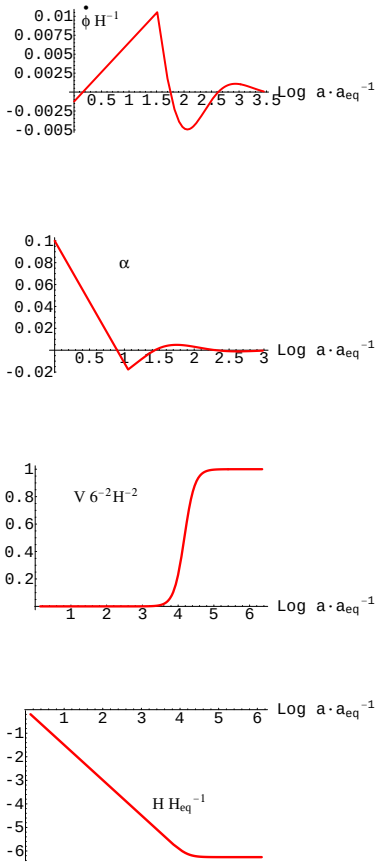

FIG. 4. *Numerical integration of eqs. (13-16) with* $m = 10^{-6} H_{\rm eq}$, $\alpha(\phi) = k(\phi - \phi_m)$, $k = 1$, $\phi_m = \phi_0$, *and an initial dilaton slightly displaced from the minimum,* $\phi_{\rm eq} = \phi_0 + 0.1$. *The other initial conditions are the same as in Fig. 2.*

An important open problem, in this approach, remains however to explain how the dilaton is attracted to the minimum of the potential in the radiation era, and why the vacuum energy is relaxed to a very small value – for instance, according to the mechanism dicussed in [19]. This problem requires a discussion of the initial conditions characterizing the (post-inflationary) cosmological configuration of our Universe, after the re-heating (and possibly pre-heating) phase, and will be addressed in a future paper.

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[1] M. B. Green, J. Schwartz and E. Witten, *Superstring theory* (Cambridge University Press, Cambridge, Ma, 1987).
[2] V. Kaplunovsky, Phys. Rev. Lett. **55**, 1036 (1985).
[3] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).
[4] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. **80**, 582 (1998); J. A. Frieman and I. Vaga, Phys. Rev. D **57**, 4642 (1998); M. S. Turner and M. White, Phys. Rev. D **56**, 4439 (1997).
[5] A. G. Riess *et al*, Astron. J. **116**, 1009 (1998); S. Perlmutter *et al*, Nature **391**, 51 (1998). N. A. Bachall, J. P. Ostriker, S. Perlmutter and P. J. Steinhardt, Science **284**, 1481 (1999), and references therein.
[6] P. Steinhardt, in *Critical problems in Physics*, ed. by V. L. Fitch and D. R. Marlow (Princeton University Press, Princeton, NJ, 1997).
[7] P. J. E. Peebles and B. Ratra, Phys. Rev. D **37**, 3406 (1988); I. Zlatev, L. Wang and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999); Phys. Rev. D **59**, 123504 (1999).
[8] P. Brax and J. Martin, Ann. Phys. (Leipzig) **11**, 507 (2000); T. Barreiro, E. J. Copeland and N. J. Nunes, Phys. Rev. D **61**, 127301 (2000).
[9] P. J. E. Peebles and A. Vilenkin, Phys. Rev. D **59**, 063505 (1999); M. Peloso and F. Rosati, JHEP **9912**, 026 (1999).
[10] L. P. Chimento, A. S. Jakubi and D. Pavon, Phys. Rev. D **62**, 063508 (2000); W. Zimdahl, D. J. Schwarz, A. B. Balakin and D. Pavon, astro-ph/0009353; S. Sen and A. A. Sen, gr-qc/0010092.
[11] A. Hebecker and C. Wetterich, Phys. Rev. Lett. **85**, 3339 (2000); L. Amendola and D. Tocchini-Valentini, astro-ph/0011243.
[12] T. Chiba, Phys. Rev. D **60**, 083508 (1999); L. Amendola, Phys. Rev. D **60**, 043501 (1999); F. Perrotta, C. Baccigalupi and S. Matarrese, Phys. Rev. D **61**, 023507 (2000); D. J. Holden and D. Wands, Phys. Rev. D **61**, 043506 (2000); N. Bartolo and M. Pietroni, Phys. Rev. D **61**, 023518 (2000); O. Bertolami and P. J. Martins, Phys. Rev. D **61**, 064007 (2000); R. De Ritis , A. A. Marino, C, Rubano and P. Scudellaro, Phys. Rev. D **62**, 043506 (2000); R. Bean and J. Magueijo, astro-ph/0007199; S. Sen and T. R. Seshadri, gr-qc/0007079; N. Banerjee and D. Pavon, Phys. Rev. D **63**, 043504 (2001); E. Gunzig *et al*, gr-qc/0012085.
[13] M. Gasperini, F. Piazza and G. Veneziano, *in preparation.*
[14] X. Chen, R. J. Scherrer and G. Steigman, astro-ph/0011531.
[15] P. Binetruy and M. K. Gaillard, Phys. Rev. D**34**, 3069 (1986).
[16] N. Kaloper and K. A. Olive, Astropart. Phys. **1**, 185 (1993).
[17] T. Damour and A. M. Polyakov, Nucl. Phys. B **423**, 532 (1994); Gen. Rel. Grav. **26**, 1171 (1994).
[18] Y. Fujii, Phys. Rev. D **62**, 044011 (2000).
[19] V. A. Rubakov, Phys. Rev. D **61**, 061501 (2000).

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Violations of the equivalence principle in a dilaton-runaway scenario

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Abstract

We explore a version of the cosmological dilaton-fixing and decoupling mechanism in which the dilaton-dependence of the low-energy effective action is extremized for infinitely large values of the bare string coupling $g_s^2 = e^{\phi}$. We study the efficiency with which the dilaton ϕ runs away towards its "fixed point" at infinity during a primordial inflationary stage, and thereby approximately decouples from matter. The residual dilaton couplings are found to be related to the amplitude of the density fluctuations generated during inflation. For the simplest inflationary potential, $V(\chi) = \frac{1}{2}\, m_\chi^2(\phi)\, \chi^2$, the residual dilaton couplings are shown to predict violations of the universality of gravitational acceleration near the $\Delta a/a \sim 10^{-12}$ level. This suggests that a modest improvement in the precision of equivalence principle tests might be able to detect the effect of such a runaway dilaton. Under some assumptions about the coupling of the dilaton to dark matter and/or dark energy, the expected time-variation of natural "constants" (in particular of the fine-structure constant) might also be large enough to be within reach of improved experimental or observational data.

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I. INTRODUCTION

All string theory models predict the existence of a scalar partner of the spin 2 graviton: the dilaton ϕ, whose vacuum expectation value (VEV) determines the string coupling constant $g_s = e^{\phi/2}$ [1]. At tree level, the dilaton is massless and has gravitational-strength couplings to matter which violate the equivalence principle [2]. This is in violent conflict with present experimental tests of general relativity. It is generally assumed that this conflict is avoided because, after supersymmetry breaking, the dilaton might acquire a (large enough) mass (say $m_\phi \gtrsim 10^{-3}\,\mathrm{eV}$ so that observable deviations from Einstein's gravity are quenched on distances larger than a fraction of a millimeter). However, Ref. [3] (see also [4]) has proposed a mechanism which can naturally reconcile a *massless* dilaton with existing experimental data. The basic idea of Ref. [3] was to exploit the string-loop modifications of the (four dimensional) effective low-energy action (we use the signature $-+++$)

$$S = \int d^4x \sqrt{\tilde{g}} \left(\frac{B_g(\phi)}{\alpha'} \tilde{R} + \frac{B_\phi(\phi)}{\alpha'} [2\tilde{\Box}\phi - (\tilde{\nabla}\phi)^2] - \frac{1}{4} B_F(\phi) \tilde{F}^2 - V + \cdots \right) , \tag{1.1}$$

i.e. the ϕ-dependence of the various coefficients $B_i(\phi)$, $i = g, \phi, F, \ldots$, given in the weak-coupling region ($e^\phi \to 0$) by series of the form

$$B_i(\phi) = e^{-\phi} + c_0^{(i)} + c_1^{(i)}\, e^{\phi} + c_2^{(i)}\, e^{2\phi} + \cdots , \tag{1.2}$$

coming from genus expansion of string theory: $B_i = \Sigma_n\, g_s^{2(n-1)} c_n^{(i)}$, with $n = 0, 1, 2, \ldots$. It was shown in [3] that, if there exists a special value ϕ_m of ϕ which extremizes all the (relevant) coupling functions $B_i^{-1}(\phi)$, the cosmological evolution of the graviton-dilaton-matter system naturally drives ϕ towards ϕ_m. This provides a mechanism for fixing a massless dilaton at a value where it decouples from matter ("Least Coupling Principle"). A simple situation where the existence of a universally extremizing dilaton value ϕ_m is guaranteed is that of S duality, i.e. a symmetry $g_s \leftrightarrow 1/g_s$, or $\phi \to -\phi$ (so that $\phi_m = 0$).

It has been recently suggested [5] that the infinite-bare-coupling limit $g_s \to \infty$ ($\phi \to +\infty$) might yield smooth *finite* limits for all the coupling functions, namely

$$B_i(\phi) = C_i + \mathcal{O}(e^{-\phi}) . \tag{1.3}$$

Under this assumption, the coupling functions are all extremized at infinity, i.e. $\phi_m = +\infty$. The late-time cosmology of models satisfying (1.3) has been recently explored [6]. In the "large N"-type toy model of [5] it would be natural to expect that the $\mathcal{O}(e^{-\phi})$ term in Eq. (1.3) be *positive*, so that $B_i(\phi)$ be *minimized* at infinity. This would correspond to

couplings $\lambda_i(\phi) \sim B_i^{-1}(\phi) = C_i^{-1} - \mathcal{O}(e^{-\phi})$ which are *maximized* at infinity. Note, however, that the most relevant cosmological coupling for this work, the coupling to the inflaton, $\lambda(\phi)$, contained in V (see Eq. (2.11) below) is closer to a B_i than to its inverse. Thus $\lambda(\phi)$ is naturally *minimized* at infinity (see further discussion of this point below), a crucial property for the attractor mechanism of [3,4].

In this paper* we shall consider in detail the early-time cosmology of models satisfying (1.3). More precisely, our main aims will be: (i) to study the efficiency with which a primordial inflationary stage drives ϕ towards the "fixed point" at infinity $\phi_m = +\infty$ (thereby generalizing the work [8] which considered the inflationary attraction towards a local extremum ϕ_m), and (ii) to give quantitative estimates of the present violations of the equivalence principle (non universality of free fall, and variation of "constants"). Our most important conclusion is that the runaway of the dilaton towards strong-coupling (under the assumption (1.3)) naturally leads to equivalence-principle violations which are rather large, in the sense of not being much smaller than the presently tested level $\sim 10^{-12}$. This gives additional motivation for the currently planned improved tests of the universality of free fall. Within our scenario, most of the other deviations from general relativity ("post-Einsteinian" effects in gravitationally interacting systems: solar system, binary pulsars, ...) are too small to be of phenomenological interest. However, under some assumptions about the coupling of ϕ to dark matter and/or dark energy, the time variation of the natural "constants" (notably the fine-structure constant) predicted by our scenario might be large enough to be within reach of improved experimental and/or observational data. The phenomenologically interesting conclusion that equivalence-principle violations are generically predicted to be rather large after inflation (in sharp contrast with the results of [8]) is due to the fact that the attraction towards an extremum at infinity is much less effective than the attraction towards a (finite) local extremum as originally contemplated in [3]. This reduced effectiveness was already pointed out in Ref. [4] within the context of equivalence-principle-respecting tensor-scalar theories (à la Jordan-Fierz-Brans-Dicke).

II. DILATON RUNAWAY

In this section we study the dilaton's runaway during the various stages of cosmological evolution. We first show (subsection II A) that, like in the case of a local extremum [8], inflation is particularly efficient in pushing ϕ towards the fixed point. We will then argue

*The main results of this work have been recently summarized in a short note [7].

(subsection II B) that the order of magnitude of the bare string coupling $e^{\phi} \simeq e^{c\varphi}$ does not suffer further appreciable changes during all the subsequent evolution.

A. The inflationary period

Assuming some primordial inflationary stage driven by the potential energy of an inflaton field $\widetilde{\chi}$, and taking into account generic couplings to the dilaton ϕ, we consider an effective action of the form

$$S = \int d^4x\sqrt{\widetilde{g}}\left(\frac{B_g(\phi)}{\alpha'}\,\widetilde{R} + \frac{B_\phi(\phi)}{\alpha'}\,[2\widetilde{\Box}\phi - (\widetilde{\nabla}\phi)^2] - \frac{1}{2}\,B_\chi(\phi)(\widetilde{\nabla}\widetilde{\chi})^2 - \widetilde{V}(\widetilde{\chi},\phi)\right). \tag{2.1}$$

In this string-frame action, the dilaton dependence of all the functions $B_i(\phi)$, $\widetilde{V}(\widetilde{\chi},\phi)$ is assumed to be of the form (1.2). It is convenient to replace the (σ-model) string metric $\widetilde{g}_{\mu\nu}$ by the conformally related Einstein metric $g_{\mu\nu} = C\,B_g(\phi)\,\widetilde{g}_{\mu\nu}$, and the dilaton field by the variable

$$\varphi = \int d\phi\left[\frac{3}{4}\left(\frac{B_g'}{B_g}\right)^2 + \frac{B_\phi'}{B_g} + \frac{1}{2}\,\frac{B_\phi}{B_g}\right]^{\frac{1}{2}}\,,\ B' \equiv \partial B/\partial\phi\,. \tag{2.2}$$

The normalization constant C is chosen so that the string units coincide with the Einstein units when $\phi \to +\infty$: $C\,B_g(+\infty) = 1$. [Note that $C = 1/C_g$ in terms of the general notation of Eq. (1.3).] Introducing the (modified) Planck mass

$$\widetilde{m}_P^2 = \frac{1}{4\pi G} = \frac{4}{C\alpha'}\,, \tag{2.3}$$

and replacing also the inflaton by the dimensionless variable $\chi = C^{-1/2}\,\widetilde{m}_P^{-1}\,\widetilde{\chi}$, we end up with an action of the form

$$S = \int d^4x\sqrt{g}\left[\frac{\widetilde{m}_P^2}{4}\,R - \frac{\widetilde{m}_P^2}{2}\,(\nabla\varphi)^2 - \frac{\widetilde{m}_P^2}{2}\,F(\varphi)(\nabla\chi)^2 - \widetilde{m}_P^4\,V(\chi,\varphi)\right], \tag{2.4}$$

where

$$F(\varphi) = B_\chi(\phi)/B_g(\phi)\,,\ V(\chi,\varphi) = C^{-2}\,\widetilde{m}_P^{-4}\,B_g^{-2}(\phi)\,\widetilde{V}(\widetilde{\chi},\phi)\,. \tag{2.5}$$

In view of our basic assumption (1.3), note that, in the strong-coupling limit $\phi \to +\infty$, $d\varphi/d\phi$ tends, according to Eq. (2.2), to the constant $(C_\phi/2C_g)^{1/2}$, while the dilaton-dependent factor $F(\varphi)$ in front of the inflaton kinetic term tends to the constant C_χ/C_g. The toy model of

Ref. [5] suggests that the various (positive) constants C_i in Eq. (1.3) are all largish and comparable to each other. We shall therefore assume that the various ratios C_i/C_j are of order unity. The most important such ratio for the following is $c \equiv (2C_g/C_\phi)^{1/2}$ which gives the asymptotic behaviour of the bare string coupling as

$$g_s^2 = e^\phi \simeq e^{c\varphi} \,. \tag{2.6}$$

In view of the fact that, in the strong-coupling limit we are interested in, the factor $F(\varphi)$ in Eq. (2.4) quickly tends to a constant, we can simplify our analysis (without modifying the essential physics) by replacing it by a constant (which can then be absorbed in a redefinition of χ). Henceforth, we shall simply take $F(\varphi) = 1$. [See, however, the comments below concerning the self-regenerating inflationary regime.]

Following [4,9] it is then useful to combine the Friedmann equations for the scale factor $a(t)$ during inflation ($ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i\,dx^j$) with the equations of motion of the two scalar fields $\chi(t)$, $\varphi(t)$, to write an autonomous equation describing the evolution of the two scalars in terms of the parameter

$$p = \int H dt = \int \frac{\dot a}{a}\, dt = \ln\, a + \text{const} \tag{2.7}$$

measuring the number of e-folds of the expansion. For any multiplet of scalar fields, $\boldsymbol{\varphi} = (\varphi^a)$, this yields the simple equation [4,9]

$$\frac{2}{3 - \boldsymbol{\varphi}'^2}\, \boldsymbol{\varphi}'' + 2\, \boldsymbol{\varphi}' = -\nabla_{\boldsymbol{\varphi}} \ln |V(\boldsymbol{\varphi})| \,, \tag{2.8}$$

where $\boldsymbol{\varphi}' \equiv d\boldsymbol{\varphi}/dp$, and where all operations on $\boldsymbol{\varphi}$ are covariantly defined in terms of the σ-model metric defining the scalar kinetic terms ($d\sigma^2 = \gamma_{ab}(\varphi)\, d\varphi^a\, d\varphi^b$). In our simple model (with $F(\varphi) = 1$), we have a flat metric $d\sigma^2 = d\varphi^2 + d\chi^2$. [Note that, when $\gamma_{ab}(\varphi)$ is curved the acceleration term $\boldsymbol{\varphi}''$ involves a covariant derivative.]

The generic solution of Eq. (2.8) is easily grasped if one interprets it as a mechanical model: a particle with position $\boldsymbol{\varphi}$, and velocity-dependent mass $m(\boldsymbol{\varphi}') = 2/(3 - \boldsymbol{\varphi}'^2)$, moves, in the "time" $p = \ln a + \text{cst}$, in the manifold $d\sigma^2$ under the influence of an external potential $\ln |V(\boldsymbol{\varphi})|$ and a constant friction force $-2\,\boldsymbol{\varphi}'$. If the curvature of the effective potential $\ln |V(\boldsymbol{\varphi})|$ is sufficiently small the motion of $\boldsymbol{\varphi}$ rapidly becomes slow and friction-dominated:

$$2\, \frac{d\boldsymbol{\varphi}}{dp} \simeq -\nabla_{\boldsymbol{\varphi}} \ln V(\boldsymbol{\varphi}) \,. \tag{2.9}$$

Eq. (2.9) is equivalent to the usual "slow roll" approximation.

Consistently with our general assumption (1.3), we consider potentials allowing a strong-coupling expansion of the form:

$$V(\chi,\varphi) = V_0(\chi) + V_1(\chi)e^{-c\varphi} + \mathcal{O}(e^{-2c\varphi})\,, \tag{2.10}$$

where $V_0(\chi)$ is a typical chaotic-inflation potential with $V_0(0) = 0$, while $V_1(0) = v_1 \geq 0$ can possibly provide (if $v_1 > 0$) the effective cosmological constant driving today's acceleration in the scenario of [6]. For the sake of simplicity we shall discuss mainly the "factorized" power-law case $V_0(\chi) \sim V_1(\chi) \sim \chi^n$ for which we can conveniently write V in the form:

$$V(\chi,\varphi) = \lambda(\varphi)\,\frac{\chi^n}{n}\,, \tag{2.11}$$

with a dilaton-dependent coupling constant $\lambda(\varphi)$ of the form

$$\lambda(\varphi) = \lambda_\infty(1 + b_\lambda\, e^{-c\varphi})\,. \tag{2.12}$$

This example belongs to the class of the two-field inflationary potentials discussed in [10]. We have checked that our results remain qualitatively the same for the more general potential (2.10) provided that $V_0(\chi)$ and $V_1(\chi)$ are not extremely diffent and given the fact that v_1 is phenomenologically constrained to be very small. [Note that, within the simplified model (2.12), the ratio $V_1(\chi)/V_0(\chi)$ is equal to the constant coefficient b_λ.]

The universal (positive) constant c appearing in the exponential $e^{-c\varphi}$ is the same as in Eq. (2.6) [i.e. $c \equiv (2C_g/C_\phi)^{1/2}$, which is expected to be of order unity]. The coefficient b_λ in (2.12) is such that $b_\lambda\, e^{-c\varphi} \simeq b_\lambda\, e^{-\phi}$ roughly corresponds to a combination of terms $\sim \pm\, C_i^{-1}\,\mathcal{O}(e^{-\phi})$ coming from the strong-coupling asymptotics of several $B_i(\phi)$, Eq. (1.3) (see Eq. (2.5)). In the toy model of [5] one would therefore expect b_λ to be smallish. Anyway, we shall see that in final results only the ratios of such b_i coefficients enter. More important than the magnitude of b_λ is its sign. It is crucial for the present strong-coupling attractor scenario to assume that $b_\lambda > 0$, i.e that $\lambda(\varphi)$ reaches a *minimum* at strong-coupling, $\varphi \to +\infty$. Note again that this behaviour is consistent with the simple "large N"-type idea of [5] if we assimilate $\lambda(\varphi)$ to one of the inverse couplings B_i appearing in (1.1) (for instance $B_F \sim g_F^{-2}$, where g_F is a gauge coupling), rather than to the coupling itself. If the latter were the case, $\lambda(\varphi)$ would reach a *maximum* as $\phi \to +\infty$, and the attractor mechanism of [3] would drive ϕ towards weak coupling ($\phi \to -\infty$). However, the Einstein-frame ϕ-dependence of $V(\chi)$ gets contributions from several $B_i^{\pm n}(\phi)$, Eq. (2.5), which might conspire to minimize it at strong coupling. This feature is also probably necessary in order to solve the cosmological-constant problem through some argument by which the vacuum at infinity has vanishing energy density.

Substituting the potential (2.10) into the slow roll equation (2.9) and assuming (for simplicity) that $V_1(\chi)e^{-c\varphi}$ is significantly smaller than $V_0(\chi)$ leads to a decoupled set of evolution equations for χ and φ (where $V' \equiv \partial V/\partial\chi$):

$$\frac{d\chi}{dp} = -\frac{1}{2}\,\frac{V_0'}{V_0}\,, \tag{2.13}$$

$$\frac{d\varphi}{dp} = \frac{1}{2}\,c\,e^{-c\varphi}\frac{V_1}{V_0}\,. \tag{2.14}$$

Given some "initial" conditions $\chi_{\rm in}$, $\varphi_{\rm in}$ (discussed below) at some starting point, say $p = 0$, the solution of Eqs. (2.13), (2.14) is simply

$$p = \,2\int_{\chi}^{\chi_{\rm in}} d\bar{\chi}\,\bar{\chi}\left(\frac{V_0(\bar{\chi})}{\bar{\chi}\,V_0'(\bar{\chi})}\right)\,, \tag{2.15}$$

$$e^{c\varphi} = e^{c\varphi_{\rm in}} + \,\frac{c^2}{2}\,\int dp\,\frac{V_1(\chi(p))}{V_0(\chi(p))}\,, \tag{2.16}$$

which simply become:

$$p = \frac{1}{n}\left(\chi_{\rm in}^2 - \chi^2\right)\,, \quad e^{c\varphi} + \frac{b_\lambda\,c^2}{2n}\,\chi^2 = \text{const.} = e^{c\varphi_{\rm in}} + \frac{b_\lambda\,c^2}{2n}\,\chi_{\rm in}^2\,, \tag{2.17}$$

in the simplified case of eqs. (2.11), (2.12).

Equations (2.17) show that, in order for the string coupling $g_s^2 \simeq e^{c\varphi}$ to have reached large values at the end of inflation, a large total number of e-folds must have occurred while the (dimensionless) inflaton field χ decreases from a large initial value, to a value of order unity (in Planck units). To get a quantitative estimate of the string coupling at the end of inflation we need to choose the initial conditions $\chi_{\rm in}$, $\varphi_{\rm in}$. A physically reasonable way (which is further discussed below) of choosing $\chi_{\rm in}$ is to start the classical evolution (2.13)-(2.17) at the exit of the era of self-regenerating inflation (see [10] and references therein). We will now show how to relate the exit from self-regenerating inflation to the size of density fluctuations generated by inflation.

Let us recall (see [10] and references therein) that the density fluctuation $\delta \equiv \delta\rho/\rho$ on large scales (estimated in the one-field approximation where the inflaton χ is the main contributor) is obtained by evaluating the expression

$$\delta(\chi) \simeq \frac{4}{3}\,\frac{1}{\pi}\left(\frac{2}{3}\right)^{\frac{1}{2}}\,\frac{V^{3/2}}{\partial_\chi V} \tag{2.18}$$

at the value $\chi = \chi_\times$, at which the physical scale we are considering crossed the horizon outwards during inflation. For the scale corresponding to our present horizon this usually corresponds to a value $\chi_\times(H_0)$ (χ_H for short) reached some 60 e-folds before the end of slow-roll. From Ref. [10], $\chi_H \simeq 5\sqrt{n}$ for the model (2.11) (and with our modified definition of χ). The numerical value of $\delta_H \equiv \delta(\chi_H)$ which is compatible with cosmological data (structure formation and cosmic microwave background) is $\delta_H \simeq 5 \times 10^{-5}$. In the model (2.11) the function $\delta(\chi)$ defined by (2.18) scales with χ as $\chi^{\frac{n+2}{2}}$. Putting together this information we obtain a relation between $\chi_{\rm in}$ and $\delta(\chi_{\rm in})$, which involves the value of the observable horizon-size fluctuations $\delta_H \equiv \delta(\chi_H)$:

$$\frac{\delta(\chi_{\rm in})}{\delta(\chi_H)} = \left(\frac{\chi_{\rm in}}{\chi_H}\right)^{\frac{n+2}{2}} , \tag{2.19}$$

i.e.

$$\chi_{\rm in} \simeq \chi_H \left(\frac{\delta_{\rm in}}{\delta_H}\right)^{\frac{2}{n+2}} \simeq 5\sqrt{n} \left(\frac{\delta_{\rm in}}{\delta_H}\right)^{\frac{2}{n+2}} , \tag{2.20}$$

where we introduced the short-hand notation $\delta_{\rm in} \equiv \delta(\chi_{\rm in})$.

Inserting Eq. (2.20) into Eq. (2.16) we then obtain the following estimate of the string coupling constant after inflation as a function of $\varphi_{\rm in}$ and $\delta(\chi_{\rm in})$

$$e^{c\varphi_{\rm end}} - e^{c\varphi_{\rm in}} \simeq \frac{c^2}{2}\langle V_1/V_0\rangle p \sim \frac{c^2}{2n}\langle V_1/V_0\rangle \chi_{\rm in}^2 \sim \frac{25c^2}{2}\langle V_1/V_0\rangle \left(\frac{\delta_{\rm in}}{\delta_H}\right)^{\frac{4}{n+2}} , \tag{2.21}$$

where $\langle V_1/V_0\rangle$ denotes the average value of V_1/V_0: $\langle V_1/V_0\rangle \equiv \int dp (V_1/V_0)/\int dp$ [note that this average ratio is equal to b_λ in the simplified model (2.12)].

To get a quantitative estimate of $e^{c\varphi_{\rm end}}$ we still need to estimate the value of $\delta(\chi_{\rm in})$ corresponding to the chosen "initial" value of the inflaton. As we will now check, taking for $\chi_{\rm in}$ the value corresponding to the exit from self-regenerating inflation corresponds simply to taking $\delta(\chi_{\rm in}) \sim 1$. Indeed, let us first recall that, during inflation, each (canonically normalized) scalar field (of mass smaller than the expansion rate H) undergoes typical quantum fluctuations of order $H/(2\pi)$, per Hubble time (see, e.g., [10]). This implies (for our dimensionless fields) that the value of χ at the exit from self-regeneration, say $\chi_{\rm ex}$, is characterized by $\widehat{H}_{\rm ex}/(2\pi) \approx [\partial_\chi V/(2V)]_{\rm ex}$, where $\widehat{H} \equiv H/\widetilde{m}_P$ is the dimensionless Hubble expansion rate and where the right-hand side (RHS) is the classical change of χ per Hubble time (corresponding to the RHS of Eq.(2.13)). Using Friedmann's equation (in the slow-roll approximation) $\widehat{H}^2_{\rm ex} \approx (2/3)V(\chi_{\rm ex})$, it is easily seen that that the exit from self-regeneration corresponds to $\delta(\chi_{\rm ex}) \approx 4/3 \sim 1$. It is, a posteriori, physically quite reasonable to start using

the classical evolution system only when the (formal extrapolation) of the density fluctuation $\delta(\chi)$ becomes smaller than one.

Within some approximation (see [10]), one can implement the effect of the combined quantum fluctuations of (φ, χ) by adding random terms with r.m.s. values $\widehat{H}/2\pi$ on the right hand side of equations (2.13) and (2.14), $d\chi/dp$ and $d\varphi/dp$ being precisely the shifts of the fields in a Hubble time. The system of equations becomes thus of the Langevin-type

$$\frac{d\chi}{dp} \;=\; -\frac{1}{2}\,\frac{V_0'}{V_0} \;+\; \frac{\widehat{H}}{2\pi}\,\xi_1\,, \tag{2.22}$$

$$\frac{d\varphi}{dp} \;=\; \frac{1}{2}\,c\,e^{-c\varphi}\frac{V_1}{V_0} \;+\; \frac{\widehat{H}}{2\pi}\,\xi_2\,, \tag{2.23}$$

where $\widehat{H} \approx [(2/3)V(\chi,\phi)]^{1/2}$ (in the slow-roll approximation) is the dimensionless expansion rate, and where ξ_1 and ξ_2 are (independent) normalized random white noises:

$$\langle \xi_i(p_1)\,\xi_j(p_2)\rangle \;=\; \delta_{ij}\delta(p_1-p_2), \qquad i,j=1,2\,. \tag{2.24}$$

When the random force terms dominate the evolution in either equation (2.22) or equation (2.23) the quasi-classical description (2.13) , (2.14) breaks down. The phase space of the sytem can thus be roughly divided into four regions according to whether the evolution of none, one or both of the two fields is dominated by quantum fluctuations. This is depicted in Figure 1 where such regions are delimited by dashed, thick curves in the case of a power-law potential (2.11).

Apart from factors of order one, the evolution of the inflaton χ is quasi-classical in the region under the line $\chi = \lambda_\infty^{-1/(n+2)}$. In the chaotic inflationary models [10] such an inflaton's value corresponds to the exit from the self-regenerating regime and to the beginning of the quasi-classical slow-roll inflation. As mentioned above it also corresponds to a perturbation $\delta(\chi) \sim 1$. Somewhat surprisingly, in the model at hand, however, the quasi-classical region for the inflaton evolution $\chi \lesssim \lambda_\infty^{-1/(n+2)}$ is affected by quantum fluctuations that still dominate the evolution of φ in the region above the hyperbola-like curve $\chi = b_\lambda^{2/n}\lambda_\infty^{-1/n}e^{-2c\varphi/n}$. We must therefore study, in some detail, the evolution of the system in the presence of the noise term for φ as in eq. (2.23), assuming a quasi classical evolution for χ. This is done in the appendix. The final result is that, if we start at $\chi \lesssim \lambda_\infty^{-1/(n+2)}$, (classical evolution for χ) the average value of $e^{c\varphi}$ is *multiplicatively renormalized*, by a factor of order unity, with respect to the classical trajectory $e^{c\varphi_{\rm cl}}$, given by solving equations (2.13) - (2.14), i.e. $\langle e^{c\varphi}\rangle \;=\; \mathcal{O}(1)e^{c\varphi_{\rm cl}}$. One also finds that the dispersion of $e^{c\varphi}$ around its average value is comparable to its average value.

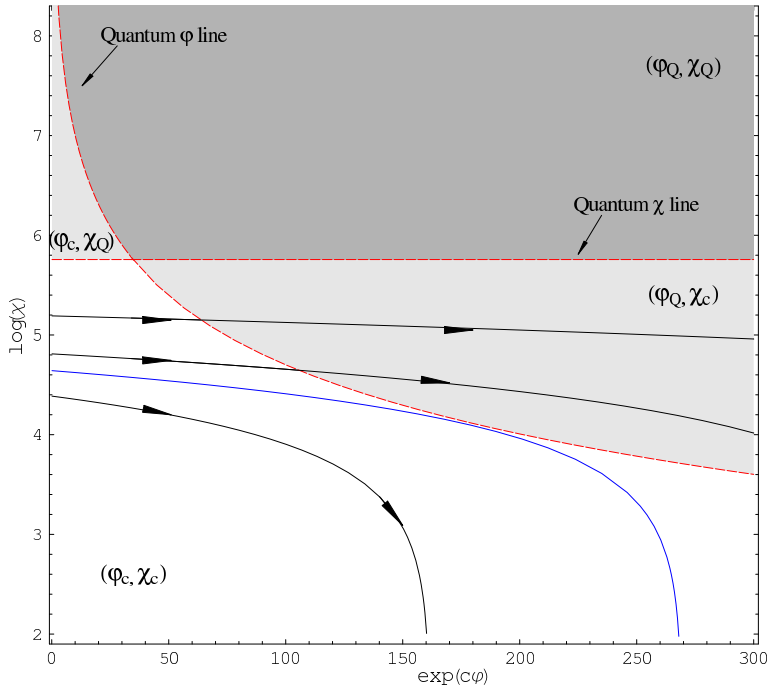

FIG. 1. *The phase space of the system is represented in the case of a power–law potential (2.11) with* $n = 2$, $b_\lambda = 0.1$ *and* $\lambda_\infty = 10^{-10}$. *The thick–dashed (red) curves delimitate the quantum behaviour of the two fields, the horizontal curve* $\chi = \lambda_\infty^{-1/(n+2)}$ *and the hyperbola-like curve* $\chi = b_\lambda^{2/n}\lambda_\infty^{-1/n}e^{-2c\varphi/n}$ *being the limit of the quantum behaviour for* χ *and* φ *respectively. In the white region both fields have a classical behaviour. The last "fully classical" trajectory has been represented by a thick (blue) curve. The bright–gray regions are those where either the* φ *or the* χ *evolution are dominated by quantum fluctuations. The fully–quantum region is the dark–gray region on the top right.*

We shall not try to discuss here what happens in the self-regenerating region $\chi \gtrsim \chi_{\rm in} \sim \lambda_\infty^{-1/(n+2)}$. Let us recall that the simple decoupled system (2.13) , (2.14) was obtained by neglecting the kinetic coupling term $F(\varphi)$ in Eq. (2.4). If we were to consider a more general model, we would have more coupling between χ and φ and we would expect that (contrary to Fig. 1 which exhibits a "classical φ region" above the "quantum χ line") the evolution in the self-regenerating region involves a strongly coupled system of Langevin equations. Then, as discussed in [10], solving such a system necessitates to give boundary conditions on all the boundaries of the problem: notably for $\chi \to \infty$, but also for $\varphi \to +\infty$ and $\varphi \to -\infty$. We leave to future work such an investigation (and a discussion of what are reasonable boundary conditions). In this work we shall content ourselves with "starting" the evolution on the quantum χ boundary line $\chi_{\rm in}$ with some value $\varphi = \varphi_{\rm in}$, assuming that $e^{c\varphi_{\rm in}}$ is smaller

than the driving effect due to inflation, i.e. than the RHS of Eq. (2.21). [This assumption is most natural in a work aimed at studying the "attracting" effect due to primordial inflation.]

Going back to our result (2.21), we can now insert, according to the preceding discussion, the values $\delta_{\rm in} = 1$ and $e^{c\varphi_{\rm in}} \ll e^{c\varphi_{\rm end}}$. Finally, in the simplified model (2.11), (2.12), we get the estimate:

$$e^{c\varphi_{\rm end}} = \mathcal{O}(1) \cdot e^{c\varphi_{\rm cl,\,end}} \sim \mathcal{O}(1) \cdot \frac{25c^2}{2}\, b_\lambda\ (\delta_H)^{-\frac{4}{n+2}}\ . \tag{2.25}$$

A more general analysis, based on the potential (2.10) leads to the same final result but with n replaced by some average value of $\frac{\chi V_{0,\chi}}{V_0}$, and with b_λ replaced by some average of the ratio V_1/V_0. Note that smaller values of the exponent n lead to larger values of $e^{c\varphi_{\rm end}}$, i.e. to a more effective attraction towards the "fixed point at infinity". The same is true if we take different exponents n_0 and n_1 (for V_0 and V_1 respectively) and assume $V_1(\chi_{\rm in}) \gg V_0(\chi_{\rm in})$ to hold as a result of $\chi_{\rm in} \gg 1$ and $n_1 > n_0$. Also note that, numerically, if we consider $n = 2$, i.e. the simplest chaotic-inflation potential $V = \frac{1}{2}\, m_\chi^2(\varphi)\, \chi^2$, Eq. (2.25) involves the large number $12.5 \times \delta_H^{-1} \sim 2.5 \times 10^5$. In the case where $n = 4$, i.e. $V = \frac{1}{4}\, \lambda(\varphi)\, \chi^4$, we have instead the number $12.5 \times \delta_H^{-2/3} \sim 0.92 \times 10^4$. To understand the phenomenological meaning of these numbers we need to relate $e^{c\varphi_{\rm end}}$ to the present, observable deviations from general relativity. This issue is addressed in Section III after having argued that the post-inflationary evolution of φ is sub-dominant.

B. Attraction of φ by the subsequent cosmological evolution

We have discussed above the efficiency with which inflation drives the dilaton towards a fixed point at infinity. We need to complete this discussion by estimating the effect of the many e-folds of expansion that took place between the end of inflation and the present time. To address this question, we need to study in more detail the coupling of a runaway dilaton to various types of matter, say a multi-component distribution of (relativistic or non-relativistic) particles. We work in the Einstein frame, with an action of the type

$$S = \int d^4x\sqrt{g}\left[\frac{\widetilde{m}_P^2}{4}R - \frac{\widetilde{m}_P^2}{2}(\nabla\varphi)^2 - \frac{1}{4}B_F(\varphi)F^2 + \ldots\right] \tag{2.26}$$
$$- \sum_A \int m_A[\varphi(x_A)]\sqrt{-g_{\mu\nu}(x_A)dx_A^\mu dx_A^\nu}\,.$$

Following [2,3], one introduces the crucial dimensionless quantity

$$\alpha_A(\varphi) \equiv \frac{\partial \ln m_A(\varphi)}{\partial \varphi}\,, \tag{2.27}$$

measuring the coupling of φ to a particle of type A. [For consistency with previous work, we keep the notation α_A but warn the reader that this should not be confused with the various gauge coupling constants, often denoted $\alpha_i = g_i^2/4\pi$.] The quantity α_A determines the effect of cosmological matter on the evolution of φ through the general equation [4,3]

$$\frac{2}{3-\varphi'^2}\,\varphi'' + \left(1 - \frac{P}{\rho}\right)\varphi' = -\sum_A \alpha_A(\varphi)\,\frac{\rho_A - 3P_A}{\rho}\,, \tag{2.28}$$

where the primes denote derivatives with respect to $p = \ln a + \text{const}$ and where $\rho = \Sigma_A\,\rho_A$ and $P = \Sigma_A\,P_A$ are the total "material" energy density and pressure respectively, both obtained as sums over the various components filling the universe *at the exception of* the kinetic energy density and pressure of φ, $\rho_k = (\widetilde{m}_P^2/2)(d\varphi/dt)^2 = (\widetilde{m}_P^2/2)H^2\varphi'^2$ and $P_k = \rho_k$. Accordingly, the Friedmann equation reads

$$3H^2 \,=\, \frac{2}{\widetilde{m}_P^2}\,\rho_{\rm tot} \,=\, \frac{2\,\rho}{\widetilde{m}_P^2} + H^2\varphi'^2. \tag{2.29}$$

Note that ρ and P may also account for the potential energy density and pressure of the scalar field, $\rho_V = V(\varphi)$, $P_V = -\rho_V$ and that one can formally extend Eq. (2.28) to the "vacuum energy" component $V(\varphi)$ by associating to the potential $V(\varphi)$ the mass scale $m_V(\varphi) \equiv V(\varphi)^{\frac{1}{4}}$ which gives $\alpha_V = \partial \ln m_V(\varphi)/\partial\varphi = \frac{1}{4}\,\partial \ln V(\varphi)/\partial\,\varphi$. Equation (2.8) is then recovered in the limit where the scalar field is the dominant component.

In the simple cases (which are quite frequent; at least as approximate cases) where one "matter" component, with known "equation of state" $P_A/\rho_A = w_A = \text{const.}$, dominates the cosmological density and pressure, Eq. (2.28) yields an autonomous equation for the evolution (with redshift) of φ. Using (2.29) one finds that the "equation of state parameter" $w_{\rm tot} \equiv P_{\rm tot}/\rho_{\rm tot}$ corresponding to the *total* energy and pressure (including now the kinetic contributions of φ; i.e. $\rho_{\rm tot} = \rho + (\widetilde{m}_P^2/2)(d\varphi/dt)^2$, $P_{\rm tot} = P + (\widetilde{m}_P^2/2)\,(d\varphi/dt)^2)$ is given in terms of the "matter" equation-of-state parameter $w \equiv P/\rho$ by

$$w_{\rm tot} = w + \frac{1-w}{3}(\varphi')^2\,. \tag{2.30}$$

The knowledge of $w_{\rm tot}$ then allows one to write explicitly the energy-balance equation $d\rho_{\rm tot} + 3(\rho_{\rm tot} + P_{\rm tot})d\ln a = 0$, which is easily solved in the simple cases where $w_{\rm tot}$ is (approximately) constant.

We see from Eq. (2.28) that, during the radiation era (starting, say, immediately after the end of inflation), i.e. when the universe is dominated by an ultra-relativistic gas ($\rho_A - 3P_A =$

0), the "driving force" on the right-hand side of (2.28) vanishes, so that φ is not driven further away towards infinity. Actually, one should take into account both the "inertial" effect of the "velocity" φ' acquired during the preceding inflationary driving of φ, and the integrated effect of the many "mass thresholds", $T_A \sim m_A$, when some component becomes non-relativistic (so that $\rho_A - 3P_A \neq 0$). Using the results of [4,3] one sees that, in our case, both these effects have only a small impact on the value of φ. Therefore, to a good approximation $\varphi \simeq \varphi_{\rm end}$ until the end of radiation era.

On the other hand, when the universe gets dominated by non-relativistic matter, one gets a non-zero driving force in Eq. (2.28). In the slow roll approximation, as the transient behaviour has died out, since $w = P/\rho$ gets negligible, we have simply

$$\varphi'_m = -\alpha_m(\varphi)\,, \tag{2.31}$$

where φ'_m stands the φ-velocity during matter domination. and $\alpha_m(\varphi)$ denotes the coupling (2.27) to dark matter.

The coupling to dark matter, $\alpha_m(\varphi)$, depends on the assumption one makes about the asymptotic behaviour, at strong bare string coupling, of the mass of the WIMPs constituting the dark matter. One natural looking, minimal assumption is that dark matter, like all visible types of matter, is coupled in a way which levels off at strong-bare-coupling, as in Eq. (1.3). In other words, one generally expects that $m_m(\varphi) \simeq m_m(+\infty)(1+b_m\, e^{-c\varphi})$ so that $\alpha_m(\varphi) \simeq -b_m\, c\, e^{-c\varphi}$. It is then easy to solve Eq. (2.31), with initial conditions $\varphi_0 = \varphi_{\rm end}$, $\varphi'_0 = 0$ (inherited from radiation era) at the beginning of the matter era. But we shall not bother to write the explicit solution because it is easily seen that the smallness of $e^{-c\varphi_{\rm end}}$ guarantees that the "driving force" $\propto \alpha_m(\varphi)$ remains always so small that the $\mathcal{O}(10)$ e-folds of matter era until vacuum-energy domination (or until the present) have only a fractionally negligible effect on φ.

A more significant evolution of φ during the matter era is provided if, as first proposed in [11] and taken up in [25,26], dark matter couples much more strongly to φ than "ordinary" matter. Such a stronger coupling to dark matter, which is not constrained by usual equivalence principle experiments, follows assuming more general quantum corrections in the dark matter sector of the theory, i.e. corrections such that the dark matter mass $m_m(\varphi)$, instead of levelling off, either vanishes or keeps increasing at strong bare coupling: $m_m(\varphi) \propto e^{c_m\varphi}$, so that $\alpha_m = c_m$ is a (negative or positive) constant. In [6] (but see also [12]) it has been shown that under the latter assumption (i.e. with a positive coupling parameter $\alpha_m > 0$) the dilaton can play the role of quintessence, leading to a late-time cosmology of accelerated expansion. By Eq. (2.31) we have $\varphi = \varphi_{\rm end} - \alpha_m p$, where p is now counted from the end of the radiation era. Given that about nine e-folds separate us from the end of the

radiation era, we see that such an evolution might (if $|\alpha_m|$ is really of order unity) have a significant effect on the present value of φ (when compared with the value at the end of inflation, i.e. $c\varphi_{\rm end} \sim \ln(1/\delta_H) \sim 10$). However, the running of φ during the matter era changes the standard recent cosmological picture and is therefore constrained by observations. In fact, by equation (2.30), the total matter-era equation of state parameter $w_{\rm tot}$ in the presence of the dilaton reads $w_{\rm tot} = (\varphi'_m)^2/3$. Accordingly, the matter density varies as $\rho \propto a^{-3(1+w_{\rm tot})} = a^{-(3+(\varphi'_m)^2)}$, possibly affecting the standard scenario of structure formation as well as the global temporal picture between now and the epoch of matter-radiation equality. The compatibility with phenomenology therefore puts constraints on the magnitude of ${\varphi'_m}^2 = \alpha_m(\varphi)^2$. In [6] $w_{\rm tot} < 0.1$, i.e. $v \equiv {\varphi'_m}^2/0.3 < 1$, was suggested to be the maximal deviation one can roughly tolerate during the matter era, the establishment of a more precise bound being presently under study [13]. We shall therefore assume either that we are in the "normal" case where the dilaton does not couple more strongly to dark matter than to ordinary matter (so that $\alpha_m(\varphi) \simeq -b_m\, c\, e^{-c\varphi} \ll 1$), or that, if it does, $\alpha_m^2 < 0.3$. This leads to a displacement of φ during matter era smaller than the dispersion $\varphi_{\rm end} - \varphi_{\rm cl,end} \sim \varphi_{\rm cl,end}$ produced by quantum fluctuations during inflation, Eq. (2.25).

In this context one should also consider the attraction effect of a negative pressure component, either in the form of a φ-dependent vacuum energy (dilatonic quintessence) or in the form of any other, φ-independent component (such as a "genuine" cosmological constant). Of course, the present recent ($z \lesssim 1$) accelerated expansion phase is very short (in "p-time") and sensible changes of the dilaton value since the end of matter domination are not expected. Still, it is crucial to estimate the present dilaton velocity φ'_0 since it is related to the cosmological variations of the coupling constants (see next section). In the general case where both non-relativistic matter and (possibly φ-dependent) vacuum energy density $V(\varphi)$ are present, the value of φ'_0 predicted by our model is obtained by applying Eq. (2.28) (in the slow-roll approximation):

$$(\Omega_m + \Omega_V)(1 - w_0)\varphi'_0 = (\Omega_m + 2\Omega_V)\varphi'_0 = -\Omega_m\alpha_m - 4\Omega_V\alpha_V. \qquad (2.32)$$

In the above expression Ω_m and Ω_V are, respectively, the non-relativistic (dark) matter- and the vacuum-fraction of critical energy density ($\rho_c \equiv (3/2)\widetilde{m}_P^2 H^2$), and the already mentioned prescriptions $\alpha_V = \frac{1}{4}\,\partial \ln V(\varphi)/\partial\,\varphi$, $P_V = -\rho_V = -V(\varphi)$ have been used.

The value of φ'_0 is therefore some combination of the values of α_m and α_V. We can have two classes of contrasting situations: In the first class, the dilaton couples "normally" (i.e. weakly) both to dark matter and to dark energy, i.e. both $\alpha_m \simeq -b_m\, c\, e^{-c\varphi} \ll 1$ and $\alpha_V \ll 1$ and Eq. (2.32) implies $\varphi'_0 \ll 1$. In the second class, the dilaton couples more strongly to some type of dark matter or energy, i.e. either (or both) α_m or/and α_V is of order unity so

that $\varphi_0' = \mathcal{O}(1)$. The second case is realized in the scenario of [6]. In the context of this scenario we have an exponential dependence of the potential on φ, $V(\varphi) \simeq V_1 e^{-c\varphi}$ so that $\alpha_V \simeq -(c/4)$ and

$$\varphi_0' = \frac{c\,\Omega_V - \alpha_m \Omega_m}{2\Omega_V + \Omega_m} \lesssim \frac{c}{3}. \tag{2.33}$$

The last inequality follows from the bound $\alpha_m > c/2$ (which is a necessary condition to have positive acceleration in the model [6]) and the reasonable bound $\Omega_m > 0.25$.

In the present work, we wish, however, to be as independent as possible from specific assumptions (as the ones used in [6]). Therefore, rather than insisting on specific (model-dependent) predictions for the present value of φ_0' we wish to find the (model-independent) upper bounds on the possible values of φ_0' set by current observational data. There are several ways of getting such phenomenological bounds, because the existence of a kinetic energy (and pressure) associated to $d\varphi/dt = H\varphi'$ has several observable consequences. A rather secure bound can be obtained by relating the value of φ' to the deceleration parameter $q \equiv -\ddot{a}a/\dot{a}^2$. In the general class of models that we consider, the cosmological energy density and pressure have (currently) three significant contributions: dark matter ($\Omega_m = \rho_m/\rho_c$), dark energy (Ω_V) and the kinetic effect of a scalar field ($\Omega_k = \rho_k/\rho_c$ with $\rho_k = (\widetilde{m}_P^2/2)(d\varphi/dt)^2 = (\widetilde{m}_P^2/2)H^2\varphi'^2$ so that $\Omega_k = \varphi'^2/3$). We assume (consistently with recent cosmic background data) that the space curvature is zero. Therefore we have the first relation

$$\Omega_m + \Omega_V + \Omega_k = 1 = \Omega_m + \Omega_V + \varphi'^2/3\,. \tag{2.34}$$

The deceleration parameter is given by the general expression $2q = \Sigma_A \Omega_A (1 + 3w_A)$. Using $w_m = 0$, $w_V = -1$ and $w_k = +1$, we get

$$2q = \Omega_m - 2\Omega_V + \frac{4}{3}\varphi'^2. \tag{2.35}$$

Using the relation (2.34) above to eliminate Ω_V we get the following expression for φ'^2 in terms of the observable quantities q and Ω_m

$$\varphi'^2 = 1 + q - \frac{3}{2}\Omega_m. \tag{2.36}$$

The supernovae Ia data [14] give a strict upper bound on the present value q_0: $q_0 < 0$. A generous lower bound on the present value of Ω_m is $\Omega_{m0} > 0.2$ [15]. Inserting these two constraints in Eq.(2.36) finally yields the safe upper bound

$${\varphi_0'}^2 < 0.7\,, \text{ i.e. } |\varphi_0'| < 0.84\,. \tag{2.37}$$

To summarize, quite different rates of evolution for the dilaton are possible. A very slow variation is expected whenever dilaton couplings to both dark energy and dark matter follow the "normal" behaviour (1.3). Otherwise, dilaton variations on the Hubble scale are expected. However, cosmological observations set the strict upper bound (2.37) on the present time variation of φ. For the purpose of the present section (evaluating the current location of the dilaton) these two alternatives do not make much difference because the vacuum-dominance era has started less than about 0.7 e-folds away ($\ln(1+z_*)$ with $z_* < 1$). Therefore, φ did not have enough "p-time" , during vacuum dominance, to move much, even if it is coupled to vacuum energy with $\alpha_V \simeq -(c/4) \sim 1$.

Finally, we conclude from this analysis that, to a good approximation (and using the fact that the phenomenology of the matter-era constrains the dark-matter couplings of the dilaton to be rather small), the value of φ now is essentially given by the value $\varphi_{\rm end}$ at the end of inflation, i.e. by Eq. (2.21).

III. DEVIATIONS FROM GENERAL RELATIVITY INDUCED BY A RUNAWAY DILATON

A. Composition-independent deviations from general relativity

The previous section has reached the conclusion that present deviations from general relativity are given, to a good approximation, by the values of the matter-coupling coefficients $\alpha_A(\varphi)$ given by Eq. (2.27) calculated at $\varphi \simeq \varphi_{\rm end}$ as given by Eq. (2.25). Let us now see the meaning of this result in terms of observable quantities.

Let us first consider the (approximately) composition-independent deviations from general relativity, i.e. those that do not essentially depend on violations of the equivalence principle. Most composition-independent gravitational experiments (in the solar system or in binary pulsars) consider the long-range interaction between objects whose masses are essentially baryonic (the Sun, planets, neutron stars). As argued in [2,3] the relevant coupling coefficient α_A is then approximately universal and given by the logarithmic derivative of the QCD confinement scale $\Lambda_{\rm QCD}(\varphi)$, because the mass of hadrons is essentially given by a pure number times $\Lambda_{\rm QCD}(\varphi)$. [We shall consider below the small, non-universal, corrections to $m_A(\varphi)$ and $\alpha_A(\varphi)$ linked to QED effects and quark masses.] Remembering from Eq. (1.1) the fact that, in the string frame (where there is a fixed cut-off linked to the string mass $\widetilde{M}_s \sim (\alpha')^{-1/2}$) the gauge coupling is dilaton-dependent ($g_F^{-2} = B_F(\varphi)$), we see that (after conformal transformation) the Einstein-frame confinement scale has a dilaton-dependence of

the form

$$\Lambda_{\rm QCD}(\varphi) \sim C^{-1/2}\, B_g^{-1/2}(\varphi) \exp[-8\pi^2\, b_3^{-1}\, B_F(\varphi)]\, \widetilde{M}_s\,, \tag{3.1}$$

where b_3 denotes the one-loop (rational) coefficient entering the Renormalization Group running of g_F. Here $B_F(\varphi)$ denotes the coupling to the SU(3) gauge fields. For simplicity, we shall assume that (modulo rational coefficients) all gauge fields couple (near the string cut off) to the same $B_F(\varphi)$. This yields the following approximately universal dilaton coupling to hadronic matter

$$\alpha_{\rm had}(\varphi) \simeq \left(\ln\left(\frac{\widetilde{M}_s}{\Lambda_{\rm QCD}}\right) + \frac{1}{2}\right) \frac{\partial \ln B_F^{-1}(\varphi)}{\partial\, \varphi}\,. \tag{3.2}$$

We recall that the quantity $\alpha_{\rm had}(\varphi)$, which measures the coupling of the dilaton to hadronic matter, should not be confused with any "strong" gauge coupling, $\alpha_s = g_s^2/4\pi$. Numerically, the coefficient in front of the R.H.S. of (3.2) is of order 40. Consistently with our basic assumption (1.3), we parametrize the φ dependence of the gauge coupling $g_F^2 = B_F^{-1}$ as

$$B_F^{-1}(\varphi) = B_F^{-1}(+\infty)\,[1 - b_F\, e^{-c\varphi}]\,. \tag{3.3}$$

Note that, like b_λ (see section II A), also b_F is expected to be smallish [$\sim B_F^{-1}(+\infty)$ or, equivalently, $\sim C_F^{-1}$ in the notations of (1.3)] and typically the ratio b_F/b_λ is of order unity. We finally obtain

$$\alpha_{\rm had}(\varphi) \simeq 40\, b_F\, c\, e^{-c\varphi}\,. \tag{3.4}$$

We can now insert the estimate (2.25) of the value of φ reached because of the cosmological evolution. Neglecting the $\mathcal{O}(1)$ renormalization factor due to quantum noise, we get the estimate

$$\alpha_{\rm had}(\varphi_{\rm end}) \simeq 3.2\, \frac{b_F}{b_\lambda\, c}\, \delta_H^{\frac{4}{n+2}}\,, \tag{3.5}$$

$$\alpha^2_{\rm had}(\varphi_{\rm end}) \simeq 10 \left(\frac{b_F}{b_\lambda\, c}\right)^2 \delta_H^{\frac{8}{n+2}}\,. \tag{3.6}$$

As said above, it is plausible to expect that the quantity c (which is a ratio) and the ratio b_F/b_λ are both of order unity. This then leads to the numerical estimate $\alpha^2_{\rm had} \sim 10\, \delta_H^{\frac{8}{n+2}}$, with $\delta_H \simeq 5 \times 10^{-5}$. An interesting aspect of this result is that the expected present value of $\alpha^2_{\rm had}$ depends rather strongly on the value of the exponent n (which entered the inflaton

potential $V(\chi) \propto \chi^n$). In the case $n = 2$ (i.e. $V(\chi) = \frac{1}{2} m_\chi^2 \chi^2$) we have $\alpha_{\rm had}^2 \sim 2.5 \times 10^{-8}$, while if $n = 4$ ($V(\chi) = \frac{1}{4} \lambda \chi^4$) we have $\alpha_{\rm had}^2 \sim 1.8 \times 10^{-5}$.

How do these numbers compare to present (composition-independent) experimental limits on deviations from Einstein's theory [16]? This question has been addressed in the literature. Concerning solar-system (post-Newtonian) tests it was shown (see, e.g., [17]) that the two main "Eddington" parameters $\gamma - 1$ and $\beta - 1$ measuring post-Newtonian deviations from general relativity are linked as follows to the dilaton coupling $\alpha_{\rm had}(\varphi)$:

$$\gamma - 1 = -2 \, \frac{\alpha_{\rm had}^2}{1 + \alpha_{\rm had}^2} \simeq -2 \, \alpha_{\rm had}^2 \, , \tag{3.7}$$

$$\beta - 1 = \frac{1}{2} \, \frac{\alpha'_{\rm had} \, \alpha_{\rm had}^2}{(1 + \alpha_{\rm had}^2)^2} \simeq \frac{1}{2} \, \alpha'_{\rm had} \, \alpha_{\rm had}^2 \, , \tag{3.8}$$

where $\alpha'_{\rm had} \equiv \partial \alpha_{\rm had}(\varphi)/\partial \varphi$.

From Eq. (3.4) we see that $\alpha'_{\rm had} \simeq -c \, \alpha_{\rm had}$, so that the deviation $\beta - 1$ is $\mathcal{O}(\alpha_{\rm had}^3)$ and thereby predicted to be too small to be phenomenologically interesting. This leaves $\gamma - 1 \simeq -2 \, \alpha_{\rm had}^2$ as the leading observable deviation. The best current solar-system limit on $\gamma - 1$ comes from Very Long Baseline Interferometry measurements of the deflection of radio waves by the Sun and is (approximately) $|\gamma - 1| \lesssim 2 \times 10^{-4}$, corresponding to $\alpha_{\rm had}^2 \lesssim 10^{-4}$ (see [16] for reviews and references). In addition to solar-system tests, we should also consider binary-pulsar tests which provide another high-precision window on possible deviations from general relativity. They have been analyzed in terms of the two quantities $\alpha_{\rm had}$ (denoted α) and $\alpha'_{\rm had}$ (denoted β) in [18]. The final conclusion is that the binary-pulsar limit on $\alpha_{\rm had}$ is of order $\alpha_{\rm had}^2 \lesssim 10^{-3}$.

At this stage it seems that the runaway scenario explored here is leading to deviations from general relativity which are much smaller than present experimental limits. However, we must turn our attention to *composition-dependent* effects which turn out to be much more sensitive tests.

B. Composition-dependent deviations from general relativity

Let us then consider situations where the non-universal couplings of the dilaton induce (apparent) violations of the equivalence principle. Let us start by considering the composition-dependence of the dilaton coupling α_A, Eq. (2.27), i.e. the dependence of α_A on

the type of matter we consider. The definition of α_A is such that, at the Newtonian approximation, the interaction potential between particle A and particle B is $-G_{AB}\, m_A\, m_B/r_{AB}$ where [3]

$$G_{AB} = G(1 + \alpha_A\, \alpha_B)\,. \tag{3.9}$$

Here, G is the bare gravitational coupling constant entering the Einstein-frame action (2.4), and $\alpha_A = \alpha_A(\varphi)$ is the strength of the dilaton coupling to A-particles, taken at the present (cosmologically determined) VEV of φ. The term $\alpha_A\, \alpha_B$ comes from the additional attractive effect of dilaton exchange. Two test masses, made respectively of A- and B-type particles will then fall in the gravitational field generated by an external mass m_E with accelerations differing by

$$\left(\frac{\Delta a}{a}\right)_{AB} \equiv 2\,\frac{a_A - a_B}{a_A + a_B} \simeq (\alpha_A - \alpha_B)\,\alpha_E\,. \tag{3.10}$$

We have seen above that in lowest approximation $\alpha_A \simeq \alpha_{\rm had}$ does not depend on the composition of A. We need, however, now to retain the small composition-dependent effects to α_A linked to the φ-dependence of QED and quark contributions to m_A. This has been investigated in [3] with the result

$$\left(\frac{\Delta a}{a}\right)_{AB} = \left(\frac{\alpha_{\rm had}}{40}\right)^2 \left[C_B\,\Delta\left(\frac{B}{M}\right) + C_D\,\Delta\left(\frac{D}{M}\right) + C_E\,\Delta\left(\frac{E}{M}\right)\right]_{AB}\,, \tag{3.11}$$

where $(\Delta X)_{AB} \equiv X_A - X_B$, where $B \equiv N + Z$ is the baryon number, $D \equiv N - Z$ the neutron excess, $E \equiv Z(Z-1)/(N+Z)^{1/3}$ a quantity linked to nuclear Coulomb effects, and where $M \equiv m/u$ denotes the mass in atomic mass unit, $u = 931.49432$ MeV. It is difficult (and model-dependent) to try to estimate the coefficients C_B and C_D. It was argued in [3] that their contributions to (3.11) is generically expected to be sub-dominant with respect to the last contribution, $\propto C_E$, which can be better estimated because it is linked to the φ-dependence of the fine-structure constant $e^2 \propto B_F^{-1}(\varphi)$. This then leads to the numerical estimate $C_E \simeq 3.14 \times 10^{-2}$ and a violation of the universality of free fall approximately given by

$$\left(\frac{\Delta a}{a}\right)_{AB} \simeq 2 \times 10^{-5}\,\alpha_{\rm had}^2 \left[\left(\frac{E}{M}\right)_A - \left(\frac{E}{M}\right)_B\right]\,. \tag{3.12}$$

The values of B/M, D/M and E/M have been computed in [19]. For mass-pairs that have been actually used in recent experiments (such as Beryllium and Copper), as well as for mass-pairs that are planned to be used in forthcoming experiments (such as Platinum

and Titanium) one finds: $(E/M)_{\rm Cu} - (E/M)_{\rm Be} = 2.56$, $(E/M)_{\rm Pt} - (E/M)_{\rm Ti} = 2.65$. Using the average estimate $\Delta(E/M) \simeq 2.6$, we get from (3.12) and (3.6) the estimate

$$\left(\frac{\Delta a}{a}\right) \simeq 5.2 \times 10^{-5}\, \alpha_{\rm had}^2 \simeq 5.2 \times 10^{-4} \left(\frac{b_F}{b_\lambda\, c}\right)^2 \delta_H^{\frac{8}{n+2}} \,. \tag{3.13}$$

Note also (from (3.7)) the link between composition-dependent effects and post-Newtonian ones

$$\left(\frac{\Delta a}{a}\right) \simeq -2.6 \times 10^{-5} (\gamma - 1) \,. \tag{3.14}$$

As current tests of the universality of free fall (UFF) have put limits in the 10^{-12} range (e.g. $(\Delta a/a)_{\rm Be\,Cu} = (-1.9 \pm 2.5) \times 10^{-12}$ from [20]), we see from Eq. (3.14) that this corresponds to limits on $\gamma - 1$ or $\alpha_{\rm had}^2$ in the 10^{-7} range. Therefore tests of the UFF put much more stringent limits on dilaton models than solar-system or binary-pulsar tests.

If we insert the estimate $\delta_H \sim 5 \times 10^{-5}$ in (3.13) we obtain a level of violation of UFF due to a runaway dilaton which is

$$\frac{\Delta a}{a} \simeq 1.3 \left(\frac{b_F}{b_\lambda\, c}\right)^2 \times 10^{-12} \quad \text{for } n = 2 \,, \tag{3.15}$$

$$\frac{\Delta a}{a} \simeq 0.98 \left(\frac{b_F}{b_\lambda\, c}\right)^2 \times 10^{-9} \quad \text{for } n = 4 \,. \tag{3.16}$$

At face value, one is tempted to conclude that a scenario with $n = 4$ (i.e. $V(\chi) \propto \chi^4$) tends to be too weak an attractor towards $\varphi = +\infty$ to be naturally compatible with equivalence-principle tests. [See, however, the discussion below.] On the other hand, the simple scenario $n = 2$ ($V(\chi) = \frac{1}{2} m_\chi^2 \chi^2$) is quite appealing in that it naturally provides enough attraction towards $\varphi = +\infty$ to be compatible with all existing experimental tests. At the same time it suggests that a modest improvement in the precision of UFF experiments might discover a violation caused by a runaway dilaton.

C. Cosmological variation of "constants"

Let us now consider another possible deviation from General Relativity and the standard model: a possible variation of the coupling constants, most notably of the fine structure constant $e^2/\hbar c$ on which the strongest limits are available. We will discuss first the effects

due to the cosmological time-variation of the homogeneous component of φ and, in the next subsection, the possible spatial (and time) variations due to quantum fluctuations of φ as they got amplified during inflation.

Consistently with our previous assumptions we expect $e^2 \propto B_F^{-1}(\varphi)$ so that, from (3.3),

$$e^2(\varphi) = e^2(+\infty)\,[1 - b_F\, e^{-c\varphi}]\,. \tag{3.17}$$

The present logarithmic variation of e^2 (using again $dp = H\,dt; \varphi' = d\varphi/dp$) is thus given by

$$\frac{d\ln e^2}{H\,dt} = \frac{d\ln e^2}{dp} \simeq b_F\, c\, e^{-c\varphi}\,\varphi_0'\,, \tag{3.18}$$

where the current value of φ', φ_0', is given in general by Eq. (2.32). Using Eq. (3.4), we can rewrite the result (3.18) in terms of the hadronic coupling:

$$\frac{d\ln e^2}{H\,dt} \simeq \frac{1}{40}\alpha_{\rm had}\varphi_0'\,. \tag{3.19}$$

As said in section II B, we have basically two alternatives concerning the current coupling of the dilaton to the dominant energy sources in the universe. These two alternatives lead to drastically different predictions for the current value of the rate of variation of the fine-structure constant. We shall consider these two alternatives in turn.

In the conservative case where the dilaton does not play any special role in the present accelerated phase of the universe ($\alpha_V \simeq 0$) nor does it have any stronger coupling to dark matter than to visible matter ($\alpha_m \simeq -b_m\, c\, e^{-c\varphi}$) the dilaton "velocity" φ' is exponentially suppressed (so that, from (2.34), $\Omega_V \simeq 1 - \Omega_m$) and by Eq. (2.32) one obtains

$$\frac{d\ln e^2}{H\,dt} \;\simeq\; -\frac{\Omega_m}{\Omega_m + 2\Omega_V}\, b_F\, c\, e^{-c\varphi}\alpha_m(\varphi) \;\simeq\; \frac{\Omega_m}{2-\Omega_m}\, b_F\, b_m\, c^2\, e^{-2c\varphi}. \tag{3.20}$$

An indicative value for the ratio $\Omega_m/(\Omega_m + 2\Omega_V) \simeq \Omega_m/(2-\Omega_m)$, by taking for instance $\Omega_m = 0.3$, is 0.18. As above, it is useful to relate (3.20) to the estimate (3.4) for $\alpha_{\rm had}$. This yields

$$\frac{d\ln e^2}{H\,dt} \;\simeq\; \frac{1}{(40)^2}\,\frac{\Omega_m}{2-\Omega_m}\,\frac{b_m}{b_F}\,\alpha_{\rm had}^2\,. \tag{3.21}$$

In terms of the UFF level $\Delta a/a$ predicted by our model in (3.13) we see also that

$$\frac{d\ln e^2}{H\,dt} \;\simeq\; 12\,\frac{\Omega_m}{2-\Omega_m}\,\frac{b_m}{b_F}\,\frac{\Delta a}{a}\,. \tag{3.22}$$

Even if the universe were completely dominated by dark matter ($\Omega_m = 1$) we see, assuming that b_m/b_F is of order unity, that current experimental limits on UFF ($\Delta a/a \lesssim 10^{-12}$) imply (within dilaton models) that $|d \ln e^2/dt| \lesssim 10^{-11}\, H \sim 10^{-21}\, \mathrm{yr}^{-1}$ (the sign of $d \ln e^2/dt$ being given by the sign of b_m/b_F). This level of variation is much smaller than the current best limit on the time variation of e^2, namely $|d \ln e^2/dt| \lesssim 5 \times 10^{-17}\, \mathrm{yr}^{-1} \sim 5 \times 10^{-7}\, H$, as obtained from an analysis of Oklo data [21]. (Note that the assumption-dependent analysis of Ref. [22] gives a limit on the variation of e^2 which is strengthened by about two orders of magnitude.)

The situation, however, is drastically different if we consider the alternative case where the dilaton coupling to the current dominant energy sources does not tend to triviality, as in the case of a φ-dependent vacuum energy $V(\varphi) = V_0 + V_1 e^{-c\varphi}$ when the first term is zero or negligible. In such a case the dilaton shares a relevant part of the total energy density and more significant (though still quite constrained by UFF data) variations of the coupling constants are generally expected. A general expression for the dilaton "velocity" is given in eq. (2.36) in terms of observable quantities. Using Eqs. (2.36) and (3.19) one can relate the expected variation of the electromagnetic coupling constant to the hadronic coupling:

$$\frac{d \ln e^2}{H\, dt} \simeq \pm \frac{\alpha_{\rm had}}{40} \sqrt{1 + q_0 - 3\Omega_m/2}\,. \tag{3.23}$$

We can also use the estimate (3.5) relating $\alpha_{\rm had}$ to the density fluctuations generated during inflation. We obtain

$$\frac{d \ln e^2}{H\, dt} \simeq \pm\, 8 \times 10^{-2} \sqrt{1 + q_0 - 3\Omega_m/2}\, \frac{b_F}{b_\lambda c}\, \delta_H^{\frac{4}{n+2}}\,. \tag{3.24}$$

However, in view of the theoretical uncertainties attached to the initial conditions $\chi_{\rm in}$ and $\varphi_{\rm in}$ used in the estimate (3.5), as well as the ones associated to the order unity ratio $b_F/(b_\lambda c)$, it is more interesting to rewrite our prediction in terms of *observable* quantities. Using again the link Eq. (3.13) between $\alpha_{\rm had}$ and the observable violation of the universality of free fall (UFF) the above result can be written in the form

$$\frac{d \ln e^2}{H\, dt} \simeq \pm\, 3.5 \times 10^{-6} \sqrt{1 + q_0 - 3\Omega_m/2} \sqrt{10^{12} \frac{\Delta a}{a}}\,. \tag{3.25}$$

Note that the sign of the variation of e^2 is in general model-dependent (as it depends both on the sign of b_F and the sign of φ_0'). Specific classes of models might, however, favour particular signs of de^2/dt. For instance, from the point of view of [5] one would expect the $\mathcal{O}(e^{-\phi})$ terms in Eq. (1.3) to be positive, which would then imply that b_F is positive. If we

combine this information with the prediction Eq. (2.33) of the model [6] implying that φ' is also positive, we would reach the conclusion that e^2 must be currently *increasing*.

Independently of this question of the sign, we see that Eq. (3.25) predicts an interesting link between the observational violation of the UFF (constrained to $\Delta a/a \lesssim 10^{-12}$), and the current time-variation of the fine-structure constant. Contrary to the relation (3.22), obtained above under the alternative assumption about the dilaton dependence of the dominant cosmological energy, which predicted a relation linear in $\Delta a/a$, we have here a relation involving the square root of the UFF violation (such a relation is similar to the result of [11] which concerned the time-variation of the Newton constant).

The phenomenologically interesting consequence of Eq. (3.25) is to predict a time-variation of constants which may be large enough to be detected by high-precision laboratory experiments. Indeed, using $H_0 \simeq 66$ km/s/Mpc, and the plausible estimates $\Omega_m = 0.3$, $q_0 = -0.4$, Eq. (3.25) yields the numerical estimate $d \ln e^2/dt \sim \pm 0.9 \times 10^{-16} \sqrt{10^{12} \Delta a/a}\, \mathrm{yr}^{-1}$. Therefore, the current bound on UFF violations ($\Delta a/a \sim 10^{-12}$) corresponds to the level $10^{-16}\mathrm{yr}^{-1}$, which is comparable to the planned sensitivity of currently developed cold-atom clocks [28]. [Present laboratory bounds are at the $10^{-14}\mathrm{yr}^{-1}$ level [29,28].] Note that if we insert in Eq. (3.25) the secure bounds $\Omega_m > 0.2$ and $q_0 < 0$ (leading to the limit Eq. (2.37)), we get as maximal estimate of the time variation of the fine-structure constant $d \ln e^2/dt \sim \pm 2.0 \times 10^{-16} \sqrt{10^{12} \Delta a/a}\, \mathrm{yr}^{-1}$. We note also that the upper limit on the variation of e^2 given by the Oklo data, i.e. $|d \ln e^2/dt| \lesssim 5 \times 10^{-17}\, \mathrm{yr}^{-1}$ [21], "corresponds" to a violation of the UFF at the level $\sim 10^{-13}$.

In this respect, it is interesting to consider not only the *present* variation of e^2 (the only one relevant for laboratory experiments), but also its variation over several billions of years. (We recall that the Oklo phenomenon took place about two billion years ago, and that astronomical observations constrain the variation of e^2 over the last ten billion years or so). In particular, an interesting question is to see whether our model could reconcile the Oklo limit (which corresponds to a redshift $z \simeq 0.14$) with the recent claim [24] of a variation $\Delta e^2/e^2 = (-0.72 \pm 0.18) \times 10^{-5}$ around redshifts $z \approx 0.5 - 3.5$ as proposed in [25,26]. The only hope of reconciling the two results would be to allow for a faster variation of e^2 for redshifts $z > 0.5$. Such recent redshifts have (apparently) been connected to a transition from matter dominance to vacuum dominance. Let us see whether taking into account this transition might allow for a large enough change of e^2 around redshifts $z \approx 0.5 - 3.5$. We must clearly assume the "strong coupling" scenario $\alpha_m = \mathcal{O}(1)$. In this scenario, the variation of φ during the matter era is given by Eq.(2.31). Neglecting, for simplicity, the transient evolution effects localized around the matter-vacuum transition (and treating both $\varphi'_m = -\alpha_m$ and $\varphi'_V = \varphi'_0$ as constants), the solution giving the recent

cosmological evolution of φ reads $\varphi - \varphi_0 = -\varphi_0' \ln(1+z)$ during the vacuum era, and $\varphi - \varphi_0 = -\varphi_0' \ln(1+z_*) - \varphi_m' \ln[(1+z)/(1+z_*)]$ during the matter era (the index 0 refers to the present epoch, i.e. $z = 0$; z_* denotes the transition redshift). Inserting this change in Eq.(3.17) leads to the following expression for the cosmological change of the fine-structure constant:

$$\frac{e^2 - e_0^2}{e_0^2} = -\mathrm{sign}(b_F)\, 3.5 \times 10^{-6} \, [\varphi_0' \ln(1+z_*) + \varphi_m' \ln \frac{1+z}{1+z_*}] \sqrt{10^{12} \frac{\Delta a}{a}} \,. \tag{3.26}$$

Here, we have written the result for the matter era. During the vacuum era the bracket is simply $[\varphi_0' \ln(1+z)]$. Remembering that the absolute value of φ_m' is (like that of φ_V') observationally constrained to be smaller than $\sqrt{0.3} \simeq 0.55$ (and that φ_0' is also constrained by $|\varphi_V'| < 0.84$), we see that there is no way, within our model, to explain a variation of e^2 as large as $\Delta e^2/e^2 = (-0.72 \pm 0.18) \times 10^{-5}$ around redshifts $z \approx 0.5 - 3.5$ [24]. In our model, even under the assumption that UFF is violated just below the currently tested level, such a change would have to correspond to a value $|\varphi_m'| > 2$, entailing observationally unacceptable modifications of standard cosmology. [For instance, in the model [6] a value as large as $\alpha_m > 1$ already leads to a pathological behaviour ("total dragging") where all the components scale like radiation.] This difficulty of reconciling the Oklo limit with the claim of [24] was addressed in [26,25] within a different class of models, namely with a field ϕ which *does not* couple universally to all gauge fields $F_{\mu\nu}$, as the dilaton φ is expected to do. The fact that the field ϕ in [26] (or ψ in [25]) is assumed to couple only to the electromagnetic gauge field drastically changes our Eq. (3.13) and allows one to satisfy the UFF limit $\Delta a/a \lesssim 10^{-12}$, for a stronger coupling of ϕ to electromagnetism than in our class of models, i.e. (in our notation) for a larger $d \ln B_F(\varphi)/d\varphi$. This explains why Ref. [26] could construct some explicit (but fine-tuned) models in which all observational limits (UFF, Oklo,...) could be met and still allow for a variation of e^2 as strong as the claim[23]. The maximal variation predicted by Eq.(3.26) for redshifts corresponding to the matter era (obtained when $\Delta a/a = 10^{-12}$ and $\varphi_m' = \pm\sqrt{0.3}$; and assuming a smaller value of φ_0' to be compatible with the Oklo constraint), is of order $\Delta e^2/e^2 = \pm 1.9 \times 10^{-6}$. This is only a factor ~ 4 below the claim [24] and is at the level of their one sigma error bar. Therefore a modest improvement in the observational precision (accompanied by an improved control of systematics) will start to probe a domain of variation of constants which, according to our scenario, corresponds to an UFF violation smaller than the 10^{-12} level.

D. Spatio-temporal fluctuations of the "constants"

We now turn to the second possible source of spatial-temporal variations for e^2 in our model, the quantum fluctuations of the dilaton generated during inflation. Within linear perturbation theory, the relevant calculation may be summarized as follows.

Consider a flat FRW universe $ds^2 = -dt^2 + a(t)^2\,\Sigma_i dx_i^2$. The dilaton fluctuations can be expanded in Fourier components $\delta\varphi_{\mathbf{k}}$ of given comoving momentum $\mathbf{k}$ as follows:

$$\delta\varphi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int d^3k\,\delta\varphi_{\mathbf{k}}(t)e^{i\mathbf{kx}}, \tag{3.27}$$

where t is the cosmological time. Each Fourier mode $\delta\varphi_{\mathbf{k}}$ "leaves" the horizon during inflation with an amplitude $\sim \widehat{H}_{\rm ex}(k)/\sqrt{2k^3}$ [27] where, by definition, $\widehat{H}_{\rm ex}(k)$ is the value of the dimensionless Hubble expansion rate as ka^{-1} equals H during inflation (note that we denote here $\widehat{H}_{\rm ex}$ what was denoted $\widehat{H}_{\times}$ above). Well after the exit ($k \ll aH$) the amplitude of each mode "freezes out", i.e. remains roughly constant, until it reenters the horizon during the post-inflationary epoch ($ka_{re}^{-1} \simeq H_{re}$). After re-entry the amplitude starts to damp out as a^{-1}. For a given Fourier mode $\delta\varphi_{\mathbf{k}}(t)$, the latter damping effect is described by the piecewise function

$$f_z(k) \equiv \begin{cases} 1 & \text{if} \qquad a_0^{-1}H_0^{-1}\,k \,<\, (z+1)^{1/2} \\ a_0^2H_0^2\,(z+1)k^{-2} & \text{if} \qquad (z+1)^{1/2} \,<\, a_0^{-1}H_0^{-1}\,k \,<\, 10^2 \\ 10^{-2}a_0H_0\,(z+1)k^{-1} & \text{if} \qquad a_0^{-1}H_0^{-1}\,k \,>\, 10^2 \end{cases} \tag{3.28}$$

Here the cosmological redshift $z = a_0/a(t) - 1$ has been introduced in replacement of the cosmological time t. The first case refers to Fourier modes that have not reentered yet at redshift z and whose amplitudes are still frozen. The second and third cases refer to modes that reenter during matter and radiation domination respectively. Putting all together, and assuming a gaussian probability distribution for the perturbations, we have:

$$\langle\delta\varphi_{\mathbf{k}}(t)^*\,\delta\varphi_{\mathbf{k}'}(t')\rangle \;=\; \frac{\widehat{H}_{\rm ex}^2(k)}{2k^3}\,f_z(k)\,f_{z'}(k)\,\delta^3(\mathbf{k}-\mathbf{k}')\,. \tag{3.29}$$

Possible spatial/temporal variations of e^2 induced by the fluctuations of the dilaton will be given by

$$\left.\frac{\Delta^{\rm fluc}e^2}{e^2}\right|_{(\mathbf{x},t;\,\mathbf{x}',t')} \;=\; \frac{d\ln e^2}{d\varphi}\,\Delta^{\rm fluc}\varphi|_{(\mathbf{x},t;\,\mathbf{x}',t')}\,, \tag{3.30}$$

where the r.m.s $\Delta^{\rm fluc}\varphi$ between two events $(\mathbf{x},t)$ and $(\mathbf{x}',t')$ is defined as follows

$$\Delta^{\rm fluc}\varphi|^2_{(\mathbf{x},t;\,\mathbf{x}',t')} \equiv \langle\,[\delta\varphi(\mathbf{x},t)-\delta\varphi(\mathbf{x}',t')]^2\rangle$$

$$= \frac{\widehat{H}^2_{\rm ex}}{(2\pi)^3}\int\frac{d^3k}{2\,k^3}\,\left[f_z(k)^2+f_{z'}(k)^2-2f_z(k)f_{z'}(k)\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\right] \tag{3.31}$$

$$= \frac{\widehat{H}^2_{\rm ex}}{(2\pi)^2}\int_0^\infty\frac{dk}{k}\,\left\{[f_z(k)-f_{z'}(k)]^2+2f_z(k)f_{z'}(k)\left[1-\frac{\sin kx}{kx}\right]\right\}. \tag{3.32}$$

Here, $x\equiv|\mathbf{x}-\mathbf{x}'|$ is the coordinate distance between the two events and, consistently with the slow-roll approximation, the Hubble expansion rate at exit has been assumed to be scale-invariant: $\widehat{H}_{\rm ex}(k)\simeq\widehat{H}_{\rm ex}\simeq 3\times10^{-5}$.

If one considers spatial fluctuations over terrestrial or solar system proper length scales $l=a_0k^{-1}\ll H_0^{-1}$ at the present time $t=t'=t_0$, the first square brackets in (3.32) vanishes and one can expand the sine function at small kx obtaining

$$\Delta^{\rm fluc}\varphi|_{l;\,z=0}\simeq\frac{\widehat{H}_{\rm ex}}{2\pi}\,\frac{H_0\,l}{\sqrt{3}}\qquad;\ \left.\frac{\Delta^{\rm fluc}e^2}{e^2}\right|_{l;\,z=0}\simeq 10^{-2}\,\alpha_{\rm had}\,\widehat{H}_{\rm ex}\,H_0\,l\,. \tag{3.33}$$

As expected, these variations are extremely small, $\Delta^{\rm fluc}e^2/e^2|_{l;\,z=0}\simeq10^{-33}\,l/{\rm km}$. It is also interesting to compare dilaton fluctuations at different redshifts along a comoving observer wordline. By putting $x\simeq0$ in (3.32) the second term in the square brackets vanishes and one has:

$$\Delta^{\rm fluc}\varphi|_{z;\,x=0}\simeq\frac{\widehat{H}_{\rm ex}}{2\pi}\frac{1}{\sqrt{2}}\left[\log(1+z)-\frac{z}{1+z}+\frac{10^{-8}}{2}z^2\right]^{1/2}\simeq\frac{\widehat{H}_{\rm ex}}{2\pi}\left[\frac{z}{2}-\frac{z^2}{3}+\ldots\right]. \tag{3.34}$$

It is slightly more complicated to compare dilaton fluctuations between "now" and events at redshift z along a null ray. Expanding in powers of z around $z=0$ one gets from (3.32), after a straightforward calculation:

$$\Delta^{\rm fluc}\varphi|_z\simeq\frac{\widehat{H}_{\rm ex}}{2\pi}\,\left[\frac{1}{2}\sqrt{\frac{7}{3}}\,z-\frac{2}{\sqrt{21}}\,z^2+\ldots\right]. \tag{3.35}$$

Numerically, at redshift $z\sim1$, the effects of dilatonic fluctuations give $\Delta^{\rm fluc}\varphi|_{z=1}\sim\widehat{H}_{\rm ex}/(2\pi)\sim5\times10^{-6}$. This is to be contrasted with the effects of the cosmic, homogeneous evolution which yields $\Delta\varphi|_{z=1}\simeq\alpha_m$. In the "normal" case where $\alpha_m\sim e^{-c\varphi}\sim\delta_H^{4/(n+2)}$, the two effects, though a priori unrelated, are related in our scenario , when $n=2$. Indeed, if $n=2$, $\delta_H^{4/(n+2)}=\delta_H\sim5\times10^{-5}$ is linked to $\widehat{H}_{\rm ex}/(2\pi)$ via $\delta(\chi_{\rm ex})=A\widehat{H}_{\rm ex}/(2\pi)$ with $A=(8/3)V/\partial_\chi V=(8/3)(\chi/n)\simeq40/(3\sqrt{n})\sim10$. On the other hand, in the case where φ is strongly coupled to dark matter, the homogeneous evolution $\Delta\varphi|_{z=1}\simeq\alpha_m\sim1$ is parametrically larger than the fluctuations $\Delta^{\rm fluc}\varphi|_{z=1}\sim\widehat{H}_{\rm ex}/(2\pi)$.

To conclude on this subsection, we see that the inhomogeneous space-time fluctuations of the fine-structure constant are typically too small to be observable (if the limits from UFF are already satisfied), being suppressed, relative to their natural values $H_0 l$, $H_0 t$, by the small factor $\alpha_{\rm had} \widehat{H}_{\rm ex}$.

IV. SUMMARY AND CONCLUSION

We have studied the dilaton-fixing mechanism of [3] within the context where the dilaton-dependent low-energy couplings are extremized at $\varphi = +\infty$, i.e. for infinitely large values of the bare string coupling $g_s^2 = e^\phi \simeq e^{c\varphi}$. [The crucial coupling to the inflaton, say $\lambda(\varphi)$ in Eq.(2.11), must be *minimized* at $\varphi \to +\infty$; the other couplings can be either minimized or maximized there.] This possibility of a fixed point at infinity (in bare string coupling space) has been recently suggested [5], and its late-cosmological consequences have been explored in [6]. We found that a primordial inflationary stage, with inflaton potential $V(\chi) = \lambda(\varphi)\, \chi^n/n$, was much less efficient in decoupling a dilaton with least couplings at infinity than in the case where the least couplings are reached at a finite value of φ (as in [3,8]). This reduced efficiency has interesting phenomenological consequences. Indeed, it predicts much larger observable deviations from general relativity. In the case of the simplest chaotic potential [10] $V(\chi) = \frac{1}{2}\, m_\chi^2(\varphi)\, \chi^2$, we find that, under the simplest assumptions about the pre-inflationary state, this scenario predicts violations of the universality of free fall (UFF) of order $\Delta a/a \sim 5 \times 10^{-4}\, \delta_H^2$ where δ_H is the density fluctuation generated by inflation on horizon scales. The observed level of large-scale density (and cosmic microwave background temperature) fluctuations fixes δ_H to be around 5×10^{-5} which finally leads to a prediction for a violation of the UFF near the $\Delta a/a \sim 10^{-12}$ level. This is naturally compatible with present experimental tests of the equivalence principle, and suggests that a modest improvement in the precision of UFF tests might be able to detect a deviation linked to dilaton exchange with a coupling reduced by the attraction towards the fixed point at infinity. Because of the presence of unknown dimensionless ratios $(c, b_F/b_\lambda)$ in our estimates, and of quantum noise in the evolution of the dilaton, we cannot give sharp quantitative estimates of $\Delta a/a$. However, we note that dilaton-induced violations of the UFF have a rather precise signature with a composition-dependence of the form (3.11), with probable domination by the last (Coulomb energy) term [3]. As explored in [19] this signature is quite distinct from UFF violations induced by other fields, such as a vector field. We note that the approved Centre National d'Etudes Spatiales (CNES) mission MICROSCOPE [30] (to fly in 2004) will explore the level $\Delta a/a \sim 10^{-15}$, while the planned National Aeronautics and Space Agency (NASA) and European Space Agency (ESA) mission STEP (Satellite Test of the Equivalence Principle)

[31] could explore the $\Delta a/a \sim 10^{-18}$ level. Our scenario gives additional motivation for such experiments and suggests that they might find a rather strong violation signal, whose composition-dependence might then be studied in detail to compare it with Eq. (3.11).

In the case of inflationary potentials $V(\chi) \propto \chi^n$ with $n > 2$ our simplest estimates predict a violation of the UFF of order $\Delta a/a \sim 5 \times 10^{-4}\, \delta_H^{\frac{8}{n+2}}$ which is larger than 10^{-12}. At face value this suggests that existing UFF experimental data can be interpreted as favouring $n \leq 2$ over $n > 2$. However, we must remember that our estimates have made several simplifying assumptions. It is possible that the large quantum fluctuations of the inflaton in the self-regenerating regime $\chi > \chi_{\rm in}$, with $\chi_{\rm in}$ defined by Eq. (2.18), can give more time for φ to run away towards large values, so that the effective value of $e^{c\varphi_{\rm in}}$ to be used in Eq. (2.21) turn out to dominate the first term in the R.H.S. that we have used for our estimates. We leave to future work a study of the system of Langevin equations describing the coupled fluctuations of ϕ and χ during the self-regenerating regime.

Finally let us note some other conclusions of our work.

We recover the conclusion of previous works on dilaton models that the most interesting experimental probes of a massless weakly coupled dilaton are tests of the UFF. The composition-independent gravitational tests (solar-system, binary-pulsar) tend to be much less sensitive probes (as highlighted by the relations (3.14), (3.21) and (3.22)).

However, a possible exception concerns the time-variation of the coupling constants. Here the conclusion depends crucially on the assumptions made about the couplings of the dilaton to the cosmologically dominant forms of energy (dark matter and/or dark energy). If these couplings are of order unity (and as large as is phenomenologically acceptable, i.e. so that $(\varphi_0')^2 = 0.7$), the present time variation of the fine-structure constant is linked to the violation of the UFF by the relation $d \ln e^2/dt \sim \pm 2.0 \times 10^{-16} \sqrt{10^{12} \Delta a/a} \mathrm{yr}^{-1}$. [The most natural sign here being +, i.e. $b_F > 0$, which corresponds to *smaller* e^2 in the past, just as suggested by the claim [24].] Such a time variation might be observable (if $\Delta a/a$ is not very much below its present upper bound $\sim 10^{-12}$) through the comparison of high-accuracy cold-atom clocks and/or via improved measurements of astronomical spectra.

More theoretical work is needed to justify the basic assumption (1.3) of our scenario. In particular, it is crucial to investigate whether it is natural to expect that the sign of the crucial coefficient b_λ in Eq. (2.12) be indeed *positive*. [Recall that the general mechanism of [3] is an attraction towards "Least Couplings" while Eq. (1.3) with $\mathcal{O}(e^{-\phi}) > 0$ leads to largest couplings at infinity.] Note in this respect that the sign of the other b_i's is not important as, once inflation has pushed $e^{c\varphi}$ to very large values $e^{c\varphi_{\rm end}}$, the subsequent cosmological evolutions tend to be ineffective in further displacing φ.

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APPENDIX A: THE STOCHASTIC EVOLUTION OF THE DILATON

In this appendix we study the stochastic evolution of the dilaton φ during inflation as described by the Langevin-type equation (2.23). We restrict our attention to the region of phase space where the evolution of the inflaton χ is classical, and to a power-law potential of the form (2.11). It follows that the inflaton evolves according to the classical slow-roll equation (2.14) whose solution reads

$$\chi^2 = \chi_{\rm in}^2 - np, \tag{A1}$$

where p, the parameter defined in (2.7), is shifted in such a way that $p_{\rm in} \equiv 0$. Equation (2.23) takes the form

$$\frac{d\varphi}{dp} = \frac{1}{2}\, b_\lambda\, c\, e^{-c\varphi} + \xi(p)\,, \tag{A2}$$

where $\xi(p)$ is a gaussian stochastic variable (GSV), with a "time-dependent" r.m.s. amplitude $\widehat{H}(p)/2\pi$:

$$\langle \xi(p_1)\, \xi(p_2) \rangle \;=\; \frac{\widehat{H}^2}{(2\pi)^2}\, \delta(p_1 - p_2), \tag{A3}$$

[the relation with the normalized random white noise term of (2.23) is $\xi(p) = \xi_2(p)\widehat{H}/2\pi$]. For any given source term $\xi(p)$, the formal solution of (A2) reads

$$e^{c\,\varphi(p)} \;=\; e^{c\,\varphi_{\rm in}}\, e^{c\,\eta(p)} + \frac{b_\lambda c^2}{2} \int_0^p dp' e^{c\,[\eta(p) - \eta(p')]}\,, \qquad \eta(p) \equiv \int_0^p dp'\, \xi(p')\,. \tag{A4}$$

Note that the classical solution in (2.17), $e^{c\,\varphi_{\rm cl}(p)} = e^{c\,\varphi_{\rm in}} + (b_\lambda c^2/2)\, p$, can be easily recovered in the small noise limit $\xi(p) \to 0$, $\eta(p) \to 0$.

It proves convenient to compare the true solution to the classical one by studying the statistical behaviour of the ratio $A(p) \equiv e^{c\,\varphi(p)}/e^{c\,\varphi_{\rm cl}(p)}$. As we will show below, $\langle e^{c\,\eta(p)}\rangle = \mathcal{O}(1)$. Moreover, we are also assuming $e^{c\,\varphi_{\rm in}} = \mathcal{O}(1)$ or, at least, $e^{c\,\varphi_{\rm in}} \ll (b_\lambda c^2/2)p$ (see Section II for details) so that the leading contribution to the first equation in (A4) is given by the integral, and we have

$$A(p) \equiv e^{c\,\varphi(p)}/e^{c\,\varphi_{\rm cl}(p)} \simeq \frac{1}{p}\int_0^p dp' e^{c\,[\eta(p)-\eta(p')]}\,. \tag{A5}$$

Since $\xi(p)$ is a GSV, also its integral $\eta(p)$ is a (centered) GSV. Moreover, if x is a GSV with $\sigma_x^2 \equiv \langle x^2\rangle - \langle x\rangle^2 = \langle x^2\rangle$, by Bloch's theorem $y = e^x$ is a new stochastic variable with $\langle y\rangle = \langle e^x\rangle = e^{\langle x^2\rangle/2}$ and $\sigma_y^2 = e^{2\langle x^2\rangle} - e^{\langle x^2\rangle}$. The average value of $A(p)$ thus reads

$$\langle A(p)\rangle \simeq \frac{1}{p}\int_0^p dp'\, e^{(c^2/2)\,\langle[\eta(p)-\eta(p')]^2\rangle}\,. \tag{A6}$$

The exponent on the right hand side of the above equation can be estimated by using (A3) and the slow-roll approximation $\widehat{H}^2 \simeq 2V(\chi,\varphi)/3 = 2\lambda(\varphi)\chi^n/3n \simeq 2\lambda_\infty\chi^n/3n$. One gets

$$\langle[\eta(p)-\eta(p')]^2\rangle = \frac{1}{(2\pi)^2}\int_{p'}^p \widehat{H}^2 dp'' \simeq \frac{n}{2(n+2)}\left[\left(\frac{\chi(p')}{\chi_{\rm in}}\right)^{n+2} - \left(\frac{\chi(p)}{\chi_{\rm in}}\right)^{n+2}\right], \tag{A7}$$

where $\chi_{\rm in}$ is the value at exit from self-regenerating inflation: $\widehat{H}(\chi_{\rm in})/2\pi = n/(2\chi_{\rm in})$ (see Section II for more details). Since we are interested in evaluating (A6) at the end of inflation, $p = p_{\rm end} \simeq \chi_{\rm in}^2/n$, we can thus write

$$\langle[\eta(p)-\eta(p')]^2\rangle \simeq \frac{n}{2(n+2)}\left[1 - \frac{p'}{p_{\rm end}}\right]^{(n+2)/2}. \tag{A8}$$

When evaluated at $p' = 0$, the above formula gives $\langle\eta(p)^2\rangle = n/(2(n+2))$. Thus the normalization factor to the initial condition in (A4) is of order one, as anticipated: $\langle e^{c\,\eta(p)}\rangle = e^{\frac{1}{2}c\langle\eta(p)^2\rangle} = \mathcal{O}(1)$. From (A6) and (A8) we have:

$$\begin{aligned}\langle A(p_{\rm end})\rangle &\simeq \frac{1}{p_{\rm end}}\int_0^{p_{\rm end}} dp' \exp\left[\frac{c^2 n}{4(n+2)}\left(1-\frac{p'}{p_{\rm end}}\right)^{(n+2)/2}\right] \\ &= \int_0^1 \exp\left[\frac{c^2 n}{4(n+2)}x^{(n+2)/2}\right]dx \\ &= \exp\left(\frac{c^2 n\theta}{4(n+2)}\right) = \mathcal{O}(1),\end{aligned} \tag{A9}$$

with $0 < \theta < 1$.

We can estimate the dispersion of the same quantity by expanding the exponential inside the integral (A6) in powers of $\xi(p)$:

$$A(p) \simeq 1 + \frac{c}{p}\int_0^p dp' \int_{p'}^p dp''\xi(p'') + \frac{1}{2}\frac{c^2}{p}\int_0^p dp' \left(\int_{p'}^p dp''\xi(p'')\right)^2 + \dots . \tag{A10}$$

At lowest order in $\xi(p)$ the variance of the above quantity calculated at $p = p_{\rm end}$ reads

$$\begin{aligned}
\sigma^2_{A(p_{\rm end})} &= \left\langle \left(\frac{c}{p}\int_0^p dp' \int_{p'}^p dp''\xi(p'')\right)^2 \right\rangle \\
&= \frac{c^2}{p^2}\int_0^p dp' \int_0^p dp'' \int_{\max(p',p'')}^p dp''' \frac{\widehat{H}^2(p''')}{(2\pi)^2} \\
&= \frac{2c^2}{p^2}\int_0^p dp' p' \int_{p'}^p dp'' \frac{\widehat{H}^2(p'')}{(2\pi)^2}.
\end{aligned} \tag{A11}$$

As in equation (A7) we can use the slow-roll approximation and obtain

$$\sigma^2_{A(p_{\rm end})} \;=\; \frac{c^2\, n}{n+2}\int_0^1 x\,(1-x)^{(n+2)/2} dx \;=\; c^2\mathcal{O}(1)\,. \tag{A12}$$

REFERENCES

[1] E. Witten, *Phys. Lett. B* **149**, 351 (1984).

[2] T.R. Taylor and G. Veneziano, *Phys. Lett. B* **213**, 450 (1988).

[3] T. Damour and A.M. Polyakov, *Nucl. Phys. B* **423**, 532 (1994); *Gen. Rel. Grav.* **26**, 1171 (1994).

[4] T. Damour and K. Nordtvedt, *Phys. Rev. Lett.* **70**, 2217 (1993); *Phys. Rev. D* **48**, 3436 (1993).

[5] G. Veneziano, *J. High Energy Phys.* **06**, 051 (2002).

[6] M. Gasperini, F. Piazza and G. Veneziano, *Phys. Rev. D* **65**, 023508 (2002).

[7] T. Damour, F. Piazza and G. Veneziano, *Phys. Rev. Lett.* **89**, 081601 (2002).

[8] T. Damour and A. Vilenkin, *Phys. Rev. D* **53**, 2981 (1996).

[9] T. Damour, in *Susy 95*, Proceedings of the 1995 International Workshop on Supersymmetry and Unification of Fundamental Interactions, edited by I. Antoniadis and H. Videau (Editions Frontières, Gif-sur-Yvette, 1996), pp. 577-584.

[10] A. Linde, *Particle Physics and Inflationary Cosmology*, (Harwood, Chur, 1990).

[11] T. Damour, G.W. Gibbons and C. Gundlach, *Phys. Rev. Lett.* **64**, 123 (1990).

[12] L. Amendola and D. Tocchini-Valentini, *Phys. Rev. D* **64**, 043509 (2001)

[13] L. Amendola, M. Gasperini, C. Ungarelli and D. Tocchini-Valentini, in preparation.

[14] S. Perlmutter et al., *Astrophys. J.* **517**, 565 (1999); A. Riess et al. *Astronom. J.* **116**, 1009 (1998).

[15] See, e.g., the review of global cosmological parameters (chap. 17) in D.E. Groom et al., *European Physical Journal* **C15**, 1 (2000); updated version available on http://pdg.lbl.gov/.

[16] D.E. Groom et al., *European Physical Journal* **C15**, 1 (2000), see Chapter 14 "Experimental tests of gravitational theory", available on http://pdg.lbl.gov/; and C.M. Will, *Living Rev. Rel.* **4**, 4 (2001).

[17] T. Damour and G. Esposito-Farèse, *Class. Quant. Grav.* **9**, 2093 (1992).

[18] T. Damour and G. Esposito-Farèse, *Phys. Rev. D* **54**, 1474 (1996); *Phys. Rev. D* **58**,

042001 (1998).

[19] T. Damour, *Class. Quant. Grav.* **13**, A33 (1996).

[20] Y. Su et al., *Phys. Rev. D* **50**, 3614 (1994).

[21] A.I. Shlyakhter, *Nature* **264**, 340 (1976); T. Damour and F. Dyson, *Nucl. Phys. B* **480**, 37 (1996).

[22] K. A. Olive, M. Pospelov, Y. Z. Qian, A. Coc, M. Casse and E. Vangioni-Flam, *Constraints on the Variations of the Fundamental Couplings*, hep-ph/0205269

[23] M. Gasperini, *Phys. Rev. D* **64** 043510 (2001).

[24] J.K. Webb et al., *Phys. Rev. Lett.* **87**, 091301 (2001).

[25] H. B. Sandvik, J. D. Barrow and J. Magueijo, *Phys. Rev. Lett.* **88** 031302 (2002).

[26] K. A. Olive and M. Pospelov, *Phys. Rev. D* **65**, 85044 (2002).

[27] A. R. Liddle and D. H. Lyth, *Cosmological inflation and large scale structure*, (Cambridge University Press, 2000).

[28] C. Salomon et al., in *Cold atom clocks on Earth and in space, Proceedings of the 17th Int. Conf. on Atomic Physics*, E. Arimondo, M. Inguscio, ed., (World Scientific, Singapore, 2001), p 23; Y. Sortais et al., *Physica Scripta*, **95**, 50 (2001).

[29] J. D. Prestage, R. L. Tjoelker, and L. Maleki, *Phys. Rev. Lett.* **74** 3511 (1995).

[30] P. Touboul et al., *C.R. Acad. Sci. Paris* **2** (série IV) 1271 (2001).

[31] P. W. Worden, in *Proceedings of the Seventh Marcel Grossmann Meeting on General Relativity*, edited by R. T. Jantzen and G. Mac Keiser (World Scientific, Singapore, 1996), pp 1569-1573.

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Runaway dilaton and equivalence principle violations

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In a recently proposed scenario, where the dilaton decouples while cosmologically attracted towards infinite bare string coupling, its residual interactions can be related to the amplitude of density fluctuations generated during inflation, and are large enough to be detectable through a modest improvement on present tests of free-fall universality. Provided it has significant couplings to either dark matter or dark energy, a runaway dilaton can also induce time-variations of the natural "constants" within the reach of near-future experiments.

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A striking prediction of all string theory models is the existence of a scalar partner of the spin 2 graviton: the dilaton ϕ, whose vacuum expectation value (VEV) determines the string coupling constant $g_s = e^{\phi/2}$ [1]. At tree level, the dilaton is massless and has gravitational-strength couplings to matter which violate the equivalence principle [2]. This is in violent conflict with present experimental tests of general relativity. It is generally assumed that this conflict is avoided because, after supersymmetry breaking, the dilaton might acquire a (large enough) mass (say $m_\phi \gtrsim 10^{-3}\,\mathrm{eV}$ so that observable deviations from Einstein's gravity are quenched on distances larger than a fraction of a millimeter). However, Ref. [3] (see also [4]) has proposed a mechanism which can naturally reconcile a *massless* dilaton with existing experimental data. The basic idea of Ref. [3] was to exploit the string-loop modifications of the (four dimensional) effective low-energy action (we use the signature $-+++$)

$$S = \int d^4x\sqrt{\widetilde{g}}\left(\frac{B_g(\phi)}{\alpha'}\,\widetilde{R} + \frac{B_\phi(\phi)}{\alpha'}\,[2\widetilde{\Box}\phi - (\widetilde{\nabla}\phi)^2] - \frac{1}{4}\,B_F(\phi)\,\widetilde{F}^2 - \dots\right), \tag{1}$$

i.e. the ϕ-dependence of the various coefficients $B_i(\phi)$, $i = g, \phi, F, \dots$, given in the weak-coupling region ($e^\phi \to 0$) by series of the form $B_i(\phi) = e^{-\phi} + c_0^{(i)} + c_1^{(i)}\,e^\phi + c_2^{(i)}\,e^{2\phi} + \cdots$, coming from genus expansion of string theory. It was shown in [3] that, if there exists a special value ϕ_m of ϕ which extremizes all the (relevant) coupling functions $B_i^{-1}(\phi)$, the cosmological evolution of the graviton-dilaton-matter system naturally drives ϕ towards ϕ_m (which is a fixed point of the Einstein-dilaton-matter system). This provides a mechanism for fixing a massless dilaton at a value where it decouples from matter ("Least Coupling Principle"). In this letter, we consider the case (recently suggested in [5]) where the coupling functions, at least in the visible sector, have a smooth *finite* limit for infinite bare string coupling $g_s \to \infty$. In this case, quite generically, we expect

$$B_i(\phi) = C_i + \mathcal{O}(e^{-\phi})\,. \tag{2}$$

Under this assumption, the coupling functions are all extremized at infinity, i.e. a fixed point of the cosmological evolution is $\phi_m = +\infty$. [See [6] for an exploration of the late-time cosmology of models satisfying (2).] We found that the "decoupling" of such a "run-away" dilaton has remarkable features: (i) the residual dilaton couplings at the present epoch can be related to the amplitude of density fluctuations generated during inflation, and (ii) these residual couplings, while being naturally compatible with present experimental data, are predicted to be large enough to be detectable by a modest improvement in the precision of equivalence principle tests (non universality of the free fall, and, possibly, variation of "constants"). This result contrasts with the case of attraction towards a finite value ϕ_m which leads to extremely small residual couplings [7].

We assume some primordial inflationary stage driven by the potential energy of an inflaton field χ. Working with the Einstein frame metric $g_{\mu\nu} = C_g^{-1}\,B_g(\phi)\,\widetilde{g}_{\mu\nu}$, and with the modified dilaton field $\varphi = \int d\phi[(3/4)(B_g'/B_g)^2 + B_\phi'/B_g + (1/2)\,B_\phi/B_g]^{1/2}$, we consider an effective action of the form

$$S = \int d^4x\sqrt{g}\left[\frac{\widetilde{m}_P^2}{4}\,R - \frac{\widetilde{m}_P^2}{2}\,(\nabla\varphi)^2 - \frac{\widetilde{m}_P^2}{2}\,F(\varphi)(\nabla\chi)^2 - \widetilde{m}_P^4\,V(\chi,\varphi)\right], \tag{3}$$

where $\widetilde{m}_P^2 = 1/(4\pi G) = 4C_g/\alpha'$, and where the dilaton dependence of the Einstein-frame action is related to its (generic) string-frame dependence (1) by $F(\varphi) = B_\chi(\phi)/B_g(\phi)\,,\ V(\chi,\varphi) = C_g^2\,\widetilde{m}_P^{-4}\,B_g^{-2}(\phi)\,\widetilde{V}(\widetilde{\chi},\phi)$.

Under our basic assumption (2), $d\varphi/d\phi$ tends, in the strong-coupling limit $\phi \to +\infty$, to the constant $1/c$, with $c \equiv (2C_g/C_\phi)^{1/2}$, so that the asymptotic behaviour of the bare string coupling is

$$g_s^2 = e^\phi \simeq e^{c\varphi}\,. \tag{4}$$

Let us consider the case where $F(\varphi) = 1$ and $V(\chi,\varphi) = \lambda(\varphi)\,\chi^n/n$ with a dilaton-dependent inflaton coupling constant $\lambda(\varphi)$ of the form

$$\lambda(\varphi) = \lambda_\infty(1 + b_\lambda\, e^{-c\varphi})\,, \tag{5}$$

where we assume that $b_\lambda > 0$, i.e that $\lambda(\varphi)$ reaches a *minimum* at strong-coupling, $\varphi \to +\infty$. It is shown in [8] that this simple case is representative of rather general cases of φ-dependent inflationary potentials $V(\chi,\varphi)$

During inflation ($ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i\,dx^j$), it is easily seen that, while χ slowly rolls down towards $\chi \sim 1$, the dilaton φ is monotically driven towards large values. The solution of the (classical) slow-roll evolution equations leads to

$$e^{c\varphi} + \frac{b_\lambda\, c^2}{2n}\,\chi^2 = \text{const.} = e^{c\varphi_{\rm in}} + \frac{b_\lambda\, c^2}{2n}\,\chi_{\rm in}^2\,. \tag{6}$$

Using the result (6), we estimate the value $\varphi_{\rm end}$ of φ at the end of inflation by inserting for the initial value $\chi_{\rm in}$ of the inflaton the value corresponding to the end of self-regenerating inflation [9]. We remark that the latter value can be related to the amplitude $\delta_H \sim 5\times 10^{-5}$ of density fluctuations, on the scale corresponding to our present horizon, generated by inflation, through $\chi_{\rm in} \simeq 5\sqrt{n}\,(\delta_H)^{-2/(n+2)}$. Finally, assuming $e^{c\varphi_{\rm in}} \ll e^{c\varphi_{\rm end}}$, we get the estimate:

$$e^{c\varphi_{\rm end}} \sim 12.5 c^2\, b_\lambda\ (\delta_H)^{-\frac{4}{n+2}}\ . \tag{7}$$

A more general study [8] of the run-away of the dilaton during inflation (including an estimate of the effect of quantum fluctuations) only modifies this result by a factor $\mathcal{O}(1)$. It is also found that the present value of the dilaton is well approximated by $\varphi_{\rm end}$.

Eq. (7) tells us that, within our scenario, the smallness of the present matter couplings of the dilaton is quantitatively linked to the smallness of the (horizon-scale) cosmological density fluctuations. To be more precise, and to study the compatibility with present experimental data, we need to estimate the crucial dimensionless quantity

$$\alpha_A(\varphi) \equiv \partial \ln m_A(\varphi)/\partial\,\varphi\,, \tag{8}$$

which measures the coupling of φ to a massive particle of type A. The definition of α_A is such that, at the Newtonian approximation, the interaction potential between particle A and particle B is $-G_{AB}\,m_A\,m_B/r_{AB}$ where [4,3] $G_{AB} = G(1+\alpha_A\,\alpha_B)$. Here, G is the bare gravitational coupling constant entering the Einstein-frame action (3), and the term $\alpha_A\,\alpha_B$ comes from the additional attractive effect of dilaton exchange.

Let us first consider the (approximately) composition-independent deviations from general relativity, i.e. those that do not essentially depend on violations of the equivalence principle. Most composition-independent gravitational experiments (in the solar system or in binary pulsars) consider the long-range interaction between objects whose masses are essentially baryonic (the Sun, planets, neutron stars). As argued in [2,3] the relevant coupling coefficient α_A is then approximately universal and given by the logarithmic derivative of the QCD confinement scale $\Lambda_{\rm QCD}(\varphi)$, because the mass of hadrons is essentially given by a pure number times $\Lambda_{\rm QCD}(\varphi)$. [We shall consider below the small, non-universal, corrections to $m_A(\varphi)$ and $\alpha_A(\varphi)$ linked to QED effects and quark masses.] Remembering from Eq. (1) the fact that, in the string frame (where there is a fixed cut-off linked to the string mass $\widetilde{M}_s \sim (\alpha')^{-1/2}$) the gauge coupling is dilaton-dependent ($g_F^{-2} = B_F(\varphi)$), we see that (after conformal transformation) the Einstein-frame confinement scale has a dilaton-dependence of the form

$$\Lambda_{\rm QCD}(\varphi) \sim C_g^{1/2}\,B_g^{-1/2}(\varphi)\exp[-8\pi^2\,b_3^{-1}\,B_F(\varphi)]\,\widetilde{M}_s\,, \tag{9}$$

where b_3 denotes the one-loop (rational) coefficient entering the Renormalization Group running of g_F. Here $B_F(\varphi)$ denotes the coupling to the SU(3) gauge fields. For simplicity, we shall assume that (modulo rational coefficients) all gauge fields couple (near the string cut off) to the same $B_F(\varphi)$. This yields the following approximately universal dilaton coupling to hadronic matter

$$\alpha_{\rm had}(\varphi) \simeq \left(\ln\left(\frac{\widetilde{M}_s}{\Lambda_{\rm QCD}}\right) + \frac{1}{2}\right)\frac{\partial \ln B_F^{-1}(\varphi)}{\partial\,\varphi}\,. \tag{10}$$

Numerically, the coefficient in front of the R.H.S. of (10) is of order 40. Consistently with our basic assumption (2), we parametrize the φ dependence of the gauge coupling $g_F^2 = B_F^{-1}$ as

$$B_F^{-1}(\varphi) = B_F^{-1}(+\infty)\,[1 - b_F\,e^{-c\varphi}]\,. \tag{11}$$

We finally obtain

$$\alpha_{\rm had}(\varphi) \simeq 40\,b_F\,c\,e^{-c\varphi}\,. \tag{12}$$

Inserting the estimate (7) of the value of φ reached because of the cosmological evolution, we get the estimate

$$\alpha_{\rm had}(\varphi_{\rm end}) \simeq 3.2\,\frac{b_F}{b_\lambda\,c}\,\delta_H^{\frac{4}{n+2}}\,. \tag{13}$$

It is plausible to expect that the quantity c (which is a ratio) and the ratio b_F/b_λ are both of order unity. This then leads to the numerical estimate $\alpha^2_{\rm had} \sim 10\,\delta_H^{\frac{8}{n+2}}$, with $\delta_H \simeq 5\times 10^{-5}$. An interesting aspect of this result is that the expected present value of $\alpha^2_{\rm had}$ depends rather strongly on the value of the exponent n (which entered the inflaton potential $V(\chi)\propto\chi^n$). In the case $n=2$ (i.e. $V(\chi)=\frac{1}{2}\,m_\chi^2\,\chi^2$) we have $\alpha^2_{\rm had}\sim 2.5\times 10^{-8}$, while if $n=4$ ($V(\chi)=\frac{1}{4}\,\lambda\,\chi^4$) we have $\alpha^2_{\rm had}\sim 1.8\times 10^{-5}$. Both estimates are compatible with present (composition-independent) experimental limits on deviations from Einstein's theory (in the solar system, and in binary pulsars). For instance, the "Eddington" parameter $\gamma-1\simeq -2\,\alpha^2_{\rm had}$ is compatible with the present best limits $|\gamma-1|\lesssim 2\times 10^{-4}$ coming from Very Long Baseline Interferometry measurements of the deflection of radio waves by the Sun [10].

Let us consider situations where the non-universal couplings of the dilaton induce (apparent) violations of the equivalence principle. This means considering the composition-dependence of the dilaton coupling α_A, Eq. (8), i.e. the dependence of α_A on the type of matter we consider. Two test masses, made respectively of A- and B-type particles will fall in the gravitational field generated by an external mass m_E with accelerations differing by

$$\left(\frac{\Delta a}{a}\right)_{AB}\equiv 2\,\frac{a_A-a_B}{a_A+a_B}\simeq(\alpha_A-\alpha_B)\,\alpha_E\,. \qquad (14)$$

We have seen above that in lowest approximation $\alpha_A\simeq\alpha_{\rm had}$ does not depend on the composition of A. We need, however, now to retain the small composition-dependent effects to α_A linked to the φ-dependence of QED and quark contributions to m_A. This has been investigated in [3] with the result that $\alpha_A-\alpha_{\rm had}$ depends linearly on the baryon number $B\equiv N+Z$, the neutron excess $D\equiv N-Z$, and the quantity $E\equiv Z(Z-1)/(N+Z)^{1/3}$ linked to nuclear Coulomb effects. Under the plausible assumption that the latter dependence is dominant, and using the average estimate $\Delta(E/M)\simeq 2.6$ (applicable to mass pairs such as (Beryllium, Copper) or (Platinum, Titanium)), one finds that the violation of the universality of free fall is approximately given by

$$\left(\frac{\Delta a}{a}\right)\simeq 5.2\times 10^{-5}\,\alpha^2_{\rm had}\simeq 5.2\times 10^{-4}\left(\frac{b_F}{b_\lambda\,c}\right)^2\,\delta_H^{\frac{8}{n+2}}\,. \qquad (15)$$

This result is one of the main predictions of our model. If we insert the observed density fluctuation $\delta_H\sim 5\times 10^{-5}$, we obtain a level of violation of the universality of free fall (UFF) due to a run-away dilaton which is $\Delta a/a\simeq 1.3(b_F/(b_\lambda\,c))^2\times 10^{-12}$ for $n=2$ (i.e. for the simplest chaotic inflationary potential $V(\chi)=\frac{1}{2}\,m_\chi^2(\phi)\,\chi^2$), and $\Delta a/a\simeq 0.98(b_F/(b_\lambda\,c))^2\times 10^{-9}$ for $n=4$ (i.e. for $V(\chi)=\frac{1}{4}\,\lambda(\phi)\,\chi^4$). The former case is naturally compatible with current tests (at the $\sim 10^{-12}$ level [12]) of the UFF. Values $n\geq 4$ of the exponent require (within our scenario) that the (unknown) dimensionless combination of parameters $(b_F/(b_\lambda\,c))^2$ be significantly smaller than one.

Let us also consider another possible deviation from general relativity and the standard model: a possible variation of the coupling constants, most notably of the fine structure constant $e^2/\hbar c$ on which the strongest limits are available. Consistently with our previous assumptions we expect $e^2\propto B_F^{-1}(\varphi)$ so that, from (11), $e^2(\varphi)=e^2(+\infty)\,[1-b_F\,e^{-c\varphi}]$. The present logarithmic variation of e^2 (introducing the derivative $\varphi'=d\varphi/dp$ with respect to the "e-fold" parameter $dp=H\,dt=da/a$) is thus given by

$$\frac{d\ln e^2}{H\,dt}\simeq b_F\,c\,e^{-c\varphi}\,\varphi_0'\simeq\frac{1}{40}\alpha_{\rm had}\varphi_0'\,. \qquad (16)$$

The current value of φ', φ_0', depends on the coupling of the dilaton to the two currently dominating energy forms in the universe: dark matter (coupling $\alpha_m(\varphi)$), and vacuum energy (coupling $\alpha_V=\frac{1}{4}\,\partial\ln V(\varphi)/\partial\,\varphi$). In the slow-roll approximation, one finds

$$(\Omega_m+2\Omega_V)\varphi_0'=-\Omega_m\alpha_m-4\Omega_V\alpha_V\,. \qquad (17)$$

where Ω_m and Ω_V are, respectively, the dark-matter- and the vacuum- fraction of critical energy density ($\rho_c\equiv(3/2)\widetilde{m}_P^2H^2$). The precise value of φ_0' is model-dependent and can vary (depending under the assumptions one makes) from an exponentially small value ($\varphi'\sim e^{-c\varphi}$) to a value of order unity. In models where either the dilaton is more strongly coupled to dark matter than to ordinary matter [11], or/and plays the role of quintessence (as suggested in [6]), φ_0' can be of order unity. Assuming just spatial flatness and saturation of the "energy budget" by non-relatistic matter and dilatonic quintessence, one can relate the value of $\varphi'=d\varphi/(Hdt)$ to Ω_m and to the deceleration parameter $q\equiv-\ddot{a}a/\dot{a}^2$: $\varphi'^2=1+q-3\Omega_m/2$. This yields the following generic, model-independent relation between the present time variation of e^2, cosmological observables and the level of UFF violation

$$\frac{d\ln e^2}{H\,dt}\simeq\pm 3.5\times 10^{-6}\,\sqrt{1+q_0-3\Omega_m/2}\,\sqrt{10^{12}\frac{\Delta a}{a}}\,. \qquad (18)$$

Note that the sign of the variation of e^2 is in general model-dependent (as it depends both on the sign of b_F and the sign of φ_0'). Specific classes of models might, however, favour particular signs of de^2/dt. For instance, within the assumptions of [5] and [6] it is natural to expect that e^2 be currently *increasing*.

The phenomenologically interesting consequence of Eq. (18) is to predict a time-variation of constants which

may be large enough to be detected by high-precision laboratory experiments. Indeed, using $H_0 \simeq 66$ km/s/Mpc, and the plausible estimates $\Omega_m = 0.3$, $q_0 = -0.4$, Eq. (18) yields the numerical estimate $d\ln e^2/dt \sim \pm 0.9 \times 10^{-16} \sqrt{10^{12}\Delta a/a}\,\mathrm{yr}^{-1}$. Therefore, the current bound on UFF violations ($\Delta a/a \sim 10^{-12}$ [12]) corresponds to the level $10^{-16}\mathrm{yr}^{-1}$, which is comparable to the planned sensitivity of currently developed cold-atom clocks [13]. [Present laboratory bounds are at the $10^{-14}\mathrm{yr}^{-1}$ level [14,13].] We note also that the upper limit on the variation of e^2 given by the Oklo data, i.e. $|d\ln e^2/dt| \lesssim 5 \times 10^{-17}\,\mathrm{yr}^{-1}$ [15], "corresponds" to a violation of the UFF at the level $\sim 10^{-13}$. Of course, present measurements are not accurate enough to exclude an almost exact cancellation occurring in $1+q_0-3\Omega_m/2$: our work points at the relevance of establishing whether or not this can be excluded.

We have also studied [8] the variation of e^2 over cosmological times. By taking into account the constraints coming from the need to be compatible with current cosmological data, we find that there is no way, within our model, to explain a variation of e^2 as large as the recent claim [16] $\Delta e^2/e^2 = (-0.72 \pm 0.18) \times 10^{-5}$ around redshifts $z \approx 0.5 - 3.5$. The largest possible variation we find (reached only if the UFF violation is just below the 10^{-12} level, and if φ is rather strongly coupled to dark matter), is of order $\Delta e^2/e^2 = \pm 1.9 \times 10^{-6}$. This is only a factor ~ 4 below the claim [16] and is at the level of their one sigma error bar. Therefore a modest improvement in the observational precision (accompanied by an improved control of systematics) will start to probe a domain of variation of constants which, according to our scenario, corresponds to an UFF violation smaller than the 10^{-12} level.

Our results suggest that the residual dilaton couplings today (as determined by a cosmological "attraction" towards the fixed point at infinite bare string coupling) are just below the level $\alpha^2_{\rm had} \sim 2.5 \times 10^{-8}$ corresponding to a violation of the UFF at the $\Delta a/a \sim 10^{-12}$ level. This gives additional motivation for improved tests of the UFF, such as the Centre National d'Etudes Spatiales (CNES) mission MICROSCOPE [17] (to fly in 2004; planned sensitivity: $\Delta a/a \sim 10^{-15}$), and the National Aeronautics and Space Agency (NASA) and European Space Agency (ESA) mission STEP (Satellite Test of the Equivalence Principle; planned sensitivity: $\Delta a/a \sim 10^{-18}$) [18]. If our estimates are correct, these experiments should find a rather strong violation signal.

Another possible observable signal of a weakly coupled run-away dilaton is the time-variation of the natural constants. Here the conclusion depends crucially on the assumptions made about the couplings of the dilaton to the cosmologically dominant forms of energy (dark matter and/or dark energy). If these couplings are of order unity (and as large as phenomenologically acceptable), the present time variation of the fine-structure constant is linked to the violation of the UFF by the relation $d\ln e^2/dt \sim \pm 2.0 \times 10^{-16} \sqrt{10^{12}\Delta a/a}\mathrm{yr}^{-1}$. Such a time variation might be observable (if $\Delta a/a$ is not very much below its present upper bound $\sim 10^{-12}$) through the comparison of high-accuracy cold-atom clocks and/or via improved measurements of astronomical spectra. The discovery of such a time variation (which is possible only if $(\Omega_m\alpha_m + 4\Omega_V\alpha_V)/(\Omega_m + 2\Omega_V)$ is not too small or, in terms of more observable quantities, if $1 + q_0 - 3\Omega_m/2$ is of order 1), would then tell us that the dilaton plays an important cosmological role, either because it is strongly coupled to dark matter ($\alpha_m \sim 1$) or/and because it plays the role of quintessence ($\alpha_V \sim 1$).

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[1] E. Witten, *Phys. Lett. B* **149**, 351 (1984).

[2] T.R. Taylor and G. Veneziano, *Phys. Lett. B* **213**, 450 (1988).

[3] T. Damour and A.M. Polyakov, *Nucl. Phys. B* **423**, 532 (1994); *Gen. Rel. Grav.* **26**, 1171 (1994).

[4] T. Damour and K. Nordtvedt, *Phys. Rev. Lett.* **70**, 2217 (1993); *Phys. Rev. D* **48**, 3436 (1993).

[5] G. Veneziano, *J. High Energy Phys.* **06**, 051 (2002)

[6] M. Gasperini, F. Piazza and G. Veneziano, *Phys. Rev. D* **65**, 023508 (2002).

[7] T. Damour and A. Vilenkin, *Phys. Rev. D* **53**, 2981 (1996).

[8] T. Damour, F. Piazza and G. Veneziano, hep-th/0205111.

[9] A. Linde, *Particle Physics and Inflationary Cosmology*, (Harwood, Chur, 1990).

[10] D.E. Groom et al., *European Physical Journal* **C15**, 1 (2000), see Chapter 14 "Experimental tests of gravitational theory", available on http://pdg.lbl.gov/; and C.M. Will, *Living Rev. Rel.* **4**, 4 (2001). 1990).

[11] T. Damour, G.W. Gibbons and C. Gundlach, *Phys. Rev. Lett.* **64**, 123 (1990).

[12] Y. Su et al., *Phys. Rev. D* **50**, 3614 (1994).

[13] C. Salomon et al., in *Cold atom clocks on Earth and in space, Proceedings of the 17th Int. Conf. on Atomic Physics*, E. Arimondo, M. Inguscio, ed., (World Scientific, Singapore, 2001), p 23; Y. Sortais et al., *Physica Scripta*, **95**, 50 (2001).

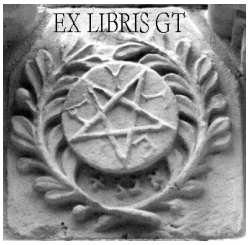
[14] J. D. Prestage, R. L. Tjoelker, and L. Maleki, *Phys. Rev. Lett.* **74** 3511 (1995).

[15] A.I. Shlyakhter, *Nature* **264**, 340 (1976); T. Damour and F. Dyson, *Nucl. Phys. B* **480**, 37 (1996).

[16] J.K. Webb et al., *Phys. Rev. Lett.* **87**, 091301 (2001).

[17] P. Touboul et al., *C.R. Acad. Sci. Paris* **2** (série IV) 1271 (2001).

[18] P. W. Worden, in *Proc. 7th Marcel Grossmann Meeting on General Relativity*, R. T. Jantzen and G. Mac Keiser eds. (World Scientific, Singapore, 1996), pp 1569-1573.



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Self-gravitating fundamental strings and black holes

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Abstract

The configuration of typical highly excited ($M \gg M_s \sim (\alpha')^{-1/2}$) string states is considered as the string coupling g is adiabatically increased. The size distribution of very massive single string states is studied and the mass shift, due to long-range gravitational, dilatonic and axionic attraction, is estimated. By combining the two effects, in any number of spatial dimensions d, the most probable size of a string state becomes of order $\ell_s = \sqrt{2\,\alpha'}$ when $g^2 M/M_s \sim 1$. Depending on the dimension d, the transition between a random-walk-size string state (for low g) and a compact ($\sim \ell_s$) string state (when $g^2 M/M_s \sim 1$) can be very gradual ($d = 3$), fast but continuous ($d = 4$), or discontinuous ($d \geq 5$). Those compact string states look like nuggets of an ultradense state of string matter, with energy density $\rho \sim g^{-2} M_s^{d+1}$. Our results extend and clarify previous work by Susskind, and by Horowitz and Polchinski, on the correspondence between self-gravitating string states and black holes.

I. INTRODUCTION

Almost exactly thirty years ago the study of the spectrum of string theory (known at the time as the dual resonance model) revealed [1] a huge degeneracy of states growing as an exponential of the mass. This led to the concept of a limiting (Hagedorn) temperature $T_{\rm Hag}$ in string theory. Only slightly more recently Bekenstein [2] proposed that the entropy of a black hole should be proportional to the area of its horizon in Planck units, and Hawking [3] fixed the constant of proportionality after discovering that black holes, after all, do emit thermal radiation at a temperature $T_{\rm Haw} \sim R_{\rm BH}^{-1}$.

When string and black hole entropies are compared one immediately notices a striking difference: string entropy[1] is proportional to the first power of mass in any number of spatial dimensions d, while black hole entropy is proportional to a d-dependent power of the mass, always larger than 1. In formulae:

$$S_s \sim \frac{\alpha' M}{\ell_s} \sim M/M_s \quad , \qquad S_{\rm BH} \sim \frac{\rm Area}{G_N} \sim \frac{R_{\rm BH}^{d-1}}{G_N} \sim \frac{(g^2\, M/M_s)^{\frac{d-1}{d-2}}}{g^2} \, , \tag{1.1}$$

where, as usual, α' is the inverse of the classical string tension, $\ell_s \sim \sqrt{\alpha' \hbar}$ is the quantum length associated with it[2], $M_s \sim \sqrt{\hbar/\alpha'}$ is the corresponding string mass scale, $R_{\rm BH}$ is the Schwarzschild radius associated with M:

$$R_{\rm BH} \sim (G_N\, M)^{1/(d-2)} \, , \tag{1.2}$$

and we have used that, at least at sufficiently small coupling, the Newton constant and α' are related via the string coupling by $G_N \sim g^2 (\alpha')^{(d-1)/2}$ (more geometrically, $\ell_P^{d-1} \sim g^2 \ell_s^{d-1}$).

Given their different mass dependence, it is obvious that, for a given set of the fundamental constants G_N, α', g^2, $S_s > S_{\rm BH}$ at sufficiently small M, while the opposite is true at sufficiently large M. Obviously, there has to be a critical value of M, M_c, at which $S_s = S_{\rm BH}$. This observation led Bowick et al. [4] to conjecture that large black holes end up their Hawking-evaporation process when $M = M_c$, and then transform into a higher-entropy string state without ever reaching the singular zero-mass limit. This reasoning is confirmed [5] by the observation that, in string theory, the fundamental string length ℓ_s should set a minimal value for the Schwarzschild radius of any black hole (and thus a maximal value for its Hawking temperature). It was also noticed [4], [6], [7] that, precisely at $M = M_c$, $R_{\rm BH} = \ell_s$ and the Hawking temperature equals the Hagedorn temperature of string theory. For any d, the value of M_c is given by:

$$M_c \sim M_s g^{-2} \, . \tag{1.3}$$

[1] The self-interaction of a string lifts the huge degeneracy of free string states. One then defines the entropy of a narrow band of string states, defined with some energy resolution $M_s \lesssim \Delta E \ll M$, as the logarithm of the number of states within the band ΔE.

[2] Below, we shall use the precise definition $\ell_s \equiv \sqrt{2\alpha' \hbar}$, but, in this section, we neglect factors of order unity.

DUALITY principle

Susskind and collaborators [6], [8] went a step further and proposed that the spectrum of black holes and the spectrum of single string states be "identical", in the sense that there be a one to one correspondence between (uncharged) fundamental string states and (uncharged) black hole states. Such a "correspondence principle" has been generalized by Horowitz and Polchinski [9] to a wide range of charged black hole states (in any dimension). Instead of keeping fixed the fundamental constants and letting M evolve by evaporation, as considered above, one can (equivalently) describe the physics of this conjectured correspondence by following a narrow band of states, on both sides of and through, the string $\rightleftharpoons$ black hole transition, by keeping fixed the entropy[3] $S = S_s = S_{\rm BH}$, while adiabatically[4] varying the string coupling g, i.e. the ratio between ℓ_P and ℓ_s. The correspondence principle then means that if one increases g each (quantum) string state should turn into a (quantum) black hole state at sufficiently strong coupling, while, conversely, if g is decreased, each black hole state should "decollapse" and transform into a string state at sufficiently weak coupling. For all the reasons mentioned above, it is very natural to expect that, when starting from a black hole state, the critical value of g at which a black hole should turn into a string is given, in clear relation to (1.3), by

g

$$g_c^2\, M \sim M_s\,, \tag{1.4}$$

and is related to the common value of string and black-hole entropy via

$$g_c^2 \sim \frac{1}{S_{\rm BH}} = \frac{1}{S_s}\,. \tag{1.5}$$

Note that $g_c^2 \ll 1$ for the very massive states ($M \gg M_s$) that we consider. This justifies our use of the perturbative relation between G_N and α'.

In the case of extremal BPS, and nearly extremal, black holes the conjectured correspondence was dramatically confirmed through the work of Strominger and Vafa [10] and others [11] leading to a statistical mechanics interpretation of black-hole entropy in terms of the number of microscopic states sharing the same macroscopic quantum numbers. However, little is known about whether and how the correspondence works for non-extremal, non BPS black holes, such as the simplest Schwarzschild black hole[5]. By contrast to BPS states whose mass is protected by supersymmetry, we shall consider here the effect of varying g on the mass and size of non-BPS string states.

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[3] One uses here the fact that, during an adiabatic variation of g, the entropy of the black hole $S_{\rm BH} \sim ({\rm Area})/G_N \sim R_{\rm BH}^{d-1}/G_N$ stays constant. This result (known to hold in the Einstein conformal frame) applies also in string units because $S_{\rm BH}$ is dimensionless.

[4] The variation of g can be seen, depending on one's taste, either as a real, adiabatic change of g due to a varying dilaton background, or as a mathematical way of following energy states.

[5] For simplicity, we shall consider in this work only Schwarzschild black holes, in any number $d \equiv D-1$ of non-compact spatial dimensions.

Although it is remarkable that black-hole and string entropy coincide when $R_{\rm BH} = \ell_s$, this is still not quite sufficient to claim that, when starting from a string state, a string becomes a black hole at $g = g_c$. In fact, the process in which one starts from a string state in flat space and increases g poses a serious puzzle [6]. Indeed, the radius of a typical excited string state of mass M is generally thought of being of order

$$R_s^{\rm rw} \sim \ell_s (M/M_s)^{1/2} \,, \tag{1.6}$$

as if a highly excited string state were a random walk made of $M/M_s = \alpha' M/\ell_s$ segments of length ℓ_s [12]. [The number of steps in this random walk is, as is natural, the string entropy (1.1).] The "random walk" radius (1.6) is much larger than the Schwarzschild radius for all couplings $g \leq g_c$, or, equivalently, the ratio of self-gravitational binding energy to mass (in d spatial dimensions)

$$\frac{G_N\, M}{(R_s^{\rm rw})^{d-2}} \sim \left(\frac{R_{\rm BH}(M)}{R_s^{\rm rw}}\right)^{d-2} \sim g^2 \left(\frac{M}{M_s}\right)^{\frac{4-d}{2}} \tag{1.7}$$

remains much smaller than one (when $d > 2$, to which we restrict ourselves) up to the transition point. In view of (1.7) it does not seem natural to expect that a string state will "collapse" to a black hole when g reaches the value (1.4). One would expect a string state of mass M to turn into a black hole only when its typical size is of order of $R_{\rm BH}(M)$ (which is of order ℓ_s at the expected transition point (1.4)). According to Eq. (1.7), this seems to happen for a value of g much larger than g_c.

Horowitz and Polchinski [13] have addressed this puzzle by means of a "thermal scalar" formalism [14]. Their results suggest a resolution of the puzzle when $d = 3$ (four-dimensional spacetime), but lead to a rather complicated behaviour when $d \geq 4$. More specifically, they consider the effective field theory of a complex scalar field χ in d (spacetime) dimensions (with period β in Euclidean time τ), with mass squared $m^2(\beta) = (4\pi^2\,\alpha'^2)^{-1}\,[G_{\tau\tau}\,\beta^2 - \beta_{\rm H}^2]$, where $G_{\mu\nu}$ is the string metric and $\beta_{\rm H}^{-1}$ the Hagedorn temperature. They took into account the effect of gravitational (and dilatonic) self-interactions in a mean field approximation. This leads to an approximate Hartree-like equation for $\chi(\boldsymbol{x})$, which admits a stable bound state, in some range $g_0 < g < g_c$, when $d = 3$. They interpret the size of the bound state "wave function" $\chi(\boldsymbol{x})$ as the "size of the string", and find that (in $d = 3$) this size is of order

$$\ell_\chi \sim \frac{1}{g^2\, M} \,. \tag{1.8}$$

They describe their result by saying that "the string contracts from its initial (large) size", when $g \sim g_0 \sim (M/M_s)^{-3/4}$, down to the string scale when $g \sim g_c \sim (M/M_s)^{-1/2}$. This interpretation of the length scale ℓ_χ, characterizing the thermal scalar bound state, as "the size of the string" is unclear to us, because of the formal nature of the auxiliary field χ which has no direct physical meaning in Minkowski spacetime. Moreover, the analysis of Ref. [13] in higher dimensions is somewhat inconclusive, and suggests that a phenomenon of hysteresis takes place (when $d \geq 5$): the critical value of g corresponding to the string $\rightleftharpoons$ black hole transition would be $g_0 \sim (M/M_s)^{(d-6)/4} > g_c$ for the direct process (string $\rightarrow$black hole), and g_c for the reverse one. Finally, they suggest that, in the reverse process, a black hole becomes an excited string in an *atypical* state.

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The aim of the present work is to clarify the string $\rightleftharpoons$ black hole transition by a direct study, in real spacetime, of the size and mass of a *typical* excited string, within the microcanonical ensemble of *self-gravitating* strings. Our results lead to a rather simple picture of the transition, in any dimension. We find no hysteresis phenomenon in higher dimensions. The critical value for the transition is (1.4), or (1.5) in terms of the entropy S, for both directions of the string $\rightleftharpoons$ black hole transition. In three spatial dimensions, we find that the size (computed in real spacetime) of a *typical self-gravitating* string is given by the random walk value (1.6) when $g^2 \leq g_0^2$, with $g_0^2 \sim (M/M_s)^{-3/2} \sim S^{-3/2}$, and by

$$R_{\rm typ} \sim \frac{1}{g^2\, M}\,, \tag{1.9}$$

when $g_0^2 \leq g^2 \leq g_c^2$. Note that $R_{\rm typ}$ smoothly interpolates between $R_s^{\rm rw}$ and ℓ_s. This result confirms the picture proposed by Ref. [13] when $d = 3$, but with the bonus that Eq. (1.9) (which agrees with Eq. (1.8)) refers to a radius which is estimated directly in physical space (see below), and which is the size of a typical member of the microcanonical ensemble of self-gravitating strings. In all higher dimensions[6], we find that the size of a typical self-gravitating string remains fixed at the random walk value (1.6) when $g \leq g_c$. However, when g gets close to a value of order g_c, the ensemble of self-gravitating strings becomes (smoothly in $d = 4$, but suddenly in $d \geq 5$) dominated by very compact strings of size $\sim \ell_s$ (which are then expected to collapse with a slight further increase of g because the dominant size is only slightly larger than the Schwarzschild radius at g_c).

Our results confirm and clarify the main idea of a correspondence between string states and black hole states [6], [8], [9], [13], and suggest that the transition between these states is rather smooth, with no apparent hysteresis, and with continuity in entropy, mass and typical size. It is, however, beyond the technical grasp of our analysis to compute any precise number at the transition (such as the famous factor 1/4 in the Bekenstein-Hawking entropy formula).

collapse

any progress since?

II. SIZE DISTRIBUTION OF FREE STRING STATES

The aim of this section is to estimate the distribution function in size of the ensemble of free string states of mass M, i.e. to count how many massive string states have a given size R. This estimate will be done while neglecting the gravitational self-interaction. The effect of the latter will be taken into account in a later section.

Let us first estimate the distribution in size by a rough, heuristic argument based on the random walk model [12] of a generic excited string state. In string units ($\ell_s \sim M_s^{-1} \sim 1$), the geometrical shape (in d-dimensional space) of a generic massive string state can be roughly identified with a random walk of M steps of unit length. We can constrain this random walk to stay of size $\lesssim R$ by considering a diffusion process, starting from a point source at the origin, in presence of an *absorbing* sphere S_R of radius R, centered on the origin. In the

[6] With the proviso that the consistency of our analysis is open to doubt when $d \geq 8$.

continuous approximation, the kernel $K_t(\boldsymbol{x},\boldsymbol{0})$ giving the conditional probability density of ending, at time t, at position $\boldsymbol{x}$, after having started (at time 0) at the origin, without having ever gone farther from the origin than the distance R, satisfies: (i) the diffusion equation $\partial_t\, K_t = \Delta\, K_t$, (ii) the initial condition $K_0(\boldsymbol{x},\boldsymbol{0}) = \delta(\boldsymbol{x})$, and (iii) the "absorbing" boundary condition $K_t = 0$ on the sphere S_R. The kernel K_t can be decomposed in eigenmodes,

$$K_t(\boldsymbol{x},\boldsymbol{0}) = \sum_n \psi_n(\boldsymbol{x})\, \psi_n(\boldsymbol{0})\, e^{-E_n\, t}\,, \tag{2.1}$$

where $\psi_n(\boldsymbol{x})$ is a normalized (real) L^2 basis ($\int d^d\, x\, \psi_n(\boldsymbol{x})\, \psi_m(\boldsymbol{x}) = \delta_{nm}$; $\sum_n \psi_n(\boldsymbol{x})\, \psi_n(\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})$) satisfying

$$\Delta\, \psi_n = -E_n\, \psi\,, \tag{2.2}$$

in the interior, and vanishing on S_R. The total conditional probability of having stayed within S_R after the time t is the integral $\int_{B_R} d^d\, x\, K_t(\boldsymbol{x},\boldsymbol{0})$ within the ball $B_R : |\boldsymbol{x}| \leq R$. For large values of t, K_t is dominated by the lowest eigenvalue E_0, and the conditional probability goes like $c_0\, e^{-E_0\, t}$, where c_0 is a numerical constant of order unity. The eigenvalue problem (2.2) is easily solved (in any dimension d), and the s-wave ground state can be expressed in terms of a Bessel function: $\psi_0(r) = N\, J_\nu(k_0\, r)/(k_0\, r)^\nu$ with $\nu = (d-2)/2$. Here, $k_0 = \sqrt{E_0}$ is given by the first zero j_ν of $J_\nu(z) : k_0 = j_\nu/R$. The important information for us is that the ground state energy E_0 scales with R like $E_0 = c_1/R^2$, where $c_1 = \mathcal{O}(1)$ is a numerical constant. This scaling is evident for the Dirichlet problem(2.2), whatever be the shape of the boundary. Finally, remembering that the number of time steps is given by the mass, $t = M$, we expect the looked for conditional probability, i.e. the *fraction* of all string states at mass level M which are of size $\lesssim R$, to be asymptotically of order

$$\boxed{f(R) \sim e^{-c_1\, M/R^2}\,.} \tag{2.3}$$

This estimate is expected to be valid when $M/R^2 \gg 1$, i.e. for string states which are much smaller (in size) than a typical random walk $R_{\rm rw}^2 \sim M$ (but still larger than the string length, $R \gtrsim 1$). In the opposite limit, $R^2 \gg M$, the kernel $K_t(\boldsymbol{x},\boldsymbol{0})$ can be approximated by the free-space value $K_t^{(0)}(\boldsymbol{x},\boldsymbol{0}) = (4\,\pi\, t)^{-d/2}\, \exp(-\boldsymbol{x}^2/(4t))$, with $t = M$, so that the fraction of string states of size $\gtrsim R \gg R_{\rm rw}$ will be of order $\sim e^{-c_2\, R^2/M}$, with $c_2 = \mathcal{O}(1)$.

As the result (2.3) will be central to the considerations of this paper, we shall now go beyond the previous heuristic, random walk argument and derive the fraction of small string states by a direct counting of quantum string states. For simplicity, we shall deal with open bosonic strings ($\ell_s \equiv \sqrt{2\,\alpha'}$, $0 \leq \sigma \leq \pi$)

$$X^\mu(\tau,\sigma) = X^\mu_{\rm cm}(\tau,\sigma) + \widetilde{X}^\mu(\tau,\sigma)\,, \tag{2.4}$$

$$X^\mu_{\rm cm}(\tau,\sigma) = x^\mu + 2\,\alpha'\, p^\mu\, \tau\,, \tag{2.5}$$

$$\widetilde{X}^\mu(\tau,\sigma) = i\,\ell_s \sum_{n\neq 0} \frac{\alpha_n^\mu}{n}\, e^{-in\tau}\, \cos\, n\,\sigma\,. \tag{2.6}$$

Here, we have explicitly separated the center of mass motion $X^\mu_{\rm cm}$ (with $[x^\mu, p^\nu] = i\,\eta^{\mu\nu}$) from the oscillatory one $\widetilde{X}^\mu$ ($[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta^0_{m+n}\,\eta^{\mu\nu}$). The free spectrum is given by $\alpha'\,M^2 = N-1$ where $(\alpha\cdot\beta \equiv \eta_{\mu\nu}\,\alpha^\mu\,\beta^\nu \equiv -\alpha^0\,\beta^0 + \alpha^i\,\beta^i)$

$$N = \sum_{n=1}^{\infty}\ \alpha_{-n}\cdot\alpha_n = \sum_{n=1}^{\infty}\ n\,N_n\,. \tag{2.7}$$

Here $N_n \equiv a^\dagger_n \cdot a_n$ is the occupation number of the $n^{\rm th}$ oscillator ($\alpha^\mu_n = \sqrt{n}\ a^\mu_n$,$[a^\mu_n, a^{\nu\dagger}_m] = \eta^{\mu\nu}\,\delta_{nm}$, with n, m positive).

The decomposition (2.4)–(2.6) holds in any conformal gauge ($(\partial_\tau\,X^\mu \pm \partial_\sigma\,X^\mu)^2 = 0$). One can further specify the choice of worldsheet coordinates by imposing

$$n_\mu\,X^\mu(\tau,\sigma) = 2\alpha'(n_\mu\,p^\mu)\,\tau\,, \tag{2.8}$$

where n^μ is an arbitrary timelike or null vector ($n\cdot n \leq 0$) [15]. Eq. (2.8) means that the n-projected oscillators $n_\mu\,\alpha^\mu_m$ are set equal to zero. The usual "light-cone" gauge is obtained by choosing a fixed, null vector n_μ. The light-cone gauge introduces a preferred ("longitudinal") direction in space, which is an inconvenience for defining the (rms) size of a massive string state. As we shall be interested in quasi-classical, very massive string states ($N \gg 1$) it should be possible to work in the "center of mass" gauge, where the vector n^μ used in Eq. (2.8) to define the τ-slices of the world-sheet is taken to be the total momentum p^μ of the string. This gauge is the most intrinsic way to describe a string in the classical limit. Using this intrinsic gauge, one can covariantly[7] define the proper rms size of a massive string state as

$$R^2 \equiv \frac{1}{d}\ \langle(\widetilde{X}^\mu_\perp\,(\tau,\sigma))^2\rangle_{\sigma,\tau}\,, \tag{2.9}$$

where $\widetilde{X}^\mu_\perp \equiv \widetilde{X}^\mu - p^\mu(p\cdot\widetilde{X})/(p\cdot p)$ denotes the projection of $\widetilde{X}^\mu \equiv X^\mu - X^\mu_{\rm cm}(\tau)$ orthogonally to p^μ, and where the angular brackets denote the (simple) average with respect to σ and τ. The factor $1/d$ in Eq. (2.9) is introduced to simplify later formulas. So defined R is the rms value of the *projected size* of the string along an arbitrary, but fixed spatial direction. [We shall find that this projected size is always larger than $\sqrt{3\,\alpha'/2}$; i.e. string states cannot be "squeezed", along any axis, more than this.]

In the center of mass gauge, $p_\mu\,\widetilde{X}^\mu$ vanishes by definition, and Eq. (2.9) yields simply

$$R^2 = \frac{1}{d}\,\ell_s^2\,\mathcal{R}\,, \tag{2.10}$$

with

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[7] In an arbitrary conformal gauge, the definition (2.9) is gauge-dependent (in spite of the use of the orthogonal projection) because both the definition of $X^\mu_{\rm cm}(\tau)$, and that of the (σ,τ)-averaging depend on the choice of world-sheet gauge. Even if we were using the (more intrinsic but more complicated) average with weight $\sqrt{-\det\,\gamma_{ab}}\,d\sigma\,d\tau = (\partial_\sigma\,X^\mu)^2\,d\sigma\,d\tau$, the dependence upon $X^\mu_{\rm cm}(\tau)$ would remain.

$$\mathcal{R} \equiv \sum_{n=1}^{\infty} \frac{\alpha_{-n} \cdot \alpha_n + \alpha_n \cdot \alpha_{-n}}{2\ n^2} \,. \tag{2.11}$$

The squared-size operator $\mathcal{R}$, Eq. (2.11), contains the logarithmically infinite contribution $\sum 1/(2n)$. Without arguing with the suggestion that this contribution may have a physical meaning (see, e.g., [16]), we note here that this contribution is state-independent. We are interested in this work in the relative sizes of various highly excited, quasi-classical states. A concept which should reduce to the well-defined, finite rms size of a classical Nambu string in the classical limit. We shall therefore discard this state-independent contribution, i.e. work with the normal-ordered operator

$$: \mathcal{R} : = \sum_{n=1}^{\infty} \frac{a_n^\dagger \cdot a_n}{n} = \sum_{n=1}^{\infty} \frac{N_n}{n} \,. \tag{2.12}$$

We shall assume that we can work both in the center-of-mass (worldsheet) gauge ($p_\mu\, \alpha_m^\mu \to 0$) and in the center-of-mass (Lorentz) frame ($(p^\mu) = (M, \mathbf{0})$). This means that the scalar product in the level occupation number N_n runs over the d spatial dimensions: $N_n = a_n^\dagger \cdot a_n = \sum_{i=1}^{d} (a_n^i)^\dagger\, a_n^i$. The "wrong sign" time oscillators α_n^0 are set equal to zero. The Virasoro constraints then imply, besides the mass formula $\alpha' M^2 = N - 1$, the usual sequence of constraints on physical states, $L_n \,|\phi\rangle = 0$, with $L_n = \frac{1}{2} \sum_m \sum_{i=1}^{d} \alpha_{n-m}^i\, \alpha_m^i$. These constraints mean that the d oscillators α_n^i at level n are not physically independent.

The problem we would like to solve is to count the number of physical states, in the Fock space of the center-of-mass oscillators α_n^i, having some fixed values of N and $\mathcal{R}$ (we henceforth work only with the normal-ordered operator (2.12) without adorning it with the : : notation). The Virasoro constraints make this problem technically quite difficult. !!!!! However, we know from the exact counting of physical states (without size restriction) in the light-cone gauge that the essential physical effect of the Virasoro constraints is simply to reduce the number of independent oscillators at any level n from $d = D - 1$ (in the center-of-mass gauge) to $d - 1 = D - 2$. If we (formally) consider d as a large parameter[8], this change in the number of effective free oscillators should have only a small fractional effect on any other coarse-grained, counting problem. We shall assume that this is the case, and solve the much simpler counting problem where the d oscillators α_n^i are considered as independent[9]. To solve this problem we pass from a microcanonical problem (fixed values

[8]It would be interesting to see if one can technically implement a large d approach to our counting problem.

[9]We tried to work in the light-cone gauge, with $d-1$ independent oscillators. However, the necessary inclusion of the longitudinal term $M^{-2}(p \cdot \widetilde{X})^2$in (2.9), which is quadratic in the longitudinal oscillators $\alpha_n^- = (p^+)^{-1}\, L_n^{\rm transverse}$, leads to a complicated, interacting theory of the $d-1$ transverse oscillators.

of N and $\mathcal{R}$) to a grand canonical one (fixed values of some thermodynamical conjugates of N and $\mathcal{R}$). Let us introduce the formal "partition function"

$$Z_d(\beta,\gamma) \equiv \sum_{\{N_n^i\}} \exp(-\beta\, N\,[N_n^i] - \gamma\, \mathcal{R}\,[N_n^i])\,, \tag{2.13}$$

where the sum runs over all sequences (labelled by $n \geq 1$ and $i = 1, \ldots, d$) of independent occupation numbers $N_n^i = (a_n^i)^\dagger\, a_n^i = 0, 1, 2, \ldots$, and where $N\,[N_n^i]$ and $\mathcal{R}\,[N_n^i]$ are defined by Eqs. (2.7), and (2.12), with $N_n \equiv \sum_{i=1}^{d} N_n^i$. Note that (2.13) is not the usual thermodynamical partition function, and that β is not the usual inverse temperature. Indeed, β is a formal conjugate to $N \simeq \alpha' M^2$ and not to the energy M. In particular, because the degeneracy grows exponentially with M (and not M^2) its Laplace transform (2.13) is defined for arbitrary values of β. We associate with the definition (2.13) that of a formal grand canonical ensemble of configurations, with the probability

$$p\,[\{N_n^i\}] = Z_d^{-1}\,(\beta,\gamma)\, \exp(-\beta\, N\,[N_n^i] - \gamma\, \mathcal{R}\,[N_n^i]) \tag{2.14}$$

of realization of the particular sequence N_n^i of occupation numbers. The mean values of $N\,[N_n^i]$ and $\mathcal{R}\,[N_n^i]$ in this ensemble are

$$\overline{N} = -\frac{\partial\, \psi_d\,(\beta,\gamma)}{\partial\, \beta}\ ,\ \overline{\mathcal{R}} = -\frac{\partial\, \psi_d\,(\beta,\gamma)}{\partial\, \gamma}\,, \tag{2.15}$$

where we denote

$$\psi_d\,(\beta,\gamma) \equiv \ln\, Z_d\,(\beta,\gamma)\,. \tag{2.16}$$

The second derivatives of the thermodynamical potential $\psi_d\,(\beta,\gamma)$ give the fluctuations of N and $\mathcal{R}$ in this grand canonical ensemble:

$$\overline{(\Delta N)^2} = \frac{\partial^2\, \psi_d\,(\beta,\gamma)}{\partial\, \beta^2}\ ,\ \overline{(\Delta \mathcal{R})^2} = \frac{\partial^2\, \psi_d\,(\beta,\gamma)}{\partial\, \gamma^2}\,. \tag{2.17}$$

Let us define as usual the entropy $S(\beta,\gamma)$ as the logarithm of the number of string configurations having values of $N$and $\mathcal{R}$ equal to $\overline{N}$ and $\overline{\mathcal{R}}$, Eqs. (2.15), within the precision of the rms fluctuations (2.17) [17]. This definition means that, in the saddle-point approximation, $Z_d\,(\beta,\gamma) \simeq \exp\,[S - \beta\,\overline{N} - \gamma\,\overline{\mathcal{R}}]$, i.e.

$$\psi_d\,(\beta,\gamma) \simeq S(\beta,\gamma) - \beta\,\overline{N} - \gamma\,\overline{\mathcal{R}}\,, \tag{2.18}$$

or

$$S \simeq \psi_d\,(\beta,\gamma) - \beta\,\frac{\partial\, \psi_d\,(\beta,\gamma)}{\partial\, \beta} - \gamma\,\frac{\partial\, \psi_d\,(\beta,\gamma)}{\partial\, \gamma}\,. \tag{2.19}$$

In other words, the entropy $S(\overline{N},\overline{\mathcal{R}})$ is the Legendre transform of $\psi_d\,(\beta,\gamma)$.

Because of the (assumed) independence of the d oscillators in (2.13), one has

$$Z_d\,(\beta,\gamma) = \prod_{n=1}^{\infty} \left[1 - e^{-(\beta n + \gamma/n)}\right]^{-d}\,, \tag{2.20}$$

i.e.

$$\psi_d\,(\beta,\gamma) = d\,\psi_1\,(\beta,\gamma)\,, \tag{2.21}$$

with

$$\psi_1\,(\beta,\gamma) = -\sum_{n=1}^{\infty} \ln\left[1 - \exp\left(-\beta n - \frac{\gamma}{n}\right)\right]\,. \tag{2.22}$$

We shall check *a posteriori* that we are interested in values of β and γ such that

$$\beta \ll \sqrt{\beta\,\gamma} \ll 1\,. \tag{2.23}$$

For such values, one can approximate the discrete sum (2.22) by a continuous integral over $x = \beta\,n$. This yields

$$\psi_1\,(\beta,\gamma) = \frac{I(\delta)}{\beta}\ ,\ \text{where}\ \delta \equiv \sqrt{\beta\,\gamma}\,, \tag{2.24}$$

and

$$I(\delta) \equiv -\int_0^{\infty} dx\ \ln\left[1 - e^{-(x\,+\,\delta^2/x)}\right]. \tag{2.25}$$

As $\delta = \sqrt{\beta\,\gamma} \ll 1$, we can try to expand $I(\delta)$ in powers of δ: $I(\delta) = I(0) + \delta\,I'(0) + o(\delta)$. [Though the integral (2.25) is expressed in terms of δ^2, its formal expansion in powers of δ^2 leads to divergent integrals.] The zeroth-order term is $I(0) = -\int_0^{\infty} dx\,\ln\left(1 - e^{-x}\right) = \pi^2/6$, while

$$I'(0) = \lim_{\delta\to 0}\left[-2\int_0^{\infty}\frac{du}{u}\,\frac{\delta}{e^{\delta(u+1/u)} - 1}\right] = -2\int_0^{\infty}\frac{du}{u^2+1} = -\pi\,.$$

Hence, using (2.21),

$$\psi_d\,(\beta,\gamma) = \frac{1}{\beta}\left[C - D\,\sqrt{\beta\,\gamma} + o\left(\sqrt{\beta\,\gamma}\right)\right] \tag{2.26}$$

with

$$C = \frac{\pi^2}{6}\,d\ ,\ D = \pi\,d\,. \tag{2.27}$$

[The notation D in (2.27) should not be confused with the space-time dimension $d+1$.] The thermodynamic potential (2.26) corresponds to the mean values

$$\overline{N} \simeq \frac{C - \frac{1}{2}\,D\,\sqrt{\beta\,\gamma}}{\beta^2}\ ,\ \overline{\mathcal{R}} \simeq \frac{D}{2\,\delta}\,, \tag{2.28}$$

and to the entropy

$$S \simeq \frac{2\,C - D\,\delta}{\beta} \simeq 2\,\sqrt{C\,\overline{N}}\left[1 - \frac{D^2}{8\,C\,\overline{\mathcal{R}}}\right] , \tag{2.29}$$

i.e.

$$S \simeq 2\,\pi\,\sqrt{\frac{d}{6}\,\overline{N}}\left[1 - \frac{3d}{4}\,\frac{1}{\overline{\mathcal{R}}}\right] \simeq 2\,\pi\left(\frac{d\,\alpha'}{6}\right)^{\frac{1}{2}}\,M\left[1 - \frac{3}{4}\,\frac{\ell_s^2}{R^2}\right] . \tag{2.30}$$

The lowest-order term $2\,\pi\,\sqrt{d\,\overline{N}/6}$ is the usual (Hardy-Ramanujan) result for d independent oscillators, without size restriction. The factor in bracket, $1 - (3/4)(\ell_s^2/R^2)$, with $\ell_s^2 = 2\,\alpha'$, gives the fractional reduction in the entropy brought by imposing the size constraint $R^2 \simeq d^{-1}\,\ell_s^2\,\overline{\mathcal{R}}$. Under the conditions (2.23) the fluctuations (2.17) are fractionally small. More precisely, Eqs. (2.17) yield

$$\frac{\overline{(\Delta N)^2}}{\overline{N}^2} \sim \beta \sim \frac{M_s}{M} , \qquad \frac{\overline{(\Delta\,\mathcal{R})^2}}{\overline{\mathcal{R}}^2} \sim \frac{\beta}{\delta} \sim \frac{(R^2/\ell_s^2)}{(M/M_s)} . \tag{2.31}$$

As said above, though we worked under the (physically incorrect) assumption of d independent oscillators at each level n, we expect the result (2.29) to be correct when $d \gg 1$. [We recall that the exact result for S in absence of size restriction is $2\,\pi\,\sqrt{(d-1)\,\overline{N}/6}$.] Note the rough physical meanings of the auxiliary quantities β, γ and δ: $\beta \sim (\overline{N})^{-1/2} \sim (M/M_s)^{-1}$, $\delta \sim \overline{\mathcal{R}}^{-1} \sim \ell_s^2/R^2$, $\gamma \sim (\overline{N})^{1/2}/\overline{\mathcal{R}}^2 \sim M\,\ell_s^5/R^4$.

Summarizing, the main result of the present section is that the number("degeneracy") of free string states of mass M and size R (within the narrow bands defined by the fluctuations (2.17)) is of the form

$$\mathcal{D}\,(M, R) \sim \exp\left[c\,(R)\,a_0\,M\right] , \tag{2.32}$$

where $a_0 = 2\,\pi\,((d-1)\,\alpha'/6)^{1/2}$ and

$$c\,(R) = \left(1 - \frac{c_1}{R^2}\right)\left(1 - c_2\,\frac{R^2}{M^2}\right) , \tag{2.33}$$

with the coefficients c_1 and c_2 being of order unity in string units. [We have added, for completeness, in Eq. (2.32) the factor $1 - c_2\,R^2/M^2$ which operates when one considers very "large" string states, $R^2 \gg R_{\rm rw}^2$ (as discussed below Eq. (2.3)).] The coefficient $c\,(R)$ gives the fractional reduction in entropy brought by imposing a size constraint. Note that (as expected) this reduction is minimized when $c_1\,R^{-2} \sim c_2\,R^2/M^2$, i.e. for $R \sim R_{\rm rw} \sim \ell_s\,\sqrt{M/M_s}$. [The absolute reduction in degeneracy is only a factor $\mathcal{O}(1)$ when $R \sim R_{\rm rw}$.] Note also that $c\,(R) \to 0$ both when $R \sim \ell_s$ and when $R \sim \ell_s(M/M_s)$. [The latter value corresponding to the vicinity of the leading Regge trajectory $J \sim \alpha'\,M^2$.]

III. MASS SHIFT OF STRING STATES DUE TO SELF-GRAVITY

In this section we shall estimate the mass shift of string states (of mass M and size R) due to the exchange of the various long-range fields which are universally coupled to

the string: graviton, dilaton and axion. As we are interested in very massive string states, $M \gg M_s$, in extended configurations, $R \gg \ell_s$, we expect that massless exchange dominates the (state-dependent contribution to the) mass shift.

The evaluation, in string theory, of (one loop) mass shifts for massive states is technically quite involved, and can only be tackled for the states which are near the leading Regge trajectory [18]. [Indeed, the vertex operators creating these states are the only ones to admit a manageable explicit oscillator representation.] As we consider states which are very far from the leading Regge trajectory, there is no hope of computing exactly (at one loop) their mass shifts. We shall resort to a semi-classical approximation, which seems appropriate because we consider highly excited configurations. As a starting point to derive the mass-shift in this semi-classical approximation we shall use the classical results of Ref. [19] which derived the effective action of fundamental strings. The one-loop exchange of $g_{\mu\nu}$, φ and $B_{\mu\nu}$ leads to the effective action

$$I^{\rm eff} = I_0 + I_1 \,, \tag{3.1}$$

where I_0 is the free (Nambu) string action ($d^2\,\sigma_1 \equiv d\sigma_1\, d\tau_1$, $\gamma_1 = -\det\,\gamma_{ab}\,(X^\mu(\sigma_1,\tau_1))$)

$$I_0 = -T \int d^2\,\sigma_1\,\sqrt{\gamma_1}\,, \tag{3.2}$$

and I_1 the effect of the one-loop interaction ($X_1^\mu \equiv X^\mu(\sigma_1,\tau_1),\ldots$)

$$I_1 = 2\pi \int\int d^2\sigma_1\, d^2\sigma_2\, G_F(X_1 - X_2)\,\sqrt{\gamma_1}\,\sqrt{\gamma_2}\; C_{\rm tot}(X_1, X_2)\,, \tag{3.3}$$

where G_F is Feynman's scalar propagator ($\Box\, G_F(x) = -\delta^D(x)$), and $C_{\rm tot}(X_1,X_2) = J_\varphi(X_1)\cdot J_\varphi(X_2) + J_g(X_1)\cdot J_g(X_2) + J_B(X_1)\cdot J_B(X_2)$ comes from the couplings of φ, $g_{\mu\nu}$ and $B_{\mu\nu}$ to their corresponding world-sheet sources (indices suppressed; spin-structure hidden in the dot product). The exchange term $C_{\rm tot}$ takes, in null (conformal) coordinates $\sigma^\pm = \tau \pm \sigma$, the simple left-right factorized form [19]

$$\sqrt{\gamma_1}\,\sqrt{\gamma_2}\; C_{\rm tot}(X_1,X_2) = 32\, G_N\, T^2\, (\partial_+ X_1^\mu\; \partial_+ X_{2\mu})(\partial_- X_1^\nu\; \partial_- X_{2\nu})\,. \tag{3.4}$$

Here, $T = (2\,\pi\,\alpha')^{-1}$ is the string tension, G_N is Newton's constant[10] and $\partial_\pm = \partial/\partial\sigma^\pm = \frac{1}{2}\,(\partial_\tau \pm \partial_\sigma)$. Let us define $P_\pm^\mu = P_\pm^\mu(\sigma^\pm)$ by ($\ell_s = \sqrt{2\,\alpha'}$ as above)

$$2\,\partial_\pm\, X^\mu = \ell_s\, P_\pm^\mu\,, \tag{3.5}$$

so that, for an open (bosonic) string (with $\alpha_0^\mu \equiv \ell_s\, p^\mu$),

$$P_\pm^\mu = \sum_{-\infty}^{+\infty}\; \alpha_n^\mu\, e^{-in\sigma^\pm}\,. \tag{3.6}$$

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[10] Normalized, in any dimension, by writing the Einstein action as $(16\,\pi\,G_N)^{-1}\int d^D\,x\,\sqrt{g}\,R(g)$.

Using the definition (3.5) and inserting the Fourier decomposition of G_F yields

$$I_1 = \frac{4\,G_N}{\pi} \int \frac{d^D\,k}{(2\pi)^D} \frac{1}{k^2 - i\,\varepsilon} \int\int d^2\sigma_1\,d^2\sigma_2\,(P_+(X_1)\cdot P_+(X_2))(P_-(X_1)\cdot P_-(X_2)) \times e^{ik\cdot(X_1-X_2)}\,, \tag{3.7}$$

where one recognizes the insertion of two gravitational vertex operators $V^{\mu\nu}(k;X) = P^\mu_+(X)\,P^\nu_-(X)\,e^{ik\cdot X}$ at two different locations on the worldsheet, and with two opposite momenta for the exchanged graviton[11]. [Note that the exchanged graviton is off-shell.] It is convenient to use the Virasoro constraints $(P^\mu_\pm(X))^2 = 0$ to replace in Eq. (3.6) $P_\pm(X_1)\cdot P_\pm(X_2) = -\frac{1}{2}\,(\Delta\,P^\mu_\pm)^2$where $\Delta\,P^\mu_\pm \equiv P^\mu_\pm(X_1) - P^\mu_\pm(X_2)$. It is important to note that the zero mode contribution α^μ_0 drops out of $\Delta\,P^\mu_\pm$ (i.e. $\Delta\,P^\mu_\pm = \Delta\,\widetilde{P}^\mu_\pm$ is purely oscillatory).

Writing that the correction I_1 to the effective action $I^{\rm eff}$ (which gives the vacuum persistence amplitude; see, e.g., Eq. (7) of [19]) must correspond to a phase shift $-\int \delta E\,dt = -\int \delta M\,dX^0_{\rm cm}$, in the center-of-mass frame of the string, yields (with the normalization (2.5)) the link $I_1 = -\ell_s^2 \int d\tau\,M\,\delta M = -\frac{1}{2}\,\ell_s^2 \int d\tau\,\delta M^2$. Let us also define $\Delta\,X^\mu \equiv X^\mu_1 - X^\mu_2$ and decompose it in its zero-mode part $\Delta\,X^\mu_{\rm cm} = \ell_s^2\,p^\mu(\tau_1-\tau_2)$ and its oscillatory part $\Delta\,\widetilde{X}^\mu = \widetilde{X}^\mu_1 - \widetilde{X}^\mu_2$. Finally, the mass-shift can be read from

$$\int d\tau\,\delta M^2 = -\frac{2\,G_N}{\pi\,\ell_s^2} \int \frac{d^D\,k}{(2\pi)^D} \frac{1}{k^2 - i\varepsilon} \int\int d^2\sigma_1\,d^2\sigma_2\,e^{i\,\ell_s^2\,k\cdot p\,(\tau_1-\tau_2)}\,W(k,1,2)\,, \tag{3.8}$$

where (1 and 2 being short-hands for (τ_1,σ_1) and (τ_2,σ_2), respectively)

$$W(k,1,2) = (\Delta\,P^\mu_+(1,2))^2\,(\Delta\,P^\nu_-(1,2))^2\,e^{ik\cdot\Delta\widetilde{X}(1,2)}\,. \tag{3.9}$$

Interpreted at the quantum level, the classical result (3.8) gives (modulo some ordering problems, which are, however, fractionally negligible when considering very massive states) the mass-shift $\delta\,M^2_N$ of the string state $|N\rangle$ when replacing $W(k,1,2)$, on the right-hand side of Eq. (3.8), by the quantum average $\langle N|\,W(k,1,2)\,|N\rangle$. Here, we shall mainly be interested in the real part of $\delta\,M^2$, obtained by replacing $(k^2-i\varepsilon)^{-1}$ by the principal part of $(k^2)^{-1}$ (denoted simply $1/k^2$), i.e. the Feynman Green's function $G_F(x)$ by the half-retarded-half-advanced one $G_{\rm sym}(x)$. [The imaginary part of $\delta\,M^2$ gives the decay rate, i.e. the total rate of emission of massless quanta.] As $L_0 - 1$ is the "Hamiltonian" that governs the τ-evolution of an open string, the vanishing of $(L_0-1)\,|N\rangle$ for any physical state ensures that $\langle N|\,W(k,1,2)\,|N\rangle$ is τ-translation invariant, i.e. that it depends only on the difference $\tau_{12} \equiv \tau_1 - \tau_2$, and not on the average $\overline{\tau} \equiv \frac{1}{2}\,(\tau_1+\tau_2)$. This means that the double world-sheet integration $d^2\sigma_1\,d^2\sigma_2 = d\tau_1\,d\sigma_1\,d\tau_2\,d\sigma_2 = d\overline{\tau}\,d\tau_{12}\,d\sigma_1\,d\sigma_2$ on the right-hand side of Eq. (3.8) contains a formally infinite infra-red "volume" factor $\int d\,\overline{\tau}$ which precisely cancels the integral $\int d\,\tau$ on the left-hand side to leave a finite answer for $\delta\,M^2$.

It is also important to note the good ultraviolet behaviour of Eq. (3.8). The ultraviolet limit $k \to \infty$ corresponds to the coincidence limit $(\tau_2,\sigma_2) \to (\tau_1,\sigma_1)$ on the world-sheet.

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[11]For simplicity, we call "graviton" the exchanged particle, which is a superposition of the graviton, the dilaton and the axion.

Let us define $u \equiv \sigma_1^+ - \sigma_2^+$, $v \equiv \sigma_1^- - \sigma_2^-$ and consider the coincidence limit $u \to 0$, $v \to 0$. In this limit the vertex insertion factors $(\Delta P_+)^2 (\Delta P_-)^2$ tend to zero like $u^2 v^2$, while the Green's function blows up like $[(\Delta X)^2]^{-(D-2)/2} \propto (uv)^{-(D-2)/2}$. The resulting integral, $\int du\, dv\, (uv)^{-(D-6)/2}$, has its first ultraviolet pole when the space-time dimension $D = d + 1 = 8$. This means probably that in dimensions $D \geq 8$ the exchange of massive modes (of closed strings) becomes important. Our discussion, which is limited to considering only the exchange of massless modes, is probably justified only when $D < 8$.

Following the (approximate) approach of Section 2 we shall estimate the average mass shift $\delta M^2(R)$ for string states of size R by using the grand canonical ensemble with density matrix

$$\rho \equiv (Z_d(\beta, \gamma))^{-1} \exp(-\beta N - \gamma : \mathcal{R} :), \tag{3.10}$$

where the operators N and $: \mathcal{R} :$, defined by Eqs. (2.7) and (2.12), belong to the Fock space built upon d sequences of string oscillators α_n^i (formal "center-of-mass" oscillators). For any quantity Q (built from string oscillators) we denote the grand canonical average as $\langle Q \rangle_{\beta,\gamma} \equiv \mathrm{tr}(Q\,\rho)$. Using the τ-shift invariance mentioned above, Eq. (3.8) yields

$$\delta M^2_{\beta,\gamma} = -\frac{2\,G_N}{\pi\,\ell_s^2} \int \frac{d^d k\; d\omega}{(2\pi)^{d+1} [\boldsymbol{k}^2 - \omega^2 - i\varepsilon]} \int d\tau_{12}\, d\sigma_1\, d\sigma_2\, e^{-i\,\ell_s^2 M \omega\, \tau_{12}} \langle W(k,1,2) \rangle_{\beta,\gamma}, \tag{3.11}$$

where we have separated k^μ in its center-of-mass components $k^0 = \omega$, $k^i = \boldsymbol{k}$, and where $\tau_{12} \equiv \tau_1 - \tau_2$ as above.

We shall estimate the grand canonical average $\langle W \rangle_{\beta,\gamma}$ in a semi-classical approximation in which we neglect some of the contributions linked to the ordering of the operator W, but take into account the quantum nature of the d ensity matrix ρ, Eq. (3.10). The discreteness of the Fock states built from the$(a_n^i)^\dagger$, and the Planckian nature of ρ will play a crucial role in the calculation below. [By contrast, a purely classical calculation would be awkward and ill-defined because of the problem of defining a measure on classical string configurations, and because of the Rayleigh-Jeans ultraviolet catastrophe.] To compute $\langle W \rangle_{\beta,\gamma}$ it is convenient to define it as a double contraction of the coefficient of $\zeta^1_{\mu_1}\, \zeta^2_{\mu_2}\, \zeta^3_{\mu_3}\, \zeta^4_{\mu_4}$ in the exponentiated version of W:

$$W_\zeta \equiv : \exp[\zeta^1_{\mu_1} \Delta P_+^{\mu_1} + \zeta^2_{\mu_2} \Delta P_+^{\mu_2} + \zeta^3_{\mu_3} \Delta P_-^{\mu_3} + \zeta^4_{\mu_4} \Delta P_-^{\mu_4} + ik \cdot \Delta \widetilde{X}] : \tag{3.12}$$

We shall define our ordering of W by working with the normal ordered exponentiated operator (3.12) (and picking the term linear in$\zeta^1_{\mu_1}\, \zeta^2_{\mu_2}\, \zeta^3_{\mu_3}\, \zeta^4_{\mu_4}$). The average $\langle W_\zeta \rangle_\beta = \mathrm{tr}(W_\zeta\, \rho)$ (where, to ease the notation, we drop the extra label γ) can be computed by a generalization of Bloch's theorem. Namely, if A denotes any operator which is linear in the oscillators α_n^i, we have the results

$$\langle e^A \rangle_\beta = \exp\left[\frac{1}{2} \langle A^2 \rangle_\beta\right] \;;\; \langle : e^A : \rangle_\beta = \exp\left[\frac{1}{2} \langle : A^2 : \rangle_\beta\right], \tag{3.13}$$

as well as their corollaries

$$\langle e^A \rangle_0 = \exp\left[\frac{1}{2} \langle A^2 \rangle_0\right] \;;\; e^A =: e^A : \exp\left[\frac{1}{2} \langle A^2 \rangle_0\right], \tag{3.14}$$

where $\langle W\rangle_0$ denotes the vacuum average (obtained in the zero temperature limit $\beta^{-1}\to 0$). The simplest way to prove these results is to use coherent-state methods [20] (see also,Ref. [15] and the Appendix 7.A of Ref. [21]). For instance, to prove the second equation (3.13) it is sufficient to consider a single oscillator and to check that (denoting $q=e^{-\epsilon}$, with $\epsilon=\beta\,n+\gamma/n$ (label n suppressed), so that $Z=(1-q)^{-1}$, and $|b)\equiv\exp\left(ba^\dagger\right)|0\rangle$)

$$\begin{aligned}\langle e^{c_1a^\dagger}\,e^{c_2a}\rangle_\beta &= Z^{-1}\,\mathrm{tr}\left(e^{c_1a^\dagger}\,e^{c_2a}\,q^{a^\dagger a}\right)=Z^{-1}\int\frac{d^2\,b}{\pi}\,e^{-b^*b}(b|\,e^{c_1a^\dagger}\,e^{c_2a}\,q^{a^\dagger a}\,|b)\\ &=(1-q)\int\frac{d^2\,b}{\pi}\,e^{-b^*b}(b|\,e^{c_1b^*}\,e^{c_2a}\,|qb)\\ &=(1-q)\int\frac{d^2\,b}{\pi}\,e^{-(1-q)b^*b}\,e^{c_1b^*+c_2qb}=e^{c_1c_2q/(1-q)}\,,\end{aligned}\tag{3.15}$$

and to recognize that $q/(1-q)=[e^\epsilon-1]^{-1}$ is the Planckian mean occupation number $\langle a^\dagger\,a\rangle_\beta$.

If we apply the second Eq. (3.13) to an expression of the type$W_\zeta =: \exp\left(\sum_{i=1}^4\zeta_i\,A_i+B\right):$, one gets a Wick-type expansion for the coefficient (say W_{1234}) of $\zeta_1\,\zeta_2\,\zeta_3\,\zeta_4$:

$$\begin{aligned}W_{1234}=e^{\frac{1}{2}[BB]}\,&\left([A_1A_2]\,[A_3A_4]+2\text{ terms}+[A_1B]\,[A_2B]\,[A_3A_4]+5\text{ terms}\right.\\ &\left.+\,[A_1B]\,[A_2B]\,[A_3B]\,[A_4B]\right),\end{aligned}\tag{3.16}$$

where $[AB]$ denotes the "thermal" contraction $[AB]\equiv\langle :AB:\rangle_\beta$.

The looked-for grand canonical average of $W(k,1,2)$, Eq. (3.9), is given by replacing $B=ik\cdot\Delta\,\widetilde{X}$ and $A_1=A_2=\Delta\,P_+^\mu$, $A_3=A_4=\Delta\,P_-^\nu$ in Eq. (3.16). This leads to

$$\langle W\rangle_{\beta,\gamma}=e^{-\frac{1}{2}\langle:(k\cdot\Delta\widetilde{X})^2:\rangle_{\beta,\gamma}}\left\{\langle:(\Delta\,P_+)^2:\rangle_{\beta,\gamma}\,\langle:(\Delta\,P_-)^2:\rangle_{\beta,\gamma}+\ldots\right\},\tag{3.17}$$

where the ellipsis stand for other contractions (which will be seen below to be subleading).

The calculation of the various contractions $[AB]\equiv\langle :AB:\rangle_\beta$ in Eqs. (3.16), (3.17) is easily performed by using the basic contractions among the oscillators a_n,$a_m^\dagger$ $(n,m>0)$ (which are easily derived from the definition (3.10) of the density matrix)

$$\langle:a_n^i\,(a_m^j)^\dagger:\rangle_{\beta,\gamma}=\langle:(a_m^j)^\dagger\,a_n^i:\rangle_{\beta,\gamma}=\frac{\delta^{ij}\,\delta_{nm}}{e^{\epsilon_n}-1}\,,\tag{3.18}$$

where $\epsilon_n=\beta\,n+\gamma/n$. The other contractions $[aa]$ and $[a^\dagger a^\dagger]$ vanish. In terms of the α-oscillators, the basic contraction reads $[\alpha_n^i\,\alpha_m^j]=\delta^{ij}\,\delta^0_{n+m}\,|n|/(\exp(\epsilon_{|n|})-1)$, where now n and m can be negative(but not zero). Using these basic contractions, and the oscillator expansion (2.6) of $\widetilde{X}^\mu$ (and noting that, in the center-of-mass frame only the spatial components of $\widetilde{X}^\mu$ survive) one gets

$$\langle:(\boldsymbol{k}\cdot\Delta\,\widetilde{\boldsymbol{X}})^2:\rangle_{\beta,\gamma}=2\,\boldsymbol{k}^2\,\ell_s^2\,\sum_{n=1}^\infty\,\frac{x_n(1,2)}{n(e^{\epsilon_n}-1)}\,,\tag{3.19}$$

with

$$x_n\,(1,2) = \cos^2\,n\,\sigma_1 + \cos^2\,n\,\sigma_2 - 2\,\cos\,n\,\sigma_1\,\cos\,n\,\sigma_2\,\cos\,n\,\tau_{12}\,. \tag{3.20}$$

Similarly, the oscillator expansion (3.6) yields

$$\langle :\Delta\,P_+)^2:\rangle_{\beta,\gamma} = 4\,d\,\sum_{n=1}^{\infty}\,\frac{n\,p_n^+(1,2)}{e^{\epsilon_n}-1}\,, \tag{3.21}$$

with

$$p_n^+(1,2) = 1 - \cos\,n(\sigma_1^+ - \sigma_2^+) = 1 - \cos\,n(\tau_{12} + \sigma_1 - \sigma_2)\,. \tag{3.22}$$

The result for $(\Delta\,P_-)^2$ is obtained by changing $\sigma^+ \to \sigma^-$ in Eq. (3.22) (i.e. $\sigma_1 - \sigma_2 \to -\sigma_1 + \sigma_2$).

We can estimate the values of the right-hand sides of Eqs. (3.19) and (3.21) by using the following "statistical" approximation. In the parameter range discussed in Section 2 the basic sums $\sum n^{\pm 1}\,(e^{\epsilon_n}-1)^{-1}$, appearing in (3.19), (3.21), see their values dominated by a large interval, $\Delta\,n \gg 1$, around some $n_0 \gg 1$, of values of n, so that one can, with a good approximation, replace the discrete sum over n by a formal continuous integral over a real parameter. In such a continuous approximation one can integrate by parts to show that any "oscillatory" integral of the type $\int dn\,f(n)\,\cos\,n\,\sigma\,=\,[n^{-1}\,f(n)\,\sin\,n\,\sigma] - \int dn\,n^{-1}\,f'(n)\,\sin\,n\,\sigma$ is, because of the factors n^{-1}, numerically much smaller than the non-oscillatory one $\int dn\,f(n)$. [Here σ denotes some combination of σ_1 and σ_2, like $2\sigma_1$, $2\sigma_2$, $\sigma_1 \pm \sigma_2$.] Alternatively, we can say that, for generic values of σ_1 and σ_2, one can treat in Eqs. (3.20) or (3.22) $\cos\,n\,\sigma_1$ and $\cos\,n\,\sigma_2$ as statistically independent random variables with zero average. Within such an approximation one can estimate (3.19) by replacing $x_n(1,2)$ by 1 (because $\cos^2\,n\,\sigma_1 + \cos^2\,n\,\sigma_2 = 1 + \frac{1}{2}\,(\cos\,2n\,\sigma_1 + \cos\,2n\,\sigma_2)$). Similarly, one can estimate (3.21) by replacing $p_n^{\pm} \to 1$. The resulting estimates of (3.19) and (3.21) introduce exactly the grand canonical averages of the quantities $:\mathcal{R}:$ and N:

$$\frac{1}{2}\,\langle :(\boldsymbol{k}\cdot\Delta\,\widetilde{\boldsymbol{X}})^2:\rangle_{\beta,\gamma} \simeq \boldsymbol{k}^2\,\langle R^2\rangle_{\beta,\gamma}\,, \tag{3.23}$$

$$\langle :(\Delta\,P_+)^2:\rangle_{\beta,\gamma} \simeq \langle :(\Delta\,P_-)^2:\rangle_{\beta,\gamma} \simeq 4\,\langle N\rangle_{\beta,\gamma} \simeq 4\,\alpha'\,M^2\,. \tag{3.24}$$

Furthermore, one can check that the other contractions (like $[\Delta\,P_+\,\Delta\,P_-]$ or $[\Delta\,P_+\;k\cdot\Delta\,\widetilde{X}]$) entering Eq. (3.17) are all of the "oscillatory" type which is expected to give subleading contributions.

Inserting the results (3.23), (3.24) into Eqs. (3.17) and (3.11) leads to a trivial integral over τ_{12} ($\int d\,\tau_{12}\,\exp(-i\,\ell_s^2\,M\,\omega\,\tau_{12}) = 2\pi\,\delta(\omega)/(\ell_s^2\,M)$) and, hence, to the following result for $\delta\,M = \delta\,M^2/(2M)$

$$\delta\,M_{\beta,\gamma} \simeq -4\,\pi\,G_N\,M^2\,\int\,\frac{d^d\,k}{(2\pi)^d}\,\frac{e^{-\boldsymbol{k}^2 R^2}}{\boldsymbol{k}^2 - i\,\varepsilon}\,. \tag{3.25}$$

The imaginary part of $\delta\,M$ is easily seen to vanish in the present approximation. Finally,

$$\delta M_{\beta,\gamma} \simeq -c_d G_N \frac{M^2}{R^{d-2}} , \tag{3.26}$$

with the (positive[12]) numerical constant

$$c_d = \left[\frac{d-2}{2} (4\pi)^{\frac{d-2}{2}} \right]^{-1} , \tag{3.27}$$

equal to $1/\sqrt{\pi}$ in $d = 3$.

The result (3.26) was expected in order of magnitude, but we found useful to show how it approximately comes out of a detailed calculation of the mass shift which incorporates both relativistic and quantum effects. It shows clearly that perturbation theory breaks down, even at arbitrarily small coupling, for sufficiently heavy and compact strings. Let us also point out that one can give a simple statistical interpretation of the calculation (3.16) of the normal-ordered vertex operator $W(k, 1, 2)$, with the basic contractions (3.18). The result of the calculation would have been the same if we had simply assumed that the oscillators a_n^i were classical, complex random variables with a Gaussian probability distribution $\propto \exp\left[-\frac{1}{2}(e^{\epsilon_n} - 1)\,|a_n^i|^2\right]$. This equivalence underlies the success of the classical random walk model of a generic excited string state. The fact that the random walk must be made of M/M_s independent steps is linked to the fact that the Planckian distribution of mean occupation numbers, $\overline{N}_n = (\exp(\beta\, n + \gamma/n) - 1)^{-1}$ is sharply cut off when $n \gtrsim \beta^{-1}$, i.e., from Eq. (2.28), when $n \gtrsim M/M_s$. More precisely, using the same "statistical" approximation as above, one finds that the slope correlator $\langle : \partial_\sigma \widetilde{X}^i(\tau, \sigma_1)\, \partial_\sigma \widetilde{X}^j(\tau, \sigma_2) : \rangle_{\beta,\gamma}$ decays quite fast when $|\sigma_2 - \sigma_1| \gtrsim M_s/M$.

Finally, let us mention that, by using the same tools as above, one can compute the imaginary part of the mass shift $\delta M = \delta M_{\rm real} - i\,\Gamma/2$, i.e. the total decay rate Γ in massless quanta. The quantity Γ is, in fact, easier to define rigorously in string theory because, using $(k^2 - i\,\varepsilon)^{-1} = PP(k^2)^{-1} + i\pi\,\delta(k^2)$ in (3.8), it is given by an integral where the massless quanta are all on-shell. When $\gamma = 0$ (a consistent approximation for a result dominated by $n \sim \beta^{-1}$) one can use the covariant formalism with $D = d + 1$ oscillators to find,after replacing a discrete sum over n by an integral over ω,

$$\Gamma = c'_d \frac{G_N}{M\,\ell_s^2} \int d\omega\, \omega^{d-2} \left(\frac{n}{e^{\beta n} - 1} \right)^2 , \tag{3.28}$$

where c'_d is a numerical constant, and where $n = M\,\ell_s^2\,\omega/2$. The spectral decomposition of the total power radiated by the β-ensemble of strings is then simply deduced from (3.28) by adding a factor $\hbar\,\omega$ in the integrand:

$$P = c'_d \frac{G_N}{M\,\ell_s^2} \int d\omega\, \omega^{d-1} \left(\frac{n}{e^{\beta n} - 1} \right)^2 . \tag{3.29}$$

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[12]The sign $\delta M < 0$ was classically clear (even when taking into account relativistic effects), say in $d = 3$, from the starting formulas (3.3), (3.4) where $G_{\rm sym}(x) = (4\pi)^{-1}\,\delta(x^2) > 0$ and $4(\partial_+ X_1 \cdot \partial_+ X_2)\,(\partial_- X_1 \cdot \partial_- X_2) = (\partial_+ \Delta X)^2\,(\partial_- \Delta X)^2 > 0$ because $\partial_\pm \Delta X^\mu$ is purely spacelike in the center-of-mass frame. The same conclusion would hold in the light-cone gauge.

The results (3.28), (3.29) agree with corresponding results (for closed strings) in the second reference [8] and in [22] (note, however, that the factor M^2 in the equation (3.2) of [22] should be M and that the constant contains G_N and powers of ℓ_s). The integrals (3.28), (3.29) are dominated by $n \sim \beta^{-1}$, i.e. $\omega \sim M_s$. This gives for the integrated quantities:

$$\Gamma \sim g^2 \, M \ , \ P \sim g^2 \, M \, M_s \, . \tag{3.30}$$

The second equation (3.30) means that the mass of a highly excited string decays exponentially, with half-evaporation time

$$\tau_{\rm evap}^{\rm string} \equiv M/P \sim \frac{\ell_s}{g^2} \, . \tag{3.31}$$

Let us anticipate on the next section and note that, at the transition $\lambda \equiv g^2 \, M/M_s \sim 1$ between string states and black hole states, not only the mass and the entropy are (in order of magnitude at least) continuous, but also the various radiative quantities: total luminosity P, half-evaporation time $\tau_{\rm evap}$, and peak of emission spectrum. Indeed, for a black hole decaying under Hawking radiation the temperature is $T_{\rm BH} \sim R_{\rm BH}^{-1}$ and

$$P_{\rm BH} \sim R_{\rm BH}^{-2} \sim \ell_s^{-2} \, \lambda^{-2/(d-2)} \ , \ \tau_{\rm evap}^{\rm BH} \sim R_{\rm BH} \, S_{\rm BH} \sim \ell_s \, g^{-2} \, \lambda^{d/(d-2)} \, . \tag{3.32}$$

IV. ENTROPY OF SELF-GRAVITATING STRINGS

In the present section we shall combine the main results of the previous sections, Eqs. (2.32) and (3.26), and heuristically extend them at the limit of their domain of validity. We consider a narrow band of string states that we follow when increasing adiabatically the string coupling g, starting from $g = 0$[13]. Let M_0, R_0 denote the "bare" values (i.e. for $g \to 0$) of the mass and size of this band of states. Under the adiabatic variation of g, the mass and size, M, R, of this band of states will vary. However, the entropy $S(M, R)$ remains constant under this adiabatic process: $S(M, R) = S(M_0, R_0)$. We assume, as usual, that the variation of g is sufficiently slow to be reversible, but sufficiently fast to be able to neglect the decay of the states. We consider states with sizes $\ell_s \ll R_0 \ll M_0$ for which the correction factor,

$$c \, (R_0) \simeq (1 - c_1 \, R_0^{-2}) \, (1 - c_2 \, R_0^2/M_0^2) \, , \tag{4.1}$$

in the entropy

$$S(M_0, R_0) = c \, (R_0) \, a_0 \, M_0 \, , \tag{4.2}$$

[13] Alternatively, we can consider the coupling as an adjustable parameter (it is so in perturbation theory) and just follow how different physical quantities change as g is varied, whithout pretending that the change takes place in physical time.

is near unity. [We use Eq. (2.32) in the limit $g \to 0$, for which it was derived.] Because of this reduced sensitivity of $c\,(R_0)$ on a possible direct effect of g on R (i.e. $R(g) = R_0 + \delta_g\, R$), the main effect of self-gravity on the entropy (considered as a function of the actual values M, R when $g \neq 0$) will come from replacing M_0 as a function of M and R. The mass-shift result (3.26) gives $\delta\, M = M - M_0$ to first order in g^2. To the same accuracy[14], (3.26) gives M_0 as a function of M and R:

$$M_0 \simeq M + c_3\, g^2\, \frac{M^2}{R^{d-2}} = M \left(1 + c_3\, \frac{g^2\, M}{R^{d-2}}\right) , \tag{4.3}$$

where c_3 is a positive numerical constant.

Finally, combining Eqs. (4.1)–(4.3) (and neglecting, as just said, a small effect linked to $\delta_g\, R \neq 0$) leads to the following relation between the entropy, the mass and the size (all considered for self-gravitating states, with $g \neq 0$)

$$S(M,R) \simeq a_0\, M \left(1 - \frac{1}{R^2}\right) \left(1 - \frac{R^2}{M^2}\right) \left(1 + \frac{g^2\, M}{R^{d-2}}\right) . \tag{4.4}$$

For notational simplicity, we henceforth set to unity the coefficients c_1, c_2 and c_3. There is no loss of generality in doing so because we can redefine ℓ_s, R and g to that effect, and use the corresponding (new) string units. The main point of the present paper is to emphasize that, for a given value of the total energy M (and for some fixed value of g), the entropy $S(M,R)$ has a non trivial dependence on the radius R of the considered string state. Eq. (4.3) exhibits two effects varying in opposite directions: (i) self-gravity favors small values of R (because they correspond to larger values of M_0, i.e. of the "bare" entropy), and (ii) the constraint of being of some fixed size R disfavors both small ($R \ll \sqrt{M}$) and large ($R \gg \sqrt{M}$) values of R. For given values of M and g, the most numerous (and therefore most probable) string states will have a size$R_*(M;g)$ which maximizes the entropy $S(M,R)$. Said differently, the total degeneracy of the complete ensemble of self-gravitating string states with total energy M (and no *a priori* size restriction) will be given by an integral (where $\Delta\, R$ is the rms fluctuation of R given by Eq. (2.17))

$$\mathcal{D}(M) \sim \int \frac{dR}{\Delta\, R}\, e^{S(M,R)} \sim e^{S(M,R_*)} \tag{4.5}$$

which will be dominated by the saddle point R_* which maximizes the exponent.

The value of the most probable size R_* is a function of M, g and the space dimension d. To better see the dependence on d, let us first consider the case (which we generically assume) where the correction factors in Eq. (4.4) (parentheses on the right-hand-side) are very close to unity so that

$$S(M,R) \simeq a_0\, M\, (1 - V(R)) , \tag{4.6}$$

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[14]Actually, Eq. (4.3) is probably a more accurate version of the mass-shift formula because it exhibits the real mass M (rather than the bare mass M_0) as the source of self-gravity.

where

$$V(R) = \frac{1}{R^2} + \frac{R^2}{M^2} - \frac{g^2\, M}{R^{d-2}}\,. \tag{4.7}$$

One can think of $V(R)$ as an effective potential for R. The most probable size R_* must minimize $V(R)$. This effective potential can be thought of as the superposition of: (i) a centrifugal barrier near $R = 0$ (coming from the result (2.30)), (ii) an harmonic potential (forbidding the large values of R), and (iii) an attractive (gravitational) potential. When g^2 is small the minimum of $V(R)$ will come from the competition between the centrifugal barrier and the harmonic potential and will be located around the value $R_*^{-2} \simeq R_*^2/M^2$, i.e. $R_* \simeq \sqrt{M} = R_{\rm rw}$. This random walk value will remain (modulo small corrections) a local minimum of $V(R)$ (i.e. a local maximum of $S(M,R)$) as long as $g^2\, M/R_*^{d-2} \ll R_*^{-2}$, i.e. for $g^2 \ll g_0^2$ with

$$g_0^2 \equiv M^{\frac{d-6}{2}}\,. \tag{4.8}$$

More precisely, working perturbatively in g^2, the minimization of $V(R)$ yields

$$R_* \simeq \sqrt{M}\left(1 - \frac{d-2}{8}\,\frac{g^2}{g_0^2}\right)\,. \tag{4.9}$$

Note that, when $g^2 \ll g_0^2$, the value of $V(R)$ at this local minimum is of order $V_{\rm min} \simeq +\,2R_*^{-2} \simeq +\,2M^{-1}$, i.e. that it corresponds to a saddle-point entropy $S(M,R_*) \simeq a_0\, M(1 - V_{\rm min}) \simeq a_0\, M - \mathcal{O}(1)$ which differs essentially negligibly from the "bare" entropy $a_0\, M$ ($\gg 1$). To study what happens when g^2 further increases let us consider separately the various dimensions $d \geq 3$. We shall see that the special value g_0^2, Eq. (4.8), is significant (as marking a pre-transition, before the transition to the black hole state) only for $d = 3$. For $d \geq 4$, the only special value of g^2 is the critical value

$$g_c^2 \sim M^{-1}\,, \tag{4.10}$$

around which takes place a transition toward a state more compact than the usual random walk one.

A. $d = 3$

Let us first consider the (physical) case $d = 3$, for which $g_0^2 \sim M^{-3/2} \ll g_c^2 \sim M^{-1}$. In that case, when g^2 becomes larger than g_0^2, the (unique) local minimum of $V(R)$ slowly shifts towards values of R lower than $R_{\rm rw}$ and determined by the competition between the centrifugal barrier $1/R^2$ and the gravitational potential $-g^2\, M/R$.

In the approximation where we use the linearized form (4.6), (4.7), and where (for $g^2 \gg g_0^2$) we neglect the term R^2/M^2, the most probable size R_* is

$$R_*^{(d=3)} \simeq \frac{2}{g^2\, M}\ ,\ \text{when}\ M^{-3/2} \ll g^2 \ll M^{-1}\,. \tag{4.11}$$

Note that as g^2 increases between $M^{-3/2}$ and M^{-1}, the most probable size $R_*^{(d=3)}$ smoothly interpolates between $R_{\rm rw}$ and a value of order unity, i.e. of order the string length. Note also that $V_{\rm min} \simeq -g^2\,M/(2R_*) \simeq -g^4\,M^2/4$ remains smaller than one when $g^2 \lesssim M^{-1}$ so that the saddle-point entropy $S(M,R_*) \simeq a_0\,M(1-V_{\rm min})$ never differs much from the "bare" value $a_0\,M$.

When g^2, in its increase, becomes comparable to M^{-1}, the radius becomes of order one and it is important to take into account the (supposedly) more exact expression (4.4) (in which the factor $(1-R^{-2})$ plays the crucial role of cutting off any size $R \leq 1$). If we neglect, as above, the term R^2/M^2 (which is indeed even more negligible in the region $R \sim 1$) but maximize the factored expression (4.4), we find that the most probable size R_* reads

$$R_*^{(d=3)} \simeq \frac{1+\sqrt{1+3\,\lambda^2}}{\lambda}\ ,\ \text{when}\ g^2 \gg M^{-3/2}\,, \tag{4.12}$$

where we recall the definition

$$\lambda \equiv g^2\,M\,. \tag{4.13}$$

When $\lambda \ll 1$, the result (4.12) reproduces the simple linearized estimate (4.11). When $\lambda \gtrsim 1$, Eq. (4.12) says that the most probable size, when g^2 increases, tends to a limiting size ($R_\infty = \sqrt{3}$) slightly larger than the minimal one ($R_{\rm min} = 1$) corresponding to zero entropy. [Note that even for the formal asymptotic value $R_\infty = \sqrt{3}$, the reduction in entropy due to the factor $1-R^{-2}$ is only 2/3.] On the other hand, the fractional self-gravity $G_N\,M/R_*$ (which measures the gravitational deformation away from flat space), or the corresponding term in Eq. (4.4), continues to increase with g^2 as

$$\frac{\lambda}{R_*} = \frac{\lambda^2}{1+\sqrt{1+3\,\lambda^2}}\,. \tag{4.14}$$

The right-hand side of Eq. (4.14) becomes unity for $\lambda = \sqrt{5} = 2.236$. The picture suggested by these results is that the string smoothly contracts, as g increases, from its initial random walk size down to a limiting compact state of size slightly larger than ℓ_s. For some value of λ of order unity (may be between 1 and 2; indeed, even for $\lambda = 1$ the size $R_* = 2$ and the self-gravity $\lambda/R_* = 0.5$ suggest one may still trust a compact string description) the self-gravity of this compact string state will become so strong that one expects it to collapse to a black hole state. We recall that, as emphasized in Refs. [4], [6], [7], [8], [9], [13], when $\lambda \sim 1$, the mass of the string state matches (in order of magnitude) that of a (Schwarzschild) black hole with Bekenstein-Hawking entropy equal to the string entropy S.

B. $d = 4$

When $d = 4$, the argument above Eq. (4.9) suggests that the random-walk size remains the most probable size up to $g^2 \lesssim g_0^2 \sim M^{-1}$, i.e. up to $\lambda \lesssim 1$. A more accurate approximation to the most probable size R_*, when $\lambda < 1$, is obtained by minimizing exactly $V(R)$, Eq. (4.7). This yields

$$R_*^{(d=4)} \simeq M^{1/2}\,(1-\lambda)^{1/4}\ ,\ \text{when}\ \lambda<1\,. \tag{4.15}$$

This shows that the size will decrease, but one cannot trust this estimate when $\lambda \to 1^-$. To study more precisely what happens when $\lambda \sim 1$ we must take into account the more exact factorized form (4.4). Let us now neglect the R^2/M^2 term and consider the approximation

$$S^{(d=4)}(M,R) \simeq a_0\, M \left(1-\frac{1}{R^2}\right)\left(1+\frac{\lambda}{R^2}\right)\,. \tag{4.16}$$

The right-hand side of Eq. (4.15) has a maximum only for $\lambda > 1$, in which case

$$R_*^{(d=4)} \simeq \left(\frac{2\lambda}{\lambda-1}\right)^{1/2} \quad \text{when}\ \lambda>1\,. \tag{4.17}$$

If we had taken into account the full expression (4.4) the two results (4.15), (4.17), valid on each side of $\lambda = 1$, would have blended in a result showing that around[15] $\lambda = 1$ the most probable size *continuously* interpolates between $R_{\rm rw}$ and a size of order ℓ_s. Note that, according to Eq. (4.17), as λ becomes $\gg 1$, $R_*^{(d=4)}$ tends to a limiting size ($R_\infty = \sqrt{2}$) slightly larger than $R_{\rm min} = 1$ (corresponding to zero entropy). When $\lambda > 1$ the fractional self-gravity of the compact string states reads

$$\frac{\lambda}{R_*^2} = \frac{\lambda-1}{2}\,. \tag{4.18}$$

As in the case $d = 3$, one expects that for some value of λ strictly larger than 1, the self-gravity of the compact string state will become so strong that it will collapse to a black hole state. Again the mass, size and entropy match (in order of magnitude) those of a black hole when $\lambda \sim 1$. The only difference between $d = 4$ and $d = 3$ is that the transition to the compact state, though still continuous, is sharply concentrated around $\lambda = 1$ instead of taking place over the extended range $M^{-1/2} \lesssim \lambda \lesssim 1$.

C. $d \geq 5$

When $d \geq 5$, the argument around Eq. (4.8) shows that the random walk size $R_{\rm rw} \simeq \sqrt{M}$ is a consistent local maximum of the entropy in the whole domain $g^2 \ll g_0^2$, i.e. for $\lambda \equiv g^2\, M \ll M^{\frac{d-4}{2}}$, which allows values $\lambda \gg 1$. However, a second, disconnected maximum of the entropy, as function of$R$, could exist. To investigate this we consider again (4.4), when neglecting the R^2/M^2 term (because we are interested in other possible solutions with small sizes):

$$\frac{S(M,R)}{a_0\, M} \simeq (1-x)\,(1+\lambda\, x^\nu) \equiv s(x)\,, \tag{4.19}$$

[15]The transition takes place in the range $|\lambda - 1| \sim M^{-2/3}$ corresponding to $R_* \sim M^{1/3}$.

where we have defined $x \equiv R^{-2}$ and $\nu \equiv (d-2)/2$. By studying analytically the maxima and inflection points of $s(x)$, one finds that, in the present case where $\nu = (d-2)/2 > 1$, there are two critical values $\lambda_1 < \lambda_2$ of the parameter $\lambda \equiv g^2 M$. The first one,

$$\lambda_1 = \left(\frac{\nu+1}{\nu-1}\right)^{\nu-1} \ , \ R_1 = x_1^{-1/2} = \left(\frac{\nu+1}{\nu-1}\right)^{1/2} > 1 \ , \ s_1 = 1 - \left(\frac{\nu-1}{\nu+1}\right)^2 \ , \tag{4.20}$$

corresponds to the birth (through an inflection point) of a maximum and a minimum of the function $s(R)$. Because $s_1 < 1$ is strictly lower than the usual random walk maximum with $s(R_{\rm rw}) \simeq 1 - 2/M \simeq 1$, the local maximum near $R \sim 1$ of the entropy, which starts to exist when $\lambda > \lambda_1$, is, at first, only metastable with respect to $R_{\rm rw}$. However, there is a second critical value of λ, $\lambda_2 > \lambda_1$, defined by

$$\lambda_2 = \nu \left(\frac{\nu}{\nu-1}\right)^{\nu-1} \ , \ R_2 = x_2^{-1/2} = \left(\frac{\nu}{\nu-1}\right)^{\frac{1}{2}} > 1 \ , \ s_2 = 1 \, . \tag{4.21}$$

When $\lambda > \lambda_2$ the local maximum near $R \sim 1$ of the entropy has $s(R) > 1$, i.e. it has become the global maximum of the entropy, making the usual random walk local maximum only metastable. Therefore,when $\lambda > \lambda_2$ the most probable string state is a very compact state of size comparable to ℓ_s. Formally, this new global maximum exists for any $\lambda \gtrsim 1$ and tends, when $\lambda \to \infty$, toward the limiting location $R_\infty = ((\nu+1)/\nu)^{1/2} > 1$, i.e. slightly (but finitely) above the minimum size $R = 1$. However, as in the cases $d \leq 4$, the self-gravity of the stable compact string state will become strong when $\lambda \gtrsim 1$, so that it is expected to collapse (for some $\lambda_c > \lambda_2$) to a black hole state. As in the cases $d \leq 4$, the mass, size and entropy of this compact string state match those of a black hole. The big difference with the cases $d \leq 4$ is that the transition between the (stable) random walk typical configuration and the (stable) compact one is discontinuous. Our present model suggests that (when $\nu \equiv (d-2)/2 > 1$) a highly excited single string system can exist, when $\lambda > \lambda_2$, in two different stable typical states: (i) a dilute state of typical size $R_{\rm rw} \simeq \sqrt{M}$ and typical mean density $\rho \sim M/R_{\rm rw}^d \sim M^{-\nu} \ll 1$, and (ii) a condensed state of typical size $R \sim 1$ and typical mean density (using $\lambda \sim 1$): $\rho \sim M \sim g^{-2} \gg 1$. We shall comment further below on the value $\rho \sim g^{-2}$ of the dense state of string matter.

V. DISCUSSION

Technically, the main new result of the present work is the (dimension independent) estimate[16] $c(R) = 1 - c_1/R^2$, with $c_1 \simeq (3/4)\, \ell_s^2 = 3\alpha'/2$, of the factor giving the decrease in the entropy ($2\pi((d-1)\,\alpha'/6)^{1/2}\, M$) of a narrow band of very massive (open[17]) string states

[16] In spite of our efforts in Section II, this result remains non rigorous and open to $\mathcal{O}(1/d)$ fractional corrections because of the difficulty to define a good quantum operator representing the mean radius of a string state.

[17] For technical simplicity, we have restricted our attention to open bosonic strings. We could have dealt with closed bosonic strings by doubling the oscillators, but the level matching condition,

$M \gg (\alpha')^{-1/2}$, when considering only string states of size R (modulo some fractionally small grand-canonical-type fluctuations). We have also justified (by dealing explicitly with relativistic and quantum effects in a semi-classical approximation) and refined (by computing the numerical coefficient, Eq. (3.27)) the naive estimate, $\delta M = -c_d\, G_N\, M^2/R^{d-2}$, of the mass shift of a massive string state due to the exchange of long range fields (graviton, dilaton and axion). [The exchange of these fields is expected to be the most important one both because very excited string states tend to be large, and because the corresponding interactions are attractive and cumulative with the mass.]

Conceptually, the main new result of this paper concerns the most probable state of a very massive single[18] self-gravitating string. By combining our estimates of the entropy reduction due to the size constraint, and of the mass shift we come up with the expression (4.4) for the logarithm of the number of self-gravitating string states of size R. Our analysis of the function $S(M,R)$ clarifies the correspondence [6], [8], [9], [13] between string states and black holes. In particular, our results confirm many of the results of [13], but make them (in our opinion) physically clearer by dealing directly with the size distribution, in real space, of an ensemble of string states. When our results differ from those of [13], they do so in a way which simplifies the physical picture and make even more compelling the existence of a correspondence between strings and black holes. For instance, [13] suggested that in $d = 5$ there was a phenomenon of hysteresis, with a critical value $g_0^2 \sim M^{-1/2}$ for the string $\to$ black hole transition, and a different critical value $g_c^2 \sim M^{-1} \ll g_0^2$ for the inverse transition: black hole $\to$ string. Also, [13] suggested that in $d > 6$, most excited string states would never form black holes. The simple physical picture suggested[19] by our results is the following: In any dimension, if we start with a massive string state and increase the string coupling g, a *typical* string state will, eventually, become more compact and will end up, when $\lambda_c = g_c^2\, M \sim 1$, in a "condensed state" of size $R \sim 1$, and mass density $\rho \sim g_c^{-2}$. Note that the basic reason why small strings, $R \sim 1$, dominate in any dimension the entropy when $\lambda \sim 1$ is that they descend from string states with bare mass $M_0 \simeq M(1 + \lambda/R^{d-2}) \sim 2M$ which are exponentially more numerous than less condensed string states corresponding to smaller bare masses.

The nature of the transition between the initial "dilute" state and the final "condensed"

$N_L = N_R$, would have complicated the definition of the grand canonical ensemble we used. We expect that our results (whichare semi-classical) apply (with some numerical changes) to open or closed superstrings.

[18] We consider states of a single string because, for large values of the mass, the single-string entropy approximates the total entropy up to subleading terms.

[19] Our conclusions are not rigourously established because they rely on assuming the validity of the result (4.4) beyond the domain ($R^{-2} \ll 1$, $g^2\, M/R^{d-2} \ll 1$) where it was derived. However, we find heuristically convincing to believe in the presence of a reduction factor of the type $1 - R^{-2}$ down to sizes very near the string scale. Our heuristic dealing with self-gravity is less compelling because we do not have a clear signal of when strong gravitational field effects become essential.

one depends on the value of the space dimension d. [As explained below Eq. (4.4), we henceforth set to unity, by suitable redefinitions of ℓ_s, R and g, the coefficients c_1, c_2 and c_3.] In $d = 3$, the transition is gradual: when $\lambda < M^{-1/2}$ the size of a typical state is $R_*^{(d=3)} \simeq M^{1/2}(1-M^{1/2}\,\lambda/8)$, when $\lambda > M^{1/2}$ the typical size is $R_*^{(d=3)} \simeq (1+(1+3\,\lambda^2)^{1/2})/\lambda$. In $d = 4$, the transition toward a condensed state is still continuous, but most of the size evolution takes place very near $\lambda = 1$: when $\lambda < 1$, $R_*^{(d=4)} \simeq M^{1/2}(1-\lambda)^{1/4}$, and when $\lambda > 1$,$R_*^{(d=4)} \simeq (2\lambda/(\lambda-1))^{1/2}$, with some smooth blending between the two evolutions around $|\lambda - 1| \sim M^{-2/3}$. In $d \geq 5$, the transition is discontinuous (like a first order phase transition between, say, gas and liquid states). Barring the consideration of metastable (supercooled) states, on expects that when $\lambda = \lambda_2 \simeq \nu^\nu/(\nu-1)^{\nu-1}$ (with $\nu = (d-2)/2$), the most probable size of a string state will jump from $R_{\rm rw}$ (when $\lambda < \lambda_2$) to a size of order unity (when $\lambda > \lambda_2$).

Let us, for definiteness, write down in more detail what happens in $d = 3$. After maximization over R, the entropy of a self-gravitating string is given, when $M^{-3/2} \ll g^2 \ll M^{-1}$, by

$$S(M) = S(M, R_*(M)) \simeq a_0\, M \left[1 + \frac{1}{4}\,(g^2\,M)^2\right] . \tag{5.1}$$

By differentiating S with respect to M, one finds the temperature of the ensemble of highly excited single string states of mass M:

$$T \simeq T_{\rm Hag} \left(1 - \frac{3}{4}\,(g^2\,M)^2\right) , \tag{5.2}$$

with $T_{\rm Hag} \equiv a_0^{-1}$. Eq. (5.2) explicitly exhibits the modification of the Hagedorn temperature due to self-gravity (in agreement with results of [13] obtained by a completely different approach). Note that, both in Eqs. (5.1) and (5.2), the self-gravity modifications are fractionally of order unity at the transition $g^2\,M \sim 1$.

One can think of the "condensed" state of (single) string matter, reached (in any d) when $\lambda \sim 1$, as an analog of a neutron star with respect to an ordinary star (or a white dwarf). It is very compact (because of self gravity) but it is stable (in some range for g) under gravitational collapse. However, if one further increases g or M (in fact, $\lambda = g^2\,M$), the condensed string state is expected (when λ reaches some $\lambda_3 > \lambda_2$, $\lambda_3 = \mathcal{O}(1)$) to collapse down to a black hole state (analogously to a neutron star collapsing to a black hole when its mass exceeds the Landau-Oppenheimer-Volkoff critical mass). Still in analogy with neutron stars, one notes that general relativistic strong gravitational field effects are crucial for determining the onset of gravitational collapse; indeed, under the "Newtonian" approximation (4.4), the condensed string state could continue to exist for arbitrary large values of λ.

It is interesting to note that the value of the mass density at the formation of the condensed string state is $\rho \sim g^{-2}$. This is reminiscent of the prediction by Atick and Witten [23] of a first-order phase transition of a self-gravitating thermal gas of strings, near the Hagedorn temperature[20], towards a dense state with energy density $\rho \sim g^{-2}$ (typical of a

[20]Note that, by definition, in our *single* string system, the formal temperature $T = (\partial S/\partial M)^{-1}$

genus-zero contribution to the free energy). Ref. [23] suggested that this transition is first-order because of the coupling to the dilaton. This suggestion agrees with our finding of a discontinuous transition to the single string condensed state in dimensions ≥ 5 (Ref. [23] work in higher dimensions, $d = 25$ for the bosonic case). It would be interesting to deepen these links between self-gravitating single string states and multi-string states.

Assuming the existence (confirmed by the present work) of a dense state of self-gravitating string matter with energy density $\rho \sim g^{-2}$, it would be fascinating to be able to explore in detail (with appropriate, strong gravity tools) its gravitational dynamics, both in the present context of a single, isolated object ("collapse problem"), and in the cosmological context (problem of the origin of the expansion of the universe).

Let us come back to the consequences of the picture brought by the present work for the problem of the end point of the evaporation of a Schwarzschild black hole and the interpretation of black hole entropy. In that case one fixes the value of g (assumed to be $\ll 1$) and considers a black hole which slowly looses its mass via Hawking radiation. When the mass gets as low as a value[21] $M \sim g^{-2}$, for which the radius of the black hole is of order one (in string units), one expects the black hole to transform (in all dimensions) into a typical string[22] state corresponding to $\lambda = g^2 M \sim 1$, which is a dense state (still of radius $R \sim 1$). This string state will further decay and loose mass, predominantly via the emission of massless quanta, with a quasi thermal spectrum with temperature $T \sim T_{\rm Hagedorn} = a_0^{-1}$ (see Eq. (3.29) and Refs. [8], [22]) which smoothly matches the previous black hole Hawking temperature. This mass loss will further decrease the product $\lambda = g^2 M$, and this decrease will, either gradually or suddenly, cause the initially compact string state to inflate to much larger sizes. For instance, if $d \geq 4$, the string state will quickly inflate to a size $R \sim \sqrt{M}$. Later, with continued mass loss, the string size will slowly shrink again toward $R \sim 1$ until a remaining string of mass $M \sim 1$ finally decays into stable massless quanta. In this picture, the black hole entropy acquires a somewhat clear statistical significance (as the degeneracy of a corresponding typical string state) *only* when M and g are related by $g^2 M \sim 1$. If we allow ourselves to vary (in a Gedanken experiment) the value of g this gives a potential statistical significance to any black hole entropy value $S_{\rm BH}$ (by choosing $g^2 \sim S_{\rm BH}^{-1}$). We do not claim, however, to have a clear idea of the direct statistical meaning of $S_{\rm BH}$ when $g^2 S_{\rm BH} \gg 1$. Neither do we clearly understand the fate of the very large space (which could be excited in many ways) which resides inside very large classical black holes of radius $R_{\rm BH} \sim (g^2 S_{\rm BH})^{1/(d-1)} \gg 1$. The fact that the interior of a black hole of given mass could be

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is always near the Hagedorn temperature.

[21] Note that the mass at the black hole $\to$ string transition is larger than the Planck mass $M_P \sim (G_N)^{-1/2} \sim g^{-1}$ by a factor $g^{-1} \gg 1$.

[22] A check on the single-string dominance of the transition black hole $\to$ string is to note that the single string entropy $\sim M/M_s$ is much larger than the entropy of a ball of radiation $S_{\rm rad} \sim (RM)^{d/(d+1)}$ with size $R \sim R_{\rm BH} \sim \ell_s$ at the transition.

arbitrarily large[23], and therefore arbitrarily complex, suggests that black hole physics is not exhausted by the idea (confirmed in the present paper) of a reversible transition between string-length-size black holes and string states.

On the string side, we also do not clearly understand how one could follow in detail (in the present non BPS framework) the "transformation" of a strongly self-gravitating string state into a black hole state.

Finally, let us note that we expect that self-gravity will lift nearly completely the degeneracy of string states. [The degeneracy linked to the rotational symmetry, i.e. $2J+1$ in $d=3$, is probably the only one to remain, and it is negligible compared to the string entropy.] Therefore we expect that the separation δE between subsequent (string and black hole) energy levels will be exponentially small: $\delta E \sim \Delta M \exp(-S(M))$, where ΔM is the canonical-ensemble fluctuation in M. Such a δE is negligibly small compared to the radiative width $\Gamma \sim g^2 M$ of the levels. This seems to mean that the discreteness of the quantum levels of strongly self-gravitating strings and black holes is very much blurred, and difficult to see observationally.

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[23]E.g., in the Oppenheimer-Snyder model, one can join an arbitrarily large closed Friedmann dust universe, with hyperspherical opening angle $0 \leq \chi_0 \leq \pi$ arbitrarily near π, onto an exterior Schwarzschild spacetime of given mass M.

REFERENCES

[1] S. Fubini and G. Veneziano, Nuovo Cim. **64 A** 811 (1969);
K.Huang and S. Weinberg, Phys. Rev. Lett **25** 895 (1970).

[2] J. D. Bekenstein, Phys. Rev. D **7** 2333 (1973).

[3] S. W. Hawking, Comm. Math. Phys. **43** 199 (1975).

[4] M. Bowick, L. Smolin and L.C.R. Wijewardhana, Gen. Rel. Grav. **19** 113 (1987).

[5] G. Veneziano, *Europhys. Lett.* **2** 199 (1986).

[6] L. Susskind, hep-th/9309145 (unpublished).

[7] G. Veneziano, in *Hot Hadronic Matter: Theory and Experiments*, Divonne, June 1994, eds. J. Letessier, H. Gutbrod and J. Rafelsky, NATO-ASI Series B: Physics, **346** (Plenum Press, New York 1995), p. 63.

[8] E. Halyo, A. Rajaraman and L. Susskind, Phys. Lett. B **392**, 319 (1997);
E.Halyo, B. Kol, A. Rajaraman and L. Susskind, Phys. Lett. B **401**,15 (1997).

[9] G.T. Horowitz and J. Polchinski, Phys. Rev. D **55**, 6189 (1997).

[10] A. Strominger and C. Vafa, Phys. Lett. B **379** 99 (1996).

[11] E.g. A. Sen, Mod. Phys. Lett. A **10** 2081 (1995);
C.G. Callan and J.M. Maldacena, Nucl. Phys. B **472**, 591 (1996);
J.C. Breckenridge et al. Phys. Lett. B **381** 423 (1996).

[12] P. Salomonson and B.S. Skagerstam, Nucl. Phys. B **268**,349 (1986);
Physica A **158**, 499 (1989);
D. Mitchell and N. Turok, Phys. Rev. Lett. **58**, 1577 (1987);
Nucl.Phys. B **294**, 1138 (1987).

[13] G.T. Horowitz and J. Polchinski, Phys. Rev. D **57**,2557 (1998).

[14] B. Sathiapalan, Phys. Rev. D **35**, 3277 (1987);
I.A. Kogan, JETP Lett. **45**, 709 (1987);
J.J. Atick and E. Witten, Nucl. Phys. B **310**, 291 (1988).

[15] J. Scherk, Rev. Mod. Phys. **47**, 123 (1975).

[16] M. Karliner, I. Klebanov and L. Susskind, Int. J. Mod.Phys. **A3**, 1981 (1988).

[17] L.D. Landau and E.M. Lifshitz, *Statistical Physics*, part 1, third edition (Pergamon Press, Oxford, 1980).

[18] H. Yamamoto, Prog. Theor. Phys. **79**, 189 (1988);
K. Amano and A. Tsuchiya, Phys. Rev. D **39**, 565 (1989);
B. Sundborg, Nucl. Phys. B **319**, 415 (1989); and **338**, 101 (1990).

[19] A. Buonanno and T. Damour, Phys. Lett. B **432**, 51(1998).

[20] V. Alessandrini, D. Amati, M. Le Bellac and D. Olive, Phys. Rep. **C1**, 269 (1971).

[21] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, Volume 1, (Cambridge University Press, Cambridge, 1987).

[22] D. Amati and J.G. Russo, Phys. Lett. B **454**, 207 (1999); hep-th/9901092.

[23] J.J. Atick and E. Witten, Nucl. Phys. B **310**, 291(1988).

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Seeds of large-scale anisotropy in string cosmology

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Pre-big bang cosmology predicts tiny first-order dilaton and metric perturbations at very large scales. Here we discuss the possibility that other – more copiously generated – perturbations may act, at second order, as scalar seeds of large-scale structure and CMB anisotropies. We study, in particular, the cases of electromagnetic and axionic seeds. We compute the stochastic fluctuations of their energy-momentum tensor and determine the resulting contributions to the multipole expansion of the temperature anisotropy. In the axion case it is possible to obtain a flat or slightly tilted blue spectrum that fits present data consistently, both for massless and for massive (but very light) axions.

I. INTRODUCTION

String theory has recently motivated the study of a cosmological scenario in which the universe, starting from the string perturbative vacuum, evolves through an early inflationary "pre-big bang" phase [1], until a transition to the radiation-dominated, decelerated evolution occurs.

In spite of some attractive aspects of the pre-big bang picture, such as the underlying duality symmetry [2], which naturally selects perturbative initial conditions and automatically leads to inflation [1,3], it is fair to say that such a cosmological scenario is far from being understood in all of its aspects. In particular, on the more theoretical side, one is lacking a complete and consistent description of the high-curvature, strong coupling regime, where the transition from the pre- to the post-big bang era is expected to take place [4]. Furthermore, opinions vary [3,5] as to whether or not the pre-big bang scenario needs a large amount of fine-tuning. On a more phenomenological side, the main outstanding problem is to reproduce the observed amplitude and slope of the large-scale temperature anisotropy [6] and of large-scale density perturbations. The difficulty is that, unlike in the more conventional (de-Sitter-like) inflationary picture, the amplification of scalar and tensor metric perturbations here leads to primordial spectra that grow with frequency [7], and whose energy density is normalized to an almost critical value at some short scale [8] (typically the GHz); in this way, too little power is left at scales that are relevant for anisotropies in the Cosmic Microwave Background (CMB) [6] or to the problem of large-scale structure (unless the high-curvature phase is long enough and characterized by an almost constant dilaton field [9]).

In this paper we address this problem and we discuss a possible solution, based on the contribution of "seeds" [10] to density fluctuations and to the anisotropy of the CMB radiation. The seeds are produced, in our context, by the amplification of quantum fluctuations of some other fields, which are present in string theory, but are not part of the homogeneous background whose perturbations we wish to study.

We shall consider two examples, in which the seed inhomogeneity spectrum is due, respectively, to vacuum fluctuations of the electromagnetic (EM) [11] and of the (Kalb-Ramond) axion (AX) [12] field. Both cases are typical of string cosmology, since no inhomogeneity is produced, in either case, in a conventional scenario based on Einstein's equations, without axion and dilaton. The spectra of EM and AX perturbations can be much flatter than those of scalar and tensor perturbations of the metric and of the dilaton field.

The idea of using the EM fluctuations as seeds was already discussed in a previous paper [13], using however the perfect fluid approximation for the EM stress tensor. Here we will compute the scalar components of the energy-momentum-tensor fluctuations due to the EM and AX seeds including an important anisotropic stress term, and will relate them to the primordial spectral energy distributions. When these seed inhomogeneities are inserted in the perturbed Einstein equations they generate scalar-metric fluctuations which are largely controlled, for seeds with

small enough anisotropic stresses, on super-horizon scales, by the so-called compensation mechanism [14]. Finally, scalar-metric perturbations can be converted in a standard manner into temperature fluctuations $\Delta T/T$ via the Sachs-Wolfe effect [15]. We will discuss whether the metric perturbation spectrum induced by seeds can be flat enough to match present observations, consistently with the COBE normalization of the amplitude on large scales, and with the high-frequency normalization of the primordial seed spectrum.

The paper aims at being rather self-contained and readable by non-specialists in string and/or cosmological perturbation theory, and is organized as follows. In Section II we set up the relevant equations needed to study super-horizon perturbations in the presence of seeds, and give their generic solution for seeds with "small" or "large" anisotropic stresses. We also discuss the way the perturbations enter the multipole expansion of $\Delta T/T$ via the Sachs-Wolfe effect. In Section III, after recalling known results about scalar, tensor, electromagnetic and axion perturbations in the pre-big bang scenario, we estimate the contribution of the two latter sources to the fluctuations of the energy-momentum tensor, including the case of massive axions. In Section IV we combine the results of the previous two sections and compute the contribution of EM and AX seeds to $\Delta T/T$. Using COBE data, we finally discuss, in the various cases, whether the seed mechanism alone is able to give a satisfactory explanation of large-scale temperature anisotropies. Section V contains our conclusions. Some technical details are relegated to the three appendices.

Notation: The Friedmann metric is given by $a^2(-d\eta^2 + \gamma_{ij}dx^i dx^j)$, where a denotes the scale factor and η is conformal time. Spatial indices, $1, 2, 3$ are denoted by latin letters while spacetime indices, $0, 1, 2, 3$ are denoted by greek letters. A dot denotes derivative with respect to η.

II. LARGE-SCALE PERTURBATIONS IN THE PRESENCE OF SEEDS

Before calculating CMB anisotropies for specific examples in the context of string cosmology, we derive a general formula for large-scale CMB anisotropies in models with seed perturbations.

A. Cosmological Perturbation Theory with Seeds

In this subsection we give a brief reminder of gauge-invariant perturbation theory with seeds. More details can be found in Refs. [10,16]. By seeds we mean an inhomogeneously distributed form of energy, which contributes only a small fraction to the total energy density of the universe and can thus be considered as a perturbation. Furthermore, we consider seeds that interact only gravitationally with the cosmic fluid.

We restrict our discussion to scalar perturbations, which are of primary interest here. The corresponding equations for vector and tensor perturbations can be found in [16]. The metric of a perturbed Friedmann universe is

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + a^2 h_{\mu\nu} \ , \tag{2.1}$$

where $g^{(0)}$ denotes the unperturbed metric:

$$g^{(0)}_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)(-d\eta^2 + \gamma_{ij}dx^i dx^j) \ . \tag{2.2}$$

Here a is the scale factor, η denotes conformal time and γ represents a metric of constant curvature $K = \pm 1, 0$. Since we will be interested in a Friedmann universe that has undergone substantial inflation, we neglect K in the sequel, setting $\gamma_{ij} = \delta_{ij}$.

For scalar perturbations, a Fourier component of $h_{\mu\nu}$ with wave vector $\mathbf{k}$ can by parametrized by 4 scalar functions A, B, H_L and H_T, defined by

$$\begin{aligned} h(k) = h_{\mu\nu}(k)dx^\mu dx^\nu = -2A(k)(d\eta)^2 - 2i\frac{k_j}{k}B(k)d\eta dx^j \\ +2\left[H_L(k) + \frac{1}{3}H_T(k)\right]\delta_{lj}dx^l dx^j - 2\frac{k_l k_j}{k^2}H_T(k)dx^l dx^j \ . \end{aligned} \tag{2.3}$$

These four functions are gauge-dependent, *i.e.* they depend on the choice of coordinates. In order to define gauge-independent metric variables, we first make use of two geometric quantities: the spatial part of the scalar curvature of the perturbed metric, δR, and the shear (traceless) part of the extrinsic curvature, $K^{(aniso)}$. An elementary calculation gives [16]:

$$\delta R = 4k^2 a^{-2}\mathcal{R} \ , \qquad\qquad \mathcal{R} = H_L + \frac{1}{3}H_T \ , \tag{2.4}$$

$$K_{ij}^{(aniso)} = ak\left(\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij}\right)\sigma \ , \qquad \sigma = \dot{H}_T/k - B \ . \tag{2.5}$$

Studying the gauge transformation properties of A, $\mathcal{R}$, and σ, one easily finds that the following variables, called the (Fourier components of the) Bardeen potentials, are gauge-invariant (see [17,18]):

$$\Phi = \mathcal{R} - (\dot{a}/a)k^{-1}\sigma \ , \tag{2.6}$$

$$\Psi = A - (\dot{a}/a)k^{-1}\sigma - k^{-1}\dot{\sigma} \ . \tag{2.7}$$

(Note that, throughout this paper, we shall always express the Bardeen potentials in momentum space, even without indicating their k dependence explicitly.)

Next, we discuss the perturbations of the energy-momentum tensor. Let us define the perturbed energy density $\rho^{(pert)}$ and 4-velocity field u as the time-like eigenvalue and eigenvector of the energy-momentum tensor:

$$T_\mu{}^\nu u^\mu = -\rho^{(pert)} u^\nu \ , \qquad u^2 = -1 \ . \tag{2.8}$$

The Fourier components of the perturbations in the density and velocity field are determined by

$$\rho^{(pert)} = \rho(1+\delta) \ , \tag{2.9}$$

$$u^0 = (1-A) \ , \qquad \frac{u^j}{u^0} = -i\frac{k^j}{k}v \ , \tag{2.10}$$

where ρ denotes the unperturbed background density. The temporal component u^0 is fixed by the normalization condition. We project the stress tensor onto the 3-space orthogonal to u:

$$\tau_{\mu\nu} = P_\mu^\rho P_\nu{}^\sigma T_{\rho\sigma}, \qquad P_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu, \tag{2.11}$$

and define the scalar perturbations of τ by:

$$\tau_i{}^j = p\left[\left(1 + \pi_L + \frac{1}{3}\pi_T\right)\delta_i{}^j - \frac{k_i k^j}{k^2}\pi_T\right] \ . \tag{2.12}$$

The variable π_L describes the pressure perturbation, π_T is the potential of the anisotropic stresses and p is the unperturbed background pressure. Studying the behaviour of the quantities δ, v, π_L and π_T under gauge transformations [19], one finds the gauge-invariant variables:

$$\begin{aligned} &\Pi = \pi_T, \quad \Gamma = \pi_L - (c_s^2/w)\delta, \quad V = v - k^{-1}\dot{H}_T, \\ &D = \delta + 3(1+w)(\dot{a}/a)k^{-1}(V+\sigma), \quad D_g = \delta + 3(1+w)\mathcal{R} \ . \end{aligned} \tag{2.13}$$

Here Π is the anisotropic stress potential, Γ is the entropy perturbation, V is the peculiar velocity potential, D and D_g are different choices for a gauge-invariant density perturbation variable (for a physical interpretation of these variables, see [18,19]). Finally, $w = p/\rho$ denotes the enthalpy and $c_s^2 = \dot{p}/\dot{\rho}$ stands for the adiabatic speed of sound. In this paper we shall limit ourselves to adiabatic perturbations ($\Gamma = 0$).

The perturbation of Einstein's equations and of energy-momentum conservation can be expressed in terms of these gauge-invariant variables (a derivation can be found in [18,19]). We obtain two constraint equations:

$$4\pi G a^2 \rho D = k^2\Phi, \tag{2.14}$$

$$4\pi G a^2 (\rho + p)V = k\left[(\dot{a}/a)\Psi - \dot{\Phi}\right]; \tag{2.15}$$

two dynamical equations:

$$-8\pi G a^2 p\Pi = k^2(\Phi + \Psi), \tag{2.16}$$

$$\begin{aligned} &8\pi G a^2 p\left[\Gamma + (c_s^2/w)D_g + (2/3)k^2\Pi\right] = \\ &= \frac{\dot{a}}{a}\left\{\dot{\Psi} - \left[a^{-1}\left(\frac{a^2\Phi}{\dot{a}}\right)^\bullet\right]^\bullet\right\} + \left[\frac{2}{a}\left(\frac{\dot{a}}{a^2}\right)^\bullet + 3\left(\frac{\dot{a}}{a^2}\right)^2\right]\left[\Psi - a^{-1}\left(\frac{a^2\Phi}{\dot{a}}\right)^\bullet\right]; \end{aligned} \tag{2.17}$$

and two conservation equations:

$$\dot{D}_\alpha - 3w_\alpha(\dot{a}/a)D_\alpha = -k\left[(1+w_\alpha)V_\alpha + 2(\dot{a}/a)w_\alpha k^{-1}\Pi_\alpha\right] + 3(1+w_\alpha)4\pi G a^2(\rho+p)(V-V_\alpha)\ , \tag{2.18}$$

$$\dot{V}_\alpha + (\dot{a}/a)V_\alpha = \frac{c_\alpha^2}{1+w_\alpha}kD_\alpha + \frac{w_\alpha}{1+w_\alpha}k\Gamma_\alpha + k\Psi - \frac{2w_\alpha}{3(1+w_\alpha)}k\Pi_\alpha\ . \tag{2.19}$$

The above conservation equations hold for any component α of the fluid stress-energy tensor which interacts with the other components of the cosmic fluid only gravitationally. The variables c_α and w_α denote the adiabatic speed of sound and the enthalpy of the fluid component, respectively. The total perturbations are defined as the sums:

$$\rho D = \sum_\alpha \rho_\alpha D_\alpha\ \ ,\quad (\rho+p)V = \sum_\alpha(\rho_\alpha+p_\alpha)V_\alpha\ \ ,\ \ \text{etc.} \tag{2.20}$$

For interacting matter, the corresponding equations can be found in [18].

In order to complete the above analysis we also need equations of state for the matter sources, which relate for instance Γ and Π to D and V. Due to the Bianchi identities, the conservation equations for the total cosmic fluid follow from the field equations (2.14)–(2.17). Thus, we need not make explicit use of both dynamical equations, but we can use, say, (2.16) and one of the conservation equations (2.18), (2.19) for the total fluid.

We now add to the perturbation equations an inhomogeneous energy-momentum distribution, $T^{(s)}_{\mu\nu}$, generated by seed fields that do not interact with the cosmic fluid other than gravitationally.

Since, by definition, seeds do not contribute as sources of the homogeneous background, the energy-momentum tensor $T^{(s)}_{\mu\nu}$ is gauge-invariant by itself [20], and can be calculated by solving the field equations for the seeds in the unperturbed background geometry. Let us assume that we can express the Fourier components of $T^{(s)}_{\mu\nu}$ in terms of four scalar "seed-functions" f_ρ, f_p, f_v and f_π (we just neglect vector and tensor contributions; since they are decoupled from density perturbations, in the linear approximation, this will not affect our results for scalar perturbations):

$$T^{(s)}_{00}(\mathbf{k},\eta) = a^2\rho^{(s)} = M^2 f_\rho(\mathbf{k},\eta)\ , \tag{2.21}$$

$$T^{(s)}_{j0}(\mathbf{k},\eta) = -i\frac{k_j}{k}a^2 v^{(s)} = -iM^2 k_j f_v(\mathbf{k},\eta)\ , \tag{2.22}$$

$$T^{(s)}_{ij}(\mathbf{k},\eta) = a^2\left[\left(p^{(s)}+\frac{1}{3}\Pi^{(s)}\right)\gamma_{ij} - \frac{k_i k_j}{k^2}\Pi^{(s)}\right] = M^2\left[\left(f_p(\mathbf{k},\eta)+\frac{k^2}{3}f_\pi(\mathbf{k},\eta)\right)\gamma_{ij} - k_i k_j f_\pi(\mathbf{k},\eta)\right]\ . \tag{2.23}$$

Note that f_ρ and f_p have dimension ℓ^{-2}, while f_v has dimension ℓ^{-1} and f_π is dimensionless. Here M denotes an arbitrary mass scale, introduced for dimensional reasons, which will eventually drop out in physical predictions.

Given an energy-momentum tensor $T_{\mu\nu}$, which in general contains vector and tensor contributions, the scalar parts f_v and f_π are determined by the identities:

$$ik^j T^{(s)}_{0j} = M^2 k^2 f_v,$$
$$-T^{(s)}_{ij}k^i k^j + \frac{1}{3}k^2\gamma^{kl}T^{(s)}_{kl} = \frac{2}{3}M^2 k^4 f_\pi\ . \tag{2.24}$$

On the other hand, f_v and f_π are related to f_ρ and f_p, by the conservation equations $\nabla^\nu T^{(s)}_{\mu\nu} = 0$:

$$\dot{f}_\rho + k^2 f_v + (\dot{a}/a)(f_\rho + 3f_p) = 0, \tag{2.25}$$

$$\dot{f}_v + 2(\dot{a}/a)f_v - f_p + (2/3)k^2 f_\pi = 0\ . \tag{2.26}$$

In the presence of seeds, and in the approximation in which perturbations are treated linearly, the total geometric perturbations can be separated into a part induced by the seeds, Ψ_s, Φ_s, and a part induced by the perturbations of the cosmic fluid, Ψ_m, Φ_m. The perturbed Einstein's equations (2.14) and (2.16) become

$$k^2\Phi = 4\pi G\rho a^2 D + \epsilon\left[f_\rho + 3(\dot{a}/a)f_v\right]\ , \tag{2.27}$$

$$\Phi + \Psi = -8\pi G a^2 k^{-2} p\Pi - 2\epsilon f_\pi\ , \tag{2.28}$$

where $\epsilon \equiv 4\pi G M^2$. If we define

$$\Psi = \Psi_s + \Psi_m, \qquad \Phi = \Phi_s + \Phi_m \tag{2.29}$$

with:

$$k^2\Phi_s = \epsilon\left[f_\rho + 3(\dot{a}/a)f_v)\right], \qquad \Phi_s + \Psi_s = -2\epsilon f_\pi, \tag{2.30}$$

we easily find

$$\Phi_m = 4\pi G\rho a^2 k^{-2}\left[D_g + 3(1+w)\left(\frac{\dot{a}}{a}\right)\frac{V}{k} - 3(1+w)\Phi\right], \tag{2.31}$$

$$\Psi_m = -\Phi_m - 8\pi G a^2 p\Pi k^{-2}\ . \tag{2.32}$$

Equation (2.31) has been written in terms of the gauge-invariant density perturbation D_g, because this choice will simplify our final equations. Physically, D_g corresponds to the density perturbation in the flat slicing. The evolution of D_g and V is described by the conservation equations (2.18) and (2.19), which read explicitly:

$$\dot{D}_g + 3(c_s^2 - w)\frac{\dot{a}}{a}D_g = -(1+w)kV, \tag{2.33}$$

$$\dot{V} + \frac{\dot{a}}{a}(1-3c_s^2)V = k(\Psi - 3c_s^2\Phi) + k\frac{c_s^2}{1+w}D_g - \frac{2w}{3(1+w)}k\Pi\ . \tag{2.34}$$

To simplify the analysis, we will assume $w = c_s^2 =$ constant. The unperturbed background equations are then solved by $a \propto \eta^r$, with $r = 2/(3w+1)$. Since we are interested in very large scale perturbations in the cosmic microwave background, we concentrate our discussion on super-horizon scales, such that $k\eta \ll 1$. Eqs. (2.31) and (2.29) then lead to

$$\Phi = \frac{1}{3(1+w)}D_g + \frac{r}{k\eta}V + \frac{2}{9r^2(1+w)}(k\eta)^2\Phi_s\ , \tag{2.35}$$

where $r = 1$ for the radiation-dominated era, and $r = 2$ for the matter-dominated era.

The evolution equation for D_g, Eq. (2.33), implies $dD_g/d(k\eta) = -(1+w)V$. In the physical picture we have in mind, metric perturbations are triggered by the presence of the seeds alone, and we do not want to include an arbitrary contribution from the perturbations of the homogeneous sources. We thus require $D_g(0) = 0$, which implies $D_g \sim k\eta V$. Hence, we may neglect the D_g-term in Eq. (2.35) for $k\eta \ll 1$.

Combining Eqs. (2.35), (2.34), (2.28) we find, on super-horizon scales,

$$\Psi = \frac{dV}{d(k\eta)} + \frac{r}{k\eta}V + \frac{2w}{3r^2(1+w)}(k\eta)^2\Phi_s + \frac{2(k\eta)^2}{9r^2(1+w)}(2\epsilon f_\pi + \Psi + \Phi)\ . \tag{2.36}$$

The two equations (2.35) and (2.36) relate the three variables Ψ, Φ and V once the seeds are given. To proceed, we need an equation of state to close the system. For single component fluids this equation usually takes the form $\Pi = \Pi(D_g, V)$. We are interested in large-scale CMB anisotropies, which are induced at recombination and later, when the universe is already matter-dominated, with $p \ll \rho$. Thus, in what follows, we will consider the case $\Pi = 0$, which implies

$$\Phi + \Psi = -2\epsilon f_\pi. \tag{2.37}$$

Furthermore, in a matter-dominated Friedmann universe, $r = 2$ and $w = 0$. The equation of motion for V, obtained by combining Eqs. (2.35), (2.36), (2.37), then reads

$$\frac{dV}{d(k\eta)} + \frac{4}{k\eta}V = -\frac{1}{18}(k\eta)^2\Phi_s - 2\epsilon f_\pi\ = -\frac{1}{18}\eta^2\epsilon\left[f_\rho + 3(\dot{a}/a)f_v\right] - 2\epsilon f_\pi\ . \tag{2.38}$$

In the next subsection, we shall see that the large-scale anisotropies of the CMB are determined by the combination Ψ–Φ. Using Eqs. (2.35), (2.37) and (2.38), we find immediately:

$$\Psi - \Phi = \frac{dV}{d(k\eta)} - \frac{1}{18}\eta^2\epsilon\left[f_\rho + 3(\dot{a}/a)f_v\right]\ . \tag{2.39}$$

Modulo numbers of order unity, which can be computed case by case, we finally arrive at the estimates:

$$\Psi - \Phi \sim \frac{dV}{d(k\eta)} \sim \frac{V}{k\eta} \sim \max\left\{\epsilon f_\pi, \epsilon\eta^2\left(f_\rho + 3\frac{\dot{a}}{a}f_v\right)\right\} \ . \tag{2.40}$$

Depending on whether $\eta^2\left(f_\rho + 3(\dot{a}/a)f_v\right)$ or f_π dominates in Eq. (2.40), we can distinguish between seeds with small and large anisotropic stresses. We will discuss in Section III to which case our string-cosmology seeds belong.

If the term $\epsilon\eta^2\left(f_\rho + 3(\dot{a}/a)f_v\right) = x^2\Phi_s$ dominates, we conclude from Eqs. (2.38),(2.39) that

$$\Phi \sim \Psi \sim (k\eta)^2\Phi_s \sim (k\eta)^2\Psi_s \ll \Phi_s \sim \Psi_s \ , \tag{2.41}$$

on super-horizon scales. This suppression of the total geometric perturbations, if compared with the source perturbations alone, is known under the name of "compensation" [14]. The conservation equations (2.18), (2.19) show that the presence of seeds induces matter perturbations that try to compensate the gravitational potential of the seeds. Since anisotropic stresses in the seeds cannot be compensated by a perfect fluid, compensation is not effective, if anisotropic stresses dominate. But, as shown here (see also [14]), the phenomenon of compensation is quite generic and, to a large extent, independent of the spectrum of seed fluctuations.

B. The Seed Contribution to CMB anisotropies

In this subsection we calculate the CMB anisotropies for models where perturbations are induced by seeds, and their contribution to $\Delta T/T$ via the Sachs-Wolfe effect [15]. We first discuss in general the motion of photons in a perturbed Friedmann universe.

We make use of the fact that the equations of motion of photons are conformally invariant. More precisely, two metrics that are conformally equivalent,

$$d\bar{s}^2 = a^2 ds^2 \ , \tag{2.42}$$

have the same light-like geodesics, only the corresponding affine parameters are different. Let us denote the two affine parameters by $\bar{\lambda}$ and λ respectively, and the tangent vectors to the geodesic by

$$n = \frac{dx}{d\lambda}, \quad \bar{n} = \frac{dx}{d\bar{\lambda}} \ , \quad n^2 = \bar{n}^2 = 0 \ , \ \ n^0 = 1 \ , \ \ \mathbf{n}^2 = 1. \tag{2.43}$$

Setting $n^0 = 1 + \delta n^0$, the geodesic equation for the perturbed metric

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{2.44}$$

yields, to first order,

$$\delta n^0|_i^f = \left[h_{00} + h_{0j}n^j\right]_i^f - \frac{1}{2}\int_i^f \dot{h}_{\mu\nu}n^\mu n^\nu d\lambda \ . \tag{2.45}$$

On the other hand, the ratio of the energy of a photon measured by some observer at t_f to the energy emitted at t_i is

$$\frac{E_f}{E_i} = \frac{(\bar{n}\cdot u)_f}{(\bar{n}\cdot u)_i} = \frac{T_f}{T_i}\frac{(n\cdot u)_f}{(n\cdot u)_i} \ , \tag{2.46}$$

where u_f and u_i are the four-velocities of the observer and emitter respectively, and the factor T_f/T_i is the usual (unperturbed) redshift, which relates n and $\bar{n}$. The velocity field of observer and emitter is given by

$$u = (1 - A)\partial_\eta + v^i\partial_i \ . \tag{2.47}$$

An observer measuring a temperature T_0 receives photons that were emitted at the time η_{dec} of decoupling of matter and radiation, at the fixed temperature T_{dec}. In first-order perturbation theory, we find the following relation between the unperturbed temperatures T_f, T_i, the measurable temperatures T_0, T_{dec}, and the photon density perturbation:

$$\frac{T_f}{T_i} = \frac{T_0}{T_{dec}}\left(1 - \frac{\delta T_f}{T_f} + \frac{\delta T_i}{T_i}\right) = \frac{T_0}{T_{dec}}\left(1 - \frac{1}{4}\delta^{(\gamma)}|_i^f\right) \ , \tag{2.48}$$

where $\delta^{(\gamma)}$ is the intrinsic density perturbation in the radiation and we used $\rho^{(\gamma)} \propto T^4$ in the last equality. Inserting the above equation and Eq. (2.45) into Eq. (2.46), and using Eq. (2.3) for the definition of $h_{\mu\nu}$, one finds, after integration by parts [16]:

$$\frac{E_f}{E_i} = \frac{T_0}{T_{dec}} \left\{ 1 - \left[\frac{1}{4} D_g^{(\gamma)} + V_j^{(m)} n^j + \Psi - \Phi \right]_i^f + \int_i^f (\dot{\Psi} - \dot{\Phi}) d\lambda \right\} . \tag{2.49}$$

Here $D_g^{(\gamma)}$ denotes the density perturbation in the radiation fluid, and $V^{(m)}$ is the peculiar velocity of the baryonic matter component (the emitter and observer of radiation). The final time values in the square bracket of Eq. (2.49) give rise only to monopole contributions and to the dipole due to our motion with respect to the CMB, and will be neglected in what follows.

Evaluating Eq. (2.49) at final time η_0 (today) and initial time η_{dec}, we obtain the temperature difference of photons coming from different directions $\mathbf{n}$ and $\mathbf{n}'$

$$\frac{\Delta T}{T} \equiv \frac{\delta T(\mathbf{n})}{T} - \frac{\delta T(\mathbf{n}')}{T}, \tag{2.50}$$

with temperature perturbation

$$\frac{\delta T(\mathbf{n})}{T} = \left[\frac{1}{4} D_g^{(\gamma)} + V_j^{(m)} n^j + \Psi - \Phi \right] (\eta_{dec}, \mathbf{x}_{dec}) + \int_{\eta_{dec}}^{\eta_0} (\dot{\Psi} - \dot{\Phi})(\eta, \mathbf{x}(\eta)) d\eta , \tag{2.51}$$

where $\mathbf{x}(\eta) = \mathbf{x}_0 - (\eta_0 - \eta)\mathbf{n}$ is the unperturbed photon position at time η for an observer at $\mathbf{x}_0$, and $\mathbf{x}_{dec} = \mathbf{x}(\eta_{dec})$. The first term in Eq. (2.51) describes the intrinsic inhomogeneities on the surface of the last scattering, due to acoustic oscillations prior to decoupling. In general, it also contains contributions to the geometrical perturbations. This is especially important in the case of adiabatic inflationary models [21]. However, for perturbations induced by seeds, which satisfy the initial condition $D_g(k, \eta) \to 0$ for $\eta \to 0$, the geometrical contributions to D_g can be neglected. The second term describes the relative motions of emitter and observer. This is the Doppler contribution to the CMB anisotropies. It appears on the same angular scales as the acoustic term, and we thus call the sum of the acoustic and Doppler contributions "acoustic peaks".

The last two terms are due to the inhomogeneities in the spacetime geometry; the first contribution determines the change in the photon energy due to the difference of the gravitational potential at the position of emitter and observer. Together with the part contained in $D_g^{(r)}$ they represent the "ordinary" Sachs-Wolfe effect. The second term accounts for red-shift or blue-shift caused by the time dependence of the gravitational field along the path of the photon, and represents the so-called Integrated Sachs-Wolfe (ISW) effect. The sum of the two terms is the full Sachs-Wolfe contribution (SW).

On angular scales $0.1^\circ \lesssim \theta \lesssim 2^\circ$, the main contribution to the CMB anisotropies comes from the acoustic peaks, while the SW effect is dominant on large angular scales. On scales smaller than about 0.1°, the anisotropies are damped by the finite thickness of the recombination shell, as well as by photon diffusion during recombination (Silk damping). Baryons and photons are very tightly coupled before recombination, and oscillate as a one-component fluid. During the process of decoupling, photons slowly diffuse out of over-dense regions into under-dense ones. To fully account for this process, one has to solve the Boltzmann equation for the photons (see, e.g. [16]).

The angular power spectrum of CMB anisotropies is expressed in terms of the dimensionless coefficients C_ℓ, which appear in the expansion of the angular correlation function in terms of the Legendre polynomials P_ℓ:

$$\left\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \right\rangle_{(\mathbf{n}\cdot\mathbf{n}' = \cos\vartheta)} = \frac{1}{4\pi} \sum_\ell (2\ell + 1) C_\ell P_\ell(\cos\vartheta) . \tag{2.52}$$

Here the brackets denote spatial average, or expectation values if perturbations are quantized.

To determine the C_ℓ we Fourier-transform Eq. (2.51), defining

$$\varphi(\mathbf{k}) = \frac{1}{\sqrt{V}} \int_V \varphi(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x , \tag{2.53}$$

and using the identity

$$e^{iz\cos\vartheta} = \sum_\ell (2\ell + 1) i^\ell j_\ell(z) P_\ell(\cos\vartheta) \tag{2.54}$$

(where j_ℓ is the spherical Bessel function of order ℓ). For the coefficients C_ℓ of Eq. (2.52) we obtain:

$$C_\ell = \frac{2}{\pi}\int \frac{\langle|\Delta_\ell(\mathbf{k})|^2\rangle}{(2\ell+1)^2}k^2 dk \ , \tag{2.55}$$

where

$$\begin{aligned}\frac{\Delta_\ell}{2\ell+1} &= j_\ell(k\eta_0)\left[\frac{1}{4}D_g^{(r)}(\mathbf{k},\eta_{dec}) + (\Psi-\Phi)(\mathbf{k},\eta_{dec})\right] - j'_\ell(k\eta_0)\mathbf{V}_r(\mathbf{k},\eta_{dec}) \\ &\quad + \int_{\eta_{dec}}^{\eta_0}(\dot\Psi-\dot\Phi)(\mathbf{k},\eta')j_\ell\left(k\eta_0 - k\eta'\right)d\eta' \\ &= \frac{1}{4}D_g^{(r)}(\mathbf{k},\eta_{dec})j_\ell(k\eta_0) - j'_\ell(k\eta_0)\mathbf{V}_r(\mathbf{k},\eta_{dec}) \\ &\quad + k\int_{\eta_{dec}}^{\eta_0}(\Psi-\Phi)(\mathbf{k},\eta')j'_\ell\left(k\eta_0 - k\eta'\right)d\eta' \ , \end{aligned} \tag{2.56}$$

and j'_ℓ stands for the derivative of j_ℓ with respect to its argument. On large angular scales, $k\eta_{dec} \ll 1$ (which corresponds to $\ell \ll 100$), the SW contribution dominates:

$$C_\ell^{SW} = \frac{2}{\pi}\int k^4 dk \left\langle\left[\int_{\eta_{dec}}^{\eta_0}(\Psi-\Phi)(\mathbf{k},\eta)j'_\ell\left(k\eta_0 - k\eta\right)d\eta\right]^2\right\rangle . \tag{2.57}$$

Let us approximate the Bardeen potentials on super-horizon scales by a power-law spectrum:

$$\langle|\Psi-\Phi|^2\rangle = C^2(k)\ (k\eta)^{2\gamma} \ . \tag{2.58}$$

Furthermore, we consider models where the seed contribution does not grow in time on sub-horizon scales. In this case the Bardeen potentials, inside the horizon, are dominated by the cold dark matter contribution, which leads to time-independent Φ and Ψ. We can thus approximate the Bardeen potentials by

$$\Psi - \Phi \approx \begin{cases} C(k)(k\eta)^\gamma & , \ k\eta \ll 1 \\ C(k) & , \ k\eta \gg 1 \ . \end{cases} \tag{2.59}$$

We further assume that also $C(k)$ is given by a simple power law. Thus, for dimensional reasons, it has the form

$$C(k) = \begin{cases} \mathcal{N}k^{-3/2}(k/k_1)^\alpha \ , & k \le k_1 \\ 0 \ , & k > k_1 \ , \end{cases} \tag{2.60}$$

where $\mathcal{N}$ is a dimensionless constant, and k_1 denotes a comoving cutoff scale, i.e. the maximal amplified frequency determined by the explicit mechanism of seed production (in the case $\alpha = 0$ no cutoff is needed). Inserting this in Eq. (2.57),

$$C_\ell^{SW} \approx \mathcal{N}^2\frac{2}{\pi}\int_0^{k_1}\frac{dk}{k}\left(\frac{k}{k_1}\right)^{2\alpha}|I(k)|^2 \ , \tag{2.61}$$

where, setting $x = k\eta, x_0 = k\eta_0, x_{dec} = k\eta_{dec}$,

$$I(k) = \int_{x_{dec}}^1 dx x^\gamma j'_\ell(x_0 - x) + \int_1^{x_0} dx j'_\ell(x_0 - x) \tag{2.62}$$

$$= \int_{x_{dec}}^1 dx x^\gamma j'_\ell(x_0 - x) + j_\ell(x_0 - 1) \ . \tag{2.63}$$

We can see explicitely from this equation that the relevant contribution of each mode to the CMB anisotropy comes while the mode is still outside the horizon ($k\eta < 1$). We now distinguish two cases.

If $\gamma > -1$ the lower bound in eq. (2.63) can be safely extended to 0, and the integral is dominated by the region $k\eta \sim 1$, so that:

$$I(k) \sim j_\ell(x_0 - x_{dec}) \sim j_\ell(x_0), \quad x_{dec} \ll 1 < x_0 \ . \tag{2.64}$$

Inserting this in Eq. (2.61), the integral can be performed exactly (assuming $\eta_0 k_1 \gg \ell$), with the result, for $\alpha < 1$,

$$C_\ell^{SW} \approx \mathcal{N}^2 (k_1\eta_0)^{-2\alpha} \frac{\Gamma(2-2\alpha)}{4^{(1-\alpha)}\Gamma(3/2-\alpha)} \frac{\Gamma(\ell+\alpha)}{\Gamma(\ell+2-\alpha)}, \qquad \alpha < 1 \tag{2.65}$$

(if $\alpha > 1$, the integral grows towards large k and is dominated by the contributions at $k \sim k_1$, leading to an ℓ-independent result of order $(\mathcal{N}/k_1\eta_0)^2$). Comparing the above equation with the standard inflationary result [22],

$$C_\ell^{SW} \propto \frac{\Gamma(\ell - 1/2 + n/2)}{\Gamma(\ell + 5/2 - n/2)} \ , \tag{2.66}$$

where n denotes the usual spectral index, we find that α is related to n by $\alpha = (n-1)/2$. The scale-invariant spectrum, as it has been observed by the DMR experiment aboard the COBE satellite [24], requires

$$0.8 \leq n \leq 1.4 \tag{2.67}$$

so that, allowing for generous error bars, the COBE observations imply

$$-0.1 \leq \alpha \leq 0.2 \ , \qquad \gamma > -1 \ . \tag{2.68}$$

Consider now the second case, $\gamma + 1 \leq 0$. The integral (2.63) is now dominated by its value at the lower boundary and we get

$$|I(k)|^2 \approx \frac{1}{(\gamma+1)^2} x_{dec}^{2(\gamma+1)} \left[\frac{\ell}{2\ell+1} j_{\ell-1}(x_0) - \frac{\ell+1}{2\ell+1} j_{\ell+1}(x_0) \right]^2 \ . \tag{2.69}$$

If also $\alpha + \gamma < 0$, the k-integral converges and we obtain (see Appendix A):

$$\begin{aligned} & C_\ell^{SW} \approx \\ & \frac{\mathcal{N}^2}{2^{2(\alpha+\gamma)}(\gamma+1)^2} \frac{\Gamma(-2(\alpha+\gamma))}{\Gamma(1/2-(\alpha+\gamma))^2} \left(\frac{\eta_{dec}}{\eta_0}\right)^{2(\gamma+1)} (k_1\eta_0)^{-2\alpha} \frac{\Gamma(\ell+1+\alpha+\gamma)}{\Gamma(\ell+1-\alpha-\gamma)} \frac{1}{(2\ell+1)^2} \\ & \times \left[\frac{\ell^2(\ell-\alpha-\gamma)}{\ell+\alpha+\gamma} + \frac{2\ell(\ell+1)(1/2+\alpha+\gamma)}{(1/2-\alpha-\gamma)} + \frac{(\ell+1)^2(\ell+1+\alpha+\gamma)}{(\ell+1-\alpha-\gamma)} \right] \ . \end{aligned} \tag{2.70}$$

Comparing again this last result with that of standard inflation, Eq. (2.66), and neglecting the weak ℓ-dependence of $(2\ell+1)^{-2}[\cdots]$ in Eq. (2.70), we obtain

$$n \sim 3 + 2(\alpha+\gamma) \ , \qquad \alpha + \gamma < 0 \ . \tag{2.71}$$

(If, on the contrary, $\alpha+\gamma > 0$, the coefficients C_ℓ are dominated by the large k (i.e. small-scale) contribution, even for the very low values of ℓ. In this case the small-scale perturbations become too large, which is excluded observationally by the fact that the spectrum, for CMB and matter perturbations, must be close to the Harrison-Zel'dovich spectrum [25]).

The observational limits on n thus impose

$$-0.1 < \gamma + 1 + \alpha < 0.2, \qquad \gamma \leq -1 \ , \qquad n \simeq 3 + 2(\alpha+\gamma) \tag{2.72}$$

and

$$-0.1 < \alpha < 0.2, \qquad \gamma > -1 \ , \qquad n = 1 + 2\alpha. \tag{2.73}$$

In the following sections we will apply these findings to electromagnetic and axionic seeds produced in string cosmology. In the axions case we will discuss separately massless and massive perturbations.

III. SEEDS FROM STRING COSMOLOGY

In this section we compute the seed functions f_ρ, f_v, f_π, and we estimate the Bardeen potentials for electromagnetic and axion perturbations, including the case of massive axions.

A. Amplification of quantum fluctuations

We start by recalling the form of the (string-frame) low-energy string effective action [26]:

$$\Gamma^S_{eff} = \int d^D x \sqrt{|g|} e^{-\phi} \left(R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12} g^{\mu\rho} g^{\nu\sigma} g^{\alpha\beta} H_{\mu\nu\alpha} H_{\rho\sigma\beta} - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right), \tag{3.1}$$

where we have included the antisymmetric tensor $H_{\mu\nu\alpha} = \partial_{[\mu} B_{\nu\alpha]}$ and the $U(1)$ gauge field $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. Note that this gauge field is typical of what emerges from heterotic string compactification. For gauge fields originating *à la* Kaluza-Klein, the action and the spectra are somewhat different, as discussed in [27].

Upon compactification down to four dimensions, and after introduction of the axion field σ by the duality transformation:

$$H^{\mu\nu\alpha} = e^{-\phi} \epsilon^{\mu\nu\alpha\beta} \partial_\beta \sigma, \tag{3.2}$$

one easily arrives at the dimensionally reduced action:

$$\Gamma^S_{eff} = \int d^4 x \sqrt{|g|} e^{-\phi} \left(R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2} e^{2\phi} g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right) . \tag{3.3}$$

The study of tensor (T), scalar-dilaton (SD), electromagnetic (EM) and axion (AX) perturbations is conveniently performed defining the external "pump field", responsible for their amplification. To this aim, we first identify for each perturbation the canonical variables ψ^i, which diagonalize the perturbed action expanded up to second order [28]. In a purely metric-dilaton background, such variables are easily found from (3.3) to be:

$$\begin{aligned} \psi^T = a e^{-\phi/2} h^{TT} \equiv a_E h^{TT}, \quad \psi^{SD} = a e^{-\phi/2}\phi + \dots , \\ \psi^{EM} = e^{-\phi/2} A_\mu, \quad \psi^{AX} = a e^{\phi/2}\sigma \equiv a_A \sigma. \end{aligned} \tag{3.4}$$

Here h^{TT} denotes the transverse-traceless part of the metric perturbations, the dots in the equation for ψ^{SD} represent the additional scalar-metric terms needed to reproduce the gauge-invariant scalar perturbation [28], a_E is the scale factor in the Einstein frame, and a_A in the axion frame [12]. By varying the perturbed action, we find that the Fourier modes $\psi_k(\eta)$ of each of these four perturbations satisfy decoupled, linear equations of the type:

$$\ddot{\psi}_k + \left(k^2 - \frac{\ddot{P}}{P} \right) \psi_k = 0 \ , \tag{3.5}$$

where $P(\eta)$ is the pump field, obtained for each case from eq. (3.4) as:

$$P^T = P^{SD} = a_E \ ; \qquad P^{EM} = e^{-\phi/2} \ ; \qquad P^{AX} = a_A \ . \tag{3.6}$$

At the beginning of the inflationary era, characterized by an accelerated evolution of the pump field, every perturbation is well inside the horizon and Eq. (3.5) has oscillating solutions, which can be consistently normalized to a vacuum fluctuation spectrum. During the whole pre-big bang phase the general solution can be written in terms of Hankel functions [29], with a Bessel index determined by the power that characterizes the background evolution (in conformal time) of the pump field. This behaviour has to be matched with the one after the pre-big bang phase when, as we assume, the universe becomes radiation-dominated and the dilaton freezes to its present value. In all four cases this implies a free Klein-Gordon equation for the canonical variable after the period of accelerated evolution. By matching the pre-big bang and radiation solutions of the perturbation equations, we eventually obtain the final amplified perturbations during the radiation era.

For T and SD perturbations the time evolution of the background leads to a spectrum that is in general too steep [7] (see also [30]) to be expected to give any significant contribution to very large scale structures, or to temperature anisotropies on the COBE scale. The only way to achieve a reasonable contribution would be to have a very long string phase with an almost constant dilaton [9], which is not excluded, in principle, either theoretically or phenomenologically, but which looks somewhat unlikely, from both points of view.

For EM perturbations, however, the situation seems to be more interesting. Consider in fact the transition from a pre-big bang phase, with growing dilaton ($\phi = -2\beta \log|\eta|$), to the standard radiation-dominated phase with $\phi =$ const, and call η_1 the transition time scale. The electromagnetic fluctuations are directly coupled to the dilaton background, in such a way that each polarization mode ψ_k satisfies at all times, in momentum space and in the radiation gauge, the evolution equation:

$$\ddot{\psi}_k + \left[k^2 - e^{\phi/2}\left(e^{-\phi/2}\right)^{\bullet\bullet}\right]\psi_k = 0. \tag{3.7}$$

In the pre-big bang phase, the general solution of this equation, normalized to a vacuum fluctuation spectrum, can be written in terms of Hankel functions of the second kind as:

$$\psi_k = \eta^{1/2} H_\mu^{(2)}(|k\eta|), \qquad \mu = \left|\beta - \frac{1}{2}\right|, \qquad \eta < \eta_1. \tag{3.8}$$

In the radiation era we have instead the free plane-wave solution,

$$\psi_k = \frac{1}{\sqrt{k}}\left[c_+(k)e^{-ik\eta} + c_-(k)e^{ik\eta}\right], \qquad \eta > \eta_1. \tag{3.9}$$

By matching the two solutions at the transition time η_1 we easily obtain, for $|k\eta_1| \ll 1$ and $\eta > \eta_1$,

$$c_\pm = \pm c(k)e^{\pm ik\eta}, \qquad \psi_k = \frac{c(k)}{\sqrt{k}}\sin k|\eta - \eta_1|, \qquad |c(k)| \simeq (k/k_1)^{-\mu-1/2}, \tag{3.10}$$

where $k_1 = 1/|\eta_1|$ represents the maximal amplified frequency (higher-frequency modes are unaffected by the background transition). The associated energy-density distribution of the produced photons is then [11]:

$$\frac{d\rho(k)}{d\log k} \simeq \left(\frac{k}{a}\right)^4 |c_-(k)|^2 \simeq \left(\frac{k_1}{a}\right)^4 \left(\frac{k}{k_1}\right)^{3-2\mu}, \quad k < k_1\ , \qquad \mu < 3/2, \tag{3.11}$$

where $\mu < 3/2$ to avoid photon overproduction which would destroy the homogeneity of the classical background, and where the amplitude $c(k)$ has been estimated modulo numerical factors of order 1. At large times $\eta \gg |\eta_1|$ we thus obtain, in string cosmology, a cosmic background of electromagnetic fluctuations that, for a long enough pre-big bang phase with $\beta \lesssim 2$, are characterized by a rather flat spectrum, and could provide the long-sought origin of the galactic magnetic fields [11]. The amplified fluctuations satisfy stochastic correlation functions, as a consequence of their quantum origin.

Correspondingly, if we consider axionic perturbations, we are led to the canonical equation

$$\ddot{\psi}_k + \left(k^2 - \frac{\ddot{a}_A}{a_A}\right)\psi_k = 0, \tag{3.12}$$

very similar to Eq. (3.7). The same procedure as in the electromagnetic case then leads to the spectrum (3.11) with $\mu = |r|$, where r parametrizes the three-dimensional axion scale factor as $a_A(\eta) \sim \eta^{r+1/2}$. For $r = -3/2$, in particular, the axion metric describes a de Sitter inflationary expansion, and the energy density of a massless axion background has a flat spectral distribution, $d\rho/d\log k \simeq (k_1/a)^4$, as first noted in [12]. The value of r depends on the number and on the kinematics of the internal dimensions, and the value $-3/2$ can be obtained, in particular, for a ten-dimensional background with special symmetries [27].

In the axion case, however, the low frequency tail of the spectrum is further affected by the radiation $\rightarrow$ matter transition, as the axion pump field a_A is not a constant (unlike the dilaton) in the matter-dominated era, where $a_A = a \propto \eta^2$. This has important consequences that will be discussed in detail in subsection III C.

After these preliminary observations we shall now estimate the form of the seed functions for both EM and AX seeds.

B. Electromagnetic seeds

Here we determine the spectral components of the inhomogeneous stress tensor, for a stochastic background obtained by amplifying the quantum EM fluctuations of the vacuum, as discussed in the previous subsection. However, independently of the production mechanism, the results of this section can be applied to any EM fluctuation background parametrized by a vector potential that, in momentum space and in the radiation gauge, takes the form

$$A_i(\mathbf{k},\eta) = \frac{c_i(\mathbf{k})}{\sqrt{k}}\sin k\eta, \qquad k_i A_i = 0, \qquad A_0 = 0\,. \tag{3.13}$$

A_i is a Gaussian random variable which obeys the stochastic average condition:

$$\langle A_i(\mathbf{k})A_j^*(\mathbf{k}')\rangle = \frac{(2\pi)^3}{2}\delta^3(k-k')\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)|\mathbf{A}(\mathbf{k},\eta)|^2\,. \tag{3.14}$$

The above condition has been normalized in such a way that

$$\sum_i \langle A_i(\mathbf{k})A_i^*(\mathbf{k}')\rangle = (2\pi)^3\delta^3(k-k')\,|\mathbf{A}(\mathbf{k},\eta)|^2\,. \tag{3.15}$$

Taking into account that the electric component of the stochastic background is rapidly dissipated, because of the conductivity of the cosmic plasma [31], the seed stress tensor can be expressed in terms of the magnetic field only. Setting $B_i(k) = i\epsilon_{ijl}k_jA_l(k)$, the condition (3.14) implies

$$\langle B_i(\mathbf{k})B_j^*(\mathbf{k}')\rangle = \frac{(2\pi)^3}{2}\delta^3(k-k')\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)|\mathbf{B}(\mathbf{k},\eta)|^2\,, \tag{3.16}$$

where

$$|\mathbf{B}(\mathbf{k},\eta)|^2 = k^2\,|\mathbf{A}(\mathbf{k},\eta)|^2 = k\,|\mathbf{c}(\mathbf{k})|^2\sin^2 k\eta. \tag{3.17}$$

In a process of photon production, the coefficient $|\mathbf{c}(\mathbf{k})|^2$ represents the Bogoliubov coefficient [28] fixing the average photon number density, $\langle n(k)\rangle$, and is linked to the spectral energy distribution by

$$\frac{d\rho(k)}{d\log k} = \left(\frac{k}{a}\right)^4\frac{\langle n(k)\rangle}{\pi^2} \simeq \left(\frac{k}{a}\right)^4\frac{|\mathbf{c}(\mathbf{k})|^2}{\pi^2}. \tag{3.18}$$

In what follows we shall use for $|\mathbf{c}(\mathbf{k})|^2$ a power-law spectrum, characterized by a cut-off frequency k_1,

$$|\mathbf{c}(\mathbf{k})|^2 = \begin{cases} (k/k_1)^{-2\mu-1}\,, & k\le k_1, \quad \mu \le 3/2 \\ 0\,, & k > k_1\,. \end{cases} \tag{3.19}$$

This reproduces in particular the spectral distribution (3.11), where μ is fixed by the dilaton growth rate.

We shall now compute the two-point correlation functions, for the various components of the inhomogeneous stress tensor T_μ^ν, associated with the electromagnetic background:

$$\xi_\mu^\nu(x,x') = \langle T_\mu^\nu(x)T_\mu^\nu(x')\rangle - \langle T_\mu^\nu(x)\rangle\langle T_\mu^\nu(x')\rangle \tag{3.20}$$

(no sum over μ,ν, and the angular brackets denote stochastic average). The Fourier transform of ξ is related to the scalar seed functions f_ρ, f_v, f_π, defined in the previous section. For ξ_0^0 we have, for instance,

$$\xi_0^0(x,x') = \left(\frac{M}{a}\right)^4\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}|f_\rho(k)|^2. \tag{3.21}$$

For $E_i = 0$, in particular, we have to compute the correlation of a sum of terms that are quadratic in the magnetic field. We start considering the energy-density correlation function,

$$\xi_0^0(x,x') = \langle\rho(x)\rho(x')\rangle - \left(\langle\rho(x)\rangle\right)^2\,, \qquad \rho = -T_0^0 = \frac{|\mathbf{B}|^2}{8\pi a^4}, \tag{3.22}$$

and compute

$$\Delta_{ij}^B(x,x') = \langle B_i^2(x)B_j^2(x')\rangle - \langle B_i^2\rangle\langle B_j^2\rangle \tag{3.23}$$

where, using the stochastic average (3.16) and the reality condition $B^*(k) = B(-k)$,

$$\langle B_i^2(x)\rangle = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,|\mathbf{B}(k)|^2\left(1-\frac{k_i^2}{k^2}\right). \tag{3.24}$$

In momentum space, the two-point correlation function for the energy density can be written as a four-point correlation function for the stochastic fields (see also [32]). We have, in particular,

$$\langle B_i^2(x)B_j^2(x')\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}')}\langle B_i(p)B_i(k-p)B_j(q)B_j(k'-q)\rangle. \tag{3.25}$$

Decomposing the four-point bracket of the Gaussian variables B_j as

$$\begin{aligned}&\langle B_i(p)B_i(k-p)\rangle\langle B_j(q)B_j(k'-q)\rangle + \\ &+\langle B_i(p)B_j(q)\rangle\langle B_i(k-p)B_j(k'-q)\rangle + \langle B_i(p)B_j(k'-q)\rangle\langle B_i(k-p)B_j(q)\rangle,\end{aligned} \tag{3.26}$$

and using Eq. (3.16), we find that the first term in the above equation is exactly cancelled by the quadratic averages $\langle B_i^2\rangle\langle B_j^2\rangle$, while the other two terms give (no sum over i, j):

$$\Delta_{ij}^B(x,x') = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}e^{i\mathbf{k}\cdot\Delta\mathbf{x}}\left|\mathbf{B}(\mathbf{p})\right|^2\left|\mathbf{B}(\mathbf{k}-\mathbf{p})\right|^2\left(\delta_{ij}-\frac{p_ip_j}{p^2}\right)\left(\delta_{ij}-\frac{(k-p)_i(k-p)_j}{|\mathbf{k}-\mathbf{p}|^2}\right), \tag{3.27}$$

where $\Delta x = x - x'$. By summing over the vector components we obtain:

$$\begin{aligned}&\Delta^B(x,x') = \sum_{ij}\Delta_{ij}^B(x,x') = \\ &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}e^{i\mathbf{k}\cdot\Delta\mathbf{x}}\left|\mathbf{B}(\mathbf{p})\right|^2\left|\mathbf{B}(\mathbf{k}-\mathbf{p})\right|^2\left[1+\frac{|\mathbf{p}\cdot(\mathbf{k}-\mathbf{p})|^2}{p^2|\mathbf{k}-\mathbf{p}|^2}\right] .\end{aligned} \tag{3.28}$$

According to Eq. (3.21), the energy-density spectrum of the electromagnetic seeds is thus determined by

$$|f_\rho|^2\left(\frac{M}{a}\right)^4 = \frac{1}{2(8\pi a^4)^2}\int\frac{d^3p}{(2\pi)^3}\left|\mathbf{c}(\mathbf{p})\right|^2\left|\mathbf{c}(\mathbf{k}-\mathbf{p})\right|^2 p|\mathbf{k}-\mathbf{p}|\left(1+\cos^2\alpha\right)\sin^2 p\eta\sin^2|\mathbf{k}-\mathbf{p}|\eta\ , \tag{3.29}$$

where α is the angle between $\mathbf{p}$ and $\mathbf{k}-\mathbf{p}$. Inserting the power spectrum (3.19), and defining $\mathbf{y} = \mathbf{p}/k_1$, $\mathbf{z} = \mathbf{k}/k_1$, the above integral can be written, in polar coordinates, as

$$|f_\rho|^2\left(\frac{M}{a}\right)^4 = \frac{k_1^5}{2(8\pi a^4)^2(2\pi)^2}\int_0^1 dy y^{2-2\mu}\int_{-1}^1 dx\beta^{-2\mu}\left(1+\cos^2\alpha\right)\sin^2(yk_1\eta)\sin^2(\beta k_1\eta), \tag{3.30}$$

where we defined $x = \cos\vartheta$, ϑ being the angle between $\mathbf{p}$ and $\mathbf{k}$, and

$$\beta^2 = |\mathbf{z}-\mathbf{y}|^2 = y^2+z^2-2xyz, \quad \cos^2\alpha = \beta^{-2}(y^2+x^2z^2-2xyz)\ . \tag{3.31}$$

The integral of Eq. (3.30) will be evaluated for $|k\eta| = |zk_1\eta| \ll 1$, since we are interested in the large-scale sector of the CMB anisotropy, namely in the spectrum of all modes that are still outside the horizon at the time of decoupling. For EM seeds these modes give the dominant contribution to the SW effect, as we will see in Section IV. Estimating the contributions to the integral from the regions $p\eta \ll 1$, $p\eta \sim 1$ and $p\eta \gg 1$, and recalling that $\mu \leq 3/2$ according to Eq. (3.11), we find that the dominant contribution comes from $p\eta \gg 1$ if $\mu \leq 3/4$. If $3/4 \leq \mu \leq 3/2$, the integral is dominated from its contribution at $p \sim k$, thus $p\eta < 1$ on super-horizon scales. In both cases we obtain for f_ρ a white noise spectrum, i.e. $|f_\rho(k)|^2 \sim$ constant, but in the second case there is a parametric enhancement (see Appendix B). More precisely

$$k^3\left|f_\rho\right|^2\left(\frac{M}{a}\right)^4 \simeq \begin{cases} d_\rho^2(k_1/a)^8(k/k_1)^3, & \mu \leq 3/4 \\ c_\rho^2(k_1/a)^8(k/k_1)^3(k_1\eta)^{4\mu-3}, & 3/4 \leq \mu \leq 3/2\ , \end{cases} \tag{3.32}$$

where d_ρ and c_ρ are dimensionless numbers of order 1. Consequently, the energy-density contribution of the EM seeds to the Bardeen potentials is, according to Eq. (2.40),

$$\epsilon\eta^2\left|f_\rho\right|k^{3/2} \simeq \begin{cases} 4\pi G d_\rho(a\eta)^2(k_1/a)^4(k/k_1)^{3/2}, & \mu \leq 3/4, \\ 4\pi G c_\rho(a\eta)^2(k_1/a)^4(k/k_1)^{3/2}(k_1\eta)^{2\mu-3/2}, & 3/4 \leq \mu \leq 3/2. \end{cases} \tag{3.33}$$

The contribution of the off-diagonal scalar potential f_π can be similarly obtained by computing the correlation function $\xi_i^j(x,x')$, with $i \neq j$. For purely magnetic seeds, $f_v = 0$, we find

$$f_\rho = 3f_p \sim k^2 f_\pi, \tag{3.34}$$

so that the Bardeen potentials, according to Eq. (2.40), are always dominated by f_π on super-horizon scales, as $\eta^2 f_\rho/f_\pi \sim (k\eta)^2 \ll 1$. Therefore

$$k^{3/2}\,|\Psi-\Phi| \sim \epsilon k^{3/2}\,|f_\pi| \simeq$$
$$\simeq \begin{cases} 4\pi G d_\pi (a\eta)^2 (k_1/a)^4 (k/k_1)^{-1/2} (k_1\eta)^{-2}, & \mu \leq 3/4, \\ 4\pi G c_\pi (a\eta)^2 (k_1/a)^4 (k/k_1)^{-1/2} (k_1\eta)^{2\mu-7/2}, & 3/4 \leq \mu \leq 3/2. \end{cases} , \tag{3.35}$$

where d_π and c_π are dimensionless numbers of order 1. By assuming that the universe becomes immediately radiation-dominated at the physical cut-off scale $H_1 = k_1/a_1$, such a fluctuation spectrum can be expressed in terms of $\Omega_\gamma(\eta) = (H_1/H)^2 (a_1/a)^4$, i.e. of the fraction of critical energy density in radiation at a given time η, and of $g_1 = H_1/M_p$, the transition scale in units of the Planck mass M_p. Denoting with $\omega = k/a$ the proper frequency, and using $\rho_c = 3M_p^2 H^2/8\pi$ for the critical density, we obtain

$$k^{3/2}\,|\Psi-\Phi| \sim \begin{cases} g_1^2 \Omega_\gamma(\eta) (\omega/\omega_1)^{-1/2} (\omega_1/H)^{-2}, & \mu \leq 3/4, \\ g_1^2 \Omega_\gamma(\eta) (\omega/\omega_1)^{-1/2} (\omega_1/H)^{2\mu-7/2}, & 3/4 \leq \mu \leq 3/2. \end{cases} \tag{3.36}$$

C. Axionic seeds

As a second example of seed inhomogeneities we will consider a pseudoscalar stochastic background, amplified according to the perturbation equation (3.12).

In the initial, higher-dimensional pre-big bang phase, i.e. for $\eta < \eta_1$, the solution for the canonical variable ψ can be written as in Eq. (3.8), with $\mu = |r| \leq 3/2$, as discussed previously. In the radiation era, i.e. for $\eta_1 < \eta < \eta_{eq}$, the effective potential $\ddot{a}_A/a_A$ is vanishing, as $\phi =$const and $a \sim \eta$, and ψ is given by the plane-wave solution (3.9). In the final matter-dominated era, i.e. for $\eta > \eta_{eq}$, we have $a \sim \eta^2$, and $\ddot{a}_A/a_A = 2/\eta^2$. The plane-wave solution is still valid for modes with $k > k_{eq} = \eta_{eq}^{-1}$, that are unaffected by the last transitions. Modes with $k < k_{eq}$ feel instead the effect of the potential in the matter era, and the general solution of Eq. (3.12), for those modes, can be written as

$$\psi_k(\eta) = \frac{\sqrt{k\eta}}{\sqrt{k}} \left(A H^{(2)}_{3/2} + B H^{(1)}_{3/2} \right)$$
$$= \frac{\sqrt{k\eta}}{\sqrt{k}} \left[(A+B) J_{3/2} - i(A-B) Y_{3/2} \right], \qquad k < k_{eq}, \quad \eta > \eta_{eq}. \tag{3.37}$$

Here $J_{3/2}$ and $Y_{3/2}$ are Bessel functions of argument $k\eta$ (we follow the conventions of [29]).

The matching of the solutions at η_1 determines the coefficients $c_\pm(k)$ as in eq. (3.10). The matching at η_{eq} gives

$$A + B \sim c(\mathbf{k}) \,(k\eta_{eq})^{-1}, \qquad A - B \sim c(\mathbf{k}) \,(k\eta_{eq})^{2}, \tag{3.38}$$

so that the contribution of $J_{3/2}$ to ψ_k is always dominant with respect to the $Y_{3/2}$ contribution, both for $k\eta > 1$ and $k\eta < 1$. In the matter-dominated era, i.e. for $\eta > \eta_{eq}$, we can thus approximate the produced stochastic axion background as follows:

$$\sigma(\mathbf{k},\eta) \simeq \frac{c(\mathbf{k})}{a\sqrt{k}} \sin k\eta, \qquad\qquad k > k_{eq},$$
$$\simeq \frac{c(\mathbf{k})}{a\sqrt{k}} \left(\frac{k}{k_{eq}} \right)^{-1} (k\eta)^2, \qquad k < k_{eq}, \quad k\eta < 1,$$
$$\simeq \frac{c(\mathbf{k})}{a\sqrt{k}} \left(\frac{k}{k_{eq}} \right)^{-1}, \qquad\qquad k < k_{eq}, \quad k\eta > 1. \tag{3.39}$$

The correlation functions for the various components of the stress tensor,

$$T^\nu_\mu = \partial_\mu \sigma \partial^\nu \sigma - \frac{1}{2} \delta^\nu_\mu \left(\partial_\alpha \sigma \right)^2 \tag{3.40}$$

can be computed by exploiting the stochastic average conditions of the Gaussian variables $\sigma, \dot{\sigma}$ and $\sigma_j = \partial_j \sigma$,

$$\langle\sigma(\mathbf{k})\sigma^*(\mathbf{k}')\rangle = (2\pi)^3\delta^3(k-k')\Sigma_1(\mathbf{k},\eta),$$
$$\langle\dot\sigma(\mathbf{k})\dot\sigma^*(\mathbf{k}')\rangle = (2\pi)^3\delta^3(k-k')\Sigma_2(\mathbf{k},\eta),$$
$$\langle\sigma_i(\mathbf{k})\sigma_j^*(\mathbf{k}')\rangle = k_ik_j(2\pi)^3\delta^3(k-k')\Sigma_1(\mathbf{k},\eta),$$
$$\langle\sigma_j(\mathbf{k})\dot\sigma^*(\mathbf{k}')\rangle = -\langle\dot\sigma(\mathbf{k})\sigma_j^*(\mathbf{k}')\rangle = ik_j(2\pi)^3\delta^3(k-k')\Sigma_3(\mathbf{k},\eta)\ , \tag{3.41}$$

where, according to Eq. (3.39),

$$\begin{aligned}\Sigma_1(\mathbf{k},\eta) &\simeq \frac{|c(\mathbf{k})|^2}{ka^2}, && k>k_{eq},\\ &\simeq \frac{|c(\mathbf{k})|^2}{ka^2}\left(\frac{k}{k_{eq}}\right)^{-2}(k\eta)^4, && k<k_{eq},\quad k\eta<1,\\ &\simeq \frac{|c(\mathbf{k})|^2}{ka^2}\left(\frac{k}{k_{eq}}\right)^{-2}, && k<k_{eq},\quad k\eta>1\end{aligned} \tag{3.42}$$

$$\begin{aligned}\Sigma_2(\mathbf{k},\eta) &\simeq k\frac{|c(\mathbf{k})|^2}{a^2}, && k>k_{eq},\\ &\simeq 0, && k<k_{eq},\quad k\eta<1,\\ &\simeq k\frac{|c(\mathbf{k})|^2}{a^2}\left(\frac{k}{k_{eq}}\right)^{-2}, && k<k_{eq},\quad k\eta>1,\end{aligned} \tag{3.43}$$

$$\begin{aligned}\Sigma_3(\mathbf{k},\eta) &\simeq \frac{|c(\mathbf{k})|^2}{a^2}, && k>k_{eq},\\ &\simeq 0, && k<k_{eq},\quad k\eta<1,\\ &\simeq \frac{|c(\mathbf{k})|^2}{a^2}\left(\frac{k}{k_{eq}}\right)^{-2}, && k<k_{eq},\quad k\eta>1,\end{aligned} \tag{3.44}$$

Following the same procedure as the one used for EM seeds, and collecting all contributions to the correlation function of the axion energy density,

$$\rho_\sigma = \frac{1}{2a^2}\left[\dot\sigma^2 + (\partial_i\sigma)^2\right], \tag{3.45}$$

we obtain from $\xi_0^0(x,x')$ that the energy density spectrum is determined by

$$\begin{aligned}k^3|f_\rho|^2\left(\frac{M}{a}\right)^4 &= \frac{2k^3}{(2a^2)^2}\int\frac{d^3p}{(2\pi)^3}\Bigg[\Sigma_2(\mathbf{p})\Sigma_2(\mathbf{k}-\mathbf{p}) + |\mathbf{p}\cdot(\mathbf{k}-\mathbf{p})|^2\,\Sigma_1(\mathbf{p})\Sigma_1(\mathbf{k}-\mathbf{p})\\ &\quad - 2\mathbf{p}\cdot(\mathbf{k}-\mathbf{p})\Sigma_3(\mathbf{p})\Sigma_3(\mathbf{k}-\mathbf{p})\Bigg].\end{aligned} \tag{3.46}$$

In order to evaluate this integral outside the horizon, in the region $k\eta\le 1$, we must distinguish two cases, $\mu<3/4$ and $\mu>3/4$. In both cases, by separate integration in the ranges $0<p<\eta^{-1}$, $\eta^{-1}<p<k_{eq}$, $k_{eq}<p<k_1$, we find a white noise spectrum,$|f_\rho|\sim$ const. In particular (see Appendix B):

$$k^{3/2}\,|f_\rho|\left(\frac{M}{a}\right)^2 = \begin{cases} d_\rho^\sigma(k_1/a)^4(k/k_1)^{3/2}\left[1+\delta_\rho^\sigma(k_{eq}/k_1)^2(k_1\eta)^{2\mu+1/2}\right], & \mu\le 3/4\\ c_\rho^\sigma(k_1/a)^4(k/k_1)^{3/2}(k_{eq}/k_1)^2(k_1\eta)^{2\mu+1/2}, & 3/4\le\mu\le 3/2,\end{cases} \tag{3.47}$$

where $c_\rho^\sigma, d_\rho^\sigma, \delta_\rho^\sigma$ are dimensionless numbers of order 1. The same power spectrum is also obtained for the scalar velocity potential f_v, associated to the axion seeds. An explicit computation gives in fact $kf_v\sim k\eta f_\rho$ so that the contribution of f_ρ and f_v to the Bardeen potential are both of the same order, namely:

$$\begin{aligned}&\epsilon\eta^2\,|f_\rho|\,k^{3/2}\sim\epsilon\eta^2\frac{\dot a}{a}\,|f_v|\,k^{3/2} =\\ &= \begin{cases} 4\pi G d_\rho^\sigma(a\eta)^2(k_1/a)^4(k/k_1)^{3/2}\left[1+\delta_\rho^\sigma(k_{eq}/k_1)^2(k_1\eta)^{2\mu+1/2}\right], & \mu\le 3/4\\ 4\pi G c_\rho^\sigma(a\eta)^2(k_1/a)^4(k/k_1)^{3/2}(k_{eq}/k_1)^2(k_1\eta)^{2\mu+1/2}, & 3/4\le\mu\le 3/2.\end{cases}\end{aligned} \tag{3.48}$$

We will now consider the anisotropic stress potential f_π, defined according to (2.23) by:

$$\nabla^4 f_\pi = \frac{3}{2M^2}\partial_i\partial_j\left[\sigma_i\sigma_j - \frac{1}{3}\delta_{ij}(\partial_k\sigma)^2\right], \qquad \nabla^2 = \delta_{ij}\partial_i\partial_j. \tag{3.49}$$

Summing all contributions in the two point correlation function, we find

$$\langle\nabla^4 f_\pi(x)\nabla^4 f_\pi(x')\rangle - \left(\langle\nabla^4 f_\pi\rangle\right)^2 = \frac{9}{2M^4}\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\Delta\mathbf{x}}k^4$$
$$\int\frac{d^3p}{(2\pi)^3}p^2|\mathbf{k}-\mathbf{p}|^2\Sigma_1(\mathbf{p})\Sigma_1(\mathbf{k}-\mathbf{p})\left(\cos^2\vartheta\cos^2\gamma - \frac{1}{3}\cos\vartheta\cos\gamma\cos\alpha + \frac{1}{9}\cos^2\alpha\right), \tag{3.50}$$

where ϑ, α and γ are, respectively, the angles between $\mathbf{p}$ and $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{k}-\mathbf{p}$ and $\mathbf{k}$ and $\mathbf{k}-\mathbf{p}$. The integral over p is of the same type as the integral for the energy density spectrum (see Eq. (3.46)), and gives for $k^2 f_\pi$ the same white noise spectrum (3.47) as for f_ρ (modulo numbers of order 1) , since

$$k^{3/2}\left|f_\pi(k)\right|\left(\frac{k}{a}\right)^2 M^2 \sim k^{3/2}\left|f_\rho(k)\right|\left(\frac{M}{a}\right)^2. \tag{3.51}$$

On super-horizon scales the contribution of f_π to the Bardeen potentials is always dominant with respect to the f_ρ contribution since, from the above equation,

$$\eta^2 f_\rho \sim (k\eta)^2 f_\pi. \tag{3.52}$$

In the whole range $k\eta \leq 1$ we can thus estimate the scalar perturbation spectrum, induced by massless axion seeds, through the f_π contribution to the Bardeen potentials. We find

$$k^{3/2}\left|\Psi-\Phi\right| \sim \epsilon k^{3/2}\left|f_\pi\right| =$$
$$\begin{cases} 4\pi G d_\pi^\sigma (a\eta)^2(k_1/a)^4(k/k_1)^{-1/2}(k_1\eta)^{-2}\left[1+\delta_\rho^\sigma(k_{eq}/k_1)^2(k_1\eta)^{2\mu+1/2}\right], & \mu \leq 3/4, \\ 4\pi G c_\pi^\sigma (a\eta)^2(k_1/a)^4(k/k_1)^{-1/2}(k_{eq}/k_1)^2(k_1\eta)^{2\mu-3/2}, & 3/4 \leq \mu \leq 3/2, \end{cases} \tag{3.53}$$

$$\sim \begin{cases} g_1^2\Omega_\gamma(\eta)(\omega/\omega_1)^{-1/2}(\omega_1/H)^{-2}\left[1+\delta_\rho^\sigma(\omega_{eq}/\omega_1)^2(\omega_1/H)^{2\mu+1/2}\right], & \mu \leq 3/4, \\ g_1^2\Omega_\gamma(\eta)(\omega/\omega_1)^{-1/2}(\omega_{eq}/\omega_1)^2(\omega_1/H)^{2\mu-3/2}, & 3/4 \leq \mu \leq 3/2. \end{cases} \tag{3.54}$$

where $c_\pi^\sigma, d_\pi^\sigma, \delta_\pi^\sigma$ are dimensionless numbers of order 1. As we will see in Section IV, the dominant contribution to the SW effect now comes, for each mode, from the time of reentry $\eta \sim 1/k$.

Let us finally discuss the case of massive axions, with

$$T_\mu^\nu = \partial_\mu\sigma\partial^\nu\sigma - \frac{1}{2}\delta_\mu^\nu\left[(\partial_\alpha\sigma)^2 - m^2\sigma^2\right], \tag{3.55}$$

and a primordial distribution again characterized by the index μ, as in Eq. (3.19). The mass term directly contribute to f_ρ and f_p, and only indirectly to the off-diagonal potentials f_v, f_π. We are interested in the axion perturbations that may be relevant to the large-scale CMB anisotropy, namely in the modes that are outside the horizon at the decoupling era, $k < aH_{dec}$. If, for these modes, the mass contribution is negligible, $ma < k < aH_{dec}$, then the AX seed functions and the corresponding Bardeen potentials are the same as in the massless case (see before). We will thus concentrate our discussion on the case in which the axion mass is large enough, so that all modes outside the horizon at the equilibrium epoch are already non-relativistic:

$$ma > aH_{eq} > k. \tag{3.56}$$

In this case we may neglect the effects of an additional axion production in the matter-dominated era, since $a^2m^2 > \ddot{a}/a$ at $\eta \geq \eta_{eq}$. The axion fluctuations are amplified by the inflation $\rightarrow$ radiation transition, but are to be evaluated in the non-relativistic regime ($\eta > \eta_{eq}$), where the mass contribution is already important.

For non-relativistic, super-horizon modes, the Fourier components of the axion field become (see the non-trivial calculation reported in Appendix C):

$$\sigma(\mathbf{k},\eta) = \frac{c(\mathbf{k})}{a\sqrt{ma}}\left(\frac{k}{k_1}\right)^{1/2}\left(\frac{H_1}{m}\right)^{1/4}\sin\left(\frac{m}{H}\right), \qquad k < k_m = k_1\left(\frac{m}{H_1}\right)^{1/2}, \tag{3.57}$$

where the initial distribution $c(k)$ is still given by Eq. (3.19). Here k_m is the limiting frequency re-entering the horizon at the same time as it becomes non-relativistic, i.e. $k_m/a_m = H_m = m$. Indeed, we are assuming that at the transition scale H_1 the mass term is completely negligible, $m \ll H_1$, and all modes are relativistic. As the proper momentum is red-shifted, the modes become non-relativistic when $m = k/a = \omega$, and re-enter the horizon when $H = \omega$.

For the axion field (4.14) the stochastic conditions (3.41) are still valid, but the squared amplitude (3.42), averaged over time scales $m/H \gg 1$, now become

$$\Sigma_1(\mathbf{p},\eta) = \frac{|c(\mathbf{k})|^2}{2ma^3}\left(\frac{k}{k_1}\right)\left(\frac{H_1}{m}\right)^{1/2} = \frac{1}{m^2a^2}\Sigma_2(\mathbf{p},\eta) = \frac{1}{ma}\Sigma_3(\mathbf{p},\eta). \tag{3.58}$$

For the case we are considering, the contribution of f_π to the Bardeen potentials is always negligible with respect to $\eta^2 f_\rho$. An explicit computation gives, in fact,

$$\eta^2 f_\rho/f_\pi \simeq m/H \gg 1, \tag{3.59}$$

where the last inequality is a consequence of (4.13). In addition, the mass contribution to the AX energy density dominates with respect to the momentum contribution, since $m > k/a$. The energy density correlation function thus becomes:

$$\xi_0^0(x,x') = m^4\left(\langle\sigma^2(x)\sigma^2(x')\rangle - \langle\sigma^2(x)\rangle^2\right) \tag{3.60}$$

(as $|\dot\sigma(k)| = ma|\sigma(k)|$), and gives, using Eq. (4.15):

$$\begin{aligned} k^3\,|f_\rho|^2\left(\frac{M}{a}\right)^4 &= m^4k^3\int\frac{d^3p}{(2\pi)^3}\Sigma_1(\mathbf{p})\Sigma_1(\mathbf{k}-\mathbf{p}) \\ &= \frac{mH_1}{8\pi^2}\left(\frac{k_1}{a}\right)^6\left(\frac{k}{k_1}\right)^3\int_0^1 dy y^{2-2\mu}\int_{-1}^1 dx\beta^{-2\mu} \end{aligned} \tag{3.61}$$

where x, y and β are defined in Section III B.

It should be noted that the above expression for the spectrum is only valid if $\mu > 3/4$. Only in that case, in fact, is the integral over $y$dominated by the contribution of the lower boundary, $p/k_1 \to 0$, and is the use of Eq. (3.58) for the axion spectrum appropriate. In the opposite case, we have to take into account the different spectrum of non-relativistic sub-horizon modes, for $p > k_m$, and possibly of relativistic modes in the high-frequency limit $p \to k_1$. In both cases we obtain, for $\mu < 3/4$, a white noise spectrum and a negligible contribution to the large-scale anisotropy, as we will see in the next section.

We will thus concentrate on the case $3/4 < \mu \leq 3/2$. For this case the integral (3.61) is estimated in Appendix B, and we obtain

$$k^3\,|f_\rho|^2\left(\frac{M}{a}\right)^4 = c_m^2 mH_1\left(\frac{k_1}{a}\right)^6\left(\frac{k}{k_1}\right)^{6-4\mu}, \qquad 3/4 < \mu \leq 3/2\,, \tag{3.62}$$

where c_m is a dimensionless number of order 1. The corresponding Bardeen spectrum is:

$$\begin{aligned} &k^{3/2}\,|\Psi| \sim k^{3/2}\,|\Phi| \sim \epsilon\eta^2\,|f_\rho|\,k^{3/2} = \\ &= 4\pi G c_m (a\eta)^2\,(mH_1)^{1/2}\left(\frac{k_1}{a}\right)^3\left(\frac{k}{k_1}\right)^{3-2\mu} \\ &\sim g_1^2 c_m\left(\frac{m}{H_1}\right)^{1/2}\left(\frac{H_1}{H}\right)^2\left(\frac{a_1}{a}\right)^3\left(\frac{\omega}{\omega_1}\right)^{3-2\mu} \sim \Omega_\sigma(\omega). \end{aligned} \tag{3.63}$$

We may note that ΨH^2 evolves in time like a^{-3}, so that, during the matter-dominated era (when $H^2 \propto a^{-3}$), the Bardeen potential Ψ remains frozen at the value reached at the time η_{eq} of matter-radiation equilibrium. Using $(H_1/H_{eq})^2(a_1/a_{eq})^3 = (a_{eq}/a_1) = (H_1/H_{eq})^{1/2}$, we obtain for $\eta > \eta_{eq}$,

$$k^{3/2}\,|\Psi| \sim k^{3/2}\,|\Phi| \sim c_m g_1^2\left(\frac{m}{H_{eq}}\right)^{1/2}\left(\frac{\omega}{\omega_1}\right)^{3-2\mu}. \tag{3.64}$$

The CMB anisotropy induced by the EM and AX seeds discussed here will be analysed in the next section.

IV. CMB FLUCTUATIONS FROM PRE-BIG BANG SEEDS

For electromagnetic seeds, with the assumption that the electric field is already dissipated away at recombination, we find that the seeds are generically suppressed by a factor $(k\eta_{dec})^2$, and the anisotropic stress f_π dominates over the density contribution f_ρ (see the discussion at the end of Section II A). By contrast, for massless axionic perturbations, there is no $(k\eta_{dec})^2$ suppression for f_ρ, while there is one for f_π. For large wave numbers which enter the horizon before matter and radiation equality, EM and AX seeds lead to similar amplitudes. Consequently, if the convolution leading to f_π is dominated by small scale contributions, $\mu < 3/4$, the two cases give similar geometric scalar perturbations Ψ, Φ, through Eq. (2.40).

However, on large scales, $k\eta_{eq} < 1$, the additional axion production during the matter-dominated era leads to an enhancement by the factor $(\eta/\eta_{eq})^2$. This changes the time-dependence of the Bardeen potentials and has important consequences as we will see below.

A. Electromagnetic seeds

The scalar metric perturbation spectrum induced by EM seeds is reproduced in Eqs. (3.35) and (3.36). Comparing with our parametrization in terms of α and γ (see Eqs. (2.59), (2.60)) we find

$$\gamma = \begin{cases} -4, & \mu \leq 3/4 \\ 2\mu - 11/2, & 3/4 \leq \mu \leq 3/2 \end{cases} \tag{4.1}$$

$$\alpha = \begin{cases} 7/2, & \mu \leq 3/4 \\ 5 - 2\mu, & 3/4 \leq \mu \leq 3/2 \end{cases} \tag{4.2}$$

and

$$\mathcal{N} = c_\pi \left(\frac{g_1}{4\pi}\right)^2 (k_1\eta_{eq})^2 \tag{4.3}$$

in both cases $\mu < 3/4, \mu > 3/4$ (modulo numbers of order 1).

Since $\gamma + 1 < 0$, in both cases the seeds decay fast enough outside the horizon, and our analysis of Section II applies. However, in both cases $\gamma + \alpha = -0.5$, which leads to the spectral index $n = 2$, i.e. to a spectrum that grows too fast with frequency to fit the results of COBE observations, see Eqs. (2.71), (2.72).

The quadrupole amplitude is given by $Q_{rms-PS} = \sqrt{(5/4\pi)C_2}T_0$, which has been measured [33] to be $Q_{rms-PS} = (18 \pm 2)\mu K$. This leads to

$$C_2 = (1.09 \pm 0.23) \times 10^{-10} \ . \tag{4.4}$$

From Eq. (2.70), using $\alpha + \gamma = -1/2$, $k_1\eta_{eq} = (H_1/H_{eq})^{1/2}$, $g_1 = H_1/M_p$, and setting $\ell = 2$, we obtain:

$$C_2^{SW} \approx \frac{c_\pi^2 g_1^{6-\alpha}}{10(4\pi)^4(\gamma+1)^2} \left(\frac{M_p}{H_{eq}}\right)^{2-\alpha} \left(\frac{\eta_{dec}}{\eta_0}\right)^{2(\gamma+1)} \left(\frac{\eta_{eq}}{\eta_0}\right)^{2\alpha} \ . \tag{4.5}$$

Compatibility with the COBE normalization, $C_2 \lesssim 10^{-10}$, thus implies

$$(6-\alpha)\log_{10} g_1 \lesssim 55(\alpha - 2) - 6 + \log_{10}(\gamma+1)^2 - \log_{10} c_\pi^2 \tag{4.6}$$

(we have used $H_{eq}/M_p \sim 10^{-55}$, and $\eta_{dec} \sim \eta_{eq} \sim 10^{-2}\eta_0$). This important constraint is easily satisfied by a growing seed spectrum, $\mu < 3/2$, i.e. $\alpha > 2$. In the limiting (and most unfavorable) case $\mu = 3/2, \quad \alpha = 2, \quad \gamma = -5/2$, the above condition reduces to

$$\log_{10} g_1 \lesssim \ -1.4 - 0.5 \log_{10} c_\pi \ . \tag{4.7}$$

Even in this limiting case there are no stringent constraints on the typical inflation scale of the "minimal" pre-big bang scenario [1,7,8], expected to approach the string mass scale M_s as $g_1 = H_1/M_p \sim M_s/M_p$. Indeed, the limiting condition (4.7) is marginally compatible even with the maximal expected value $H_1 \sim M_s$, since [34]

$$10^{-2} \lesssim M_s/M_p \lesssim 10^{-1}. \tag{4.8}$$

To conclude, the EM fluctuations seem to lead to a scalar perturbation spectrum that grows too fast with frequency to contribute in a significant way to the observed large-scale anisotropy. The positive aspect of our result is that there are no significant constraints from the COBE normalization to the production of seeds for galactic magnetic fields, which remains allowed as discussed in [11].

B. Axionic seeds

Let us first consider massless axions. If $\mu < 3/4$, the situation is like in the electromagnetic case. The CMB fluctuations induced have the wrong spectrum, but their amplitude is sufficiently low to avoid conflict with observations.

If $3/4 \leq \mu \leq 3/2$ the situation becomes radically different. Comparing Eq. (3.53) with the ansatz (2.59), (2.60) we obtain, due to the additional factor $(\eta/\eta_{eq})^2$,

$$\gamma = 2\mu - 7/2, \tag{4.9}$$

$$\alpha = -\gamma - 1/2 = -2\mu + 3\ . \tag{4.10}$$

In the limiting case $\mu = 3/2$ this yields $\gamma = -1/2$ and $\alpha = 0$, which corresponds to a Harrison-Zel'dovich spectrum of CMB fluctuations, according to Eq. (2.73), with an amplitude

$$\mathcal{N} \simeq g_1^2 \tag{4.11}$$

(we have absorbed into g_1 all dimensionless numerical coefficients of order one appearing in the spectrum (3.53)). Note that f_ρ leads to a Bardeen potential with the same α, but with $\gamma = 2\mu - 3/2$. However, since again $\gamma > -1$, the contribution to the SW effect is the same for f_ρ and f_π (see Section II).

The normalization of the axion spectrum to the COBE amplitude (4.4), according to Eq. (2.65), imposes the condition

$$C_2^{SW} \simeq \mathcal{N}^2 \left(k_1\eta_0\right)^{-2\alpha} \simeq g_1^4 \left(\frac{\omega_0}{\omega_1}\right)^{6-4\mu} \simeq 10^{-10}, \tag{4.12}$$

which implies

$$\log_{10} g_1 \simeq \frac{164 - 116\mu}{1 + 2\mu} \tag{4.13}$$

(again we have absorbed numerical coefficients into g_1, and we have used $\omega_0 \sim 10^{-18}$ Hz, $\omega_1 \sim g_1^{1/2} 10^{11}$ Hz, according to [7,8]). On the other hand, the allowed range for the spectral index (see Eq. (2.73), combined with the condition $\mu \leq 3/2$ (required to prevent over-critical axion production), implies

$$1.4 < \mu < 1.5. \tag{4.14}$$

The combination of (4.13), (4.14) leads to

$$3 \times 10^{-3} \lesssim g_1 = (H_1/M_p) \lesssim 2.6, \tag{4.15}$$

which is perfectly compatible with the identification $H_1 \sim M_s$ (see Eq, (4.8).

A stochastic background of massless axions, produced in the context of the pre-big bang scenario, is thus a possible viable candidate for a consistent explanation of the large-scale anisotropy observed by COBE. The important difference between AX and EM seeds is the non-conformal coupling of the axions to the metric, that leads to an additional amplification of perturbations after the matter-radiation equality.

Another interesting case is that of a massive axion background, for which the f_π contribution to the Bardeen potentials is negligible when the super-horizon modes are already non-relativistic at the time of decoupling, $m > H_{dec}$. As seen in the previous section, one then obtains constant Bardeen potentials, with $\gamma = 0$, $\alpha = (n-1)/2 = 3 - 2\mu$ and

$$\mathcal{N} = g_1^2 \left(\frac{m}{H_{eq}}\right)^{1/2} \tag{4.16}$$

(see Eq. (3.64), where we have set $c_m = 1$). A flat Harrison-Zel'dovich spectrum is again possible in the limiting case $\mu = 3/2$. The amplitude of perturbations, however, is enhanced by the factor $(m/H_{eq})^{1/2}$, so that the value of the axion mass has to be bounded, to avoid conflicting with the COBE normalization (4.4).

The allowed range for the spectral index, $0.8 \leq n \leq 1.4$, and the condition $\mu < 3/2$ to avoid over-critical axion production, again imply for the parameter μ the allowed range (4.14). In addition, the present axion energy density is constrained by the critical density bound, $\Omega_\sigma(\eta_0) \leq 1$, imposed at the peak frequency ω_m of non-relativistic modes (see Appendix C). Actually, an even stronger condition is required for the validity of our perturbative approach, which

neglects the back-reaction of the axionic seeds on the expansion of the universe. Using Eq. (C25) we thus impose the condition

$$\Omega_\sigma(\omega_m,\eta_0)\sim g_1^2\left(\frac{H_1}{H_{eq}}\right)^{1/2}\left(\frac{m}{H_1}\right)^{2-\mu}\lesssim 0.1, \tag{4.17}$$

which implies

$$(2-\mu)\left[\log_{10}(m/H_{eq})-\log_{10} g_1-55\right]+\frac{5}{2}\log_{10} g_1+\frac{55}{2}<-1. \tag{4.18}$$

In order to find a possible AX mass window compatible with the COBE data, we now impose the normalization $C_2\simeq 10^{-10}$ on the massive axion spectrum Eq. (3.64). From Eq. (2.65) we obtain

$$C_2^{SW}\simeq \mathcal{N}^2(k_1\eta_0)^{-2\alpha}\simeq g_1^4\left(\frac{m}{H_{eq}}\right)\left(\frac{\omega_0}{\omega_1}\right)^{6-4\mu}\simeq 10^{-10}, \tag{4.19}$$

from which

$$\mu\simeq\left[164-\log_{10}(m/H_{eq})-4\log_{10} g_1\right]/116 \tag{4.20}$$

(we have used $\omega_1/\omega_0\sim 10^{29}$, neglecting the weak dependence of ω_1 on the transition scale g_1). By eliminating μ in terms of m and g_1, according to the above equation, the constraints (4.14) and (4.18), plus the condition $m>H_{dec}\sim H_{eq}$ (assumed for the validity of the spectrum (3.64)), determine an allowed region in the plane (m,H_1) as follows:

$$\begin{cases} 10^{-10}\left(M_p/H_1\right)^4\lesssim m/H_{eq}\lesssim 10^{1.6}\left(M_p/H_1\right)^4, & m\gtrsim H_{eq},\\ \left[68+\log_{10}(m/H_{eq})+4\log_{10} g_1\right]\left[\log_{10}(m/H_{eq})-55-\log_{10} g_1\right]+58\left(55+5\log_{10} g_1\right)\lesssim -1. \end{cases} \tag{4.21}$$

For a typical string cosmology scale, $H_1\sim M_s\sim(10^{-1}-10^{-2})M_p$, we thus obtain the maximal allowed window:

$$10^{-27}\text{ eV}\lesssim m\lesssim 10^{-17}\text{ eV}. \tag{4.22}$$

As illustrated in Fig. 1, the window is shifted towards higher values of mass as the final inflation scale is lowered, and the seed condition (4.17) becomes important only at low inflation scales, $H_1/M_p\lesssim 10^{-7}$. The stringent upper limit we obtained for the mass can be traced back to the simplest model of background used in this paper, that gives the same slope for the axion spectrum at low and high frequency (see eqs. (C24), (C26)). It is not excluded that higher values of the mass may become possible in a more complicated model of background, giving a steeper high frequency spectrum.

V. CONCLUSIONS

In this paper we have considered the possibility that, in a string cosmology context, the large-scale temperature anisotropies may arise from the contribution of seeds to the metric fluctuations, and *not* from the direct amplification of the metric fluctuations themselves, as in the conventional inflationary scenario. We have discussed, in particular, two cases: one in which the seeds are EM vacuum fluctuations amplified by the growth of the dilaton field, and one in which the seeds are AX vacuum fluctuations amplified by the time evolution of a higher-dimensional background.

In the case of EM perturbations we have found that the induced angular power spectrum of $\Delta T/T$ grows too fast to be compatible with COBE observations. However, the contribution of the seeds to the large-scale anisotropy may be consistently imposed to be negligible, without constraining in a significant way the basic parameters of the pre-big bang models.

Massless AX perturbations, unlike EM perturbations, are also affected by the radiation $\rightarrow$ matter transitions. This changes the time dependence of the seed contribution to the Bardeen potentials and, due to the integrated Sachs-Wolfe effect, a flat or slightly tilted blue spectrum of temperature anisotropies can be induced, compatible with present COBE observations. Scale-invariant massless axion seeds thus appear as possible promising candidates for structure formation. Determining in more details the CMB anisotropy spectrum also on smaller angular scales requires however numerical simulations, which we defer to a future research project.

For massive AX seeds the situation is qualitatively different if the mass is such that all modes outside the horizon at the time of decoupling are already non-relativistic. In that case the contribution to $\Delta T/T$ is controlled by the axion

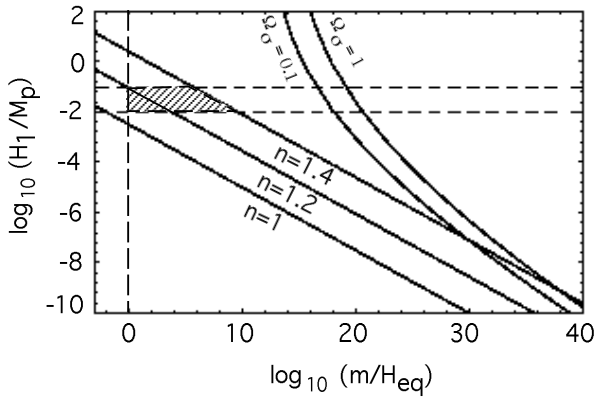

FIG. 1. *The phenomenologically allowed region is to the left of the curve $\Omega_\sigma = 0.1$, to the right of the vertical dashed line $m = H_{eq}$, and lies within the full lines $n = 1$, $n = 1.4$, to avoid conflicting with present COBE observations ($n < 1$ is excluded by over-critical axion production). The shaded area defines the allowed mass window for an inflation scale $H_1 = M_s$, typical of string cosmology.*

mass, and a slightly tilted blue spectrum is still compatible with the amplitude and the slope measured by COBE, provided the axion mass is inside an appropriate window, in the ultra-light mass region. Higher values of masses may become possible in models with more complicated backgrounds.

At smaller angular scales, an axionic origin of CMB anisotropies should lead to acoustic peaks in the spectrum, with a structure different from that of the standard inflationary scenario. This may in principle allow a test of models with axionic seeds through the very accurate observations of the CMB anisotropy planned in the near future [35]. A thorough investigation of this possibility is postponed to a future paper.

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APPENDIX A: SACHS-WOLFE COEFFICIENTS FOR POWER-LAW SPECTRA

Assume that the Bardeen potentials are given by power-law spectra as in Eq. (2.58),

$$\Psi - \Phi = \begin{cases} C(k)x^\gamma, & x \ll 1 \\ C(k), & x \gg 1 \end{cases} \quad , \qquad C(k) = \mathcal{N} k^{-3/2} (k/k_1)^\alpha \ , \tag{A1}$$

where $x = k\eta$, $x_0 = k\eta_0$, $x_{dec} = k\eta_{dec}$. The SW contribution to the angular coefficients C_ℓ is given by

$$C_\ell^{SW} = \mathcal{N}^2 \frac{2}{\pi} \int_0^{k_1} \frac{dk}{k} \left(\frac{k}{k_1}\right)^{2\alpha} |I(k)|^2, \tag{A2}$$

where

$$I(k) = j_\ell(x_0 - 1) + \int_{x_{dec}}^1 x^\gamma j_\ell'(x_0 - x) dx \ , \tag{A3}$$

and a prime stands for the derivative of j_ℓ with respect to its argument.

We concentrate here on the case where $\gamma + 1 < 0$. Furthermore, we are interested in the situation where the integral in Eq. (A2) is dominated by large scales (small values of k), and therefore $x_{dec} \ll 1$. In that case the integral $I(k)$ is dominated by its value at the lower bound:

$$I(k) \approx \frac{1}{|1+\gamma|} x_{dec}^{\gamma+1} j_\ell'(x_0) = \frac{1}{|1+\gamma|} x_{dec}^{\gamma+1} \left[\frac{\ell}{2\ell+1} j_{\ell-1}(x_0) - \frac{\ell+1}{2\ell+1} j_{\ell+1}(x_0) \right]. \tag{A4}$$

This leads to the following expression for the C_ℓ's:

$$\begin{aligned} C_\ell^{SW} &= \mathcal{N}^2 \frac{2}{\pi} \left(\frac{\eta_{dec}}{\eta_0} \right)^{2(\gamma+1)} \frac{1}{|1+\gamma|^2} (k_1\eta_0)^{-2\alpha} \int_0^\infty \frac{dx_0}{x_0} x_0^{2(\alpha+\gamma+1)} \\ &\quad \times \left[\frac{\ell^2}{(2\ell+1)^2} j_{\ell-1}^2(x_0) - \frac{2\ell(\ell+1)}{(2\ell+1)^2} j_{\ell-1}(x_0) j_{\ell+1}(x_0) + \frac{(\ell+1)^2}{(2\ell+1)^2} j_{\ell+1}^2(x_0) \right] \\ &= \frac{\mathcal{N}^2}{|1+\gamma|^2} \frac{2}{\pi} \left(\frac{\eta_{dec}}{\eta_0} \right)^{2(\gamma+1)} (k_1\eta_0)^{-2\alpha} \\ &\quad \times \left[\frac{\ell^2}{(2\ell+1)^2} I_\ell^{(1)} - \frac{2\ell(\ell+1)}{(2\ell+1)^2} I_\ell^{(2)} + \frac{(\ell+1)^2}{(2\ell+1)^2} I_\ell^{(3)} \right], \end{aligned} \tag{A5}$$

where, setting $j_\ell = \sqrt{\pi/x} J_{\ell-1/2}$, we find (Ref. [36], number 6.574) for $\alpha + \gamma < 0$,

$$\begin{aligned} I_\ell^{(1)} &= \frac{\pi}{2} \int_0^\infty dx x^{2(\alpha+\gamma)} J_{\ell-1/2}^2(x) \\ &= \frac{\pi}{2} \frac{\Gamma(-2(\alpha+\gamma))\Gamma(\ell+\alpha+\gamma)}{2^{-2(\alpha+\gamma)}[\Gamma(-(\alpha+\gamma)+1/2)]^2\Gamma(\ell-(\alpha+\gamma))}\ ; \end{aligned} \tag{A6}$$

$$\begin{aligned} I_\ell^{(2)} &= \frac{\pi}{2} \int_0^\infty dx x^{2(\alpha+\gamma)} J_{\ell-1/2}(x) J_{\ell+3/2}(x) \\ &= \frac{\pi}{2} \frac{\Gamma(-2(\alpha+\gamma))\Gamma(\ell+1+\alpha+\gamma)}{2^{-2(\alpha+\gamma)}\Gamma(-(\alpha+\gamma)-1/2)\Gamma(3/2-(\alpha+\gamma))\Gamma(\ell+1-(\alpha+\gamma))}\ ; \end{aligned} \tag{A7}$$

$$\begin{aligned} I_\ell^{(3)} &= \frac{\pi}{2} \int_0^\infty dx x^{2(\alpha+\gamma)} J_{\ell+3/2}^2(x) \\ &= \frac{\pi}{2} \frac{\Gamma(-2(\alpha+\gamma))\Gamma(\ell+2+\alpha+\gamma)}{2^{-2(\alpha+\gamma)}[\Gamma(-(\alpha+\gamma)+1/2)]^2\Gamma(\ell+2-(\alpha+\gamma))}\ . \end{aligned} \tag{A8}$$

Finally, combining the above results, we obtain the result given in Eq. (2.70):

$$\begin{aligned} C_\ell^{SW} &= \frac{\mathcal{N}^2}{2^{-2(\alpha+\gamma)}(\gamma+1)^2} \frac{\Gamma(-2(\alpha+\gamma))}{\Gamma(1/2-\alpha-\gamma)^2} \left(\frac{\eta_{dec}}{\eta_0} \right)^{2(\gamma+1)} (k_1\eta_0)^{-2\alpha} \frac{\Gamma(\ell+1+\alpha+\gamma)}{\Gamma(\ell+1\alpha-\gamma)} \\ &\times \left[\frac{\ell^2}{(2\ell+1)^2} \frac{\ell-\alpha-\gamma}{\ell+\alpha+\gamma} + \frac{2\ell(\ell+1)}{(2\ell+1)^2} \frac{1/2+\alpha+\gamma}{1/2-\alpha-\gamma} + \frac{(\ell+1)^2}{(2\ell+1)^2} \frac{\ell+1+\alpha+\gamma}{\ell+1-\alpha-\gamma} \right]. \end{aligned} \tag{A9}$$

It is interesting to note that, for $\alpha + \gamma = -1/2$, the mixed term $I_\ell^{(2)}$ vanishes, which is indeed what happens in the case of electromagnetic seeds (see Section IV).

APPENDIX B: THE SEED FUNCTIONS

1. Electromagnetic Seeds

For purely magnetic seeds, all the seed functions can be approximately determined by the energy density correlation function ξ_0^0, which leads to Eq. (3.30). The contribution of super-horizon modes ($k\eta \ll 1$) to the spectrum can be estimated in the limit $z = k/k_1 \to 0$. In this limit $\beta \to y$, $\cos^2\alpha \to 1$, and the integral (3.30) reduces to

$$I = \frac{k^3 k_1^5}{a^8} \int_0^1 dy\ y^{2-4\mu} \sin^4(y k_1 \eta), \qquad \mu \leq 3/2. \tag{B1}$$

The dominant region of integration is easily shown to be $y \sim 1$ for $\mu \leq 3/4$ and $y k_1 \eta \sim 1$ for $3/4 \leq \mu \leq 3/2$. This gives

$$I = \begin{cases} (k_1/a)^8 (k/k_1)^3, & \mu \leq 3/4 \\ (k_1/a)^8 (k/k_1)^3 (k_1\eta)^{4\mu-3}, & 3/4 \leq \mu \leq 3/2 \end{cases}, \tag{B2}$$

modulo numerical factors of order one. This coincides with the result reported in Eq. (3.32).

2. Massless Axions

For massless axions, the seed spectral functions are determined by the integrals (3.46), (3.50). The various terms appearing in the integrands turn out to give comparable contributions, so let us concentrate on the typical term

$$I = \frac{k^3}{a^4} \int d^3p \; p^2 |\mathbf{k} - \mathbf{p}|^2 \Sigma_1(\mathbf{p}) \Sigma_1(\mathbf{k} - \mathbf{p}). \tag{B3}$$

We distinguish different integration regions: $0 < p < k$, $k < p < \eta^{-1}$, $\eta^{-1} < p < k_{eq}$, $k_{eq} < p < k_1$. The dominant integration regions depend on the value of μ but, for all $\mu \leq 3/2$, they always lie at $p \geq \eta^{-1} > k$. This is the reason why we always obtain a white noise spectrum. On the other hand, the behaviour in η depends on which region of p dominates. Specifically we find:

1) For $3/4 \leq \mu \leq 3/2$ the leading contribution to I comes from $p \sim \eta^{-1}$, and gives the single term appearing in eq. (3.47).

2) For $\mu < 3/4$ the leading contribution comes either from $p \sim k_1$ (giving the first term in the square brackets of (3.47)), or (for μ very close to 3/4) from $p \sim \eta^{-1}$ (giving the second term in the same brackets).

3. Massive Axions

For massive actions, the energy density spectrum is determined by Eq. (3.61), with $3/4 < \mu \leq 3/2$. This integral is dominated by the region $p \sim k$, and its rough behaviour can be easily obtained this way. For a more precise evaluation we proceed as follows: the angular integration gives

$$k^3 |f_\rho|^2 \left(\frac{M}{a}\right)^4 = \frac{m H_1}{16\pi^2 z(\mu - 1)} \left(\frac{k_1}{a}\right)^6 \left(\frac{k}{k_1}\right)^3 \int_0^1 dy \; y^{1-2\mu} \left[(z-y)^{2-2\mu} - (z+y)^{2-2\mu}\right]. \tag{B4}$$

Defining $t = y/z$ we obtain

$$k^3 |f_\rho|^2 \left(\frac{M}{a}\right)^4 = \frac{m H_1}{16\pi^2 (\mu - 1)} \left(\frac{k_1}{a}\right)^6 \left(\frac{k}{k_1}\right)^3 z^{3-4\mu} \, (A - B) \tag{B5}$$

where, after some manipulation [36],

$$\begin{aligned} A &= \int_0^\infty dt \; t^{1-2\mu} \left[(1-t)^{2-2\mu} - (1+t)^{2-2\mu}\right] = \\ &= \frac{2^{4\mu-4}}{\sqrt{\pi}} \Gamma(2-2\mu) \Gamma(2\mu - 3/4) \left[\cos 2\pi(\mu - 1) - 1\right] \end{aligned} \tag{B6}$$

and

$$B = \int_{1/z}^\infty dt \; t^{1-2\mu} \left[(1-t)^{2-2\mu} - (1+t)^{2-2\mu}\right]. \tag{B7}$$

By evaluating this second integral in the limit $z \to 0$, we obtain

$$B \sim z^{4\mu - 3} \ll A. \tag{B8}$$

so that

$$k^3 |f_\rho|^2 \left(\frac{M}{a}\right)^4 = \frac{m H_1 A}{16\pi^2 (\mu - 1)} \left(\frac{k_1}{a}\right)^6 \left(\frac{k}{k_1}\right)^{6-4\mu}, \qquad 3/4 < \mu < 3/2, \tag{B9}$$

as reported in eq. (3.62). Note that there is no singularity for $\mu = 1$, as

$$\lim_{\mu \to 1} \frac{\Gamma(2-2\mu)}{(\mu - 1)} \left[\cos 2\pi(\mu - 1) - 1\right] = \frac{4\pi^2}{(\mu - 1)^2} (\mu - 1)^2 = \text{const} \tag{B10}$$

APPENDIX C: NON-RELATIVISTIC CORRECTIONS TO THE AXION SPECTRUM

For a massive-axion perturbation σ, the string frame action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}e^{\phi}\left[(\partial_\mu\sigma)^2 - m^2\sigma^2\right], \tag{C1}$$

in a conformally flat background, can be written in terms of the canonical variable

$$\psi = z\sigma, \qquad\qquad z = ae^{\phi/2}, \tag{C2}$$

as

$$S = \frac{1}{2}\int d^3x d\eta\left[\dot{\psi}^2 - (\partial_i\psi)^2 + \frac{\ddot{z}}{z}\psi^2 - m^2a^2\psi^2\right] \tag{C3}$$

(the dot denotes differentiation with respect to the conformal time η). The Fourier modes ψ_k satisfy the perturbation equation

$$\ddot{\psi}_k + \left(k^2 - \frac{\ddot{z}}{z} + m^2a^2\right)\psi_k = 0. \tag{C4}$$

We shall consider the background transition at $\eta = \eta_1$ from an initial pre-big bang phase in which the axion is massless, to a final radiation-dominated phase in which the dilaton freezes to its present value, and the axion acquires a small (in string units) mass. For $\eta > \eta_1$ the solution of Eq. (C4) depends on the kinematics of the pump field z and, after normalization to an initial vacuum spectrum, it can be written in terms of the second-kindHankel functions [29] as:

$$\psi_k(\eta) = \eta^{1/2}H_\mu^{(2)}(k\eta). \tag{C5}$$

In the radiation era, $\eta > \eta_1$, the "effective potential" $\ddot{z}/z$ is vanishing, and the perturbation equation reduces to

$$\ddot{\psi}_k + \left(k^2 + \alpha^2\eta^2\right)\psi_k = 0, \tag{C6}$$

where we have put

$$m^2a^2 = \alpha^2\eta^2, \qquad\qquad \alpha = mH_1a_1^2, \tag{C7}$$

using the time behaviour of the scale factor, $a \sim \eta$.

Assuming that the mass term is negligible at the transition scale, $m \ll k/a$, we can match the solution (C5) to the plane-wave solution

$$\psi_k = \frac{1}{\sqrt{k}}\left[c_+(k)e^{-ik\eta} + c_-(k)e^{ik\eta}\right], \tag{C8}$$

and obtain:

$$c_\pm = \pm c(k)e^{\pm ik\eta}, \qquad |c(k)| \sim (k/k_1)^{-\mu-1/2}. \tag{C9}$$

(We are neglecting, for simplicity, numerical factors of order 1, which are not very significant in view of the many approximations performed. Their contribution will be included into an overall numericalcoefficient in front of the final spectrum.) In the relativistic regime, the amplified axion perturbation then takes the form:

$$\sigma(\mathbf{k},\eta) = \frac{c(\mathbf{k})}{a\sqrt{k}}\sin(k\eta), \tag{C10}$$

used in Section III C for the massless-axion case.

In the radiation era the proper momentum is red-shifted with respect to the rest mass, and all axion modes tend to become non-relativistic. When the mass term is no longer negligible, the general solution of Eq. (C6) can be written in terms of parabolic cylinder functions [29]. For an approximate estimate of the axion field in the non-relativistic regime, however, it is convenient to distinguish two cases, depending on whether a mode k becomes non-relativistic

inside or outside the horizon. Defining as k_m the limiting comoving frequency of a mode that becomes non-relativistic ($k_m = ma_m$) at the time it re-enters the horizon ($k_m = H_m a_m$), we find, in the radiation era,

$$k_m = k_1 \left(\frac{m}{H_1}\right)^{1/2}. \tag{C11}$$

We will thus consider the two cases $k \gg k_m$ and $k \ll k_m$.

In the first case, we rewrite the perturbation equation (C6) as

$$\frac{d^2\psi_k}{dx^2} + \left(\frac{x^2}{4} - b\right)\psi_k = 0, \qquad x = \eta(2\alpha)^{1/2}, \qquad -b = k^2/2\alpha, \tag{C12}$$

and we give the general solution in the form

$$\psi = AW(b,x) + BW(b,-x)\ , \tag{C13}$$

where $W(b,x)$ are the Weber parabolic cylinder functions (see [29], chap. 19). In order to fix the integration constants A and B we shall match the solutions (C13) and (C10) in the relativistic limit

$$\frac{k^2}{m^2a^2} = \frac{k^2}{\alpha^2\eta^2} = \frac{-4b}{x^2} \gg 1. \tag{C14}$$

In this limit, as we are considering modes that become non-relativistic when they are already inside the horizon,

$$\left(\frac{k}{k_m}\right)^2 \sim \frac{k^2}{\alpha} \sim (-b) \gg 1, \tag{C15}$$

we can expand the W functions for b large with x moderate [29]. Matching to the plane-wave solution (C10), we obtain $A = 0$, and

$$\psi_k \simeq \frac{c(\mathbf{k})}{\alpha^{1/4}} W(b,-x). \tag{C16}$$

In the opposite, non-relativistic limit $x^2 \gg |4b|$, the expansion of the Weber functions gives [29]

$$\psi_k \simeq \frac{c(\mathbf{k})}{(\alpha\eta)^{1/2}} \sin\left(\frac{m}{H}\right) \tag{C17}$$

(we have used $x^2/4 = ma\eta/2 \sim m/H$). The corresponding axion field is (inside the horizon)

$$\sigma(\mathbf{k},\eta) = \frac{c(\mathbf{k})}{a\sqrt{ma}} \sin\left(\frac{m}{H}\right), \qquad\qquad k > k_m. \tag{C18}$$

Consider now the case of a mode that becomes non-relativistic when it is still outside the horizon, $k \ll k_m$. In this case, we cannot use the large $|b|$ expansion as $|b| < 1$, and it is convenient to express the general solution of Eq. (C12) as

$$\psi = Ay_1(b,x) + By_2(b,x)\ , \tag{C19}$$

where y_1 and y_2 are the even and odd parts of the parabolic cylinder functions [29]. Matching to (C10), in the relativistic limit $x \to 0$, gives $A = 0$ and

$$\psi_k \simeq c(\mathbf{k}) \left(\frac{k}{2\alpha}\right)^{1/2} y_2(b,x). \tag{C20}$$

In the non-relativistic limit $x^2 \gg |b|$ we use the relation [29]

$$y_2 \sim [W(b,x) - W(b,-x)] \sim \frac{1}{\sqrt{x}} \sin\frac{x^2}{4}, \tag{C21}$$

which leads to

$$\psi_k \simeq \frac{c(\mathbf{k})}{(\alpha\eta)^{1/2}} \left(\frac{k^2}{\alpha}\right)^{1/4} \sin\left(\frac{m}{H}\right). \tag{C22}$$

Using Eqs. (C15) and (C11) for k^2/α, we finally arrive at the non-relativistic axion field presented in Eq. (4.14):

$$\sigma(\mathbf{k},\eta) = \frac{c(\mathbf{k})}{a\sqrt{ma}} \left(\frac{k}{k_1}\right)^{1/2} \left(\frac{H_1}{m}\right)^{1/4} \sin\left(\frac{m}{H}\right), \qquad k < k_m. \tag{C23}$$

For later use, it is also convenient to define the spectral energy density in critical units, $\Omega_\sigma(\omega) = d(\rho/\rho_c)/d\ln\omega$, associated with the stochastic axion background in the three different regimes defined before.

For relativistic modes we find, from Eq. (C10),

$$\Omega_\sigma(\omega) \sim g_1^2 \left(\frac{\omega}{\omega_1}\right)^{3-2\mu} \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4, \qquad m < \omega < \omega_1. \tag{C24}$$

For modes that becomes non-relativistic after re-entry we find, from Eq. (C18),

$$\Omega_\sigma(\omega) \sim g_1^2 \frac{m}{H_1} \left(\frac{\omega}{\omega_1}\right)^{2-2\mu} \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3, \qquad \omega_m < \omega < m. \tag{C25}$$

For modes that becomes non-relativistic before re-entry we find, from Eq. (C23),

$$\Omega_\sigma(\omega) \sim g_1^2 \left(\frac{m}{H_1}\right)^{1/2} \left(\frac{\omega}{\omega_1}\right)^{3-2\mu} \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3, \qquad \omega < \omega_m. \tag{C26}$$

The last two spectral distributions are constant during the matter-dominated era, and the last one corresponds to the spectrum of the Bardeen potentials, as given in Eq. (3.63).

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[1] G. Veneziano, *Phys. Lett. B* **265**, 287 (1991); M. Gasperini and G. Veneziano, *Astropart. Phys.* **1**, 317 (1993); *Mod. Phys. Lett. A* **8**, 3701 (1993); *Phys. Rev. D* **50**, 2519 (1994). An updated collection of papers on the pre-big bang scenario is available at `http://www.to.infn.it/~gasperin/`.

[2] R. Brandenberger and C. Vafa, *Nucl. Phys. B* **316**, 391 (1989); G. Veneziano, Ref. [1]; A. A. Tseytlin, *Mod. Phys. Lett. A* **6**, 1721 (1991); K. A. Meissner and G. Veneziano, *Phys. Lett. B* **267**, 33 (1991); *Mod. Phys. Lett. A* **6**, 3397 (1991); A. Sen, *Phys. Lett. B* **271**, 295 (1991) ; S. F. Hassan and A. Sen, *Nucl. Phys. B* **375**, 103 (1992); A. A. Tseytlin and C. Vafa, *Nucl. Phys. B* **372**, 443 (1992); M. Gasperini and G. Veneziano, *Phys. Lett. B* **277**, 256 (1992).

[3] G. Veneziano, *Phys. Lett. B* **406**, 297 (1997); A. Buonanno et al., *Phys. Rev. D* **57**, 2543 (1998).

[4] R. Brustein and G. Veneziano, *Phys. Lett. B* **329**, 429 (1994); N. Kaloper, R. Madden and K. A. Olive, *Nucl. Phys. B* **452**, 677 (1995); *Phys. Lett. B* **371**, 34 (1996); R. Easther, K. Maeda and D. Wands, *Phys. Rev. D* **53**, 4247 (1996); R. Easther and K. Maeda, *Phys. Rev. D* **54**, 7252 (1996); M. Gasperini, J. Maharana and G. Veneziano, *Nucl. Phys. B* **472**, 349 (1996); S. J. Rey, *Phys. Rev. Lett.* **77**, 1929 (1996); M. Gasperini and G. Veneziano, *Phys. Lett. B* **387**, 715 (1996); M. Gasperini, M. Maggiore and G. Veneziano, *Nucl. Phys. B* **494**, 315 (1997); R. Brustein and R. Madden, *Phys. Lett. B* **410**, 110 (1997); *Phys. Rev. D* **57**, 712 (1988).

[5] M. S. Turner and E. J. Weinberg, *Phys. Rev. D* **56**, 4604 (1997); M. Maggiore and R. Sturani, *Phys. Lett. B* **415**, 335 (1997); N. Kaloper, A. Linde and R. Bousso, hep-th/9801073.

[6] G. F. Smoot et al., *Ap. J.* **396**, L1 (1992); C. L. Bennet et al., *Ap. J.* **430**, 423 (1994).

[7] M. Gasperini and M. Giovannini, *Phys. Lett. B* **282**, 36 (1991); *Phys. Rev. D* **47**, 1519 (1993); M. Gasperini and G. Veneziano, Ref. [1]; R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov and G. Veneziano, *Phys. Rev. D* **51**, 6744 (1995).

[8] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Lett. B* **361**, 45 (1995); R. Brustein, M. Gasperini and G. Veneziano, *Phys. Rev. D* **55**, 3882 (1997).

[9] M. Gasperini, *Phys. Rev. D* **56**, 4815 (1997).

[10] R. Durrer, *Phys. Rev. D* **42**, 2533 (1990).

[11] M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Rev. Lett.* **75**, 3796 (1995); D. Lemoine and M. Lemoine, *Phys. Rev. D* **52**, 1955 (1995).

[12] E. J. Copeland, R. Easther and D. Wands, *Phys. Rev. D* **56**, 874 (1997); E. J. Copeland, J. E. Lidsey and D. Wands, *Nucl. Phys. B* **56**, 506 (1997).
[13] M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Rev. D* **52**, 6651 (1995).
[14] R. Durrer and M. Sakellariadou, *Phys. Rev. D* **56**, 4480 (1997).
[15] R. K. Sachs and A. M. Wolfe, *Ap. J.* **147**, 73 (1967).
[16] R. Durrer, *Fund. of Cosmic Physics* **15**, 209 (1994).
[17] J. Bardeen, *Phys. Rev. D* **22**, 1882 (1980).
[18] H. Kodama and M. Sasaki, *Prog. Theor. Phys. Suppl.* **78**, 1(1984).
[19] R. Durrer and N. Straumann, *Helv. Phys. Acta* **61**, 1027 (1988).
[20] J. M. Steward and M. Walker, *Proc. R. Soc. London A* **341**, 49 (1974).
[21] R. Durrer, A. Gangui and M. Sakellariadou, *Phys. Rev. Lett.* **76**, 579 (1996).
[22] R. Bond and G. Efstathiou, *MNRAS* **226**, 655 (1987).
[23] P. J. E. Peebles, *The Large Scale Structure of the Universe* (Princeton University Press, Princeton, 1980).
[24] G. F. Smoot and D. Scott, in L. Montanet et al., *Phys. Rev. D* **50**, 1173 (1994) (update 1996).
[25] H. R. Harrison, *Phys. Rev. D* **1**, 2726 (1970); Y. B. Zel'dovich, *Mon. Not. Roy. Ast. Soc.* **160**, 1 (1972).
[26] R. R. Metsaev and A. A. Tseytlin, *Nucl. Phys. B* **293**, 385 (1987).
[27] A. Buonanno et al., *JHEP01*, 004 (1998).
[28] V. F. Mukhanov, A. Feldman and R. Brandenberger, *Phys. Rep.* **215**, 203 (1992).
[29] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions* (Dover, New York, 1972).
[30] M. Gasperini, *Relic dilatons in string cosmology*, in "Proc. of the 12th Int. Conf. on General Relativity and Gravitational Physics", Rome, 1996, editors M. Bassan et al. (World Scientific, Singapore, 1997), p. 181.
[31] M. S. Turner and L. M. Widrow, *Phys. Rev. D* **37**, 2743 (1988).
[32] M. Giovannini, *Teoria quantistica delle perturbazioni in cosmologia di stringa*, PhD Thesis, Turin University (January 1996).
[33] A. J. Banday et al., *Astrophys. J.* **475**, 393 (1997).
[34] V. Kaplunowsky, *Phys. Rev. Lett.* **55**, 1036 (1985).
[35] M. Bersanelli et al., *COBRAS/SAMBA, Report on the phase A study*, ESA Document D/SCI(96) 3 (1996).
[36] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1965).

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Remarks on Non-Abelian Duality

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ABSTRACT

A class of two-dimensional globally scale-invariant, but not conformally invariant, theories is obtained. These systems are identified in the process of discussing global and local scaling properties of models related by duality transformations, based on non-semisimple isometry groups. The construction of the dual partner of a given model is followed through; non-local as well as local versions of the former are discussed.

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1 Introduction and Discussion

The large amount of constraints imposed by conformal invariance in two dimensions has enabled the exact solution of many such systems. The question has arisen time and again whether the same symmetry and results could be obtained by requiring only the global scale symmetry. In this paper we deal with various aspects of this issue in two-dimensional field theories, aspects that have emerged rather unexpectedly in the process of mapping one theory into another by discrete transformations of target-space-duality type.

Duality relates two different geometries by establishing an isomorphism between the sets of harmonic maps from S^2 into the two manifolds. A standard procedure exists for discovering the dual partner of a given manifold, when the latter possesses a continuous group of isometries [1].

In ref. [2], a flat manifold was discovered for which the dual partner resulting from this procedure, when applied with respect to a certain group of isometries, is a manifold whose corresponding two-dimensional sigma model is not conformal. An explicit calculation of the β-functions corresponding to this model shows not only that they do not vanish, but also that they cannot be cancelled by an appropriate dilaton term. This shows that duality, in the above sense of correspondence between classical sigma model solutions, is not sufficient for the equivalence of the two quantum models on a curved worldsheet background.

It was pointed out in ref. [3] that the group of isometries used in this example is non-semisimple, and contains traceful structure constants. It was also suggested that these features are related to an anomaly. The anomaly was identified in ref. [4] as a mixed gravitational-gauge or conformal-gauge anomaly. Its origin is in the possible dependence of the Jacobians – related to the passage from the original coordinates to the dual ones – on the worldsheet metric and the traceful isometry generators.

In this paper we show that the Jacobian appears in the form

$$\det(MN)/(\det M \det N),$$

and therefore, to have an overall anomaly it is crucial to have a multiplicative anomaly [5], *i.e.*, $\det(MN) \neq \det M \det N$. It turns out that, while the

dependence survives for the mixed gauge-conformal anomaly, for the pure conformal anomaly it cancels. As a result, in the dual sigma-model action the duality process generates an additional non-local term proportional to the trace of the generator and to $\frac{1}{\Box}R^{(2)}$, where $R^{(2)}$ is the worldsheet curvature. (This term is essentially the additional term which duality induces from the anomaly term of [4].)

We use these results to study the worldsheet-metric dependence of models emerging from such an anomalous duality process. Let A be the original model with the invariance group of isometries, and B' the sigma model resulting from the duality procedure (the "prime" on B will be explained later). We first stress that, since the only difference between B' and A is the calculable and well-controlled non-local anomalous term, the extra dependence of B' on the conformal background should cancel exactly the background dependence of this term. In particular, if A is conformal, the ordinary geometric calculation of the β-function of B' should be cancelled by the variation of the anomaly term with respect to the background conformal factor. We check this in detail for a few examples, including that studied in [2].

On a flat worldsheet background, the sigma models A and B' are already equivalent. In particular, their spectra are identical. This is confirmed by the result that the anomaly is proportional to $R^{(2)}$. In the case in which A is conformal, the spectrum of B' must be massless as well. Next we recall some general features of massless 2-d sigma models.

The simplest situation is when the model is truly conformal, *i.e.*, when there exists a local, traceless energy-momentum tensor $T_{\alpha\beta}$. The β-function equations give a vanishing result. As a consequence, in this case the model can be coupled to a two-dimensional background metric in a Weyl-invariant manner at the quantum level. It could happen that such a local traceless tensor does not exist. Since the spectrum is massless, there should exist a local, conserved dilation current D_α. From the relation between $T_{\alpha\beta}$ and D_α it follows [6] that in this case the trace of the energy-momentum tensor must be a total derivative. The β-function equations no longer give zero; however, the deviation from zero can be absorbed in a "wave function renormalization", *i.e.*, a reparametrization of the fields, and in a gauge transformation of the

antisymmetric tensor which enters the sigma model [7, 6]. Explicitly, the β-function with respect to the target-space metric $G_{\mu\nu}$ has the form:

$$\beta_{G_{\mu\nu}} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{1.1}$$

where ξ_μ defines the reparametrization. For a general ξ_μ one cannot define a local Weyl-invariant coupling to the background metric. If, however, ξ_μ has the special form:

$$\xi_\mu = \partial_\mu \Phi, \tag{1.2}$$

where Φ is the "dilaton" field, a local coupling can be defined involving the curvature, and an "improved" traceless energy-momentum tensor exists.

Our case of a sigma model B', resulting by an anomalous duality process from an originally conformal theory, is of the type of a massless *nonconformal* theory. In fact, the representation of its β-function as resulting from the variation of the anomalous term with respect to the conformal worldsheet factor, makes it explicit that these β-functions are total derivatives representable in the form of eq. (1.1).

We will study two such examples. One will turn out to be improvable by a dilaton term. The other example, that of [2], develops a β-function which is a total derivative, but not improvable by a local background-dependent term. We have thus found a genuine example of a two-dimensional, scale-invariant, interacting non-conformal model. In ref. [6] it was proved that for a unitary theory in $d = 2$ with a discrete operator spectrum, an improved energy-momentum tensor always exists. Our example is, necessarily, a non-compact sigma model. Similar situations arise in higher dimensions at the tree level for gauge theories based on n-forms [8].

Note, however, that this apparently non-locally improvable B' model is actually improvable in a more general sense. One can "improve" it by adding to its action the non-local background-dependent anomaly term, and then re-express it in a local form by passing to the language of the A-model which, in the presence of the anomaly term, is completely equivalent to it.

The paper is organized as follows: in section 2, we discuss in detail the procedure of non-Abelian duality [9] (for an isometry group acting with no fixed points [3]), in particular the appearance of the anomaly. In section

3, we show how the original A-model can be recovered. In section 4, we discuss the B-model related to it, *i.e*, the B' sigma model together with the (non-local) improvement. In Section 5, we discuss an example where the reparametrization needed is of the form (1.2) and, therefore, models A and B are equivalent after a dilaton-type correction to B' is made. In section 6, we discuss the model proposed in [2]: model A is conformal but model B' is just dilation invariant. In section 7, we discuss how the anomaly can be understood in terms of Ward identities in a flat worldsheet background. In section 8, we discuss some alternative local representations of the mixed anomaly term, in a higher dimensional sigma model. A general form of the anomaly for a non-compact group and a general proof of the dilaton corrections [10] are discussed in two appendices.

2 The general case (without isotropy)

As in ref. [3], we can consider a target space with coordinates g that transform as $g \to ug$ for u in some group G, and further coordinates x^i that are inert. A general action can be written in the form

$$\begin{aligned} S[g,x] = \frac{1}{2\pi}\int d^2z \Big(E_{ab}(x)(g^{-1}\partial g)^a(g^{-1}\bar\partial g)^b &+ F^R_{aj}(x)(g^{-1}\partial g)^a\bar\partial x^j \\ + F^L_{ib}(x)\partial x^i(g^{-1}\bar\partial g)^b &+ F_{ij}(x)\partial x^i\bar\partial x^j \\ &- \Phi(x)\partial\bar\partial\sigma\Big)\ , \end{aligned} \tag{2.1}$$

where

$$(g^{-1}\partial g)^a \equiv tr(\tilde T^a g^{-1}\partial g) \quad \Leftrightarrow \quad g^{-1}\partial g = (g^{-1}\partial g)^a T_a\ , \quad etc. \tag{2.2}$$

The generators T_a, $a = 1, ..., dim(G)$, obey

$$[T_a, T_b] = f^c_{ab}T_c, \tag{2.3}$$

and the "dual generators" $\tilde T^a$ are defined by the condition

$$tr(T_a\tilde T^b) = \delta^b_a. \tag{2.4}$$

In eq. (2.1) the background matrices $E_{ab}, F^L_{ib}, F^R_{aj}, F_{ij}$ and the dilaton Φ depend only on the coordinates x^i. Here $z = (\zeta_1 + i\zeta_2)/\sqrt{2}$, $\bar{z} = (\zeta_1 - i\zeta_2)/\sqrt{2}$ are complex worldsheet coordinates, $\partial \equiv \partial/\partial z = (\partial_1 - i\partial_2)/\sqrt{2}$, $\bar{\partial} \equiv \partial/\partial\bar{z} = (\partial_1 + i\partial_2)/\sqrt{2}$, and $\sigma(z,\bar{z})$ is the worldsheet conformal factor, *i.e.*

$$\partial\bar{\partial}\sigma = \frac{1}{4}\sqrt{h}R^{(2)}, \qquad h_{z\bar{z}} = e^{-2\sigma}, \tag{2.5}$$

where h is the worldsheet metric (in the conformal gauge) and $R^{(2)}$ is the worldsheet curvature.

The quantum field theory is defined by the functional integral

$$Z_O = \int D_L g \; Dx \; e^{-S[g,x]} \tag{2.6}$$

(for future use, we label this partition function as Z_O). Here $D_L g$ is the left invariant measure (which is required for consistency with the isometry that acts from the left, $g \to ug$) [6]

$$D_L g \equiv \prod_{a,z,\bar{z}} (g^{-1}dg)^a, \tag{2.7}$$

and Dx is the rest of the sigma-model measure ($Dx \equiv \prod_{z,\bar{z}} \sqrt{\det(F_{ij})\,\det(E_{ab})}dx$, if $F^L = F^R = 0$).

We now rewrite the theory (2.6) by inserting the identity $I = \int D_L\bar{g}\,\delta_L(g,\bar{g})$. The (left-invariant) delta-function sets $\bar{g} = g$ and allows, in particular, the replacement of $g^{-1}\bar{\partial}g$ by $\bar{g}^{-1}\bar{\partial}\bar{g}$:

$$Z_O = \int D_L g \; D_L\bar{g} \; \delta_L(g,\bar{g}) \; Dx \; e^{-S[g,\bar{g},x]}, \tag{2.8}$$

where

$$\begin{aligned} S[g,\bar{g},x] \;&=\; \frac{1}{2\pi}\int d^2z \Big(E_{ab}(x)(g^{-1}\partial g)^a(\bar{g}^{-1}\bar{\partial}\bar{g})^b \;+\; F^R_{aj}(x)(g^{-1}\partial g)^a\bar{\partial}x^j \\ &+\; F^L_{ib}(x)\partial x^i(\bar{g}^{-1}\bar{\partial}\bar{g})^b \;+\; F_{ij}(x)\partial x^i\bar{\partial}x^j \\ &-\; \frac{1}{4}\Phi(x)\sqrt{h}R^{(2)}\Big)\,. \end{aligned} \tag{2.9}$$

[6] For semi-simple groups there is no difference between left and right invariant measures; however, for general groups they may differ.

At this point we define a vector field $A, \bar{A}$ in terms of the group variables $g, \bar{g} \in G$:

$$A = g^{-1}\partial g, \qquad \bar{A} = \bar{g}^{-1}\bar{\partial}\bar{g}. \tag{2.10}$$

Changing the integration variables from $g, \bar{g}$ to $A, \bar{A}$ one gets

$$\begin{aligned} Z_O \to Z &= \int DA\ D\bar{A}\ \det(DA/D_L g)^{-1}\ \det(D\bar{A}/D_L \bar{g})^{-1} \\ &\times\ \delta(F)\ \det(DF/D_L g)|_{\bar{g}=g}\ e^{-S[A,\bar{A},x]}, \end{aligned} \tag{2.11}$$

where $S[A, \bar{A}, x]$ is given by inserting eq. (2.10) into $S[g, \bar{g}, x]$ (2.9). In eq. (2.11) F is the field strength,

$$\begin{aligned} F(A, \bar{A}) &= \partial\bar{A} - \bar{\partial}A + [A, \bar{A}] = g^{-1}\partial(f^{-1}\bar{\partial}f)g = \bar{g}^{-1}\bar{\partial}(\partial f f^{-1})\bar{g}, \\ f &\equiv \bar{g}g^{-1}, \end{aligned} \tag{2.12}$$

and we have used the equality[7]

$$\delta_L(g, \bar{g}) = \delta(F) \det N, \qquad N = (DF/D_L g)|_{\bar{g}=g}. \tag{2.13}$$

Calculating N (by using eqs. (2.7) and (2.12)) we obtain formally

$$N = M(A)\bar{M}(\bar{A}), \tag{2.14}$$

where

$$M(A) = \partial + [A, \cdot], \qquad \bar{M}(\bar{A}) = \bar{\partial} + [\bar{A}, \cdot]. \tag{2.15}$$

Next we replace the $F = 0$ delta-function with the constraint imposed by a Lagrange multiplier term in the action, namely,

$$\delta(F) = \int D\lambda\ e^{-\frac{1}{2\pi}\int d^2z\, tr\lambda F}. \tag{2.16}$$

[7] The equality (2.13) is only correct locally: $\delta(F)$ constrains the curvature to vanish, but does not force the connection A to be trivial on a non-zero genus worldsheet. Therefore, to correct eq. (2.13), we need to sum $\delta_L(g, \bar{g})$ over all flat connections. We will not address such global issues [11, 3, 12] here. Moreover, as $F = 0$, one could involve in the process of duality any function of F which would vanish for $F = 0$. Such terms lead, in general, to very complicated non-local theories, all equivalent. This structure can, however, be removed by applying non-local field redefinition.

Here

$$F \equiv F^a T_a, \quad \lambda \equiv \lambda_a \tilde{T}^a \;\Rightarrow\; tr\lambda F = \lambda_a F^a. \tag{2.17}$$

Finally, we should deal with the Jacobian of the change of variables from $g, \bar{g}$ to $A, \bar{A}$:

$$\det(DA/D_L g)\det(D\bar{A}/D_L\bar{g}) = \det M(A)\det\bar{M}(\bar{A}), \tag{2.18}$$

where $M(A), \bar{M}(\bar{A})$ are given in (2.15). The precise manner of calculating these determinants is given in [4].

Altogether, the partition function takes the form:

$$\begin{aligned} Z &= \int D\lambda\; Dx\; DA\; D\bar{A}\; e^{J(A,\bar{A})}\; e^{-S[A,\bar{A},\lambda,x]}, \\ e^{J(A,\bar{A})} &= \frac{\det(M(A)\bar{M}(\bar{A}))}{\det M(A)\det\bar{M}(\bar{A})}, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} S[A,\bar{A},\lambda,x] &= \frac{1}{2\pi}\int d^2z \Big(E_{ab}(x)A^a\bar{A}^b \;+\; F^R_{aj}(x)A^a\bar{\partial}x^j \\ &+ \; F^L_{ib}(x)\partial x^i\bar{A}^b \;+\; F_{ij}(x)\partial x^i\bar{\partial}x^j \;-\; \frac{1}{4}\Phi(x)\sqrt{h}R^{(2)} \\ &+ \; \lambda_a(\partial\bar{A}^a - \bar{\partial}A^a + f^a_{bc}A^b\bar{A}^c)\Big)\,. \end{aligned} \tag{2.20}$$

In the following, we discuss the evaluation of the determinants term in Z (2.19): $\exp J(A,\bar{A})$. Naively the determinants would cancel; however, there is a possibility of a multiplicative anomaly [5]. It is convenient to use a covariant notation and to replace the determinants by actions involving bosonic ghosts $\beta_\alpha, \gamma, \gamma'$ (here $\alpha = 1, 2$ is a worldsheet index) and fermionic ghosts b, c:

$$\begin{aligned} e^{J(A,\bar{A})} &= \int D\beta\; D\gamma\; D\gamma'\; Db\; Dc \\ &\times \; \exp\int d^2\zeta\; tr[\sqrt{h}h^{\alpha\delta}(\beta_\alpha D_\delta\gamma + bD_\alpha D_\delta c) + \beta_\alpha\tilde{D}^\alpha\gamma'], \end{aligned} \tag{2.21}$$

where

$$\beta_\alpha \equiv \beta_{\alpha a}\tilde{T}^a, \qquad \gamma \equiv \gamma^a T_a, \qquad \gamma' \equiv \gamma'^a T_a, \qquad b \equiv b_a\tilde{T}^a, \qquad c \equiv c^a T_a, \tag{2.22}$$

$$D_\alpha \equiv \partial_\alpha + [g^{-1}\partial_\alpha g, \cdot], \qquad \tilde{D}^\alpha \equiv \epsilon^{\alpha\beta} D_\beta. \tag{2.23}$$

In deriving (2.21) we used the fact that

$$(D^\alpha + \tilde{D}^\alpha)(D_\alpha - \tilde{D}_\alpha) = 2D^\alpha D_\alpha \tag{2.24}$$

if the field entering D_α is a pure gauge.

The ghost actions are formally conformal invariant as well as gauge invariant under $g \to gu$, with all ghost fields transforming in the adjoint representation: $c \to u^{-1}cu$, $b \to u^{-1}bu$. A new type of anomaly may appear coupling the background worldsheet metric to the gauge field [3, 4]. This anomaly can be studied by considering for the free part of (2.21) the correlator between the energy-momentum tensor and the vector current V_a^α coupled to the gauge field (note that by giving a negative intrinsic parity to γ', its gauge coupling can also be considered vectorlike):

$$\frac{\delta^2 J}{\sqrt{h}\delta h_{\alpha\beta}(\zeta)\sqrt{h}\delta A^a_\gamma(\xi)} = \langle T^{\alpha\beta}(\zeta) V^\gamma_a(\xi)\rangle \equiv S^{\alpha\beta\gamma}_a(\zeta - \xi), \tag{2.25}$$

$$\begin{aligned} T^{\alpha\beta} &= \frac{1}{2}\sqrt{h}\ tr\Big(\beta^\alpha D^\beta \gamma - \frac{1}{2}h^{\alpha\beta}\beta^\delta D_\delta \gamma\Big) + (\alpha \leftrightarrow \beta), \\ V^\alpha_a &\equiv \frac{1}{\sqrt{h}}\frac{\delta J}{\delta A^a_\alpha} = f^e_{ad}(\beta^\alpha_e \gamma^d + \epsilon^{\delta\alpha}\beta_{\delta e}\gamma'^d + 2b_e \nabla^\alpha c^d). \end{aligned} \tag{2.26}$$

Note that $T^{\alpha\beta}$ in (2.26) does not involve γ'. This energy-momentum tensor is invariant under the group G. Taking the expectation value of the commutator of $T^{\alpha\beta}V^\gamma_a$ with an isometry generator Q_b, and using $[Q_b, T^{\alpha\beta}] = 0$, and $Q_b|0\rangle = 0$ which implies $\langle [Q_b, T^{\alpha\beta}V^\gamma_a]\rangle = 0$, one obtains

$$\langle T^{\alpha\beta}[Q_b, V^\gamma_a]\rangle = 0. \tag{2.27}$$

Therefore, the correlator (2.25) can be non-zero only for a group index which does not appear on the right-hand side of a commutator $[Q_b, V_a]$. Note that all the generators which do appear on this right-hand side necessarily have zero trace (because the trace of a commutator is zero). The necessary and sufficient condition to have such "quasi-Abelian" directions is for the group to be non-semisimple [13].

The kinematical constraints on the correlator (2.25) on a flat background are very simple to analyse: the correlator depends on $\zeta - \xi$ and, therefore, its Fourier transform $\tilde{S}$ depends on a single momentum q^α, and its general kinematical decomposition has the form

$$\tilde{S}_a^{\alpha\beta\gamma} = A_a \eta^{\alpha\beta} q^\gamma + B_a(\eta^{\alpha\gamma} q^\beta + \eta^{\beta\gamma} q^\alpha) + C_a q^\alpha q^\beta q^\gamma, \tag{2.28}$$

where A_a, B_a, C_a are invariant amplitudes depending on q^2 only. Using eq. (2.28) it is easy to show that at least two out of the three Ward identities, following from the conservation and tracelessness of $T^{\alpha\beta}$ and the conservation of V_a^α, should be violated. In particular, if the conservation of $T^{\alpha\beta}$ is imposed, $\tilde{S}_a^{\alpha\beta\gamma}$ has a unique form in any dimension d:

$$\tilde{S}_a^{\alpha\beta\gamma} = K_a(\eta^{\alpha\beta} q^2 - q^\alpha q^\beta)(q^2)^{(d-4)/2} q^\gamma, \tag{2.29}$$

where K_a are dimensionless constants. The expression (2.29) violates the conservation of the vector current. It follows that, if dimensional regularization can be used, the correlator will vanish identically. Therefore, the only possibility to obtain an anomalous correlator is to have in the action a γ_5 or $\epsilon^{\alpha\beta}$. We conclude, therefore, that the b, c action corresponding to $\det(M(A)\bar{M}(\bar{A}))$ does not have an anomaly [8]. In order to calculate the coefficient K we isolate the convergent C amplitude in the Feynman diagram, the other amplitudes being determined by the choice of which Ward identity is preserved. We obtain: $K_a = 1/4\pi\ (tr\ T_a)$.

On a general background, covariantizing (2.29) and using eq. (2.25), the simultaneous variation of J with respect to the background metric and gauge field is

$$\begin{aligned} \delta J &= \frac{1}{4\pi}(tr\ T_a) \int d^2\zeta d^2\xi\ S_a^{\alpha\beta\gamma}(\zeta - \xi)\sqrt{h}\delta h_{\alpha\beta}(\zeta)\sqrt{h}\delta A_\gamma^a(\xi) \\ &= \frac{1}{4\pi}(tr\ T_a) \int d^2\zeta d^2\xi \sqrt{h(\zeta)}\sqrt{h(\xi)} \end{aligned}$$

[8] If we studied $\det(M(A)\bar{M}(\bar{A}))$ in a general A-background, the anomaly could appear only in the unnatural parity part; for this type of anomaly the current conservation can be imposed, and therefore the anomaly vanishes for a pure gauge configuration, leading again to the same conclusion.

$$\times \quad \Big(\frac{h^{\alpha\beta}}{\sqrt{h}}\delta^2(\zeta-\xi) - \frac{\nabla^\alpha\nabla^\beta}{\Box}\Big)\delta\nabla^\gamma A^a_\gamma(\xi)\delta h_{\alpha\beta}(\zeta). \tag{2.30}$$

In the conformal gauge, $h_{\alpha\beta} = \exp(-2\sigma)\eta_{\alpha\beta}$ (2.5), one finds

$$J = -\frac{1}{2\pi}(trT_a)\int d^2\zeta\sqrt{h}h^{\alpha\beta}\ \nabla_\alpha A^a_\beta\ \sigma. \tag{2.31}$$

We remark that, potentially, there could be an anomalous contribution involving just the metric (the standard Polyakov anomaly, $\sqrt{h}R^{(2)}\frac{1}{\Box}R^{(2)}$). The explicit expression (2.21) shows that we have a bosonic β-γ system, with $c = 2$, and an anticommuting complex scalar (b-c), with $c = -2$, coupled to the background worldsheet metric. Therefore, the anomalies cancel and there is no multiplicative anomaly depending just on the metric. The central charge, c_t, of the system described by eq. (2.19) is equal, by construction, to the system described by eq. (2.1). For semisimple groups, one can show that the system in eq. (2.20) already has a central charge c_t, leaving for the rest of the system, e^J in (2.19), only a zero central charge. This result was obtained above by assuming that (b, c) was a scalar pair. With such a scalar pair, one can show that the numerator in e^J (2.19) can be regularized in such a manner that it leads to no mixed anomaly.

From eq. (2.31), we see that for a non-trivial worldsheet metric and gauge fields, an anomalous (non-local) piece appears in the effective action:

$$\begin{aligned} S_{nonlocal} &= \frac{1}{8\pi}(tr\ T_a)\int d^2z\Big(\frac{1}{\partial}A^a + \frac{1}{\bar\partial}\bar A^a\Big)\sqrt{h}R^{(2)} \\ &= \frac{1}{2\pi}\int d^2z\ ln(\det M(g)\det M(\bar g))\partial\bar\partial\sigma, \end{aligned} \tag{2.32}$$

(here we are back to complex worldsheet coordinates) where $M_a{}^b(g)$ is the adjoint representation matrix, *i.e.*

$$gT_ag^{-1} = M_a{}^b(g)T_b \ \Rightarrow\ M_a{}^b(g) = tr(gT_ag^{-1}\tilde T^b). \tag{2.33}$$

The proof of the second equality in (2.32) is given in Appendix A. We should note here that $\det M$ is *not* a constant if T_a is not traceless for some a:

$$tr\ T_a \neq 0\ \Leftrightarrow\ \det\ M \neq 1. \tag{2.34}$$

Explicitly, if we parametrize

$$g(\theta) = e^{\theta^a T_a}, \tag{2.35}$$

then

$$ln\ \det\ g = tr\ ln\ g = tr\ \theta^a T_a. \tag{2.36}$$

(Obviously, the matrix g is equivalent to $M(g)$ if the generators T_a are in the adjoint representation.)

To summarize, the theory is now defined by the functional integral

$$Z\ =\ \int DA\ D\bar{A}\ D\lambda\ Dx\ e^{-(S[A,\bar{A},\lambda,x]+\frac{1}{2\pi}\int d^2z\, ln(\det M(g)\det M(\bar{g}))\partial\bar{\partial}\sigma)}, \tag{2.37}$$

where $S[A,\bar{A},\lambda,x]$ is given in (2.20).

Starting with the theory (2.37), we now consider two avenues which we call the "A-model" and the "B-model". One gets the A-model by integrating out the Lagrange multiplier. On the other hand, one gets the B-model by integrating out the gauge field[9].

3 The A-model

By construction, integrating out λ in (2.37) constrains the gauge field to be pure gauge ($g = \bar{g}$), and we get

$$Z_A\ =\ \int DA\ D\bar{A}\ \delta(F)\ Dx\ e^{-(S[A,\bar{A},\lambda,x]+S_{nonlocal}[A,\bar{A},\sigma])}$$

[9] An alternative way [11] to get (2.37) from (2.6) is to gauge the G-symmetry of the action $S[g,x]$ with non-dynamical gauge fields (*i.e.*, without a F^2 term) by minimal coupling:

$$g^{-1}\partial g \to g^{-1}(\partial + A)g\ ,\quad g^{-1}\bar{\partial} g \to g^{-1}(\bar{\partial}+\bar{A})g,$$

(here $A \equiv A^a T_a$ transforms as $A \to u(\partial + A)u^{-1}$, *etc.*), and add the Lagrange multiplier term, $tr\lambda F$, that constrains the gauge field to be (locally) pure gauge. Then, after choosing a unitary gauge, $g = 1$, and adding the non-local term to cancel the trace anomaly [3, 4], one recovers (2.37). We chose not to work with such a gauge theory, in order not to deal with the problem of a gauge-invariant measure for the gauge field in non-semisimple groups.

$$
\begin{aligned}
&= \int D_L g \; D_L \bar{g} \; \delta_L(g, \bar{g}) \; Dx \; e^{-S[g,\bar{g},x]} \\
&= \int D_L g \; Dx \; e^{-S[g,x]}. \qquad (3.1)
\end{aligned}
$$

Here $S[g,x]$ is given by (2.1) and $S_{nonlocal}[A, \bar{A}, \sigma]$ is given in (2.32). Therefore, we get that the A-model is equivalent to the original model

$$
Z_A = Z_O = \int D_L g Dx e^{-S[g,x]} \qquad (3.2)
$$

(up to global issues [3, 12] that we do not address here).

4 The B-model

To integrate out the gauge field $A, \bar{A}$ in (2.37) it is convenient to re-express the non-local part of the action in terms of $A, \bar{A}$ and the conformal factor σ. The non-local part takes the form

$$
\begin{aligned}
S_{nonlocal}[A, \bar{A}, \sigma] &= \frac{1}{2\pi} \int d^2 z \, ln(\det M(g) \det M(\bar{g})) \partial\bar{\partial}\sigma \\
&= \frac{1}{8\pi} (tr \; T_a) \int d^2 z \Big(\frac{1}{\partial} A^a + \frac{1}{\bar{\partial}} \bar{A}^a \Big) \sqrt{h} R^{(2)} \\
&= -\frac{tr T_a}{2\pi} \int d^2 z (A^a \bar{\partial}\sigma + \bar{A}^a \partial\sigma). \qquad (4.1)
\end{aligned}
$$

The action $S[A, \bar{A}, \lambda, x] + S_{nonlocal}[A, \bar{A}, \sigma]$ is bilinear in $A, \bar{A}$ (it is actually linear in A and in $\bar{A}$, separately). Therefore, integrating out $A, \bar{A}$ is simple and leads to

$$
\begin{aligned}
Z_B &= \int D\lambda \; Dx \; DA \; D\bar{A} \; e^{-(S[A,\bar{A},\lambda,x] + S_{nonlocal}[A,\bar{A},\sigma])} \\
&= \int D\lambda \; Dx \; \det N(x, \lambda) \; e^{-S_B[\lambda, x, \sigma]}, \qquad (4.2)
\end{aligned}
$$

where the *dual* action S_B is

$$
\begin{aligned}
S_B[\lambda, x, \sigma] &= \frac{1}{2\pi} \int d^2 z \Big(F_{ij} \partial x^i \bar{\partial} x^j - (\Phi - ln \det N) \partial\bar{\partial}\sigma \\
&+ (\partial\lambda_a - \partial x^i F^L_{ia} + tr T_a \partial\sigma) N^{ab} (\bar{\partial}\lambda_b + F^R_{bj} \bar{\partial} x^j - tr T_b \bar{\partial}\sigma) \Big), \qquad (4.3)
\end{aligned}
$$

and

$$N^{ab}(x,\lambda) = [(E(x) + \lambda_c f^c)^{-1}]^{ab}. \tag{4.4}$$

Here the shift of the dilaton comes from a Jacobian factor that arises from integrating over the gauge field [10] (see Appendix B), and the matrices f^c have the structure constants as components $(f^c)_{ab} = f^c_{ab}$.

Equation (4.3) is our key result: generally it is non-local, since σ appears explicitly in a form which does not allow its replacement by the curvature, without the use of the inverse Laplacian. Since by construction (4.3) is the form equivalent to model A, after the quantum corrections are taken into account we can read off the non-local corrections to the energy-momentum tensor needed to make the two models equivalent. The variation with respect to σ gives the trace of the energy-momentum. Therefore, the explicit σ-dependent terms in (4.3) give the difference at one-loop level between the traces of the energy-momentum tensors for models A and B' (recall that B' is the sigma-model part of the B model, without the (non-local) dilaton-type correction). From the explicit expressions we see that this difference will be *always* a total derivative. This means, as discussed in the introduction, that if model A was conformally invariant, then model B' will have at least a local, conserved dilation current. Moreover, if model A is conformal invariant, then (4.3) is by construction σ-independent. Therefore, since the total variation with respect to σ is obtained by the β-function equations in addition to the explicit σ dependence, we can directly obtain the β-functions of model B' by taking (with negative sign) the σ-variations in (4.3).

We discuss now two typical examples.

5 Example 1

In this section we present the first example. We discuss a two-dimensional group G_{JP} with generators T_a, $a = 1, 2$, that we write in the adjoint representation as

$$T_2 = P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad T_1 = J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.1}$$

They obey the algebra

$$[J, P] = P \;\Rightarrow\; f^P_{JP} = -f^P_{PJ} = 1, \tag{5.2}$$

and all other commutators and structure constants are 0. The dual generators $\tilde{T}^a$ are given by

$$\tilde{T}^2 = P^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{T}^1 = J^t = J. \tag{5.3}$$

It is easy to check that (2.4) is satisfied.

We parametrize elements $u(\alpha, \beta) \in G_{JP}$ by

$$u(\alpha, \beta) = e^{\alpha P} e^{\beta J} = \begin{pmatrix} 1 & 0 \\ \alpha & e^{\beta} \end{pmatrix}. \tag{5.4}$$

It is easy to check that $u(\alpha, \beta)$ obey the group product

$$u(\alpha, \beta) u(\alpha', \beta') = u(\alpha + e^{\beta}\alpha', \beta + \beta'), \tag{5.5}$$

and the inverse is

$$u^{-1}(\alpha, \beta) = u(-\alpha e^{-\beta}, -\beta) = \begin{pmatrix} 1 & 0 \\ -\alpha e^{-\beta} & e^{-\beta} \end{pmatrix}. \tag{5.6}$$

To write G_{JP} invariant actions it is convenient to parametrize $g \in G_{JP}$ by

$$g(\phi, \chi) = \begin{pmatrix} 1 & 0 \\ \phi & \chi^{-1} \end{pmatrix}. \tag{5.7}$$

Then the isometry acts as

$$g(\phi, \chi) \to g'(\phi', \chi') = u(\alpha, \beta) g(\phi, \chi) = \begin{pmatrix} 1 & 0 \\ \phi' & (\chi')^{-1} \end{pmatrix}, \tag{5.8}$$

where

$$\phi' = e^{\beta}\phi + \alpha, \qquad \chi' = e^{-\beta}\chi. \tag{5.9}$$

The G_{JP} invariant elements in the action are constructed out of

$$\begin{aligned} (g^{-1}\partial_\mu g)^P &\equiv tr(P^t g^{-1}\partial_\mu g) = \chi \partial_\mu \phi, \\ (g^{-1}\partial_\mu g)^J &\equiv tr(J^t g^{-1}\partial_\mu g) = -\chi^{-1}\partial_\mu \chi. \end{aligned} \tag{5.10}$$

For example, we will choose the action

$$\begin{aligned} S[\phi,\chi] &= \frac{1}{2\pi}\int d^2z\left((g^{-1}\partial g)^P(g^{-1}\bar\partial g)^J+(g^{-1}\partial g)^J(g^{-1}\bar\partial g)^P\right) \\ &= -\frac{1}{2\pi}\int d^2z\,(\partial\phi\bar\partial\chi+\partial\chi\bar\partial\phi). \end{aligned} \tag{5.11}$$

This action is of the form (2.1) with $E_{PJ}=E_{JP}=1$, and all other backgrounds are 0. The background is a flat Minkowski space in two dimensions and, therefore, the A-model is conformal.

To find the B-model we need the background matrix N (4.4):

$$N^{ab}(x,\lambda)=[(E(x)+\lambda_c f^c)^{-1}]^{ab}=\frac{1}{1-\lambda_P^2}\begin{pmatrix}0 & 1+\lambda_P\\ 1-\lambda_P & 0\end{pmatrix}, \tag{5.12}$$

Using

$$\det N=\frac{1}{\lambda_P^2-1}\ , \qquad trT_a=(tr\ J)\delta_a^J=\delta_a^J, \tag{5.13}$$

one finds

$$\begin{aligned} S_B[\lambda] &= \frac{1}{2\pi}\int d^2z\left((\partial\lambda_a+\delta_a^J\partial\sigma)N^{ab}(\bar\partial\lambda_b-\delta_b^J\bar\partial\sigma)+ln\ \det N\ \partial\bar\partial\sigma\right) \\ &= \frac{1}{2\pi}\int d^2z\left(\frac{1}{1-\lambda_P^2}(\partial\lambda_P\bar\partial\lambda_J+\partial\lambda_J\bar\partial\lambda_P)+\text{total derivative}\right), \end{aligned} \tag{5.14}$$

and

$$Z_B=\int\frac{D\lambda_P D\lambda_J}{\lambda_P^2-1}\ e^{-S_B[\lambda]}. \tag{5.15}$$

The background in S_B is a 2-d flat Minkowski space up to a conformal factor:

$$ds^2=e^{\rho(\lambda_P)}(\partial\lambda_P\bar\partial\lambda_J+\partial\lambda_J\bar\partial\lambda_P), \qquad \rho(\lambda_P)=-ln(1-\lambda_P^2), \tag{5.16}$$

and the curvature is

$$R=-4e^{-\rho(\lambda_P)}\frac{\partial}{\partial_{\lambda_J}}\frac{\partial}{\partial_{\lambda_P}}\rho(\lambda_J)=0. \tag{5.17}$$

Therefore, the B-model is flat. Changing coordinates to

$$d\phi=\frac{1}{\lambda_P^2-1}d\lambda_P, \qquad d\chi=d\lambda_J, \tag{5.18}$$

we get

$$Z_B = \int D\phi\ D\chi\ e^{-S[\phi,\chi]}, \qquad S[\phi,\chi] = -\frac{1}{2\pi}\int d^2z\,(\partial\phi\bar{\partial}\chi + \partial\chi\bar{\partial}\phi). \quad (5.19)$$

Therefore, the B-model (5.15) is equivalent to the A-model (5.11).

The fact that the B-model could be written in a local form, although the anomaly was present, derives from the two-dimensional target-space nature. In two dimensions any ξ appearing in eq. (1.1) is of the type (1.2). Therefore, the energy-momentum tensor is improvable by a dilaton term. Indeed, in the example above, the term induced by the anomaly is of a dilaton type (up to a total derivative), and moreover, it exactly cancels the usual dilaton term arising by duality.

6 Example 2

In this section we present the four-dimensional Bianchi V cosmological example considered in ref. [2]. We discuss a three-dimensional group G_V with generators T_a, $a = 1, 2, 3$, that we write in the adjoint representation as

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (6.1)$$

They obey the algebra

$$\begin{aligned} &[T_1, T_2] = T_2, \qquad [T_1, T_3] = T_3, \qquad [T_2, T_3] = 0 \ \Rightarrow \\ &f^2_{12} = f^3_{13} = 1 = -f^2_{21} = -f^3_{31}, \qquad f^c_{ab} = 0 \quad \text{otherwise.} \end{aligned} \quad (6.2)$$

The dual generators $\tilde{T}^a$ are given by

$$\tilde{T}^1 = \frac{1}{2}T_1, \qquad \tilde{T}^2 = (T_2)^t, \qquad \tilde{T}^3 = (T_3)^t. \quad (6.3)$$

It is easy to check that (2.4) is satisfied.

We parametrize elements $u(\alpha, \vec{\beta}) \in G_V$, $\vec{\beta} = (\beta_1, \beta_2)$ by

$$u(\alpha, \vec{\beta}) = e^{\beta_1 T_2 + \beta_2 T_3} e^{\alpha T_1} = \begin{pmatrix} 1 & 0 & 0 \\ \beta_1 & e^\alpha & 0 \\ \beta_2 & 0 & e^\alpha \end{pmatrix}. \quad (6.4)$$

It is easy to check that $u(\alpha,\vec{\beta})$ obey the group product

$$u(\alpha,\vec{\beta})u(\alpha',\vec{\beta}') = u(\alpha+\alpha',\vec{\beta}+e^{\alpha}\vec{\beta}'), \tag{6.5}$$

and the inverse is

$$u^{-1}(\alpha,\vec{\beta}) = u(-\alpha,-\vec{\beta}e^{-\alpha}) = \begin{pmatrix} 1 & 0 & 0 \\ -\beta_1 e^{-\alpha} & e^{-\alpha} & 0 \\ -\beta_2 e^{-\alpha} & 0 & e^{-\alpha} \end{pmatrix}. \tag{6.6}$$

Let $g(\phi,\vec{\chi}) \in G_V$, $\vec{\chi} = (\chi_1,\chi_2)$ be

$$g(\phi,\vec{\chi}) = \begin{pmatrix} 1 & 0 & 0 \\ \chi_1 & e^{\phi} & 0 \\ \chi_2 & 0 & e^{\phi} \end{pmatrix}. \tag{6.7}$$

Then the isometry u acts on g by a left multiplication

$$g(\phi,\vec{\chi}) \to g'(\phi',\vec{\chi}') = u(\alpha,\vec{\beta})g(\phi,\vec{\chi}) = g(\phi+\alpha, e^{\alpha}\vec{\chi}+\vec{\beta}). \tag{6.8}$$

The G_V invariant elements in the action are constructed out of

$$\begin{aligned} (g^{-1}\partial_\mu g)^1 &\equiv tr(\tilde{T}^1 g^{-1}\partial_\mu g) = \partial_\mu\phi, \\ (g^{-1}\partial_\mu g)^2 &\equiv tr(T^{2t} g^{-1}\partial_\mu g) = e^{-\phi}\partial_\mu\chi_1, \\ (g^{-1}\partial_\mu g)^3 &\equiv tr(T^{3t} g^{-1}\partial_\mu g) = e^{-\phi}\partial_\mu\chi_2. \end{aligned} \tag{6.9}$$

To write the four-dimensional Bianchi V model we choose to insert in eq. (2.1):

$$x^i \equiv \{t\}, \qquad E_{ab}(t) = a(t)^2\delta_{ab}, \qquad F(t) = -1, \qquad F^L = F^R = \Phi = 0. \tag{6.10}$$

The action takes the form

$$\begin{aligned} S[\phi,\vec{\chi},t] &= \frac{1}{2\pi}\int d^2z \left(a(t)^2\delta_{ab}(g^{-1}\partial g)^a(g^{-1}\bar{\partial}g)^b - \partial t\bar{\partial}t\right) \\ &= \frac{1}{2\pi}\int d^2z \Big(- \partial t\bar{\partial}t + a(t)^2[\partial\phi\bar{\partial}\phi + e^{-2\phi}(\partial\chi_1\bar{\partial}\chi_1 + \partial\chi_2\bar{\partial}\chi_2)]\Big). \end{aligned} \tag{6.11}$$

Conformal invariance requires

$$a(t) = t. \tag{6.12}$$

In this case the metric is flat (it is the interior of the light-cone in Minkowski space). In the following we consider the conformal model (6.11),(6.12).

To find the B-model we need the background matrix N (4.4):

$$N^{ab}(t,\lambda) = [(E(t)+\lambda_c f^c)^{-1}]^{ab} = \frac{1}{t^2(t^4+\lambda_2^2+\lambda_3^2)} \begin{pmatrix} t^4 & -\lambda_2 t^2 & -\lambda_3 t^2 \\ \lambda_2 t^2 & t^4+\lambda_3^2 & -\lambda_2\lambda_3 \\ \lambda_3 t^2 & -\lambda_2\lambda_3 & t^4+\lambda_2^2 \end{pmatrix}, \tag{6.13}$$

Using

$$\det N = \frac{1}{t^2(t^4+\lambda_2^2+\lambda_3^2)}\ , \qquad trT_a = (tr\ T_1)\delta_a^1 = 2\delta_a^1, \tag{6.14}$$

one finds

$$\begin{aligned} S_B[\lambda,t] &= \frac{1}{2\pi}\int d^2z \Big((\partial\lambda_a + 2\delta_a^1\partial\sigma)N^{ab}(\bar\partial\lambda_b - 2\delta_b^1\bar\partial\sigma) \\ &- \partial t\bar\partial t + ln\ \det N\ \partial\bar\partial\sigma\Big) \\ &= S_0[\lambda,t] + S_1[\lambda,t] + S_2[\lambda,t], \end{aligned} \tag{6.15}$$

$$S_0[\lambda,t] = \frac{1}{2\pi}\int d^2z \Big(\partial\lambda_a N^{ab}(t,\lambda)\bar\partial\lambda_b + ln\ \det N\partial\bar\partial\sigma\Big), \tag{6.16}$$

$$S_1[\lambda,t] = \frac{2}{2\pi}\int d^2z\ \sigma\Big(\bar\partial(N^{a1}(t,\lambda)\partial\lambda_a) - \partial(N^{1b}(t,\lambda)\bar\partial\lambda_b)\Big). \tag{6.17}$$

$$S_2[\lambda,t] = -\frac{2^2}{2\pi}\int d^2z\ N^{11}(t,\lambda)\partial\sigma\bar\partial\sigma. \tag{6.18}$$

The A-model is conformal and, therefore, one expects the B-model to be conformal, *i.e.*,

$$\pi\frac{\delta S_B}{\delta\sigma} = T_{z\bar z} = 0 = \pi\frac{\delta(S_0+S_1+S_2)}{\delta\sigma} = T^0_{z\bar z} + T^1_{z\bar z} + T^2_{z\bar z}. \tag{6.19}$$

Variation of S_0 with respect to the conformal factor σ gives

$$\pi\frac{\delta S_0}{\delta\sigma} = T^0_{z\bar z} = \frac{1}{2}[\beta_{G_{IJ}}(X) + \beta_{B_{IJ}}(X)]\partial X^I\bar\partial X^J + \frac{1}{2}\beta_\Phi(X)\partial\bar\partial\sigma, \tag{6.20}$$

where

$$X^I \equiv \{\lambda^a, t\}, \qquad \Phi = -ln\ \det N,$$
$$G_{ab} = \frac{1}{2}(N + N^t)^{ab}, \quad G_{tt} = -1, \qquad \text{otherwise}\ \ G_{IJ} = 0,$$
$$B_{ab} = \frac{1}{2}(N - N^t)^{ab}, \qquad \text{otherwise}\ \ B_{IJ} = 0, \tag{6.21}$$

and to one-loop order the beta-functions are [14]

$$\begin{aligned}
\beta_{G_{IJ}} &= R_{IJ} - \frac{1}{4}H^2_{IJ} - \nabla_I\nabla_J\Phi, \\
\beta_{B_{IJ}} &= -\frac{1}{2}\nabla^K H_{IJK} - \frac{1}{2}\nabla^K\Phi H_{IJK}, \\
\beta_\Phi &= R - \frac{1}{12}H^2 - 2\nabla^2\Phi - (\nabla\Phi)^2 - \frac{2(c-4)}{3},
\end{aligned} \tag{6.22}$$

where

$$H_{IJK} = \nabla_{[I}B_{JK]}, \qquad H^2_{IJ} = H_{IKL}H_J^{KL}. \tag{6.23}$$

Variation of S_1 and S_2 with respect to σ gives

$$\begin{aligned}
\pi\frac{\delta S_1}{\delta\sigma} &= T^1_{z\bar{z}} = \bar{\partial}(N^{a1}(t,\lambda)\partial\lambda_a) - \partial(N^{1b}(t,\lambda)\bar{\partial}\lambda_b) + \mathcal{O}(\sigma), \\
\pi\frac{\delta S_2}{\delta\sigma} &= T^2_{z\bar{z}} = 0 + \mathcal{O}(\sigma).
\end{aligned} \tag{6.24}$$

Here the leading order comes from naive variation with respect to σ; higher order corrections in σ require more careful regularization of (6.17),(6.18), as done in deriving $T^0_{z\bar{z}}$ (6.20). We should note that the leading order contribution to $T^1_{z\bar{z}} + T^2_{z\bar{z}}$ is a total derivative. As will be shown below, using the equation of motion

$$\partial(N^{ab}\bar{\partial}\lambda_b) + \bar{\partial}(N^{ba}\partial\lambda_b) - \frac{\delta N^{bc}}{\delta\lambda_a}\partial\lambda_b\bar{\partial}\lambda_c = 0 + \mathcal{O}(\sigma), \tag{6.25}$$

one finds that

$$T^0_{z\bar{z}} + T^1_{z\bar{z}} + T^2_{z\bar{z}} = 0 + \mathcal{O}(\sigma). \tag{6.26}$$

In the following we present the detailed calculations. The σ-model described by the action S_0 (6.16) has non-zero β-functions [2]. Explicitly, we have the metric

$$ds^2 = -dt^2 + \frac{t^2}{4x(t^4+x)}dx^2 + \frac{x}{t^2}dy^2 + \frac{t^2}{t^4+x}dz^2, \tag{6.27}$$

the antisymmetric tensor

$$B = \frac{1}{2(t^4+x)}dx \wedge dz, \tag{6.28}$$

and the dilaton

$$\Phi = ln\ t^2 + ln(t^4+x). \tag{6.29}$$

These are expressed in terms of the variables $x^\mu = \{t, x, y, z\}$ defined by:

$$\lambda_1 = z, \qquad \lambda_2 = \sqrt{x}\cos y, \qquad \lambda_3 = \sqrt{x}\sin y. \tag{6.30}$$

In our case, we have for the Ricci tensor the following non-vanishing components:

$$\begin{aligned} R_{tt} &= \frac{-6t^8+16t^4x-2x^2}{t^2(t^4+x)^2}, \qquad R_{tx} = \frac{-t^4+x}{t(t^4+x)^2}, \\ R_{xx} &= \frac{2t^8-7t^4x-x^2}{2x(t^4+x)^3}, \qquad R_{yy} = \frac{4x}{t^4+x}, \\ R_{zz} &= \frac{6t^8-10t^4x}{(t^4+x)^3}. \end{aligned} \tag{6.31}$$

For $H^2_{\mu\nu}$ we get the non-vanishing components

$$(H^2)_{tt} = \frac{32t^2x}{(t^4+x)^2}, \qquad (H^2)_{xx} = \frac{-8t^4}{(t^4+x)^3}, \qquad (H^2)_{zz} = \frac{-32t^4x}{(t^4+x)^3}. \tag{6.32}$$

Including the the $\nabla_\mu\nabla_\nu\Phi$ contributions,

$$\begin{aligned} \nabla_t\nabla_t\Phi &= \frac{2(-3t^8+4t^4x-x^2)}{t^2(t^4+x)^2}, \qquad \nabla_t\nabla_x\Phi = \frac{-(3t^4+x)}{t(t^4+x)^2}, \\ \nabla_x\nabla_x\Phi &= \frac{4t^8-t^4x-x^2}{2x(t^4+x)^3}, \\ \nabla_y\nabla_y\Phi &= \frac{4x(2t^4+x)}{t^4(t^4+x)}, \qquad \nabla_z\nabla_z\Phi = \frac{2(3t^8-3t^4x-2x^2)}{(t^4+x)^3}, \end{aligned} \tag{6.33}$$

we have for $\beta_{G_{\mu\nu}}$ in (6.22):

$$\begin{aligned}
\beta_{G_{tx}} &= \frac{2}{t(t^4+x)}, \qquad \beta_{G_{xx}} = \frac{-t^4}{x(t^4+x)^2}, \qquad \beta_{G_{yy}} = \frac{-4x}{t^4}, \\
\beta_{G_{zz}} &= \frac{4x}{(t^4+x)^2},
\end{aligned} \tag{6.34}$$

and for $\beta_{B_{\mu\nu}}$ in (6.22):

$$\begin{aligned}
\beta_{B_{\mu\nu}} &= -\frac{1}{2}\nabla^\rho H_{\mu\nu\rho} - \frac{1}{2}\nabla^\rho \Phi H_{\mu\nu\rho} \\
\beta_{B_{tz}} &= \frac{-4t^5}{(t^4+x)^2} - \frac{4tx}{(t^4+x)^2} = \frac{-4t}{t^4+x}, \\
\beta_{B_{xz}} &= \frac{4t^6}{(t^4+x)^3} - \frac{2t^2(3t^4+x)}{(t^4+x)^3} = \frac{-2t^2}{(t^4+x)^2}.
\end{aligned} \tag{6.35}$$

All other components of β_G and β_B are zero. The dilaton β-function in (6.22) turns out to be a constant:

$$\begin{aligned}
\beta_\Phi &= R - \frac{1}{12}H^2 - [2\nabla^2\Phi - (\nabla\Phi)^2] - \frac{2(c-4)}{3} \\
&= \frac{4t^4(5t^4-9x)}{t^2(t^4+x)^2} - \frac{8t^2x}{(t^4+x)^2} - \frac{4t^2(5t^4-7x)}{(t^4+x)^2} - \frac{2(c-4)}{3} \\
&= \frac{2(4-c)}{3},
\end{aligned} \tag{6.36}$$

and therefore, β_Φ vanishes if the central charge is $c = 4$, as for the A-model [10].

Although we have found non-vanishing contributions to the β-functions, the additional non-local term (6.17), resulting from the anomaly (2.32), has an extra explicit σ dependence which should be taken into account. The variation of this term with respect to σ exactly cancels the above β-functions, recovering the conformal symmetry of the original model. Indeed, from (6.17)

[10] Note that here β_G and β_B do not vanish and, therefore, the fact that β_Φ is a constant (actually zero) is not obvious. Even if the dilaton beta-function would not vanish, there could be $\mathcal{O}(\sigma)$ contributions in (6.24) that would cancel $\beta_\Phi \partial\bar{\partial}\sigma$ in $T_{z\bar{z}}$. In our case we conclude that such $\mathcal{O}(\sigma)$ corrections must vanish.

one finds

$$\begin{aligned}\delta_\sigma S_1 &= \frac{1}{\pi}\int d^2z\ \delta\sigma\Big[\bar\partial\Big(\frac{1}{t^4+x}(t^2\partial z+\frac{1}{2}\partial x)\Big)-\partial\Big(\frac{1}{t^4+x}(t^2\bar\partial z-\frac{1}{2}\bar\partial x)\Big)\Big]\\ &= \frac{1}{\pi}\int d^2z\ \delta\sigma\Big[\frac{2t(-t^4+x)}{(t^4+x)^2}(\bar\partial t\partial z-\partial t\bar\partial z)-\frac{t^2}{(t^4+x)^2}(\bar\partial x\partial z-\partial x\bar\partial z)\\ &- \frac{1}{(t^4+x)^2}[2t^3(\bar\partial t\partial x+\partial t\bar\partial x)+\bar\partial x\partial x]+\frac{1}{t^4+x}\partial\bar\partial x\Big]. \qquad (6.37)\end{aligned}$$

Substituting the equation of motion,

$$\begin{aligned}\partial\bar\partial x &= \frac{t^4+2x}{2x(t^4+x)}\partial x\bar\partial x\ +\ \frac{t^4-x}{t(t^4+x)}(\partial t\bar\partial x+\bar\partial t\partial x)\\ &+ \frac{2x(t^4+x)}{t^4}\partial y\bar\partial y\ -\ \frac{2x}{t^4+x}\partial z\bar\partial z\ +\ \frac{4tx}{t^4+x}(\partial t\bar\partial z-\bar\partial t\partial z),\end{aligned} \qquad (6.38)$$

into eq. (6.37), one finds

$$\delta_\sigma S_1=-\frac{1}{2\pi}\int d^2z\,[(\beta_{G_{\mu\nu}}+\beta_{B_{\mu\nu}})\partial x^\mu\bar\partial x^\nu]\delta\sigma, \qquad (6.39)$$

with the same β_G and β_B of eqs. (6.34), (6.35). Therefore, the total action S_0+S_1 is σ independent, as it should be.

Finally, let us emphasize that we have checked cancellation only to leading order in σ, namely, for $\sigma=0$. However, we should note that since the $\mathcal{O}(\sigma)$ corrections in (6.26) must vanish, this can be used to find the $\mathcal{O}(\sigma)$ contribution to the variation of the non-local action with respect to the conformal factor: $\delta(S_1+S_2)/\delta\sigma$.

To appreciate the possible effect of the mixed anomaly term on the actual value of the Virasoro central charge, consider the extremely simple system of a single scalar with a linear dilaton term:

$$L_A=\partial x\bar\partial x+Q\sqrt{h}R^{(2)}x. \qquad (6.40)$$

This is a conformal system with central charge $c=1+3Q^2$. Applying the above procedure we get in the new variables A and λ:

$$L=A\bar A+\lambda F+Q(\partial\bar A+\bar\partial A)\frac{1}{\Box}\sqrt{h}R^{(2)}, \qquad (6.41)$$

where now the non-local term does not result from a Jacobian anomaly but rather from the explicit Q term in the original action. Integrating out A will now give:

$$L_B = \partial\lambda\bar{\partial}\lambda + \frac{1}{2}Q^2\sqrt{h}R^{(2)}\frac{1}{\Box}\sqrt{h}R^{(2)}. \tag{6.42}$$

The scalar λ is now a normal massless scalar field with $c = 1$, and the difference between the original central charge and the final one is compensated by the explicit non-local Q^2 term in L_B. Note that this is an order σ^2 term (exact, in this example) as expected.

7 The relation between local energy-momentum tensors in models A and B

In this work we considered the two-dimensional quantum field theory (2.19). For the purpose of the discussion in this section, let us neglect the determinants term $\exp J(A, \bar{A})$ in (2.19), and consider the theory

$$\int D\lambda\ Dx\ DA\ D\bar{A}\ e^{-S[A,\bar{A},\lambda,x]}, \tag{7.1}$$

where $S[A, \bar{A}, \lambda, x]$ is given in (2.20). In a flat worldsheet background the two transformations of (7.1) leading to models A and B are valid for any isometry group G. The field theories described by A and B are equivalent, *i.e.*, there exists an exact mapping between the Hilbert spaces of the two theories which preserves the spectrum. The correspondence between the various local operators in the two theories is more involved. In particular, the relation between local energy-momentum tensors, and therefore, the coupling to a non-flat background in the two models, is the subject of this section.

We will use two methods which should be equivalent:
a) Duality transformation in a curved worldsheet background.
b) Ward identities involving the two energy-momentum tensors calculated with the action in a flat background.

Consider (7.1) in a curved background. In the conformal gauge, classically the conformal factor does not appear. Through the measure, however, a

dependence on the conformal factor may appear. This was discussed in section 2. It was shown that in a curved background the theory (7.1) does not lead to model A. If we want to get exactly model A we should add to (7.1) the anomalous piece (2.32).[11]

Equivalently, by taking functional derivatives with respect to the world-sheet metric and gauge field, we conclude that the energy-momentum tensor of model A should have an anomalous correlator with the current coupled to the gauge field when calculated with the flat background action.

As was remarked in section 2, for the "pure" Weyl anomaly, *i.e.*, the contributions to the term $\sqrt{h}R^{(2)}\frac{1}{\Box}R^{(2)}$ in the effective action, there is no multiplicative anomaly. As a consequence we expect that once the current anomaly $\nabla A\frac{1}{\Box}R^{(2)}$ is cancelled by an appropriate counter term, the A- and B-models, if conformal, will have the same central charge.

We present now an independent calculation, confirming the conclusions above.

We consider for action (7.1) the topologically conserved current [12]

$$J_{a\alpha} = \epsilon_{\alpha\beta}\partial^{\beta}\lambda_a. \tag{7.2}$$

In the following, we choose "a" to be a quasi-Abelian direction. Using the equations of motion, $J_{a\alpha}$ is given by

$$J_{a\alpha} = \frac{\delta S}{\delta A^{a\alpha}} + \epsilon_{\alpha\beta}f^{b}_{ac}A^{c\beta}\lambda_b. \tag{7.3}$$

We want to express the topological current of model B, $J_{a\alpha}$, in terms of the variables of model A. The variables λ_b can be calculated by adding to S a source term, $\lambda_b K^b$, and taking the derivative of S with respect to the source:

$$\lambda_a(\zeta) = \frac{\delta S}{\delta K^a(\zeta)} = \int d\zeta' \frac{\delta S}{\delta A^b_\alpha(\zeta')}\frac{\delta A^b_\alpha(\zeta')}{\delta K^a(\zeta)}. \tag{7.4}$$

[11] In section 2, this term was included in the theory from the beginning, since we were careful to keep the term $\exp J$ in the measure of Z (2.19).

[12] The conservation equation of this current is the "dual Bianchi identity" presented in ref. [3].

To first order in K, the gauge field is given by:

$$A^a_\alpha(\zeta) = (g^{-1}\partial_\alpha g)^a + \Big[h^{-1}(\zeta)\frac{1}{\Box_{\zeta\zeta'}}\epsilon_{\alpha\beta}\partial^\beta(gKg^{-1})_{\zeta'}g(\zeta)\Big]^a, \tag{7.5}$$

where $g^{-1}\partial_\alpha g = A_\alpha$. Using (7.5) the final result for the current is:

$$\begin{aligned} J_{a\alpha}(\zeta) &= \frac{\delta S}{\delta A^{a\alpha}(\zeta)} + \epsilon_{\alpha\beta}f^b_{ac}A^{c\beta}(\zeta) \ \times \\ &\times \ M_b{}^d(g^{-1}(\zeta))\frac{1}{\Box_{\zeta\zeta'}}\epsilon_{\gamma\delta}\partial^{\delta'}\Big[\frac{\delta S}{\delta A^{\gamma e}(\zeta')}M_d{}^e(g(\zeta'))\Big]. \end{aligned} \tag{7.6}$$

Here $M(g)$ is the adjoint representation (2.33). If S has a quadratic dependence on $A^{a\alpha}$, the second term in (7.6) contains

$$\epsilon^{\gamma\delta}\partial_\delta(gg^{-1}\partial_\gamma gg^{-1}) = \epsilon^{\gamma\delta}[g^{-1}\partial_\gamma g, g^{-1}\partial_\delta g], \tag{7.7}$$

which does not have a component in a quasi-Abelian direction. It follows that up to terms cubic in the fields,

$$J_{a\alpha}(\zeta) = \frac{\delta S}{\delta A^{a\alpha}(\zeta)}. \tag{7.8}$$

In particular, for the model based on the group G_V in section 6, $J_{a\alpha}$ has the form:

$$J_\alpha = E_{11}(t)\partial_\alpha\phi. \tag{7.9}$$

Considering at one-loop level the correlator with the energy-momentum tensor we obtain a non-zero, anomalous contribution from the term

$$T_{\alpha\beta} = E_{11}(\partial_\alpha\chi_1\partial_\beta\chi_1 + \partial_\alpha\chi_2\partial_\beta\chi_2) \tag{7.10}$$

in the presence of the interaction term

$$L_{int} = -2E_{11}\phi(\partial_\alpha\chi_1\partial^\alpha\chi_1 + \partial_\alpha\chi_2\partial^\alpha\chi_2). \tag{7.11}$$

On the other hand, if we calculate directly the correlator in model B, since the current (7.2) is topological in terms of the new local field λ_a, it cannot be anomalous: the derivative and ϵ-tensors can be taken outside the correlator

for which dimensional regularization can be used. We conclude that the energy-momentum tensors defined by (7.1) for models A and B differ by their correlators to the topological current (7.2). This current is the source of the gauge field $g^{-1}\partial_\alpha g$ and, therefore, we reproduce this way the anomaly contribution in eq. (2.32).

8 On localizing the mixed anomaly term

Let us write down the formal equality

$$\int Ds\ Dt\ Dx\ e^{-\frac{1}{2\pi}\int d^2z\left(S_0[x]+2\partial s\bar\partial t+Q(x)\frac{1}{\Box}\sqrt{h}R^{(2)}\right)}$$
$$= \int Du\ Dv\ Dx\ e^{-\frac{1}{2\pi}\int d^2z\left(S_0[x]+2\partial u\bar\partial v+4vQ(x)+\frac{1}{4}u\sqrt{h}R^{(2)}\right)}, \qquad (8.1)$$

where $\Box = 2\partial\bar\partial$. (The models are not necessarily the same.) One can show this equality by integrating out s, t on the left-hand side, and comparing with the integration over u, v on the right-hand side of eq. (8.1). Some choices of $Q(x)$ are interesting. For example, if $S_0[x]$ is a sigma model with beta-functions β_G, β_B, then with the choice

$$Q(x) = -\frac{1}{2}(\beta_{G_{\mu\nu}} + \beta_{B_{\mu\nu}})\partial x^\mu \bar\partial x^\nu, \qquad (8.2)$$

the model $S_0[x] + 2\partial s\bar\partial t + Q(x)\frac{1}{\Box}\sqrt{h}R^{(2)}$ is manifestly independent of σ, to leading order, which implies that the model $S_0[x] + 2\partial u\bar\partial v + 4vQ(x) + \frac{1}{4}u\sqrt{h}R^{(2)}$ is conformally invariant to leading order. (For a discussion about higher order corrections, including a dilaton beta-function in (8.2), see section 6.) Similar models were discussed in ref. [15].

Moreover, if $S_0 = S[A, \bar A, \lambda, x]$ (2.20), and if we choose

$$Q(A, \bar A) = \frac{1}{4}(trT_a)(\partial\bar A^a + \bar\partial A^a), \qquad (8.3)$$

then, using eq. (8.1), one finds that

$$\int Dx\ D\lambda\ DA\ D\bar A\ Ds\ Dt\ e^{-\frac{1}{2\pi}\int d^2z\, S_a[x,A,\lambda,s,t]}$$
$$= \int Dx\ D\lambda\ DA\ D\bar A\ Du\ Dv\ e^{-\frac{1}{2\pi}\int d^2z\, S_b[x,A,\lambda,u,v]}, \qquad (8.4)$$

where

$$\begin{aligned} S_a &= S[A,\bar{A},\lambda,x] + S_{nonlocal}[A,\bar{A},\sigma] + \frac{1}{2\pi}\int d^2z\ 2\partial s\bar{\partial}t, \\ S_b &= S[A,\bar{A},\lambda,x] \\ &+ \frac{1}{2\pi}\int d^2z\left(2\partial u\bar{\partial}v + v(trT_a)(\partial\bar{A}^a + \bar{\partial}A^a) + \frac{1}{4}u\sqrt{h}R^{(2)}\right). \end{aligned} \quad (8.5)$$

Here $S[A,\bar{A},\lambda,x]$ and $S_{nonlocal}[A,\bar{A},\sigma]$ are given in eqs. (2.20) and (4.1), respectively. Now, integrating out λ on the left-hand side of eq. (8.4), we recover the A-model in section 3, with an additional decoupled null kinetic term $\partial s\bar{\partial}t$. On the other hand, integrating out $A,\bar{A}$ on the right-hand side of eq. (8.4), we get a *local* dual theory with action

$$\begin{aligned} S_b[\lambda,x,u,v] &= \frac{1}{2\pi}\int d^2z\left(2\partial u\bar{\partial}v - (\Phi - u - ln\det N)\partial\bar{\partial}\sigma\right. \\ &+ (\partial\lambda_a - \partial x^i F^L_{ia} + trT_a\partial v)N^{ab}(\bar{\partial}\lambda_b + F^R_{bj}\bar{\partial}x^j - trT_b\bar{\partial}v) \\ &+ \left. F_{ij}\partial x^i\bar{\partial}x^j\right). \end{aligned} \quad (8.6)$$

Comparing to $S_B[\lambda,x,\sigma]$ in eq. (4.3), we see that there is an additional null kinetic term, $\partial u\bar{\partial}v$, the term leading to non-locality, $trT_a\partial\sigma$, has been replaced by $trT_a\partial v$, and the dilaton Φ is shifted to $\Phi - u$. Therefore, the additional null coordinates u,v provide a localization of the B-model in a higher dimensional sigma model.

Finally, we should mention that another way to localize the mixed anomaly term, within a larger sigma model framework, is to bosonize the ghosts in eq. (2.21). By integrating out the gauge field in (2.19), one will obtain a local sigma model action.

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9 Appendix A

In this Appendix we prove the second equality in (2.32), namely

$$ln(\det M(g) \det M(\bar{g})) = trT_a\Big(\frac{1}{\partial}A^a + \frac{1}{\bar{\partial}}\bar{A}^a\Big). \tag{9.1}$$

If all the generators are traceless, there is nothing to prove. Suppose not all of them are traceless. We choose a basis for the generators T_a such that

$$tr\ T_1 \neq 0\ , \qquad tr\ T_a = 0\ , \qquad a = 2, ..., D = \dim\ G\ . \tag{9.2}$$

This is always possible by redefining $T_a \to T_a - (tr\ T_a/tr\ T_1)T_1$. Therefore, we want to prove that

$$ln\ \det\ M(g) = (tr\ T_1)\frac{1}{\partial}A^1. \tag{9.3}$$

Recall that $A = g^{-1}\partial g$, and we choose to parametrize g by

$$g(\alpha_a) = g_2 g_1\ , \qquad g_1 = e^{\alpha^1 T_1}\ , \quad g_2 = e^{\sum_{a=2}^{D}\alpha^a T_a}\ . \tag{9.4}$$

Recall also that $M_a{}^b(g)T_b = gT_ag^{-1}\ \Rightarrow\ M_a{}^b(g)tr\ T_b = tr(gT_ag^{-1}) = tr\ T_a$, which implies

$$M_1{}^1(g) = 1\ , \qquad M_a{}^1(g) = 0 \quad \text{for} \quad a \neq 1. \tag{9.5}$$

Let us now find A^1 in terms of α:

$$\begin{aligned} A^1 &= tr(A\tilde{T}^1) = tr(g^{-1}\partial g\tilde{T}^1) \\ &= tr({g_1}^{-1}\partial g_1\tilde{T}^1) + tr({g_1}^{-1}{g_2}^{-1}\partial g_2 g_1\tilde{T}^1) \\ &= \partial\alpha^1\ tr(T_1\tilde{T}^1) + ({g_2}^{-1}\partial g_2)^a tr({g_1}^{-1}T_a g_1\tilde{T}^1) \\ &= \partial\alpha^1 + ({g_2}^{-1}\partial g_2)^a M_a{}^1({g_1}^{-1}) = \partial\alpha^1. \end{aligned} \tag{9.6}$$

Here we have used eqs. (2.4), (2.33), (9.4), and in the last equality we have used eq. (9.5). Moreover,

$$ln\ \det\ M(g) = ln\ \det\ M(g_1) + ln\ \det\ M(g_2) = tr\ ln\ M(g_1) = tr\ T_1\alpha^1\ . \tag{9.7}$$

Here we have used eq. (9.4), and the fact that $tr\ T_a = 0$ for $a \neq 1$ implies $\det\ M(g_2) = 1$. Inserting $\alpha_1 = (1/\partial)A^1$ (9.6) in (9.7) proves eq. (9.3).

10 Appendix B

In this Appendix we study the conformal factor-dependent terms which may appear when Gaussian integration over the A-variables is performed in (4.2). The origin of these terms is the implicit dependence on the conformal factor in the measure of A [10]. This can be made explicit by expressing the vector field $A^a, \bar{A}^b$ in terms of scalar variables $y^a, \bar{y}^{\bar{b}}$:

$$A^a \equiv \partial y^a, \qquad \bar{A}^b \equiv \bar{\partial}\bar{y}^{\bar{b}}. \tag{10.1}$$

Since the x and λ variables do not participate in this effect, and the linear terms in A do not contribute (we can complete the square in $A, \bar{A}$), in order to find the σ-dependence of the A measure (to one loop), we can replace $S[A, \bar{A}, \lambda, x] + S_{nonlocal}[A, \bar{A}]$ in (4.2),(2.20),(2.32) by the action

$$S[y, \bar{y}, x] = \frac{1}{2\pi}\int d^2z\,[E_{ab}(x)\partial y^a\bar{\partial}\bar{y}^{\bar{b}} + \partial x\bar{\partial}x], \tag{10.2}$$

where x is a single field with flat metric. The variation with respect to the conformal factor in the A-measure is obtained by calculating the trace of the energy-momentum tensor of (10.2). This is given by the general beta-function equations [14],

$$T_{z\bar{z}} = \frac{1}{2}(R_{IJ} - \frac{1}{4}H^2_{IJ} - \frac{1}{2}\nabla^K H_{IJK})\partial X^I\bar{\partial}X^J, \tag{10.3}$$

where X^I in our case include $x, y^a, \bar{y}^{\bar{b}}$, and R, H are the Ricci curvature and antisymmetric field strength calculated from the metric and torsion. From

eq. (10.2) we identify the metric G:

$$G_{xx} = 1, \qquad G_{a\bar{b}} = G_{\bar{b}a} = \frac{E_{ab}}{2}, \tag{10.4}$$

and torsion B:

$$B_{a\bar{b}} = -B_{\bar{b}a} = \frac{E_{ab}}{2}, \tag{10.5}$$

all other components being zero. Using the special (quasi-Kählerian) form of G and B and their independence of $y^a, \bar{y}^{\bar{b}}$, eq. (10.3) is easily calculated:

$$\begin{aligned} 2T_{z\bar{z}} &= \frac{\partial^2 ln\sqrt{\det G}}{\partial x^2}\partial x\bar{\partial}x + \frac{1}{2}\frac{\partial}{\partial x}(ln\sqrt{\det G})\frac{\partial E_{ab}}{\partial x}\partial y^a\bar{\partial}\bar{y}^{\bar{b}} \\ &= \partial\bar{\partial}(ln\sqrt{\det G}) \;=\; \partial\bar{\partial}(ln\ \det E), \end{aligned} \tag{10.6}$$

where in the second equality we used the equation of motion, and in the last step we used the special form of G (10.4). To get eq. (10.6) it is convenient to use:

$$\begin{aligned} R_{ij} &= \frac{\partial^2 ln\sqrt{g}}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^k}\Gamma^k_{ij} + \Gamma^m_{ik}\Gamma^k_{mj} - \Gamma^m_{ij}\frac{\partial ln\sqrt{g}}{\partial x^m}, \\ g^{ij}\frac{\partial g_{ij}}{\partial x^m} &= 2\frac{\partial ln\sqrt{g}}{\partial x^m} \end{aligned} \tag{10.7}$$

Therefore, the conformal factor dependence produces a dilaton field $\Phi(x)$ for the x variables:

$$\Phi(x) = ln\ \det E. \tag{10.8}$$

Since the result in (10.6) is given in terms of the worldsheet variables, the result clearly is not dependent on the number of x variables.

References

[1] For a review, see for example: A. Giveon, M. Porrati and E. Rabinovici, *Target Space Duality in String Theory*, hep-th/9401139, Phys. Rep. **244** (1994) 77.

[2] M. Gasperini, R. Ricci and G. Veneziano, Phys. Lett. **B319** (1993) 438.

[3] A. Giveon and M. Roček, Nucl. Phys. **B421** (1994) 173.

[4] E. Alvarez, L. Alvarez-Gaumé and Y. Lozano, *On Non-Abelian Duality*, hep-th/9403155.

[5] M. Kontsevich and S. Vishik, *Geometry of Determinants of Elliptic Operators*, hep-th/9406140, and references therein.

[6] J. Polchinski, Nucl. Phys. **B303** (1988) 226.

[7] D.H. Friedan, Ann. of Phys. **163** (1985) 318;
C.M. Hull and P.K. Townsend, Nucl. Phys. **B274** (1986) 349;
A.A. Tseytlin, Phys. Lett. **B178** (1986) 34;
G. Curci and G. Paffuti, Nucl. Phys. **B274** (1987) 399;
G.M. Shore, Nucl. Phys. **B286** (1987) 349;
H. Osborne, Nucl. Phys. **B294** (1987) 1257;
A.A. Tseytlin, Phys. Lett. **B194** (1987) 63;
J. de Boer, K. Clubok and M.B. Halpern, *Linearized Form of the Generic Affine-Virasoro Action*, hep-th/9312094.

[8] S. Deser and A. Schwimmer, *Gauge Field Improvement, Form - Scalar Duality, Conformal Invariance and Quasilocality*, hep-th/9404183.

[9] B.E. Fridling and A. Jevicki, Phys. Lett. **B134** (1984) 70;
E.S. Fradkin and A.A. Tseytlin, Ann. Phys. **162** (1985) 31;
T. Buscher, Ph.D. Thesis, May 1988, unpublished;
X.C. de la Ossa and F. Quevedo, Nucl. Phys. **B403** (1993) 377.

[10] T. Buscher, Phys. Lett. **B194** (1987) 59; Phys. Lett. **B201** (1988) 466.

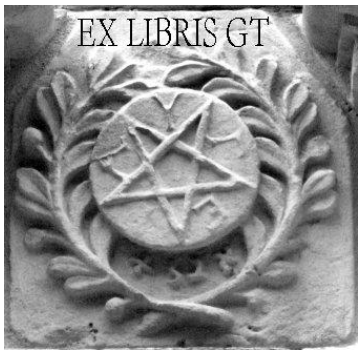
[11] M. Roček and E. Verlinde, Nucl. Phys. **B373** (1992) 630.

[12] E. Alvarez, L. Alvarez-Gaumé, J.L.F. Barbon and Y. Lozano, Nucl. Phys. **B415** (1994) 71.

[13] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag NY, 1972.

[14] C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. **B262** (1985) 593.

[15] A.A. Tseytlin, Phys. Lett. **B288** (1992) 279.



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String theory and primordial cosmology

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SUMMARY.

String cosmology aims at providing a reliable description of the very early Universe in the regime where standard-model physics is no longer appropriate, and where we can safely apply the basic ingredients of superstring models such as dilatonic and axionic forces, duality symmetries, winding modes, limiting sizes and curvatures, higher-dimensional interactions among elementary extended object. The sought target is that of resolving (or at least alleviating) the big problems of standard and inflationary cosmology like the spacetime singularity, the physics of the trans-Planckian regime, the initial condition for inflation, and so on.

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Chapter 1

String theory and primordial cosmology

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SUMMARY. String cosmology aims at providing a reliable description of the very early Universe in the regime where standard-model physics is no longer appropriate, and where we can safely apply the basic ingredients of superstring models such as dilatonic and axionic forces, duality symmetries, winding modes, limiting sizes and curvatures, higher-dimensional interactions among elementary extended object. The sought target is that of resolving (or at least alleviating) the big problems of standard and inflationary cosmology like the spacetime singularity, the physics of the trans-Planckian regime, the initial condition for inflation, and so on.

1.1 The standard "Big Bang" cosmology

In the second half of the last century the theoretical and observational study of our Universe, grounded on one hand on the Einstein theory of general relativity, and on the other hand on astronomical observations of every increasing precision, has led to the formulation (and to the subsequent completion) of the so-called *standard cosmological model* (see e.g. [1, 2, 3]).

Such a model – like every physical model – is based on various assumptions. We should mention, in particular:

1. The assumption that the large-scale spacetime geometry can be foliated by a class of three-dimensional space-like hypersurfaces which are exactly homogeneous and isotropic.

2. The assumption that the matter and the radiation filling our Universe behave exactly as a perfect fluid with negligible friction and viscosity terms.

3. The assumption that the radiation is in thermal equilibrium.

4. The assumption that the dominant source of gravity, on cosmological scales, is the so called *dark matter* component of the cosmological fluid (invisible, up to now, to all attempted detection procedures of nongravitational type); and so on.

Using such assumptions, the standard cosmological model has obtained a long and impressive series of successes and experimental confirmations, such as:

1. The geometric interpretation of the apparent recessional velocity of distant light sources, together with a precise theoretical formalization of the empirical Hubble law.

2. The prediction of a relic background of thermal radiation.

3. The explanation of the process of genesis of the light elements and of the other *building blocks* of our present macroscopic world (like the processes of nucleosynhesis and baryogenesis); and so on.

In spite of these important achievements the standard cosmological model was put in trouble when, in the 1980s, the scientific community started to investigate the problem of the origin of the observed galactic structures, and of the small (but finite) inhomogeneity fluctuations presents in the temperature T of the relic background radiation ($\Delta T/T \sim 10^{-5}$). How did originate the temperature inhomogeneities $\Delta T/T$ and, especially, the matter inhomogeneities $\Delta\rho/\rho$ which are at the grounds of the concentration and subsequent growth of the cosmic aggregates (cluster of galaxies, stars and planets) that we presently observe? No temperature fluctuation and density fluctuation should exist, on macroscopic scales, if our Universe would be exactly homogeneous and isotropic as required by the standard cosmological model.

This problem was solved by assuming that the standard cosmological model has to be modified, at some very early epoch, by the introduction of a cosmological phase – called *inflation* – characterized by an accelerated expansion rate [4, 5, 6]. During such a primordial inflationary phase the three-dimensional spatial sections of our Universe underwent a gigantic (almost exponential) growth of proper volume in few units of the Hubbe-time parameter (see see.g. [3, 7, 8]). This process was able to amplify the microscopic quantum fluctuations of the matter fields (and of the geometry), thus producing the macroscopic inhomogeneities required for the formation of the matter structures and of the temperature anisotropies we observe today (see see.g. [8, 9, 10]).

A phase of inflationary evolution like that proposed above, however, cannot be extended back in time to infinity (or, to use the standard terminology, cannot be *past eternal* [11, 12, 13]). If we go back in time to sufficiently earlier epochs we find that the inflationary phase of the standard model has a beginning at

a precise instant of time. Before that time, the Universe was in an extremely hot, dense and curved primordial state – an ultimate concentrate of matter and radiation at extremely high energy and temperature.

This means, in other words, that before starting inflating the Universe was quite close to the so-called *big bang* epoch, namely to the epoch of the huge cosmic explosion which – according to the standard model, even including the inflationary phase – gave rise to the matter and energy species we observe today, and was at the origin of the spacetime itself.

In fact, the big bang epoch of the standard model corresponds (strictly speaking) to a mathematical singularity where the energy density and the spacetime curvature blow up to infinity. We can thus say that to the question *How did the Universe begin?*, the standard cosmological model provides the answer: *The Universe was born from the initial big bang singularity.*

1.1.1 Validity restrictions of the standard cosmological model

It is well known that standard cosmology is based on the Einstein theory of general relativity, which is a *relativistic* theory of gravity, but *not* a *quantum* theory. Hence, like all classical theories, general relativity has a limited validity range. Because of those limits the standard cosmological model cannot be extrapolated to physical regimes where the energy and the spacetime curvature are too high: this prevents taking too seriously the predictions of such a model about the initial singularity.

We should recall, in fact, that a classical model is valid until the corresponding action $S = Et$ is much larger than the elementary *quantum of action* (or Planck's quantum) h. If we take a cosmological patch of the size given by the Hubble radius c/H, we can then estimate the total involved energy E by multiplying the energy density ρ of the gravitational sources by the spatial volume $(c/H)^3$, containing the contribution of all observable matter and radiation at a given time t. The typical cosmological time scale, on the other hand, is provided by the Hubble time H^{-1}, and the energy density ρ is related to the Hubble time by the Einstein equations, which imply (modulo numerical factors of order one) $\rho = c^2H^2/G$, where G is the Newton gravitational constant. By imposing the condition $Et \gg h$ we then find that the standard cosmological model may give a reliable (classical) description of the Universe provided that

$$\frac{c^5}{GH^2} \gg h. \tag{1.1}$$

(This condition, in units $h = c = 1$, can also be rephrased as $H \ll M_{\rm P}$, where $M_{\rm P} = (hc/G)^{1/2}$ is the Planck mass).

The parameters C, G, h appearing in the above equation are constant, while the Hubble parameter H is closely related to the spacetime curvature and is time dependent, $H = H(t)$. According to the standard model, in particular, H grows as we go back in time, and diverges at the time of the big-bang singularity

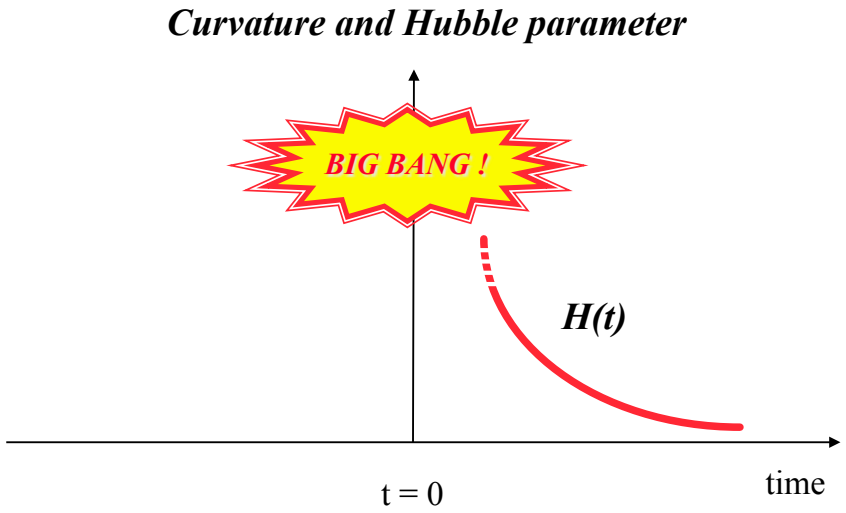

Figure 1.1: *According to the standard cosmological model, the spacetime curvature and the associated Hubble parameter $H(t)$ undergo an unbounded growth as we go back in time, and blow up at the time $t = 0$ of the initial "big bang" singularity.*

(Fig. 1.1). Correspondingly, the ratio c^5/GH^2 decreases and goes to zero at the singularity. Hence, before reaching the big bang epoch we necessarily enter the regime where the condition (1.1) is violated, and the standard cosmological model is no longer valid.

In order to provide a reliable description of the primordial Universe we should thus use a more general approach, based on a theory able to describe gravity also in the quantum regime. A possible candidate for this theory, which is complete, consistent at all energy scales and, besides gravity, also incorporates all fundamental interactions, is the so-called *theory of strings* (see, e.g., [14, 15, 16]).

1.2 String theory

The name of this theory is due to the fact that it proposes a model where the fundamental *building blocks* of our physical description of nature are one-dimensional extended object (elementary "strings", indeed), instead of elementary particles. Such strings can be open (of finite length), or closed, and the spectrum of states associated to their vibration modes can reproduce the particle states of the gravitational interaction and of all the other fundamental (electromagnetic, strong and weak) interactions.

In addition, if the string model is appropriately *supersymmetrized* – namely, if we add to each bosonic degree of freedom a corresponding fermionic partner

– we arrive at the so-called theory of *superstrings*. This model potentially describes not only all interaction fields, but also their elementary sources (quarks and leptons), and thus all possible species and states of matter [14, 15].

But there is more. A basic property of string theory – probably the most revolutionary property, comparing with the other theories – is the property of determining not only the possible form of the interaction terms (which is also done by the usual gauge theories, through the minimal coupling procedure), but also the form of the free-field (kinetic) terms (which in the other theories is always left, to some extent, arbitrary). Indeed, string theory satisfies a new symmetry (called *conformal symmetry*) which rigidly prescribes the allowed free-field dynamics, at any given order of the chosen perturbative expansion [14, 15].

Quantizing a string, and imposing that the conformal symmetry is left unbroken by the quantum corrections (i.e. imposing the absence of *conformal anomalies*), one finds, in fact, that – to lowest order – the electromagnetic field *must* satisfy the Maxwell equations, the gravitational field *must* satisfy the Einstein equations, the spinor fields *must* satisfy the Dirac equations, and so on. All field equations, laboriously discovered in the past centuries through the theoretical elaboration of a large amount of empirical data, can be simply *predicted* by string theory even in the absence of any experimental input!

Finally, as already stressed, string theory is valid for all interactions also in the quantum regime, and can thus be used at arbitrarily high energy scales. In particular, unlike general relativity, can be applied to describe the Universe at epochs arbitrarily near to the big bang epoch. In such a limiting high-energy regime the equations we obtain from string theory are different, in general, from the corresponding field-theory equations, and thus it makes sense to ask the question *What's new from string theory about cosmology?*

In particular, *What's new about the very early epochs at the beginning of the Universe?*

1.3 String cosmology

There are, in particular, two aspects of string theory which can play a relevant role in the formulation of a consistent cosmological scenario.

The first one concerns the so-called *dual symmetry*, typical of one-dimensional extended objects. If such a symmetry is respected (even at the approximate level) by the gravitational dynamics on cosmological scales, then any cosmological phase occurring at $t > 0$, and characterized by a decreasing Hubble parameter H (hence, decreasing curvature), must be associated to a *dual* partner phase, defined at $t < 0$ and characterized by growing H (see [17] for a nontechnical illustration of this duality symmetry). It follows in particular that the present cosmological phase, subsequent to the big bang epoch and well described by the standard model, must be preceded in time by an almost specularly symmetric phase occurring *before the big bang* (Fig. 1.2). Such a duality symmetry should also leave an imprint on the properties of the cosmological perturbations [18].

In Fig. 1.2 both phases are characterized by a curvature (and a Hubble

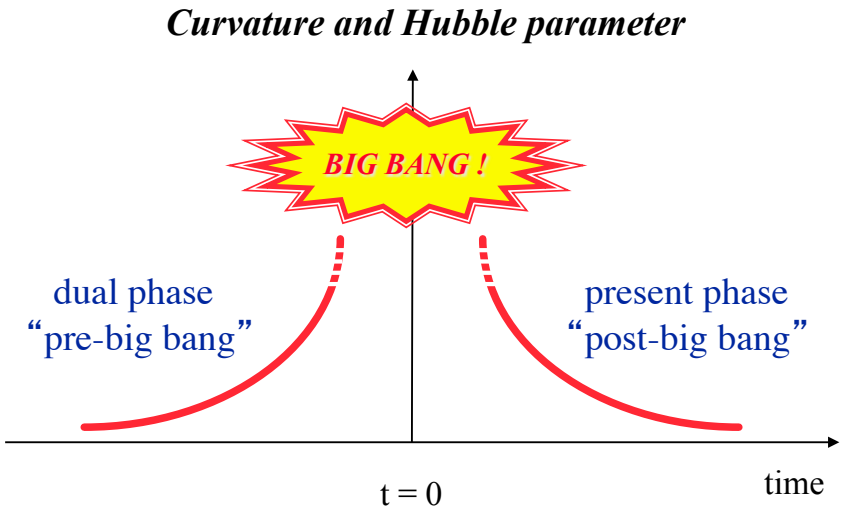

Figure 1.2: *The standard cosmological phase, of "post-big bang" type, is preceded by a (string-theory) dual phase of "pre-big bang" type.*

parameter) which diverges as t goes to zero. If that would be the case, then the two branches of the cosmological evolution would be causally disconnected by a spacetime singularity, with no chances of merging together into a single coherent model of spacetime evolution. It is here, however, that comes into play another crucial aspect of string theory.

String theory is indeed characterized by a fundamental length $\lambda_{\rm s}$, which is a constant parameter of the string action and which controls the typical size of a quantized string. The physical role played by $\lambda_{\rm s}$ is very similar to the role played by the Bohr radius for the atom, which represents the minimal allowed size of the quantum electronic orbitals. The numerical value, however, is quite different: we may expect, in fact, $\lambda_{\rm s} \sim 10^{-33}$ cm (i.e. a value of $\lambda_{\rm s}$ which is about 10 times that of the Planck length $\lambda_{\rm P} = h/M_{\rm P}c^2$), in order that string theory may include a realistic description of all fundamental interactions (different values of λ_s are possible in the presence of large extra dimensions, see below).

Aside from the particular numerical value of $\lambda_{\rm s}$, what is important, in our context, is that proper distances and sizes smaller than $\lambda_{\rm s}$ have no physical meaning in a string-inspired model. It follows that, in a string-cosmology context, the Hubble radius c/H has to be constrained by the condition $c/H \gtrsim \lambda_{\rm s}$. Since the Hubble radius is directly related to the inverse of the spacetime curvature, we can deduce that the curvature cannot blow up to infinity, because of the constraint $H \lesssim c/\lambda_{\rm s}$. Hence, when a given spacetime region has reached the limiting value $H \approx c/\lambda_{\rm s}$, its geometrical state can only evolve in two ways: it can either stabilize at such a maximum value, or start decaying towards lower curvature states after a bounce induced by appropriate "stringy" effects (see

e.g. [19]).

In such a context, the big bang singularity predicted by the standard model and sharply localized at a a given epoch (say, $t = 0$), has thus to be replaced by an extended phase of very high (but finite) maximal curvature: the so-called *string phase* (see e.g. [20, 21]). By combining the existence of the dual symmetry and of a minimal length scale, a string-based model can thus complete the standard cosmological scenario by removing the curvature singularity and extending the physical description of the Universe back in time, beyond the big bang, to infinity. The *big bang era* is still there, but it is deprived of the standard role of initial singularity: it corresponds, instead, to the epoch marking the transition between the growing curvature and the decreasing curvature regime (Fig. 1.3).

Within such a cosmological scenario (first presented in detail in [22]) the initial cosmological state is no longer localized at $t = 0$, but it is moved to the limit $t \to -\infty$, and corresponds to an asymptotic state usually called the *string perturbative vacuum*. Such a new initial state, as illustrated in Fig. 1.3, turns out to be a sort of specularly symmetric version of the final state that would be reached in the asymptotic future by a Universe which keeps expanding for ever according to the standard cosmological dynamics. Namely, a flat, empty and cold initial state, drastically different from the initial hot, explosive state, extremely curved and concentrated, proposed by the standard scenario.

There is, however, a possible asymmetry between the initial and final state of the above string-cosmology model, due to the coupling strength of the fundamental interactions: such a coupling tends to zero as $t \to -\infty$, while it may become very strong in the opposite limit, if not appropriately stabilized (see e.g. [20, 21]). This growth of the coupling can be accompanied, in principle, also by a large amount of entropy production (see e.g. [23]).

1.4 A higher-dimensional Universe

String theory, which is at the grounds of the cosmological scenario described in the previous section, can be consistently formulated only in the context of a higher dimensional spacetime manifold.

In fact, in order to consistently quantize a bosonic string without introducing *ghosts* (states of negative norm), and without violating the Lorentz symmetry, one must introduce a generalized spacetime manifold with 26 dimensions (see e.g. [14, 15]). In this way, however, one obtains a model which has still a pathology, as it contains *tachyons* (states of imaginary mass), which we believe should be absent in any realistic physical model.

In order to eliminates the tachyons, we can generalize the bosonic string model by adding fermion states and considering the so-called *superymmetric* string models, or *superstring* models. In that case, a consistent quantization requires 10 spacetime dimensions, and the number of total dimensions is to be increased up to 11 (with one time-like and 10 space-like dimensions) if we require that the five possible types of superstrings may be connected by duality

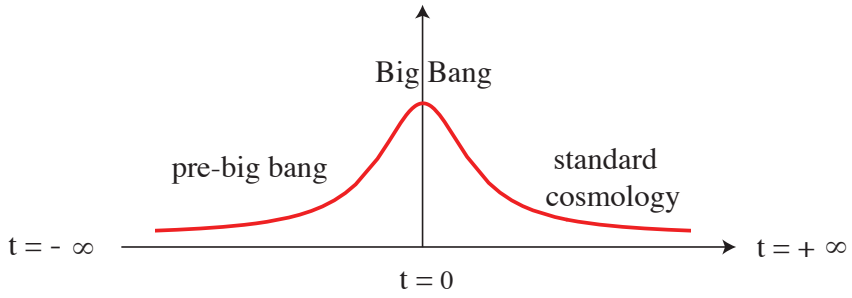

Figure 1.3: *Time evolution of the curvature scale and of the energy density in a typical example of string-cosmology scenario. The big bang epoch does not correspond to a singularity (like in the standard model) but to a phase of maximal, finite curvature. The Universe evolves starting from a flat, cold and empty state called the "string perturbative vacuum", asymptotically localized at* $t = -\infty$.

transformations, and may represent various weak-coupling regimes of a more fundamental theory, called *M-theory* [24] (see also [25]).

Hence, whatever string model is assumed to apply, it is clear that the associated string cosmology scenario must be referred to a higher dimensional Universe. On the other hand, all present phenomenological experience (including the most sensitive high-energy experiments) points at a world with one time-like and *only three* space-like dimensions. We are thus naturally led to the following questions:

If string theory is correct, and the Universe in which we live has a number $d > 3$ *of spatial dimensions, why our experience is only limited to a three-dimensional space? why we cannot detect the additional "extra" dimensions? what happened to those dimensions, if they really exist?*

There are at least two possible answers to the above questions.

There is an *old-fashioned* answer – which, for a long time, has been also the only possible answer to the previous questions – dating back to the so-called Kaluza-Klein model, formulated at the beginning of the last century [26, 27] in the context of a higher dimensional version of general relativity. According to such model, we cannot detect the extra dimensions simply because such dimensions are compactified on length scales of extremely small size (hence, they need extremely high energies in order to be experimentally resolved).

We can take, as a simple example, a long and very thin cylinder. A cylinder is a two-dimensional object but, if it is observed from a distance much larger than its radius, it may appear (in all respects) as being one-dimensional, extended in length but deprived of any sensible thickness. In the same way the spatial extension of our Universe could be largely asymmetric, with three spa-

tial dimensions macroscopically expanded on a large scale, while all the other dimensions *rolled up* in a highly compact way, and confined on a very small length scale – of order (for instance) of λ_s. If we do not have a sufficiently powerful instrument, able to resolve the required (very tiny) distance scales, we will always observe three spatial dimensions only.

Very recently, however, a new possible answer to the dimensionality problem has been suggested by theoretical studies mainly performed in the second half of the 1990s, and closely related to particular string-model configurations, called *branes* [25]. Such a new answer states that we cannot "see" the extra dimensions simply because the fundamental interactions propagate only along three spatial dimensions. All instruments we use to explore the world around us (starting from our eyes up to the more powerful and sophisticated technological tools) have indeed a working mechanism based on the fundamental (electromagnetic, nuclear, and so on) interactions. If such interactions are living only on a restricted subspace of the full spacetime manifold (like, for instance, waves which propagate on the surface of a pond, and not in the direction orthogonal to the pond surface), then the extra dimensions are hidden to our direct experience, even if they are largely (or infinitely) extended.

This second possible answer to the dimensionality problem has suggested new, interesting types of cosmological models, formulated in the context of the so-called *brane-world* scenario (see, e.g., [28]).

1.5 Brane cosmology

According to the so-called brane-world cosmology, our Universe could be a four-dimensional "slice" of a higher-dimensional *bulk* manifold. The elementary charges sourcing the gauge interactions are confined on a three-dimensional hypersurface Σ_3 associated to an object called *Dirichet* 3-*brane* (or D_3-brane), and we cannot detect the external spatial dimensions because the gauge fields of those charges can propagate only on the *world-volume* Σ_4 swept by the time evolution of the brane. (It should be recalled that the description of our Universe as a four-dimensional "domain wall" embedded in a higher-dimensional bulk spacetime was previously suggested, with different motivations, also in [29]).

In a string theory context, however, the confinement mechanism is not equally efficient for all fundamental interactions. Gravity, in particular, is not confined, or is only partially [30] confined, so that it can propagate outside the brane spacetime. This possibility is illustrated in Fig. 1.4, which shows a brane spacetime Σ_4 with two possible sources of interactions. One is a charge, source of the electromagnetic field: the associated electromagnetic waves (or photons) are strcicly confined and can propagate only on Σ_4. The other is a mass, source of the gravitational field: the associated gravitational waves (or gravitons) can leave the brane spacetime Σ_4 and propagate through the external spatial dimensions.

This property of the gravitational field is quite important because, if the higher-dimensional bulk spacetime contains two (or more) fundamental branes,

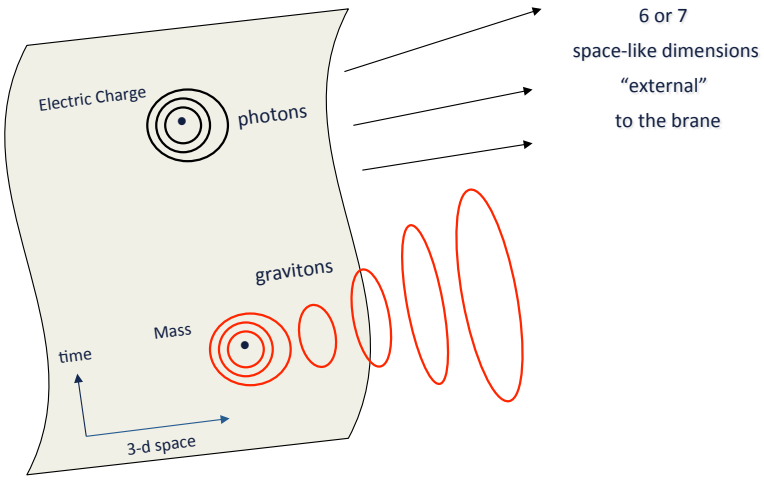

Figure 1.4: *A brane-Universe with one time-like and three space-like dimensions, embedded in an external bulk spacetime characterized by six (according to superstring theory) or seven (according to M-theory) extra spatial dimensions. Electromagnetic forces are confined on the brane spacetime, while gravitational forces propagate also in the directions external to the brane.*

they can interact among themselves gravitationally. And this possibility leads us to an interesting geometric interpretation of the big bang mechanism, namely of the high-energy process which has marked the beginning of the standard cosmological phase, bringing the Universe to the form we are presently observing.

In fact, during the high-curvature phase localized around $t = 0$ (Fig. 1.3), a higher-dimensional Universe tends to be filled by branes which are spontaneously produced in pairs from the high-energy vacuum, and which can gravitationally (and strongly) interact among themselves [21]. According to string theory, on the other hand, the total gravitational force in a higher-dimensional spacetime includes various components: we should mention, in particular, the symmetric-tensor contribution associated to the *graviton*, the scalar contribution associated to the *dilaton*, and the antisymmetric-tensor contribution associated to the *axion* (see, e.g., [20, 21]).

The first two types of forces are always attractive, while the axion force is repulsive between sources of the same sign and attractive between sources of opposite sign (like, for instance, a brane and an antibrane, characterized by opposite axionic charges). It follows, in particular, that if we have two identical branes (or antibranes) in an initial static and symmetric state, then the axion repulsion exactly cancels the attraction due to the graviton and to the dilaton, and the net resulting force is vanishing. If we have instead a brane and an antibrane then the total gravitational force is always non-vanishing and

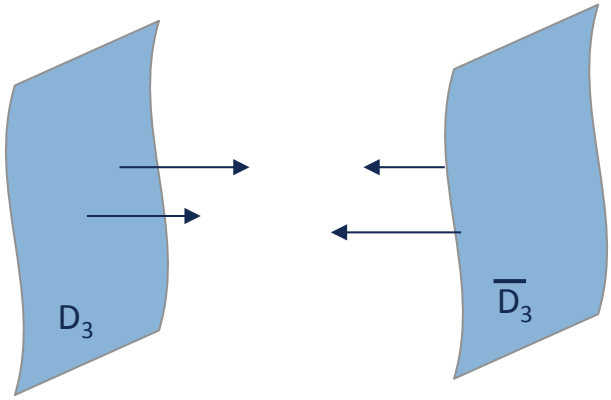

Figure 1.5: *A brane (D_3) and an antibrane ($\overline{D}_3$) tend to collide because the mutual gravitational force they experience in a higher dimensional spacetime is always of attractive type (like the electric force acting between a charged particle and the corresponding antiparticle in the usual three-dimensional space).*

attractive, quite irrespective of their initial configuration.

Because of such attractive force, branes and antibranes, copiously produced during the high-energy pre-big bang phase, tend to collide among themselves (Fig. 1.5): it could be, therefore, that it was the collision of our brane-Universe with an antibrane to simulate the big bang explosion, and trigger the transition from the pre-big bang phase to the phase of standard (post-big bang) evolution. This type of scenario is very similar to the so-called *ekpyrotic* model (first proposed in [31], and later embedded in the context of a more general type of *cyclic* cosmologies, see e.g. [32]), with the only difference that, in the ekpyrotic case, the 3-branes are *domain walls* representing the spacetime boundaries.

1.6 Conclusion

String theory, M-theory, and the related models of brane interactions suggests new and interesting scenarios for the birth of the Universe and its subsequent primordial evolution, not necessarily limited in time by a big bang singularity. They can be tested by present (or near-future) observations concerning the properties – in particular, the "blue" tilt [33] of the spectrum – of the cosmic background of relic gravitational radiation (see, e.g., [34]), the possibility of axion contributions to the CMB anisotropies (see, e.g., [35]), the evolution of the so-called dark energy (or quintessence) field dominating the large-scale dynamics (see, e.g., [36, 37]).

Some of those scenario have been briefly introduced and illustrated in the previous sections. But there are also other, equally interesting scenarios closely related to the previous ones, among which I would like to mention the string-gas [38] and brane-gas [39] cosmologies, based on the repulsive mechanism of winding modes, as well as more general bouncing cosmology models (see, e.g., [40, 41, 42]). Also, models of brane anti-brane inflation [43, 44], where the (time-varying) distance between the two branes plays the role of the inflation field.

All these models have many (and interesting) phenomenological implications, but – as usual in a cosmological context – many studies and many observational data are required before being able of selecting the model most appropriate to our Universe. Thus, we can easily predict that we still have in front of us many years of work and – maybe – of surprising findings.

Bibliography

[1] S. Weinberg: *Gravitation and Cosmology* (Wiley, New York 1972)

[2] S. Dodelson: *Modern Cosmology* (Academic Press, San Diego, CA 2003)

[3] S. Weinberg: *Cosmology* (Oxford University Press, Oxford 2008)

[4] A. Guth: *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*, Phys. Rev. D **23**, 347 (1981)

[5] A. D. Linde: *A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems*, Phys. Lett. B **108**, 389 (1982)

[6] A. Albrecht, P. J. Steinhardt: *Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking*, Phys. Rev. Lett. **48**, 1220 (1982)

[7] E. W. Kolb, M. S. Turner: *The Early Universe* (Addison Wesley, Redwod City, CA 1990)

[8] A.R. Liddle, D.H. Lyth: *Cosmological Inflation and Large-Scale Structure* (Cambridge University Press, Cambridge 2000)

[9] V.F. Mukhanov: *Physical Foundation of Cosmology* (Cambridge University Press, Cambridge 2005)

[10] R. Durrer: *The Cosmic Microwave Background* (Cambridge University Press, Cambridge 2008)

[11] A. Borde, A. Vilenkin: *Eternal inflation and the initial singularity*, Phys. Rev. Lett. **72**, 3305 (1994)

[12] A. Borde, A. Guth, A. Vilenkin: *Inflationary space-times are incomplete in past directions* Phys. Rev. Lett. **90**, 151301 (2003)

[13] A. Mithani, A. Vilenkin: *Did the Universe have a beginning?* (arXiv:1204.4658)

[14] M.B. Green, J. Schwartz, E. Witten: *Superstring Theory* (Cambridge University Press, Cambridge 1987)

[15] J. Polchinski: *String Theory* (Cambridge University Press, Cambridge 1998)

[16] B. Zwiebach: *A First Course in String Theory* (Cambridge University Press, Cambridge 2009)

[17] M. Gasperini: *The Universe before the Big Bang: cosmology and string theory* (Springer-Verlag, Berlin, Heidelberg 2008)

[18] R. Brustein, M. Gasperini, G. Veneziano: *Duality in cosmological perturbation theory*, Phys. Lett. B **431**, 277 (1998)

[19] M. Gasperini, M. Giovannini, G. Veneziano: *Cosmological perturbations across a curvature bounce*, Nucl. Phys. B **694**, 206 (2004)

[20] M. Gasperini: *Elements of String Cosmology* (Cambridge University Press, Cambridge 2007)

[21] M. Gasperini, G. Veneziano: *The Pre-big bang scenario in string cosmology*, Phys. Rep. **373**, 1 (2003)

[22] M. Gasperini, G. Veneziano: *Pre-big bang in string cosmology*, Astropart. Phys. **1**, 317 (1993)

[23] M. Gasperini, M. Giovannini: *Quantum squeezing and cosmological entropy production*, Class. Quant. Grav. **10**, L133 (1993)

[24] E. Witten: *String theory dynamics in various dimensions*, Nucl. Phys. **B443**, 85 (1995)

[25] K. Becker, M. Becker, J. E. Schwarz: *String theory and M-theory: a modern introduction* (Cambridge University Press, Cambridge 2006)

[26] T. Kaluza: *On the Problem of Unity in Physics*, Sitzungsber. Preuss. Akad. Wiss. Berlin, **1921**, 966 (1921)

[27] O. Klein: *Quantum Theory and Five-Dimensional Theory of Relativity*, Z. Phys. **37**, 895 (1926)

[28] D. Langlois: *Brane cosmology: An Introduction*, Prog. Theor. Phys. Suppl. **148**, 181 (2003)

[29] V. A. Rubakov, M. Shaposhnikov: *Do We Live Inside a Domain Wall?*, Phys. Lett. B **125**, 136 (1983)

[30] L. Randall, R. Sundrum: *An Alternative to compactification*, Phys. Rev. Lett. **83**, 4960 (1999)

[31] J. Khoury, B. A. Ovrut, P. J. Steinhard, N. Turok: *The Ekpyrotic universe: Colliding branes and the origin of the hot big bang*, Phys. Rev. D **64**, 123522 (2001)

[32] P. J. Steinhardt, N. Turok: *Cosmic evolution in a cyclic universe*, Phys. Rev. D **65**, 126003 (2002)

[33] R. Brustein, M. Gasperini, M. Giovannini, G. Veneziano: *Relic gravitational waves from string cosmology*, Phys. Lett. B **361**, 45 (1995)

[34] M. Gasperini: *Elementary introduction to pre-big bang cosmology and to the relic graviton background*, in Gravitational Waves, edited by I. Ciufolini, V. Gorini, U. Mischella, P. Frè (IOP Publishing, Bristol 2001), p. 280-337, hep-th/9907067.

[35] R. Durrer, M. Gasperini, M. Sakellariadou, G. Veneziano: *Massless (pseudo)scalar seeds of CMB anisotropy*, Phys. Lett. B **436**, 66 (1998)

[36] M. Gasperini: *Dilatonic interpretation of the quintessence?*, Phys. Rev. D **64**, 043510 (2001)

[37] L. Amendola, M. Gasperini, F. Piazza: *Fitting type Ia supernovae with coupled dark energy*, JCAP **0409**, 014 (2004)

[38] R. Brandenberger, C. Vafa: *Superstrings in the Early Universe*, Nucl. Phys. B **316**, 391 (1989)

[39] S. Alexander, R. Brandenberger, D. Easson: *Brane gases in the early universe*, Phys. Rev. D **62**, 103509 (2000)

[40] T. Biswas, A. Mazumdar, W. Siegel: *Bouncing universes in string-inspired gravity*, JCAP **0603**, 009 (2006)

[41] T. Biswas, T. Koivisto, A. Mazumdar: *Towards a resolution of the cosmological singularity in non-local higher derivative theories of gravity*, JCAP **1011**, 008 (2010)

[42] Y.-F. Cai, D. A. Easson, R. Brandenberger: *Towards a Nonsingular Bouncing Cosmology*, JCAP **1208**, 020 (2012)

[43] C. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh, R. J. Zhang: *The Inflationary brane anti-brane universe*, JHEP **0107**, 047 (2001)

[44] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAlliste, S. P. Trivedi: *Towards inflation in string theory*, JCAP **0310**, 013 (2003)

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ELEMENTARY INTRODUCTION TO PRE-BIG BANG COSMOLOGY AND TO THE RELIC GRAVITON BACKGROUND

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Abstract

This is a contracted version of a series of lectures for graduate and undergraduate students given at the *"VI Seminario Nazionale di Fisica Teorica"* (Parma, September 1997), at the Second Int. Conference *"Around VIRGO"* (Pisa, September 1998), and at the Second *SIGRAV* School on *"Gravitational Waves in Astrophysics, Cosmology and String Theory"* (Center "A. Volta", Como, April 1999). The aim is to provide an elementary, self-contained introduction to string cosmology and, in particular, to the background of relic cosmic gravitons predicted in the context of the so-called "pre-big bang" scenario. No special preparation is required besides a basic knowledge of general relativity and of standard (inflationary) cosmology. All the essential computations are reported in full details either in the main text or in the Appendices. For a deeper and more complete approach to the pre-big bang scenario the interested reader is referred to the updated collection of papers available at `http://www.to.infn.it/~gasperin/`.

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Elementary introduction to pre-big bang cosmology and to the relic graviton background

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1. Introduction

The purpose of these lectures is to provide an introduction to the background of relic gravitational waves expected in a string cosmology context, and to discuss its main properties. To this purpose, it seems to be appropriate to include a short presentation of string cosmology, in order to explain the basic ideas underlying the so-called pre-big bang scenario, which is one of the most promising scenarios for the production of a detectable graviton background of cosmological origin.

After a short, qualitative presentation of the pre-big bang models, I will concentrate on the details of the cosmic graviton spectrum: I will discuss the theoretical predictions for different models, and I will compare the predictions with existing phenomenological constraints, and with the expected sensitivities of the present gravity wave detectors. A consistent part of these lectures will thus be devoted to introduce the basic notions of cosmological perturbation theory, which are required to compute the graviton spectrum and to understand why the amplification of tensor metric perturbations, at high frequency, is more efficient in string cosmology than in the standard inflationary context.

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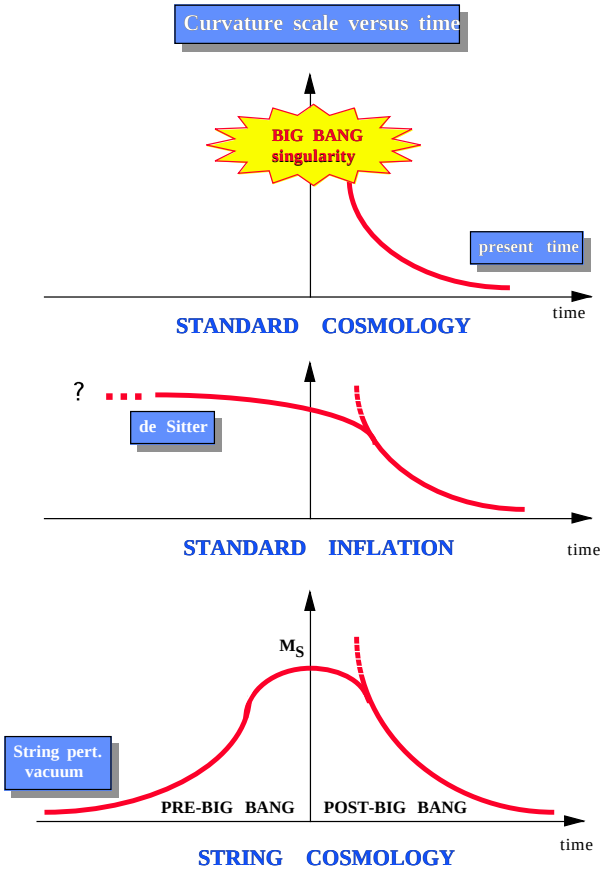

Figure 1. *Time evolution of the curvature scale in the standard cosmological scenario, in the conventional inflationary scenario, and in the string cosmology scenario.*

Let me start by noting that a qualitative, but effective representation of the main difference between string cosmology and standard, inflationary cosmology can be obtained by plotting the curvature scale of the Universe versus time, as illustrated in Fig. 1.

According to the cosmological solutions of the so-called "standard" scenario [1], the spacetime curvature decreases in time. As we go back intime the curvature grows monotonically, and blows up at the initial "big bang" singularity, as illustrated in the top part of Fig. 1 (a similar plot, in the standard scenario, also describes the behaviour of the temperature and of the energy density of the gravitational sources).

According to the standard inflationary scenario [2], on the contrary, the Universe in the past is expected to enter a de Sitter, or "almost" de Sitter phase, during which the curvature tends to stay frozen at a nearly constant value. From a classical point of view, however, this scenario has a problem, since a phase of expansion at constant curvature cannot be extended back in time for ever [3], for reasons of geodesic completeness. This point was clearly stressed also in Alan Guth's recent survey of inflationary cosmology [4]:

"... Nevertheless, since inflation appears to be eternal only into the future, but

not to the past, an important question remains open. How did all start? Although eternal inflation pushes this question far into the past, and well beyond the range of observational tests, the question does not disappear."

A possible anwer to this question, in a quantum cosmology context, is that the Universe emerges in a de Sitter state "from nothing" [5] (or from some unspecified "vacuum"), though a process of quantum tunnelling. I will not discuss the quantum approach in these lectures, but let me note that the computation of the transition probability requires an appropriate choice of the boundary conditions [6], which in the context of standard inflation are imposed *"ad hoc"* when the Universe is in an unknown state, deeply inside the non-perturbative, quantum gravity regime. In a string cosmology context, on the contrary, the initial conditions are referrred asymptotically to a low-energy, classical state which is known, and well controlled by the low-energy string effective action [7].

From a classical point of view, however, the answer to the above question – what happens to the Universe before the phase of constant curvature, which cannot last for ever – is very simple, as we are left only with two possibilities. Either the curvature starts growing again, at some point in the past (but in this case the singularity problems remains, it is simply shifted back in time), or the curvature starts decreasing.

In this second case we are just led to the string cosmology scenario, illustrated in the bottom part of Fig. 1. String theory suggests indeed for the curvature a specular behaviour (or better a "dual" behaviour, as we shall see in a moment) around the time axis. As we go back in time the curvature grows, reaches a maximum controlled by the string scale, and then decreases towards a state which is asymptotically flat and with negligible interactions (vanishing coupling constants), the so-called "string perturbative vacuum". In this scenario the phase of high, but finite (nearly Planckian) curvature is what replaces the big bang singularity of the standard scenario. It comes thus natural, in a string cosmology context, to call "pre-big bang" [8] the initial phase with growing curvature, in contrast to the subsequent, standard, "post-big bang " phase with decreasing curvature.

At this point, a number of questions may arise naturally. In particular:

- **Motivations**: why such a cosmological scenario, characterized by a "bell-like" shape of the curvature, seems to emerge in a string cosmology context and not, for istance, in the context of standard cosmology based on the Einstein equations?
- **Kinematics**: in spite of the differences, is the kinematic of the pre-big bang phase still appropriate to solve the well known problems (horizon, flatness ...) of the standard scenario? After all, we do not want to loose the main achievements of the conventional inflationary models.
- **Phenomenology**: are there phenomenological consequence that can discriminate between string cosmology models and other inflationary models? and are such effects observable, at least in principle?

In the following sections I will present a quick discussion of the three points listed above.

2. Motivations: duality symmetry

There are various motivations, in the context of string theory, suggesting a cosmological scenario like that illustrated in Fig. 1. All the motivations are however related, more or less directly, to an important property of string theory, the duality symmetry of the effective action.

To illustrate this point, let me start by recalling that in general relativity the solutions of the standard Einstein action,

$$S = -\frac{1}{2\lambda_p^{d-1}} \int d^{d+1}x \sqrt{|g|}\ R \tag{2.1}$$

(d is the number of spatial dimensions, and $\lambda_p = M_p^{-1}$ is the Planck length scale), are invariant under "time-reversal" transformations. Consider, for instance, a homogeneous and isotropic solution of the cosmological equations, represented by a scale factor $a(t)$:

$$ds^2 = dt^2 - a^2(t) dx_i^2. \tag{2.2}$$

If $a(t)$ is a solution, then also $a(-t)$ is a solution. On the other hand, when t goes into $-t$, the Hubble parameter $H = \dot{a}/a$ changes sign,

$$a(t) \to a(-t), \qquad\qquad H = \dot{a}/a \to -H. \tag{2.3}$$

To any standard cosmological solution $H(t)$, describing decelerated expansion and decreasing curvature ($H > 0$, $\dot{H} < 0$), is thus associated a "reflected" solution, $H(-t)$, describing a contracting Universe because H is negative.

This is the situation in general relativity. In string theory the action, in addition to the metric, contains at least another fundamental field, the scalar dilaton ϕ. At the tree-level, namely to lowest order in the string coupling and in the higher-derivatives (α') string corrections, the effective action which guarantees the absence of conformal anomalies for the motion of strings in curved backgrounds (see the Apppendix A) can be written as:

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|}\ e^{-\phi} \left[R + (\partial_\mu \phi)^2\right] \tag{2.4}$$

($\lambda_s = M_s^{-1}$ is the fundamental string length scale; see the Appendix B for notations and sign conventions). In addition to the invariance under time-reversal, the above action is also invariant under the "dual" inversion of the scale factor, accompanied by an appropriate transformation of the dilaton (see [9] and the first paper of Ref. [8]). More precisely, if $a(t)$ is a solution for the cosmological background (2.2) , then $a^{-1}(t)$ is also a solution, provided the dilaton transforms as:

$$a \to \tilde{a} = a^{-1}, \qquad\qquad \phi \to \tilde{\phi} = \phi - 2d \ln a \tag{2.5}$$

(this transformation implements a particular case of T-duality symmetry, usually called "scale factor duality", see the Appendix B).

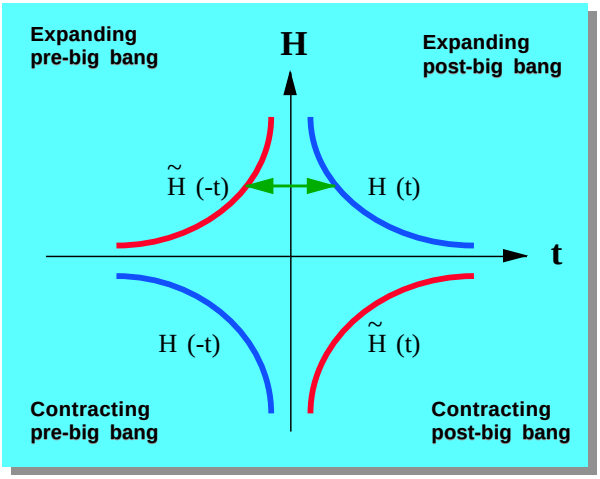

Figure 2. *The four branches of a low-energy string cosmology background.*

When a goes into a^{-1}, the Hubble parameter H again goes into $-H$ so that, to each one of the two solutions related by time reversal, $H(t)$ and $H(-t)$, is associated a dual solution, $\tilde{H}(t)$ and $\tilde{H}(-t)$, respectively (see Fig. 2). The space of solutions is thus reacher in a string cosmology context. Indeed, because of the combined invariance under the transformations (2.3) and (2.5), a cosmological solution has in general four branches: two branches describe expansion (positive H), two branches describe contraction (negative H). Also, as illustrated in Fig. 2, for two branches the curvature scale ($\sim H^2$) grows in time, with a typical "pre-big bang" behaviour, while for the other two branches the curvature scale decreases, with a typical "post-big bang" behaviour.

It follows, in this context, that to any given decelerated expanding solution, $H(t) > 0$, with decreasing curvature, $\dot{H}(t) < 0$ (typical of the standard cosmological scenario), is always associated a "dual partner" describing accelerated expansion, $\tilde{H}(-t) > 0$, and growing curvature, $\dot{\tilde{H}}(-t) > 0$. This doubling of solutions has no analogue in the context of the Einstein cosmology, where there is no dilaton, and the duality symmetry cannot be implemented.

It should be stressed, before proceeding further, that the duality symmetry is not restricted to the case of homogeneous and isotropic backgrounds like (2.2), but is expected to be a general property of the solutions of the string effective action (possibly valid at all orders [10], with the appropriate generalizations). The inversion of the scale factor, in particular, and the associated transformation (2.5), is only a special case of a more general, $O(d,d)$ symmetry of the string effective action, which is manifest already at the lowest order. In fact, the tree-level action in general contains, besides the metric and the dilaton, also a second rank antisymmetric tensor $B_{\mu\nu}$, the so-called Kalb-Ramond "universal" axion:

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|}\; e^{-\phi} \left[R + (\partial_\mu \phi)^2 - \frac{1}{12} \left(\partial_{[\mu} B_{\nu\alpha]} \right)^2 \right]. \tag{2.6}$$

Given a background, even anisotropic, but with d abelian isometries, this action is invariant under a global, pseudo-orthogonal group of $O(d,d)$ transformations which mix

in a non-trivial way the components of the metric and of the antisymmetric tensor, leaving invariant the so-called "shifted" dilaton $\overline{\phi}$:

$$\overline{\phi} = \phi - \ln \sqrt{|\det g_{ij}|}. \tag{2.7}$$

In the particular, "cosmological" case in which we are interested in, the d isometries correspond to spatial traslations (namely, we are in the case of a homogeneous, Bianchi I type metric background). For this background, the action (2.6) can be rewritten in terms of the $2d \times 2d$ symmetric matrix M, defined by the spatial components of the metric, g_{ij}, and of the axion, B_{ij}, as:

$$B \equiv B_{ij}, \qquad G \equiv g_{ij}, \qquad M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}. \tag{2.8}$$

In the cosmic time gauge, the action takes the form (see the Appendix B)

$$S = -\frac{\lambda_s}{2} \int dt\ e^{-\overline{\phi}} \left[(\dot{\overline{\phi}})^2 + \frac{1}{8} \mathrm{Tr}\ \dot{M} (M^{-1})\dot{} \right], \tag{2.9}$$

and is manifestly invariant under the set of global transformations [11]:

$$\overline{\phi} \to \overline{\phi}, \qquad M \to \Lambda^T M \Lambda, \qquad \Lambda^T \eta \Lambda = \eta, \qquad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{2.10}$$

where I is the d-dimensional unit matrix, and η is the $O(d,d)$ metric in off-diagonal form. The transformation (2.5), representing scale factor duality, is now reproduced as a particular case of (2.10) with the trivial $O(d,d)$ matrix $\Lambda = \eta$, and for an isotropic background with $B_{\mu\nu} = 0$.

This $O(d,d)$ symmetry holds even in the presence of matter sources, provided they transform according to the string equations of motion in the given background [12]. In the perfect fluid approximation, for instance, the inversion of the scale factor corresponds to a reflection of the equation of state, which preserves however the "shifted" energy $\overline{\rho} = \rho |\det g_{ij}|^{1/2}$:

$$a \to \tilde{a} = a^{-1}, \qquad \overline{\phi} \to \overline{\phi}, \qquad p/\rho \to -p/\rho, \qquad \overline{\rho} \to \overline{\rho} \tag{2.11}$$

A detailed discussion of the duality symmetry is outside the purpose of these lectures. What is important, in our context, is the simultaneous presence of duality and time-reversal symmetry: by combining these two symmetries, in fact, it is possible in principle to obtain cosmological solutions of the "self-dual" type, characterized by the conditions

$$a(t) = a^{-1}(-t), \qquad \overline{\phi}(t) = \overline{\phi}(-t). \tag{2.12}$$

They are important, as they connect in a smooth way the phase of growing and decreasing curvature, and also describe a smooth evolution from the string perturbative vacuum (i.e. the asymptotic no-interaction state in which $\phi \to -\infty$ and the string coupling is vanishing, $g_s = \exp(\phi/2) \to 0$), to the present cosmological phase in which the dilaton is frozen, with an expectation value [13] $\langle g_s \rangle = M_s/M_p \sim 0.3$ – 0.03 (see Fig. 3).

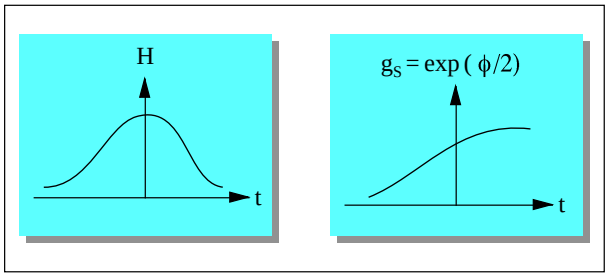

Figure 3. *Time evolution of the curvature scale H and of the string coupling $g_s = \exp(\phi/2) \simeq M_s/M_p$, for a typical self-dual solution of the string cosmology equations.*

The explicit occurrence of self-dual solutions and, more generally, of solutions describing a complete and smooth transition between the phase of pre- and post-big bang evolution, seems to require in general the presence of higher order (higher loop and/or higher derivative) corrections to the string effective action [14] (see however [15, 16]). So, in order to give only a simple example of combined {duality $\oplus$ time-reversal} transformation, let me consider here the low-energy, asymptotic regimes, which are well described by the lowest order effective action.

By adding matter sources, in the perfect fluid form, to the action (2.4), the string cosmology equations for a $d = 3$, homogeneous, isotropic and conformally flat background can be written as (see Appendix C, eqs. (C.16), (C.18), (C.15), respectively):

$$\begin{aligned} &\dot\phi^2 - 6H\dot\phi + 6H^2 = e^\phi \rho, \\ &\dot H - H\dot\phi + 3H^2 = \frac{1}{2} e^\phi p, \\ &2\ddot\phi + 6H\dot\phi - \dot\phi^2 - 6\dot H - 12H^2 = 0. \end{aligned} \tag{2.13}$$

For $p = \rho/3$, in particular, they are exactly solved by the standard solution with constant dilaton (see eqs. (C.32), (C.33)),

$$a \sim t^{1/2}, \qquad \rho = 3p \sim a^{-4}, \qquad \phi = \text{const}, \qquad t \to +\infty, \tag{2.14}$$

describing decelerated expansion and decreasing curvature scale:

$$\dot a > 0, \qquad \ddot a < 0, \qquad \dot H < 0. \tag{2.15}$$

This is exactly the radiation-dominated solution of the standard cosmological scenario, based on the Einstein equations. In string cosmology, however, to this solution is associated a "dual complement", i.e. an additional solution which can be obtained by applying on the background (2.14) a time-reversal transformation $t \to -t$, and the duality transformation (2.11):

$$a \sim (-t)^{-1/2}, \qquad \phi \sim -3\ln(-t), \qquad \rho = -3p \sim a^{-2}, \qquad t \to -\infty, \tag{2.16}$$

This is still an exact solution of the equations (2.13) (see the Appendix C), describing however accelerated (i.e. inflationary) expansion, with growing dilaton and growing

curvature scale:

$$\dot{a} > 0, \qquad \ddot{a} > 0, \qquad \dot{H} > 0. \tag{2.17}$$

We note, for future reference, that accelerated expansion with growing curvature is usually called "superinflation" [17], or "pole-inflation", to distinguish it from the more conventional power-inflation, with decreasing curvature.

The two solutions (2.14) and (2.16) provide a particular, explicit representation of the scenario illustrated in Fig. 3, in the two asympotic regimes of t large and positive, and t large and negative, respectively. The duality symmetry seems thus to provide an important motivation for the pre-big bang scenario, as it leads naturally to introduce a phase of growing curvature, and is a crucial ingredient for the "bell-like" scenario of Fig. 3.

It should be noted that pure scale factor duality, by itself, is not enough to convert a phase of decreasing into growing curvature (see for instance Fig. 2, where it is clearly shown that H and $\tilde{H}$, in the same temporal range, lead to the same evolution of the curvature scale, $H^2 \sim \tilde{H}^2$). Time reflection is thus necessarily required, if we want to invert the curvature behaviour. From this point of view, time-reversal symmetry is more important than duality.

In a thermodynamic context, however, duality by itself is able to suggests the existence of a primordial cosmological phase with "specular" properties with respect to the present, standard cosmological phase [18]. It must be stressed, in addition, that it is typically in the cosmology of extended objects that the phase of growing curvature may describe accelerated expansion instead of contraction, and that the growth of the curvature may be regularized, instead of blowing up to a singularity. For instance, it is with the string dilaton [8], or with a network of strings self-consistently coupled to the background [19], that we are naturally lead to superinflation. Also, in quantum theories of extended objects, it is the minimal, fundamental lenght scale of the theory that is expected to bound the curvature, and to drive superinflation to a phase of constant, limiting curvature [20] asymptotically approaching de Sitter, as explicitly checked in a string theory context [21]. Duality symmetries, on the other hand, are typical of extended objects (and of strings, in particular), so that it is certainly justified to think of duality as of a fundamental motivation and ingredient of the pre-big bang scenario.

Duality is an important symmetry of modern theoretical physics, and to conclude this section I would like to present an analogy with another very important symmetry, namely supersymmetry (see Table I).

According to supersymmetry, to any *bosonic state* is associated a *fermionic partner*, and viceversa. From the existence of bosons that we know to be present in Nature, if we believe in supersymmetry, we can predict the existence of fermions not yet observed, like the photino, the gravitino, and so on.

In the same way, according to duality and time-reversal, to any *geometrical state* with decreasing curvature is associated a *dual partner* with growing curvature. On the other hand, our Universe, at present, is in the standard post-big bang phase,

	SUPERSYMMETRY	**DUALITY + TIME-REVERSAL**
Pair of partners	{bosons, fermions}	{growing curvature, decreasing curvature}
Known states	photons, gravitons, ...	decelerated, standard post-big bang
Predicted	photinos, gravitinos, ...	accelerated, inflationary pre-big bang

Table 1. Analogy between supersymmetry and duality.

with decreasing curvature. If we believe that duality has to be implemented, even approximately, in the course of the cosmological evolution, we can then predict the existence of a phase, in the past, characterized by growing curvature and by a typical pre-big bang evolution.

3. Kinematics: shrinking horizons

If we accept, at least as a working hypothesis, the possibility that our Universe had in the past a "dual" complement, with growing curvature, we are lead to the second of the three questions listed in Section 1: is the kinematics of the pre-big bang phase still appropriate to solve the problems of the standard inflationary scenario? The anwer is positive, but in a non-trival way.

Consider, for instance, the present cosmological phase. Today the dilaton is expected to be constant, and the Universe should be appropriately described by the Einstein equations. The gravitational part of such equations contains two types of terms: terms controlling the geometric curvature of a space-like section, evolving in time like a^{-2}, and terms controlling the gravitational kinetic energy, i.e. the spacetime curvature scale, evolving like H^2. According to present observations the spatial curvature term is non-dominant, i.e.

$$r = \frac{a^{-2}}{H^2} \sim \frac{\text{spatial curvature}}{\text{spacetime curvature}} \lesssim 1. \tag{3.1}$$

According to the standard cosmological solutions, on the other hand, the above ratio must grow in time. In fact, by putting $a \sim t^\beta$,

$$r \sim \dot{a}^{-2} \sim t^{2(1-\beta)}, \tag{3.2}$$

so that r keeps growing both in the matter-dominated ($\beta = 2/3$) and in the radiation-dominated ($\beta = 1/2$) era. Thus, as we go back in time, r becomes smaller and smaller, and when we set initial conditions (for instance, at the Planck scale) we have to impose an enormous fine tuning of the spatial curvature term, with respect to the other terms of the cosmological equations. This is the so-called flatness problem.

The problem can be solved if we introduce in the past a phase (usually called inflation), during which the value of r was decreasing, for a time long enough to compensate the subsequent growth during the phase of standard evolution. It is important to stress that this requirement, in general, can be implemented by two physically different classes of backgrounds.

Consider for simplicity a power-law evolution of the scale factor in cosmic time, with a power β, so that the time-dependence of r is the one given in eq. (3.2). The two possible classes of backgrounds corresponding to a decreasing r are then the following:

- Class I: $a \sim t^\beta, \beta > 1, t \to +\infty$. This class of backgrounds corresponds to what is conventionally called "power inflation", describing a phase of accelerated expansion and decreasing curvature scale, $\dot{a} > 0, \ddot{a} > 0, \dot{H} < 0$. This class contains, as a limiting case, the standard de Sitter inflation, $\beta \to \infty$, $a \sim e^{Ht}$, $\dot{H} = 0$, i.e. accelerated exponential expansion at constant curvature.
- Class II: $a \sim (-t)^\beta, \beta < 1, t \to 0_-$. This is the class of backgrounds corresponding to the string cosmology scenario. There are two possible subclasess:

  IIa : $\beta < 0$, describing superinflation, i.e. accelerated expansion with growing curvature scale, $\dot{a} > 0, \ddot{a} > 0, \dot{H} > 0$;

  IIb : $0 < \beta < 1$, describing accelerated contraction and growing curvature scale, $\dot{a} < 0, \ddot{a} < 0, \dot{H} < 0$.

A phase of growing curvature, if accelerated like in the pre-big bang scenario, can thus provide an unconventional, but acceptable, inflationary solution of the flatness problem (the same is true for the other standard kinematical problems, see [8]). It is important to stress, in particular, that the two subclasses *IIa*, *IIb*, do not correspond to different models, as they are simply different kinematical representation of the *same* scenario in two *different frames*, the string frame (S-frame), in which the effective action takes the form (2.4),

$$S(g,\phi) = -\int d^{d+1}x\sqrt{|g|}\; e^{-\phi}\left[R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \tag{3.3}$$

and the Einstein frame (E-frame), in which the dilaton is minimally coupled to the metric, and has a canonical kinetic term:

$$S(\tilde{g},\tilde{\phi}) = -\int d^{d+1}x\sqrt{|\tilde{g}|}\;\left[\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}\right]. \tag{3.4}$$

In order to illustrate this point, we shall proceed in two steps. First we will show that, through a field redefinition $g = g(\tilde{g},\tilde{\phi})$, $\phi = \phi(\tilde{g},\tilde{\phi})$,, it is always possible to move from the S-frame to the E-frame; second, we will show that, by applying such a redefinition, a superinflationary solution obtained in the S-frame becomes an accelerated contraction in the E-frame, and viceversa.

We shall consider, for simplicity, an isotropic, spatially flat background with d spatial dimensions, and we set:

$$g_{\mu\nu} = \mathrm{diag}\left(N^2, -a^2\delta_{ij}\right), \qquad \phi = \phi(t), \tag{3.5}$$

where $g_{00} = N^2$ is to be fixed by an arbitrary choice of gauge. For this background we get:

$$\Gamma_{0i}{}^{j} = H\delta_i^j, \qquad \Gamma_{ij}{}^{0} = \frac{a\dot{a}}{N^2}\delta_{ij}, \qquad \Gamma^0_{00} = \frac{\dot{N}}{N} = F$$
$$R = \frac{1}{N^2}\left[2dFH - 2d\dot{H} - d(d+1)H^2\right], \tag{3.6}$$

and the S-frame action (3.3) becomes

$$S(g,\phi) = -\int d^{d+1}x \frac{a^d e^{-\phi}}{N}\left[2dFH - 2d\dot{H} - d(d+1)H^2 + \dot{\phi}^2\right] \tag{3.7}$$

Modulo a total derivative, we can eliminate the first two terms, and the action takes the quadratic form

$$S(g,\phi) = -\int d^{d+1}x \frac{a^d e^{-\phi}}{N}\left[\dot{\phi}^2 - 2dH\dot{\phi} + d(d-1)H^2\right] \tag{3.8}$$

where, as expected, N plays the role of a Lagrange multiplier (no kinetic term in the action).

In the E-frame the variables are $\tilde{N}, \tilde{a}, \tilde{\phi}$, and the action(3.4), after integration by part, takes the canonical form

$$S(\tilde{g},\tilde{\phi}) = -\int d^{d+1}x \frac{\tilde{a}^d}{\tilde{N}}\left[-\frac{1}{2}\dot{\tilde{\phi}}^2 + d(d-1)H^2\right] \tag{3.9}$$

A quich comparison with eq. (3.8) leads finally to the field redefinition (no coordinate transformation!) connecting the Einstein and String frame:

$$\tilde{a} = ae^{-\phi/(d-1)}, \qquad \tilde{N} = Ne^{-\phi/(d-1)}, \qquad \tilde{\phi} = \phi\sqrt{\frac{2}{d-1}}. \tag{3.10}$$

In fact, the above transformation gives

$$\tilde{H} = H - \frac{\dot{\phi}}{d-1} \tag{3.11}$$

and, when inserted into eq. (3.9), exactly reproduces the S-frame action (3.8).

Consider now a superinflationary, pre-big bang solution obtained in the S-frame, for instance the isotropic, d-dimensional vacuum solution

$$a = (-t)^{-1/\sqrt{d}}, \qquad e^{\phi} = (-t)^{-(\sqrt{d}+1)}, \qquad t < 0, \qquad t \to 0_- \tag{3.12}$$

(aee Appendix B, eqs. (B.21), (B.22)), and look for the corresponding E-frame solution. The above solution is valid in the syncronous gauge, $N = 1$, and if we choose, for instance, the syncronous gauge also in the E-frame, we can fix $\tilde{N}$ by the condition:

$$\tilde{N}dt \equiv Ne^{-\phi/(d-1)}dt = d\tilde{t}, \tag{3.13}$$

which defines the E-frame cosmic time, $\tilde{t}$, as:

$$d\tilde{t} = e^{-\phi/(d-1)}dt. \tag{3.14}$$

After integration

$$t \sim \tilde{t}^{\frac{d-1}{d+\sqrt{d}}}, \tag{3.15}$$

and the transformed solution takes the form:

$$\tilde{a} = (-\tilde{t})^{1/d}, \qquad e^{\tilde{\phi}} = (-\tilde{t})^{-\sqrt{\frac{2(d-1)}{d}}}, \qquad \tilde{t} < 0, \qquad \tilde{t} \to 0_-. \tag{3.16}$$

One can easily check that this solution describes accelerated contraction with growing dilaton and growing curvature scale:

$$\frac{d\tilde{a}}{d\tilde{t}} < 0, \qquad \frac{d^2\tilde{a}}{d\tilde{t}^2} < 0, \qquad \frac{d\tilde{H}}{d\tilde{t}} < 0, \qquad \frac{d\tilde{\phi}}{d\tilde{t}} > 0. \tag{3.17}$$

The same result applies if we transform other isotropic solutions from the String to the Einstein frame, for instance the perfect fluid solution of Appendix C, eq. (C.32). We leave this simple exercise to the interested reader.

Having discussed the "dynamical" equivalence (in spite of the kinematical differences) of the two classes of string cosmology metrics, *IIa* and *IIb*, it seems appropriate at this point to stress the main dynamical difference between standard inflation, *Class I* metrics, and pre-big bang inflation, *Class II* metrics. Such a difference can be conveniently illustrated in terms of the proper size of the event horizon, relative to a given comoving observer.

Consider in fact the the proper distance, $d_e(t)$, between the surface of the event horizon and a comoving observer, at rest at the origin of an isotropic, conformally flat background [22]:

$$d_e(t) = a(t) \int_t^{t_M} dt' a^{-1}(t'), \tag{3.18}$$

Here t_M is the maximal allowed extension, towards the future, of the cosmic time coordinate for the given background manifold. The above integral coverges for all the above classes of accelerated (expanding or contracting) scale factors. In the case of *Class I* metrics we have, in particular,

$$d_e(t) = t^\beta \int_t^\infty dt' t'^{-\beta} = \frac{t}{\beta - 1} \sim H^{-1}(t) \tag{3.19}$$

for power-law inflation ($\beta > 1, t > 0$), and

$$d_e(t) = e^{Ht} \int_t^\infty dt' e^{-Ht'} = H^{-1} \tag{3.20}$$

for de Sitter inflation. For *Class II* metrics ($\beta < 1, t < 0$) we have instead

$$d_e(t) = (-t)^\beta \int_t^0 dt' (-t')^{-\beta} = \frac{(-t)}{1 - \beta} \sim H^{-1}(t). \tag{3.21}$$

In all cases the proper size $d_e(t)$ evolves in time like the so-called Hubble horizon (i.e. the inverse of the Hubble parameter), and then like the inverse of the curvature scale. The size of the horizon is thus constant or growing in standard inflation (*Class I*), decreasing in pre-big bang inflation (*Class II*), both in the S-frame and in the E-frame.

Such an important difference is clearly illustrated in Fig. 4 and Fig. 5, where the dashed lines represent the evolution of the horizon, the full lines the evolution of the scale factor. The shaded area at time t_0 represents the portion of Universe inside our present Hubble radius. As we go back in time, according to the standard scenario, the

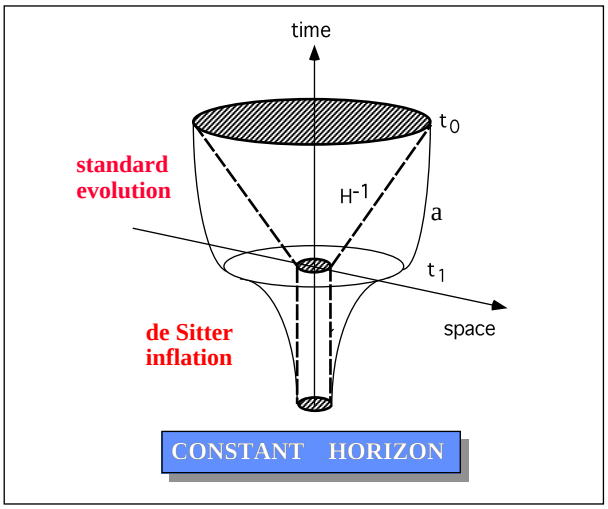

Figure 4. *Qualitative evolution of the Hubble horizon (dashed curve) and of the scale factor (full curve) in the standard inflationary scenario.*

horizon shrinks linearly, ($H^{-1} \sim t$), but the decrease of the scale factor is slower so that, at the beginning of the phase of standard evolution ($t = t_1$), we end up with a causal horizon much smaller than the portion of Universe that we presently observe. This is the well known "horizon problem" of the standard scenario.

In Fig. 4 the phase of standard evolution is preceeded in time by a phase of standard de Sitter inflation. Going back in time, for $t < t_1$, the scale factor keeps shrinking, and our portion of Universe "re-enters" inside the Hubble radius during a phase of constant (or slightly growing in time) horizon.

In Fig. 5 the standard evolution is preceeded in time by a phase of pre-big bang inflation, with growing curvature . The Universe "re-enters" the Hubble radius during a phase of shrinking horizon. To emphasize the difference, I have plotted the evolution of the scale factor both in the expanding S-frame, $a(t)$, and in the contracting E-frame, $\tilde{a}(t)$. Unlike in standard inflation, the proper size of the initial portion of the Universe may be very large in strings (or Planck) units, *but not larger than the initial horizon* itself [23], as emphasized in the picture. The initial horizon is large because the initial curvature scale is small, in string units, $H_i \ll 1/\lambda_s$.

This is a basic consequence of the choice of the initial state which, in the pre-big bang scenario, approaches the flat, cold and empty string perturbative vacuum [8], and which is to be contrasted to the extremely curved, hot and dense initial state of the standard scenario, characterizing a Universe which starts inflating at the Planck scale, $H_i \sim 1/\lambda_p$.

4. Open problems and phenomenological consequences

In order to give a honest presentation of the pre-big bang scenario, it is fair to say that the string cosmology models are not free from various (more or less important)

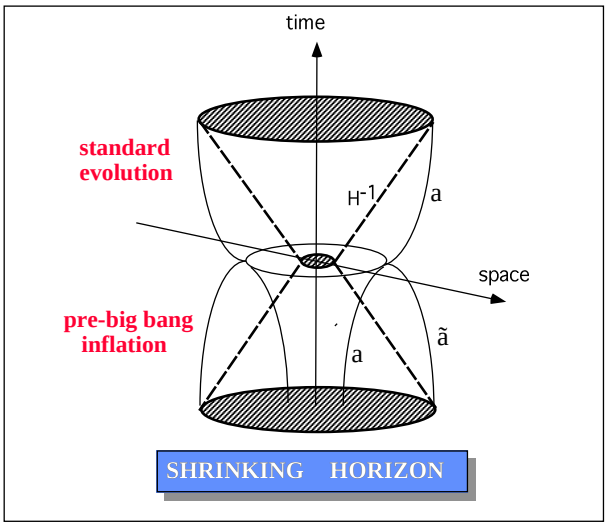

Figure 5. *Qualitative evolution of the Hubble horizon (dashed curve) and of the scale factor (full curve) in the pre-big bang inflationary scenario, in the S-frame, $a(t)$, and in the E-frame, $\tilde{a}(t)$.*

difficulties, and that many aspects of the scenario are still unclear. A detailed discussion of such aspects is outside the purpose of this paper, but I would like to mention here at least three important open problems. Presented in "time-ordered" form (from the beginning to the end of inflation) they are the following.

- The first concerns the initial conditions, and in particular the decay of the string perturbative vacuum. The question is whether or not the "switching on" of a long inflationary phase requires fine-tuning. Originally raised in [24], this problem was recently re-proposed as a fundamental difficulty of the pre-big bang scenario [25] (see however [23, 26, 27, 28]).
- The second concerns the transition from the pre- to the post-big bang phase, which is expected to occur in the high curvature and strong coupling regime. There is a quantum cosmology approach, based on the scattering of the Wheeler-De Witt wave function in minisuperspace [7], but the problem seems to require, in general, the introduction of higher derivative (α') and quantum loop corrections [21, 29] in the string effective action (see however [15, 16]).
- The third problem concerns the final matching to the standard Friedman-Robertson-Walker phase, with a transition from the dilaton-dominated to the radiation-dominated regime, and all the associated problems of dilaton oscillations, re-heating, pre-heating, particle production, entropy production [30], and so on.

All these problems are under active investigation, and further work is certainly needed for a final answer. However, even assuming that all the problems will be solved in a satisfactory way, we are left eventually with a further question, the third one listed in the Introduction, which is the basic question (in my opinion). Are there phenomenological consequences that can discriminate string cosmology from the other

inflationary scenarios? and, in particular, are such consequences observable (at least in principle) ?

The answer is positive. There are many phenomenological differences, even if all the differences seem to have the same "common denominator", i.e. the fact that the quantum fluctuations of the background fields are amplified in different models with different spectra. The spectrum, in particular, tends to follow the behaviour of the curvature scale during the phase of inflation. In the standard scenario the curvature is constant or decreasing, so that the spectrum tends to be flat, or decreasing with frequency. In string cosmology the curvature is growing, and the spectrum tends to grow with frequency.

In the following sections I will discuss in detail this effect for the case of tensor metric perturbations. Here I would like to note that the phenomenological consequences of the pre-big bang scenario can be classified into three different types, depending on the possibility of their observation: *Type I* effects, referring to observations to be performed in a not so far future (20-30 years?); *Type II* effects, referring to observations to be performed in a near future (a few years); *Type III* effects, referring to observations already (in part) performed. To conclude this very quick presentation of the pre-big bang scenario, let me give one example for each type of phenomenological effect.

- *Type I*: the production of a relic graviton background that, in the frequency range of conventional detectors ($\sim 10^2 - 10^3$ Hz), is much higher (by 8 – 9 orders of magnitude) than the background expected in conventional inflation [8, 31, 32, 33]. The sensitivity of the presently operating gravitational antennas is not enough to detect it, however, and we have to wait for the advanced, second generation interferometric detectors (LIGO [34], VIRGO [35]), or for interferometers in space (LISA [36]).
- *Type II*: the large scale CMB anisotropy "seeded" by the inhomogeneous fluctuations of a massless [37] or massive [38] axion background. Metric fluctuations are indeed too small, on the horizon scale, to be responsible for the temperature anisotropies detected by COBE [39]; the axion spectrum, on the contrary, can be sufficiently flat [40] for that purpose. Such a different origin of the anisotropy may lead to non-gaussianity, or to differences (with respect to the standard inflationary scenario) in the height and position of the first Doppler peak of the spectrum [41]. Such differencs could be soon confirmed, or disproved, by the planned satellite observations (MAP [42], PLANCK [43], ...).
- *Type III*: the production of primordial magnetic fields strong enough to "seed" the galactic dynamo, and to explain the origin of the cosmic magnetic fields observed on a large (galactic, intergalactic) scale [44]. In the standard inflationary scenario, in fact, the amplification of the vacuum fluctuations of the electromagnetic field is not efficient enough [45], because of the conformal invariance of the Maxwell equations. In string cosmology, on the contrary, the electromagnetic field is also coupled to the dilaton, and the fluctuations are amplified by the accelerated growth

of the dilaton during the phase of pre-big bang evolution.

Finally, I wish to mention a further important phenomenogical effect, typical of string cosmology (and that I do not know how to classify within the tree types defined above, however): dilaton production, i.e. the amplification of the dilatonic fluctuations of the vacuum, and the formation of a cosmic background of relic dilatons [46].

The possibility of detecting such a background is strongly dependent on the value of the dilaton mass, that we do not known, at present. If dilatons are massless [47], then the amplitude and the spectrum of the relic background should be very similar to those of the graviton background, and the relic dilatons could be possibly detected, in the future, by gravitational antennas able to respond to scalar modes, unless their coupling to bulk matter is too small [47], of course.

If dilatons are massive, the mass has to be large enough to be compatible with existing tests of the equivalence principle and of macroscopic gravitational forces. In addition, there is a rich phenomenology of cosmological bounds, which leaves open only two possible mass windows [46]. Interestingly enough, however, in the allowed light mass sector the dilaton lifetime is longer than the present age of the universe, and the dilaton fraction of critical energy density ranges from 0.01 to 1: in this context, the dilaton becomes a new, interesting dark matter candidate (see [48] for a detailed discussion of the allowed mass windows, and of the possibility that light but non-relativistic dilatons could represent today a significant fraction of dark matter on a cosmological scale). I have no idea, however, of how to detect directly such a massive dilaton background, because the mass is light, but it is heavy enough ($\gtrsim 10^{-4}$eV) to be far outside the sensitivity range of resonant gravitational detectors.

The rest of this lecture will be devoted to discuss various theoretical and phenomenological aspects of graviton production, in a general cosmological context and, in particular, in the context of the pre-big bang scenario. Let me start by recalling some basic notions of cosmological perturbation theory, which are required for the computation of the graviton spectrum.

5. Cosmological perturbation theory

The standard approach to cosmological perturbation theory is to start with a set of non-perturbed equations, for instance the Einstein or the string cosmology equations,

$$G_{\mu\nu} = T_{\mu\nu}, \tag{5.1}$$

to expand the meric and the matter fields around a given background solution,

$$g_{\mu\nu} \to g^{(0)}_{\mu\nu} + \delta^{(1)} g_{\mu\nu}, \qquad T_{\mu\nu} \to T^{(0)}_{\mu\nu} + \delta^{(1)} T_{\mu\nu}, \qquad G^{(0)}_{\mu\nu} = T^{(0)}_{\mu\nu}, \tag{5.2}$$

and to obtain, to first order, a linearized set of equations describing the classical evolution of perturbations,

$$\delta^{(1)} G_{\mu\nu} = \delta^{(1)} T_{\mu\nu}. \tag{5.3}$$

In principle, the procedure is simple and straightforward. In practice, however, we have to go through a series of formal steps, that I list here in "chronological" order:

- choice of the "frame";
- choice of the "gauge";
- normalization of the amplitude;
- computation of the spectrum.

5.1. Choice of the frame

The choiche of the frame is the choice of the basic set of fields (metric included) used to parametrize the action. The action, in general, can be expressed in terms of different fields. In string cosmology, for instance, there is a preferred frame, the S-frame, in which the lowest order gravi-dilaton action takes the form (3.3). It is preferred because the metric appearing in the action is the same as the sigma-model metric to which test strings are minimally coupled (see Appendix A): with respect to this metric, the motion of free strings is then geodesics. It is always possible, however, through the field redefinition

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} e^{-2\phi/(d-1)}, \qquad \tilde{\phi} = \sqrt{\frac{2}{d-1}}\phi, \tag{5.4}$$

to introduce the more conventional E-frame (3.4) in which the dilaton is minimally coupled to the metric, with a canonical kinetic term.

In the two frames the field equations are different, and the perturbation equations are also different. This seems to rise a potential problem: which frame is to be used to evaluate the physical effects of the cosmological perturbations?

The problem is only apparent, however, because physical observables (like the perturbation spectrum) are the same in both frames. The reason is that there is a compensation between the different perturbation equations and the different background solution around which we expand. A general proof of this result can be given by using the notion of canonical variable (see the Subsection 5.3). Here I will give only an explicit example for tensor perturbations in a $d = 3$, isotropic and spatially flat background.

Let me start in the E-frame, with the background equations:

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi, \tag{5.5}$$

referring to the "tilded" variables (5.4) (I will omit the "tilde", for simplicity, and I will explicitly reinsert it at the end of the computation). Consider the transverse, traceless part of metric perturbations:

$$\delta^{(1)}\phi = 0, \quad \delta^{(1)}g_{\mu\nu} = h_{\mu\nu}, \quad \delta^{(1)}g^{\mu\nu} = -h^{\mu\nu}, \quad \nabla_\nu h_\mu{}^\nu = 0 = h_\mu{}^\nu \tag{5.6}$$

(∇_μ denotes covariant differentiation with respect to the unperturbed metric g, and the indices of h are also raised and lowered with g). The perturbation of the background equations gives:

$$\delta^{(1)}R_\mu{}^\nu = 0. \tag{5.7}$$

We can work in the syncronous gauge, where

$$g_{00} = 1, \qquad g_{0i} = 0, \qquad g_{ij} = -a^2\delta_{ij},$$
$$h_{00} = 0, \qquad h_{0i} = 0, \qquad g^{ij}h_{ij} = 0, \qquad \partial_j h_i{}^j = 0. \tag{5.8}$$

To first order in h we get

$$\delta^{(1)}\Gamma_{0i}{}^j = \frac{1}{2}\dot{h}_i{}^j, \qquad \delta^{(1)}\Gamma_{ij}{}^0 = -\frac{1}{2}\dot{h}_{ij},$$
$$\delta^{(1)}\Gamma_{ij}{}^k = \frac{1}{2}\left(\partial_i h_j{}^k + \partial_j h_i{}^k - \partial^k h_{ij}\right). \tag{5.9}$$

The $(0,0)$ component of eq. (5.7) is trivially satisfied (as well as the perturbation of the scalar field equation); the (i,j) components, by using the identities (see for instance [54])

$$g^{jk}\dot{h}_{ik} = \dot{h}_i{}^j + 2Hh_i{}^j,$$
$$g^{jk}\ddot{h}_{ik} = \ddot{h}_i{}^j + 2\dot{H}h_i{}^j + 4H\dot{h}_i{}^j + 4H^2h_i{}^j, \tag{5.10}$$

give

$$\delta^{(1)}R_i{}^j = -\frac{1}{2}\left(\ddot{h}_i{}^j + 3H\dot{h}_i{}^j - \frac{\nabla^2}{a^2}h_i{}^j\right) \equiv -\frac{1}{2}\Box h_i{}^j = 0. \tag{5.11}$$

In terms of the conformal time coordinate, $d\eta = dt/a$, this wave equation can be finally rewritten, for each polarization mode, as

$$\tilde{h}'' + 2\frac{\tilde{a}'}{\tilde{a}}\tilde{h}' - \nabla^2\tilde{h} = 0, \tag{5.12}$$

(where I have explicitly re-inserted the tilde, and where a prime denotes differentiation with respect to the conformal time, which is the same in the Einstein and in the String frame, according to eqs. (3.10), (3.14)).

Let us now repeat the computation in the S-frame, where the background equations for the metric (eq. (C.12) with no contribution from $H_{\mu\nu\alpha}, T_{\mu\nu}, V$ and σ) can be written explicitly as

$$R_\mu{}^\nu + g^{\nu\alpha}\left(\partial_\mu\partial_\alpha\phi - \Gamma_{\mu\alpha}{}^\rho\partial_\rho\phi\right) = 0. \tag{5.13}$$

Perturbing to first order,

$$\delta^{(1)}R_\mu{}^\nu - \left(\delta^{(1)}g^{\nu\alpha}\Gamma_{\mu\alpha}{}^0 + g^{\nu\alpha}\delta^{(1)}\Gamma_{\mu\alpha}{}^0\right)\dot{\phi} = 0. \tag{5.14}$$

The $(0,0)$ component, as well as the perturbation of the dilaton equation, are trivially satisfied. The (i,j) components, using again the identities (5.10), lead to [31]:

$$\Box h_i{}^j - \dot{\phi}\dot{h}_i{}^j = 0. \tag{5.15}$$

In conformal time, and for each polarization component,

$$h'' + \left(2\frac{a'}{a} - \phi'\right)h' - \nabla^2 h = 0 \tag{5.16}$$

This last equation seems to be different frm the E-frame equation (5.12). Recalling, however, the relation (3.10) between a and $\tilde{a}$, we have

$$2\frac{\tilde{a}'}{\tilde{a}} = 2\frac{a'}{a} - \phi', \tag{5.17}$$

so that we have the same equation for h and $\tilde{h}$, the same solution, and the same spectrum when the solution is expanded in Fourier modes. The perturbation analysis is thus *frame-independent*, and we can safely choose the more convenient frame to compute the spectrum.

5.2. Choice of the gauge

The second step is the choice of the gauge, i.e. the choice of the coordinate system within a given frame. The perturbation spectrum is of course gauge-independent, but the the perturbative analysis *is not*, in general. It is possible, in fact, that the validity of the linear approximation is broken in a given gauge, but still valid in a different, more appropriate gauge.

Since this effect is particularly important, let me give, in short, an explicit example for the scalar perturbations of the metric tensor in a $d = 3$, isotropic and conformally flat background, in the E-frame (I will omit the tilde, for simplicity). The perturbed metric, in the so-called longitudinal gauge, depends on the two Bardeen potentials Φ and Ψ as [49]:

$$ds^2 = a^2 \left[(1 + 2\Phi)\, d\eta^2 - (1 - 2\Psi)\, dx_i^2 \right]. \tag{5.18}$$

By perturbing the Einstein equations (5.5), the dilaton equation, and combining the results for the various components, one obtains to first order that $\Phi = \Psi$, and that the metric fluctuations satisfy the equation:

$$\Phi'' + 6\frac{a'}{a}\Phi' - \nabla^2\Phi = 0. \tag{5.19}$$

We now consider the particular, exact solution of the vacuum string cosmology equations in the E-frame,

$$a(\eta) = |\eta|^{1/2}, \qquad \phi(\eta) = -\sqrt{3}\ln|\eta|, \qquad \eta \to 0_-, \tag{5.20}$$

corresponding to a phase of accelerated contraction and growing dilaton (i.e. the pre-big bang solution (3.16), written in conformal time, for $d = 3$). For this background, the perturbation eq. (5.19) becomes a Bessel equation for the Fourier modes Φ_k,

$$\Phi_k'' + \frac{3}{\eta}\Phi_k' + k^2\Phi_k = 0, \qquad \nabla^2\Phi_k = -k^2\Phi_k, \tag{5.21}$$

and the asymptotic solution, for modes well outside the horizon ($|k\eta| \ll 1$),

$$\varphi_k = A_k \ln|k\eta| + B_k|k\eta|^{-2} \tag{5.22}$$

contains a growing part which blows up ($\sim \eta^{-2}$) as the background approaches the high curvature regime ($\eta \to 0_-$). In this limit the linear approximation breaks down, so that the longitudinal gauge is not in general consistent with the perturbative expansion around a homogeneous, inflationary pre-big bang background, as scalar inhomogeneities may become too large.

In the same background (5.20) the problem is absent, however, for tensor perturbations, since their growth outside the horizon is only logarithmic. From eq. (5.12) we have in fact the asymptotic solution

$$h_k = A_k + B_k \ln|k\eta|, \qquad |k\eta| \ll 1. \tag{5.23}$$

This may suggests that the breakdown of the linear approximation, for scalar perturbations, is an artefact of the longitudinal gauge. This is indeed confirmed by the fact that, in a more appropriate off-diagonal (also called "uniform curvature" [50]) gauge,

$$ds^2 = a^2\left[(1+2\varphi)\,d\eta^2 - dx_i^2 - 2\partial_i B dx^i d\eta\right], \tag{5.24}$$

the growing mode is "gauged down", i.e. it is suppressed enough to restore the validity of the linear approximation [51] (the off-diagonal part of the metric fluctuations remains growing, but the growth is suppressed in such a way that the amplitude, normalized to the vacuum fluctuations, keeps smaller than one for all scales k, provided the curvature is smaller than one in string units). This result is also confirmed by a covariant and gauge invariant computation of the spectrum, according to the formalism developed by Bruni and Ellis [52].

It should be stressed, however, that the presence of a growing mode, and the need for choosing an appropriate gauge, is a problem typical of the pre-big bang scenario. In fact, let us come back to tensor perturbations, in the E-frame: for a generic accelerated background the scale factor can be parametrized in conformal time with a power α, as follows:

$$a = (-\eta)^\alpha, \qquad \eta \to 0_-, \tag{5.25}$$

and the perturbation equation (5.12) gives, for each Fourier mode, the Bessel equation

$$h_k'' + \frac{2\alpha}{\eta} h_k' + k^2 h_k = 0, \tag{5.26}$$

with asymptotic solution, outside the horizon ($|k\eta| \ll 1$):

$$h_k = A_k + B_k \int^\eta \frac{d\eta'}{a^2(\eta')} = A_k + B_k\,|\eta|^{1-2\alpha}. \tag{5.27}$$

The solution tends to be constant for $\alpha < 1/2$, while it tends to grow for $\alpha > 1/2$. It is now an easy exercise to re-express the scal factor (5.25) in cosmic time,

$$dt = a d\eta, \qquad a(t) \sim |t|^{\alpha/(1+\alpha)}, \tag{5.28}$$

and to chech that, by varying α, we can parametrize all types of accelerated backgrounds introduced in Section 3: accelerated expansion (with decreasing, constant and growing curvature), and accelerated contraction, with growing curvature (see Table II).

In the standard, inflationary scenario the metric is expanding, $\alpha < 0$, so that the amplitude h_k is frozen outside the horizon. In the pre-big bang scenario, on the contrary, the metric is contracting in the E-frame, so that h_k may grow if the contraction is fast enough, i.e. $\alpha > 1/2$ (in fact, the growing mode problem was first pointed out in the

$\alpha < -1$	power-inflation	$\dot{a} > 0, \ddot{a} > 0, \dot{H} < 0$
$\alpha = -1$	de Sitter	$\dot{a} > 0, \ddot{a} > 0, \dot{H} = 0$
$-1 < \alpha < 0$	super-inflation	$\dot{a} > 0, \ddot{a} > 0, \dot{H} > 0$
$\alpha > 0$	accelerated contraction	$\dot{a} < 0, \ddot{a} < 0, \dot{H} < 0$

Table 2. The four classes of accelerated backgrounds.

context of Kaluza-Klein inflation and dynamical dimensional reduction [53], where the internal dimensions are contracting). For the low-energy string cosmology background (5.20) we have $\alpha = 1/2$, the growth is simply logarithmic (see (5.23)), and the linear approximation can be applied consistently, provided the curvature remains bounded by the string scale [51]. But for $\alpha > 1/2$ the growth of the amplitude may require a different gauge for a consistent linearized description.

5.3. Normalization of the amplitude

The linearized equations describing the classical evolution of perturbations can be obtained in two ways:

- by perturbing directly the background equations of motion;
- by perturbing the metric and the matter fields to first order, by expanding the action up to terms quadratic in the first order fluctuations,

$$g \to g + \delta^{(1)} g, \qquad \delta^{(2)} S \equiv S\left[\left(\delta^{(1)} g\right)^2\right], \tag{5.29}$$

  and then by varying the action with respect to the fluctuations.

The advantage of the second method is to define the so-called "normal modes" for the oscillation of the system {gravity + matter sources}, namely the variables which diagonalize the kinetic terms in the perturbed action, and satisfy canonical commutation relations when the fluctuations are quantized. Such canonical variables are required, in particular, to normalize perturbations to a spectrum of quantum, zero-point fluctuations, and to study their amplification from the vacuum state up to the present state of the Universe.

Let us apply such a procedure to tensor perturbations, in the S-frame, for a $d = 3$ isotropic background. In the syncronous gauge, the transverse, traceless, first order metric perturbations $h_{\mu\nu} = \delta^{(1)} g_{\mu\nu}$ satisfy eq. (5.8). We expand all terms of the low energy gravi-dilaton action (3.3) up to order h^2:

$$\delta^{(1)} g^{\mu\nu} = -h^{\mu\nu}, \qquad \delta^{(2)} g^{\mu\nu} = h^{\mu\alpha} h_\alpha{}^\nu,$$

$$\delta^{(1)}\sqrt{-g}=0, \qquad \delta^{(2)}\sqrt{-g}=-\frac{1}{4}\sqrt{-g}h_{\mu\nu}h^{\mu\nu}, \tag{5.30}$$

and so on for $\delta^{(1)}R_{\mu\nu}$, $\delta^{(2)}R_{\mu\nu}$ (see for instance [54]). By using the background equations, and integrating by part, we finally arrive at the quadratic action

$$\delta^{(2)}S=\frac{1}{4}\int d^4x a^3 e^{-\phi}\left(\dot{h}_i^j\dot{h}_j^i+h_i^j\frac{\nabla}{a^2}h_j^i\right). \tag{5.31}$$

By separating the two physical polarization modes, i.e. the standard "cross" and "plus" gravity wave components,

$$h_i^j h_j^i=2\left(h_+^2+h_\times^2\right), \tag{5.32}$$

we get, for each mode (now generically denoted with h), the effective scalar action

$$\delta^{(2)}S=\frac{1}{2}\int d^4x a^3 e^{-\phi}\left(\dot{h}^2+h\frac{\nabla}{a^2}h\right), \tag{5.33}$$

which can be rewritten, using conformal time, as

$$\delta^{(2)}S=\frac{1}{2}\int d^3x d\eta a^2 e^{-\phi}\left(h'^2+h\nabla h\right). \tag{5.34}$$

The variation with respect to h gives finally eq. (5.16), i.e. the same equation obtained by perturbing directly the background equations in the S-frame.

The above action describes a scalar field h, non-minimally coupled to a time-dependent external field, $a^2e^{-\phi}$ (also called "pump field"). In order to impose the correct quantum normalization to vacuum fluctuations, we introduce now the so-called "canonical variable" ψ, defined in terms of the pump field as

$$\psi=zh, \qquad\qquad z=ae^{-\phi/2}. \tag{5.35}$$

With such a definition the kinetic term for ψ appears in the standard canonical form: for each mode k, in fact, we get the action

$$\delta^{(2)}S_k=\frac{1}{2}\int d\eta\left(\psi_k'^2-k^2\psi_k^2+\frac{z''}{z}\psi_k^2\right), \tag{5.36}$$

and the corresponding canonical evolution equation:

$$\psi_k''+\left[k^2-V(\eta)\right]\psi_k=0, \qquad\qquad V(\eta)=\frac{z''}{z}, \tag{5.37}$$

which has the form of a Schrodinger-like equation, with an effective potential depending on the external pump field. This form of the canonical equation, by the way, is the same for all types of perturbations (with different potentials, of course). What is important, in our context, is that for an accelerated inflationary background $V(z)\to 0$ as $\eta\to-\infty$. This means that, asymptotically, the canonical variable satisfies the free-field oscillating equation

$$\eta\to-\infty, \qquad\qquad \psi_k''+k^2\psi_k=0, \tag{5.38}$$

and can be normalized to an initial vacuum fluctuation spectrum,

$$\eta\to-\infty, \qquad\qquad \psi_k=\frac{1}{\sqrt{2k}}e^{-ik\eta}, \tag{5.39}$$

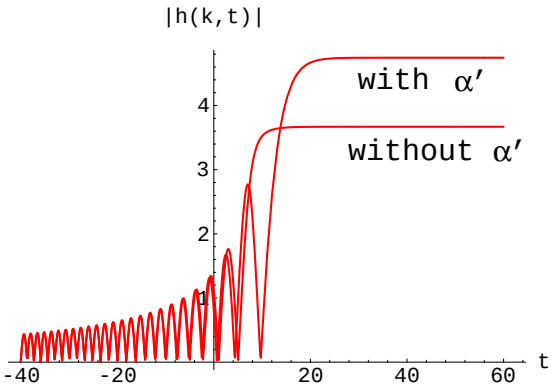

Figure 6. Amplification of tensor fluctuations, with and without the higher-curvature corrections included in the canonical perturbation equation.

in such a way as to satisfy the free field canonical commutation relations, $[\psi_k, \psi_j^{*'}] = i\delta_{kj}$. The normalization of ψ_k then fixes the normalization of the metric variable $h_k = \psi_k/z$.

It is important to stress that there is no need to introduce the canonical variable to study the classical evolution of perturbations, but that such variable is needed for the initial normalization to a vacuum fluctuation spectrum. We can also normalize perturbation in a different way, of course but in that case we are studying the amplification not of the vacuum fluctuations, but of a different spectrum [55].

At this point, two remarks are in order. The first concerns the frame-independence of the spectrum. The above procedure can also be applied in the E-frame, to define a canonical variable $\tilde{\psi}$: one then obatins for $\tilde{\psi}_k$ the canonical equation (5.37), with a pump field that depends only on the metric, $\tilde{z} = \tilde{a}$. However, by using the conformal transformation connecting the two frames, it turns out that the two pump fields are the same, $\tilde{z} = \tilde{a} = ae^{-\phi/2} = z$, so that for ψ and $\tilde{\psi}$ we have the same potential, the same evolution equation, the same solution, and thus the same spectrum.

The second remark is that the canonical procedure can be applied to any action, and in particular to the string effective action including higher curvature corrections of order α', which can be written as [21]:

$$S = \int d^4x\sqrt{-g}e^{-\phi}\left\{-R - \partial_\mu\phi\partial^\mu\phi + \frac{\alpha'}{4}\left[R^2_{GB} - (\partial_\mu\phi\partial^\mu\phi)^2\right]\right\} \tag{5.40}$$

where $R^2_{GB} \equiv R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the Gauss-Bonnet invariant (we have chosen a convenient field redefinition that removes terms with higher-than-second derivatives from the equations of motion, see Appendix A). From the quadratic perturbed action we obtain α' corrections to the pump fields. The canonical equation turns out to be the same as before, but with a k-dependent effective potential [54], and such an equation can be used to estimate the effects of the higher curvature corrections on the amplification of tensor perturbations. A numerical integration [54], in which the metric fluctuations are expanded around the high-curvature background solution of ref. [21], leads in particular to the results illustrated in Fig. 6.

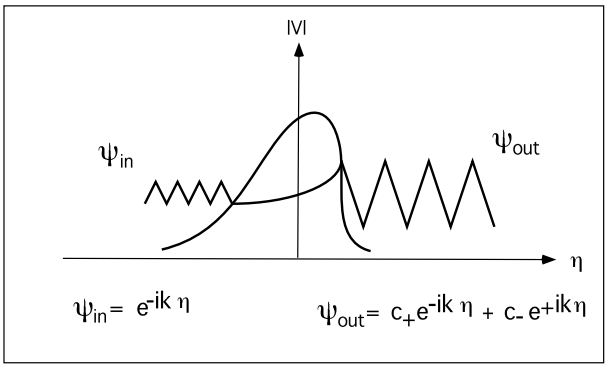

Figure 7. Scattering and amplification of the canonical variable.

The qualitative behaviour is similar, both with and without α' corrections in the perturbed equations: the fluctuations are oscillating inside the horizon and frozen outside the horizon, as usual. However, the final amplitude is enhanced when α' corrections are included, and this suggests that the energy spectrum of the gravitational radiation, computed with the low-energy perturbation equation, may represent a sort of lower bound on the total amount of produced gravitons.

5.4. Computation of the spectrum

The final, amplified perturbation spectrum is to be obtained from the solutions of the canonical equation (5.37). In order to solve such an equation we need explicitly the effective potential $V[z(\eta)]$ which, in general, vanishes asymptotically at large postive and negative values of the conformal time. Consider, for instance, the tensor perturbation equation in the E-frame, so that the pump field is simply the scale factor. The typical cosmological bckground in which we are interested in should describe a transition fom an initial accelerated, inflationary evolution,

$$\eta \to -\infty, \qquad a \sim |\eta|^{\alpha}, \qquad V \sim \eta^{-2}, \tag{5.41}$$

to a final standard, radiation-dominated phase,

$$\eta \to +\infty, \qquad a \sim \eta, \qquad V = 0. \tag{5.42}$$

In this context, the evolution of fluctuations, initially normalized as in eq. (5.39), can be described as a scattering of the canonical variable by an effective potential, according to the Schrodinger-like perturbation equation (5.37) (see Fig. 7).

However, the differential variable in eq. (5.37) is (conformal) time, not space. As a consequence, the eigenfrequencies represent (comoving) energies, not momenta. Thus, even normalizing the initial state to a positive frequency mode, as in eq. (5.39), the final state is in general a mixture of positive and negative frequency modes, i.e. of positive and negative energy states,

$$\eta \to +\infty, \qquad \psi_{\rm out} \sim c_+ e^{-ik\eta} + c_- e^{+ik\eta}. \tag{5.43}$$

In a quantum field theory context, such a mixing represents a process of pair production from the vacuum. The coefficients $c_\pm$ are the so-called Bogoliubov coefficients, parametrizing a unitary transformation between $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ states. In matrix form, they connect the set of $|\mathrm{in}\rangle$ annihilation and creation operators, $\{\psi_{\mathrm{in}}, b_k, b_k^\dagger\}$ to the out ones $\{\psi_{\mathrm{out}}, a_k, a_k^\dagger\}$, as follows:

$$a_k = c_+ b_k + c_-^* b_{-k}^\dagger, \qquad a_{-k}^\dagger = c_- b_k + c_+^* b_{-k}^\dagger. \tag{5.44}$$

Thus, even starting from the vacuum,

$$\overline{n}_{\mathrm{in}} = \langle 0|b^\dagger b|0\rangle = 0, \tag{5.45}$$

we end up with a final number of produced pairs which is nonzero, in general, and is controlled by the Bogoliubov coefficient c_- as

$$\overline{n}_{\mathrm{out}} = \langle 0|a^\dagger a|0\rangle = |c_-|^2 \neq 0. \tag{5.46}$$

In a second quantization approach, the amplification of perturbations can thus be seen as a process of pair production from the vacuum (or from any otherwise specified initial state), under the action of a time-dependent external field (the gravi-dilaton background, in the string cosmology case). Equivalently, the process can be described as a "squeezing" of the initial state [56] (this description is useful to evaluate the associated entropy production [57]), or, in a semiclassical language, as a "parametric amplification" [58] of the wave function ψ_k, which is scattered by an effective potential barrier through an "anti-tunnelling" process [59]. Quite independently of the adopted language, the differential energy density of the produced radiation, for each mode k, depends on the number of produced pairs, and can be written as

$$d\rho_k = 2k\overline{n}_k \frac{d^3k}{(2\pi)^3}, \qquad \overline{n}_k = |c_-(k)|^2. \tag{5.47}$$

The computation of the so-called energy spectrum, defined as the spectral energy density per logarithmic interval of frequency,

$$\frac{d\rho_k}{d\ln k} \equiv k\frac{d\rho_k}{dk} = \frac{k^4}{\pi^2}|c_-(k)|^2, \tag{5.48}$$

thus requires the computation of $c_-(k)$, and then the knowledge of the asymptotic solution of the canonical pertubation equation at large positive times.

To give an explicit example we shall consider here a very simple model consisting of two cosmological phases, an initial accelerated evolution up to the time η_1, and a subsequent radiation-dominated evolution for $\eta > \eta_1$:

$$\begin{aligned} &a \sim (-\eta)^\alpha, && \eta < \eta_1, \\ &a \sim \eta, && \eta > \eta_1. \end{aligned} \tag{5.49}$$

The effective potential for tensor perturbations in the E-frame, $|a''/a|$, starts from zero at $-\infty$, grows like η^{-2}, reaches a maximum $\sim \eta_1^{-2}$, and vanishes in the radiation phase. We must solve the canonical perturbation equation for $\eta < \eta_1$ and $\eta > \eta_1$. In the first phase the equation reduces to a Bessel equation:

$$\psi_k'' + \left[k^2 - \frac{\alpha(\alpha-1)}{\eta^2}\right]\psi_k = 0, \tag{5.50}$$

with general solution [60]

$$\psi_k = |\eta|^{1/2}\left[AH_\nu^{(2)}(|k\eta|) + BH_\nu^{(1)}(|k\eta|)\right], \qquad \nu = |\alpha - 1/2|, \tag{5.51}$$

where $H_\nu^{(1,2)}$ are the first and second kind Hankel functions, of argument $k\eta$ and index $\nu = |\alpha - 1/2|$ determined by the kinematics of the background. By using the large argument limit of the Hankel functions for $\eta \to -\infty$,

$$H_\nu^{(2)}(k\eta) \sim \frac{1}{\sqrt{k\eta}} e^{-ik\eta}, \qquad H_\nu^{(1)}(k\eta) \sim \frac{1}{\sqrt{k\eta}} e^{+ik\eta}, \tag{5.52}$$

we choose initially a positive frequency mode, normalizing the solution to a vacuum fluctuation spectrum,

$$A = 1/2, \qquad B = 0. \tag{5.53}$$

In the second phase $V = 0$, and we have the free oscillating solution :

$$\psi_k = \frac{1}{\sqrt{k}}\left(c_+ e^{-ik\eta} + c_- e^{+ik\eta}\right). \tag{5.54}$$

The matching of ψ and ψ' at $\eta = \eta_1$ gives now the coefficients $c_\pm$ (more precisely, the matching would require the continuity of the perturbed metric projected on a spacelike hypersurface containing η_1, and the continuity of the extrinsic curvature of that hypersurface [61]; but in many cases these conditions are equivalent to the continuity of the canonical variable ψ, and of its first time derivative).

For an approximate determination of the spectrum, which is often sufficient for pratical purposes, it is convenient to distinguish two regimes, in which the comoving frequency k is much higher or much lower than the frequency associated to the top of the effective potential barrier, $|V(\eta_1)|^{1/2} \simeq \eta_1^{-1}$. In the first case, $k \gg 1/|\eta_1| \equiv k_1$, we can approximate the Hankel functions with their large argument limit, and we find that there is no particle production,

$$|c_+| \simeq 1, \qquad |c_-| \simeq 0. \tag{5.55}$$

In practice, c_- is not exactly zero, but is exponentially suppressed as a function of the frequency, just like the quantum reflection probability for a wave with a frequency well above the top of a potential step. We will neglect such an effect here, as we are mainly interested in a qualitative estimate of the perturbation spectrum.

In the second case, $k \ll 1/|\eta_1| \equiv k_1$, we can use the small argument limit of the Hankel functions,

$$H_\nu^{(2)} \sim a(k\eta_1)^\nu - ib(k\eta_1)^{-\nu}, \qquad H_\nu^{(1)} \sim a(k\eta_1)^\nu + ib(k\eta_1)^{-\nu}, \tag{5.56}$$

and we find

$$|c_+| \simeq |c_-| \simeq |k\eta_1|^{-\nu-1/2}, \tag{5.57}$$

corresponding to a power-law spectrum:

$$\frac{d\rho_k}{d\ln k} = \frac{k^4}{\pi^2}|c_-(k)|^2 \simeq \frac{k_1^4}{\pi^2}\left(\frac{k}{k_1}\right)^{3-2\nu}, \qquad k < k_1, \tag{5.58}$$

	scale factor	Bessel index	spectrum
de Sitter, constant curvature	$\alpha = -1$	$3 - 2\nu = 0$	flat
power-inflation, decreasing curvature	$\alpha < -1$	$3 - 2\nu < 0$	decreasing
pre-big bang inflation, growing curvature	$\alpha > -1$	$3 - 2\nu > 0$	increasing

Table 3. Slope of the graviton spectrum.

with a cut-off frequency $k_1 = \eta_1^{-1}$ controlled by the height of the effective potential.

For a comparison with present observations, it is finally convenient to express the spectrum in terms of the proper frequency, $\omega(t) = k/a(t)$, and in units of critical energy density, $\rho_c(t) = 3M_p^2 H^2(t)/8\pi$. We then obtain the dimensionless spectral distribution,

$$\Omega(\omega, t) = \frac{\omega}{\rho_c(t)} \frac{d\rho(\omega)}{d\omega} \simeq \frac{8}{3\pi} \frac{\omega_1^4}{M_p^2 H^2} \left(\frac{\omega}{\omega_1}\right)^{3-2\nu}$$
$$\simeq g_1^2 \Omega_\gamma(t) \left(\frac{\omega}{\omega_1}\right)^{3-2\nu}, \qquad \omega < \omega_1, \tag{5.59}$$

where

$$\omega_1 = \frac{k_1}{a} = \frac{1}{a\eta_1} \simeq \frac{H_1 a_1}{a} \tag{5.60}$$

is the maximal amplified proper frequency, $g_1 = H_1/M_p$, and

$$\Omega_\gamma(t) = \frac{\rho_\gamma}{\rho_c} = \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \tag{5.61}$$

is the energy density (in critical units) of the radiation that becomes dominant at $t = t_1$, rescaled down at a generic time t (today, $\Omega_\gamma(t_0) \sim 10^{-4}$).

It is important to stress that the amplitude of the spectrum is controlled by $g_1 = H_1/M_p$, i.e. by the curvature scale in Planck units at the time of the transition t_1 (a fundamental parameter of the given inflationary model). The slope of the spectrum, $3 - 2\nu$, is instead controlled by the kinematics of the background. In fact, it depends on the Bessel index ν which, in its turn, depends on α, the power of the scale factor (eq. (5.51). The behaviour in frequency of the graviton spectrum, in particular, tends to follow the behaviour of the curvature scale during the epoch of accelerated evolution, see Table III.

The standard inflationary scenario is thus characterized by a flat or decreasing graviton spectrum; in string cosmology, instead, we must expect a growing spectrum. This has important phenomenological implications, that will be discussed in the following section.

6. The relic graviton background

As discussed in the previous Section, one of the most firm predictions of all inflationary models is the amplification of the traceless, transverse part of the quantum fluctuations of the metric tensor, and the formation of a primordial, stochastic background of relic gravitational waves, distributed over a quite large range of frequencies (see [62] for a discussion of the stochastic properties of such a background, and [63] for a possible detection of the associated "squeezing" [64]).

In a string cosmology context, the expected graviton background has been already discussed in a number of detailed review papers [59, 65, 66]. Here I will summarize the main properties of the background predicted in the context of the pre-big bang scenario.

For a phenomelogical discussion of the spetrum, it is convenient to consider the plane $\{\Omega_G, \omega\}$. In this plane there are three main phenomenological constraints:

- A first constraint comes from the large scale isotropy of the CMB radiation. The degree of anisotropy measured by COBE imposes a bound on the energy density of the graviton background at the scale of the present Hubble radius [67],
$$\Omega_G(\omega_0) \lesssim 10^{-10}, \qquad \omega_0 \sim 10^{-18} \text{ Hz}. \tag{6.1}$$
- A second constraint comes from the absence of distortion of the pulsar timing-data [68], and gives the bound
$$\Omega_G(\omega_p) \lesssim 10^{-8}, \qquad \omega_p \sim 10^{-8} \text{ Hz}. \tag{6.2}$$
- A third constraint comes from nucleosynthesys [69], which implies that the total graviton energy density, integrated over all modes, cannot exceed the energy density of one massless degree of freedom in thermal equilibrium, evaluated at the nucleosynthesis epoch. This gives a bound for the peak value of the spectrum [33],
$$h_{100} \int d \ln \omega \Omega_G(\omega, t_0) \lesssim 0.5 \times 10^{-5},$$
$$h_{100} = H_0/(100 \text{ km sec}^{-1}\text{Mpc}^{-1}), \tag{6.3}$$
which applies to all scales.

A further bound can be obtained by considering the production of primordial black holes [70]. The production of gravitons, in fact, could be associated to the formation of black holes, whose possible evaporation, at the present epoch, is constrained by a number of astrophysical observations. The absence of evaporation imposes an indirect upper limit on the graviton background. In a string cosmology context, however, and in the frequency range of interest for observations, this upper limit is roughly of the same order as the nucleosynthesis bound [70].

For flat or decreasing spectra it is now evident that the more constraining bound is the low-frequency one, obtained from the COBE data. In the standard inflationary scenario, characterized by flat or decreasing spectra (see Table III), the maximal allowed graviton background can thus be plotted as in Fig. 8. The flat spectrum corresponds

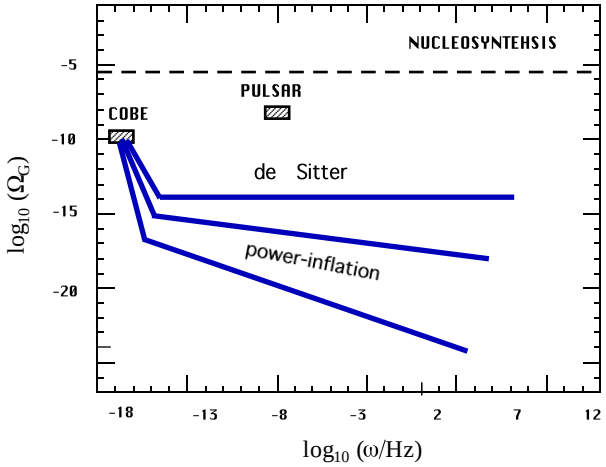

Figure 8. Graviton spectra in the standard inflationary scenario.

to de Sitter inflation, the decreasing spectra to power inflation. The breakdown in the spectrum, around $\omega_e q \sim 10^{-16}$ Hz, is due to the transition from the radiation-dominated to the matter-dominated phase, which only affects the low-frequency part of the spectrum, namely those modes re-entering the horizon in the matter-dominated era. For such modes there is an additional potential barrier in the canonical perturbation equation, which induces an additional amplification $\sim (\omega_e q/\omega)^2 > 1$, with respect to the flat de Sitter spectrum.

However, this break of the spectrum is not important for our purposes. What is important is the fact that the observed anisotropy constrains the maximal amplitude of the spectrum. But the amplitude depends on the inflation scale, as stressed in the previous Section. From the COBE bound (6.1), imposed on the modified de Sitter spectrum at the Hubble scale,

$$\Omega_G(\omega_0, t_0) = g_1^2 \Omega_\gamma(t)(\omega_e q/\omega)^2 \lesssim 10^{-10}, \tag{6.4}$$

we thus obtain a direct constraint on the inflation scale:

$$H_1/M_p \lesssim 10^{-5} \tag{6.5}$$

(for power-inflation the bound is even stronger [31]).

This bound applies to all models characterized by a flat or decreasing spectrum. The bound can be evaded, however, if the spectrum is growing, like in the string cosmology context. To illustrate this point, let me consider the simplest class of the so-called "minimal" pre-big bang models, characterized by three main kinematic phases [32, 44]: an initial low-energy, dilaton-driven phase, an intermediate high-energy "string" phase, in which α' and loop corrections become important, and a final standard, radiation-dominated phase (see Fig. 9). The time scale η_s marks the transition to the high curvature phase, and the time scale η_1, characterized by a final curvature of order one in string units, marks the transition to the radiation-dominated cosmology.

By computing graviton production in this background [32] we find that the spectrum is characterized by two branches: a high frequency branch, for modes crossing

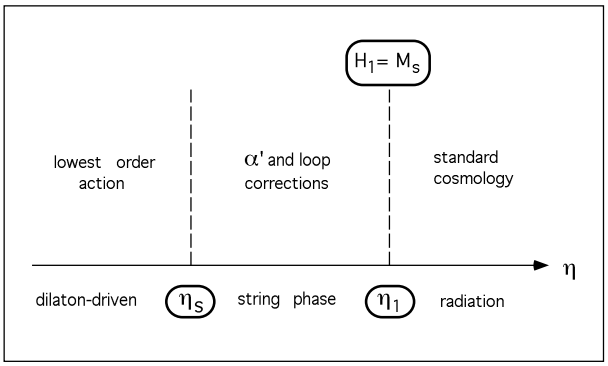

Figure 9. A minimal model of pre-big bang background.

the horizon (or "hitting" the barrier) in the string phase, $\omega > \omega_s = (a\eta_s)^{-1}$; and a low frequency branch, for modes crossing the horizon in the initial dilaton phase, $\omega < \omega_s$. The slope is cubic at low frequency, and flatter at high frequency, and the spectrum can be parametrized as follows:

$$\begin{aligned} \Omega_G(\omega, t_o) &\simeq g_1^2 \Omega_\gamma(t_0) \left(\frac{\omega}{\omega_1}\right)^{3-2\nu}, && \omega_s < \omega < \omega_1, \\ &\simeq g_1^2 \Omega_\gamma(t_0) \left(\frac{\omega_1}{\omega_s}\right)^{2\nu} \left(\frac{\omega}{\omega_1}\right)^3, && \omega < \omega_s \end{aligned} \tag{6.6}$$

(modulo logarithmic corrections). There are two main parameters: the transition frequency ω_s, and the Bessel index ν, for the high frequency part of the spectrum. These parameters represent our present ignorance about the duration and the kinematic details of the high curvature phase.

In spite of this uncertainty, however, there is a rather precise prediction for the height and the position of the peak of the spectrum, which turns out to be fixed in terms of the fundamental ratio M_s/M_p as:

$$\Omega_G(\omega_1) \sim 10^{-4}(M_s/M_p)^2, \qquad \omega_1 \sim 10^{11}(M_s/M_p)^{1/2} \text{ Hz}. \tag{6.7}$$

The behaviour of the spectrum, in this class of models, is illustrated in Fig. 10. A precise computation [33] shows that, given the maximal expected value of the string scale [13] ($M_s/M_p \simeq 0.1$), the peak value is automatically compatible with the nucleosynthesis bound, as well as with bounds from the production of primordial black holes.

In the minimal models the position of the peak is fixed. Actually, what is really fixed, in string cosmology, is the maximal height of the peak, but not necessarily the position in frequency, and it is not impossible, in more complicated non-minimal models, to shift the peak at lower frequencies.

In minimal models, in fact, the beginning of the radiation phase coincides with the end of the string phase. We may also consider models, however, in which the dilaton coupling $g_s^2 = \exp(\phi)$ is still small at the end of the high curvature phase, and the radiation era begins much later, when $g_s^2 \sim 1$, after an intermediate dilaton-dominated regime. The main difference between the two cases is that in the second case the effective potential which amplifies tensor perturbations is non-monotonic [59], so that there are

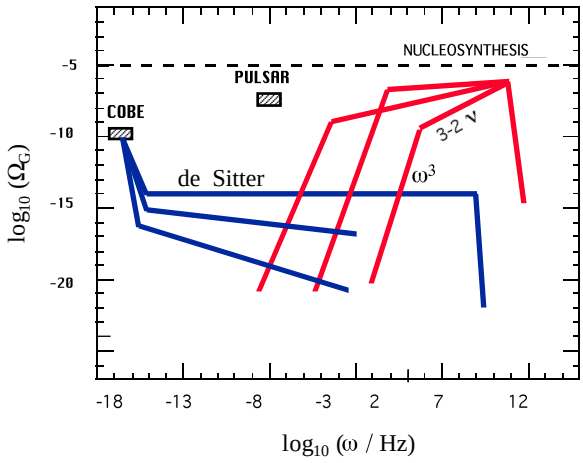

Figure 10. Graviton spectra in minimal pre-big bang models.

high frequency modes re-entering the horizon before the radiation era. As a consequence, the perturbation spectrum is also non-monotonic, and the peak does not coincide any longer, in general, with the end point of the spectrum (see also [71]), as illustrated in Fig. 11.

For minimal models the peak is around 100 GHz, for non minimal models it could be at lower frequencies. These are good news from an experimental point of view, of course, but non-minimal models seem to be less natural, at least from a theoretical point of view. The boxes around the peak, appearing in Fig. 11, represent the uncertainty in the position of the peak due to our present ignorance about the precise value of the ratio M_s/M_p (for the illustrative purpose of Fig. 11, the ratio is assumed to vary in the range $0.1 - 0.01$). The wavy line, in the high frequency branch of the spectrum, represents the fact that the spectrum associated to the string phase could be monotonic on the average, but locally oscillating [72]. Finally, the lower strip, labelled by $\delta s = 99\%$, represents the fact that even the height of the peak could be lower than expected, if the produced gravitons have been diluted by some additional reheating phase, occurring well below the string scale, during the standard evolution.

This last effect can be parameterized in terms of δs, which is the fraction of present entropy density in radiation, due to such additional, low-scale reheating. The position of the peak then depends on δs as [33]:

$$\omega_1(t_0) \simeq T_0 \left(\frac{M_s}{M_P}\right)^{1/2} (1-\delta s)^{1/3},$$
$$\Omega_G(\omega_1, t_0) \simeq 7 \times 10^{-5} h_{100}^{-2} \left(\frac{M_s}{M_P}\right)^{2} (1-\delta s)^{4/3}, \tag{6.8}$$

where $T_0 = 2.7^0 K \simeq 3.6 \times 10^{11}$Hz. Such a dependence is not dramatic, however, because even for $\delta s = 99\%$ the peak keeps well above the standard inflationary prediction, represented by the line labelled "de Sitter" in Fig. 11.

Given the various theoretical uncertainties, the best we can do, at present, is to define the maximal allowed region for the expected graviton background, i.e. the region

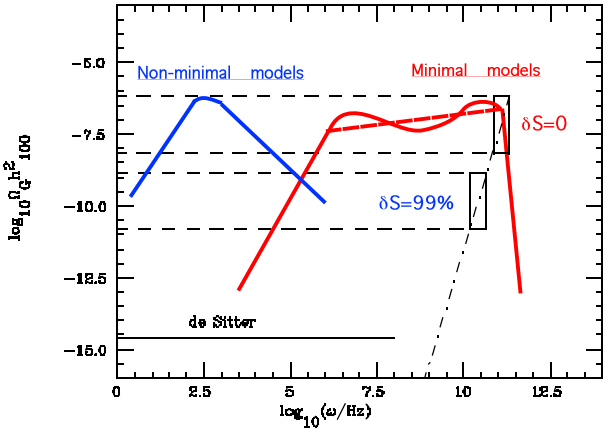

Figure 11. Peak and end point of the spectrum in minimal and non-minimal models.

spanned by the spectrum when all its parameters are varied. Such a region is illustrated in Fig. 12, for the phenomenologically interesting high frequency range. The figure emphasizes the possible, large enhancement (of about eight orders of magnitude) of the intensity of the background in string cosmology, with respect to the standard inflationary scenario.

It may be useful to stress again the reason of such enhancement. In the standard inflationary scenario the graviton spectrum is decreasing, the normalization is imposed at low frequency, and the peak value is contolled by the anisotropy of the CMB radiation, $\delta T/T \lesssim 10^{-5}$. Thus, at low frequency,

$$\Omega_G(t_0) \lesssim \Omega_\gamma(t_0) \left(\frac{\Delta T}{T}\right)^2_{COBE} \sim 10^{-14}. \tag{6.9}$$

In string cosmology the spectrum is growing, the normalization is imposed at high frequency, and the peak value is controlled by the fundamental ratio $M_s/M_p \lesssim 0.1$. Thus

$$\Omega_G(t_0) \lesssim \Omega_\gamma(t_0) \left(\frac{M_s}{M_p}\right)^2 \lesssim 10^{-6}. \tag{6.10}$$

The graviton background obtained from the amplification of the vacuum fluctuations, in string cosmology and in standard inflation, is compared in Fig. 12 with other, more unconventional graviton spectra. In particular: the graviton spectrum obtained from cosmic strings and topological defects [73], from bubble collision at the end of a first order phase transition [74], and from a phase of parametric resonance of the inflaton oscillations [75]. Also shown in Fig. 12 is the spectrum from models of quintessential inflation [76], and a thermal black body spectrum for a temperature of about one kelvin. All these cosmological backgrounds are higher than the background expected from the vacuum fluctuations in standard inflation, but not in a string cosmology context.

It may be interesting, at this point, to recall the expected sensitivities of the present, and near future, gravitational antennas, referred to the plots of Fig. 12.

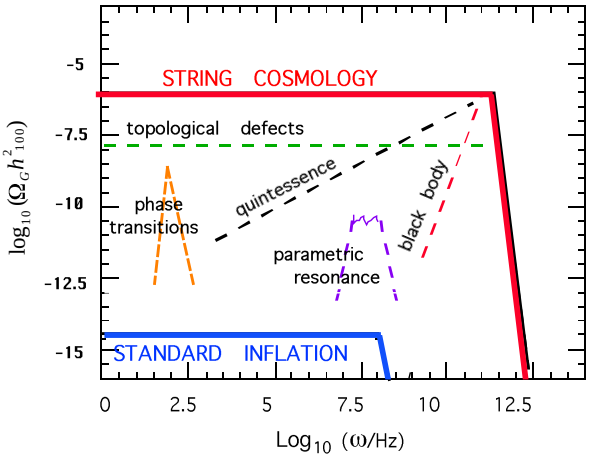

Figure 12. Allowed region for the spectrum of vacuum fluctuations in string cosmology and in standard inflation, compared with other relic spectra of primordial origin.

At present, the best direct, experimental upper bound on the energy of a stochastic graviton background comes from the cross-correlation of the data of the two resonant bars NAUTILUS and EXPLORER [77]:

$$\Omega_G h_{100} \lesssim 60, \qquad \nu \simeq 907 \text{ Hz} \tag{6.11}$$

(similar sensitivities are also reached by AURIGA [78]). Unfortunately, the bound is too high to be significant for the plots of Fig. 12. However, a much better sensitivity, $\Omega_G \sim 10^{-4}$ around $\nu \sim 10^3$ Hz, is expected from the present resonant bar detectors, if the integration time of the data is extended to about one year. A similar, or slightly better sensitivity, $\Omega_G \sim 10^{-5}$ around $\nu \sim 10^2$ Hz, is expected from the first operating version of the interferometric detectors, such as LIGO and VIRGO. At high frequency, from the kHz to the MHz range, a promising possibility seems to be the use of resonant electromagnetic cavities as gravity wave detectors [79]. Work is in progress [80] to attempt to improve their sensitivity.

The present, and near future, available sensitivities of resonant bars and interferometers, therefore, are still outside the allowed region of Fig. 12, determined by the border line

$$\Omega_G h_{100} \simeq 10^{-6}. \tag{6.12}$$

Such sensitivities are not so far from the border, after all, but to get really inside we have to wait for the cross-correlation of two spherical resonant-mass detectors [81], expected to reach $\Omega_G \sim 10^{-7}$ in the kHz range, or for the advanced interferometers, expected to reach $\Omega_G \sim 10^{-10}$ in the range of 10^2 Hz. At lower frequencies, around $10^{-2} - 10^{-3}$ Hz, the space interferometer LISA [36] seem to be able to reach very high sensitivities, up to $\Omega_G \sim 10^{-11}$. Work is in progress, however, for a more precise computation of their sensitivity to a cosmic stochastic background [82].

Detectors able to reach, and to cross the limiting sensitivity (6.12), could explore for the first time the parameter space of string cosmology and of Planck scale physics. The detection of a signal from a pre-big bang background, extrapolated to the GHz range, could give a first experimental indication on the value of the fundamental ratio M_s/M_p. Even the absence of a signal, inside the allowed region, would be significant, as we could exclude some portion of parameter space of the string cosmlogy models, obtaining in such a way direct experimental information about processes occuring at (or very near to) the string scale.

7. Conclusion

The conclusion of these lectures is very short and simple.

There is a rich structure of stochastic, gravitational wave backgrounds, of cosmological origin, in the frequency range of present (or planned for the future) gravity wave detectors.

Among such backgrounds, the stronger one seems to be the background possibly predicted in the context of the pre-big bang scenario, in a string cosmology context, originating at (or very near to) the fundamental string scale. Also, the maximal predicted intensity of the background seems to be accessible to the sensitivity of the future advanced detectors.

If this is the case, the future gravity wave detectors will be able to test string theory models, or perhaps models referring to some more fundamental unified theory, such as D-brane theory, M-theory, and so on. In any case, such detectors will give direct experimental information on Planck scale physics.

Acknowledgments

It is a pleasure to thank the organizers of the *Second SIGRAV School*, and all the staff of the Center "A. Volta", in Villa Olmo, for the pleasant hospitality and the perfect organization of this interesting School.

Appendix A. The string effective action

The motion of a point particle in an external gravitational field, $g_{\mu\nu}$, is governed by the action

$$S = -\frac{m}{2}\int d\tau \dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x), \tag{A.1}$$

where $x^\mu(\tau)$ are the spacetime coordinates of the particle, and τ is an affine parameter along the particle world-line.

The time evolution of a one-dimensional object like a string describes a world-surface, or "world-sheet", instead of a world-line, and the action governing its motion is given by the surface integral

$$S = -\frac{M_s^2}{2}\int d\tau d\sigma \sqrt{-\gamma}\gamma^{ij}\partial_i x^\mu \partial_j x^\nu g_{\mu\nu}(x), \tag{A.2}$$

where $\partial_i \equiv \partial/\partial\xi^i$, and $\xi^i = (\tau, \sigma)$ are respectively the timelike and spacelike coordinate on the string world-sheet ($i, j = 1, 2$). The coordinates $x^\mu(\tau, \sigma)$ are the fields governing the embedding of the string world-sheet in the external (also called "target") space. The parameter M_s^2 defines (in units $h/2\pi = 1 = c$) the so-called string tension (the mass per unit length), and its inverse defines the fundamental length scale of the theory (often called, for hystorical reasons, the α' parameter):

$$M_s^2 \equiv \frac{1}{\lambda_s^2} \equiv \frac{1}{2\pi\alpha'}. \tag{A.3}$$

In a curved metric background $g_{\mu\nu}$ depends on x^μ, and the nonlinear action (A.2) represents what is called a "sigma model" defined on the string world sheet.

For the point particle action (A.1) the variation with respect to x^μ leads to the well known geodesic equations of motion,

$$\ddot{x}^\mu + \Gamma_{\alpha\beta}{}^\mu \dot{x}^\alpha \dot{x}^\beta = 0. \tag{A.4}$$

The string equations of motion are similarly obtained by varying with respect to x^μ the action (A.2): we get then the Eulero-Lagrange equations

$$\partial_i \frac{\partial L}{\partial(\partial_i x^\mu)} = \frac{\partial L}{\partial x^\mu}, \qquad L = \sqrt{-\gamma}\gamma^{ij}\partial_i x^\mu \partial_j x^\nu g_{\mu\nu}, \tag{A.5}$$

which can be written explicitly as

$$\Box x^\mu + \gamma^{ij}\partial_i x^\alpha \partial_j x^\beta \Gamma_{\alpha\beta}{}^\mu = 0, \qquad \Box \equiv \frac{1}{\sqrt{-\gamma}}\partial_i \sqrt{-\gamma}\gamma^{ij}\partial_j, \tag{A.6}$$

These equations describe the geodesic evolution of a test string in a given external metric. The variation with respect to γ_{ik} imposes the so-called "costraints", i.e. the vanishing of the world sheet stress tensor T_{ik},

$$T_{ij} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma^{ij}} = \partial_i x^\mu \partial_j x^\nu g_{\mu\nu} - \frac{1}{2}\gamma_{ij}\partial_k x^\mu \partial^k x_\mu = 0. \tag{A.7}$$

It is important to note, at this point, that for a classical string it is always possible to impose the so-called "conformal gauge" in which the world sheet metric is flat, $\gamma_{ij} = \eta_{ij}$. In fact, in an appropriate basis, the two-dimensional metric tensor can always be set in a diagonal form, $\gamma_{ij} = \mathrm{diag}(a, b)$, and then, by using reparametrization invariance on the world sheet, $b^2 d\sigma^2 = a^2 d\sigma'^2$, the metric can be set in a conformally flat form, $\gamma_{ij} = a^2\eta_{ij}$. Since the action (A.2) in invariant under the conformal (or Weyl) transformation $\gamma_{ij} \to \Omega^2(\xi_k)\gamma_{ij}$,

$$\sqrt{-\gamma}\gamma^{ij} \to \sqrt{\Omega^4}\ \Omega^{-2}\sqrt{-\gamma}\gamma^{ij}, \tag{A.8}$$

we can always eliminate the conformal factor a^2 in front of the Minkowski metric, by choosing $\Omega = a^{-1}$. In the conformal gauge the equations of motion (A.6) reduce to

$$\ddot{x}^\mu - x''^\mu + \Gamma_{\alpha\beta}{}^\mu \left(\dot{x}^\alpha + x'^\alpha\right)\left(\dot{x}^\beta - x'^\beta\right) = 0, \tag{A.9}$$

where $\dot{x} = dx/d\tau$, $x' = dx/d\sigma$, and the constraints (A.7) become

$$g_{\mu\nu}\left(\dot{x}^\mu\dot{x}^\nu + x'^\mu x'^\nu\right) = 0, \qquad g_{\mu\nu}\dot{x}^\mu x'^\nu = 0. \tag{A.10}$$

We now come to the crucial observation which leads to the effective action governing the motion of the background fields. The conformal transformation (A.8) is an invariance of the classical theory. Let us require that there are no "anomalies", i.e. no quantum violations of this classical symmetry. By imposing such a constraint, we will obtain a set of differential equations to be satisfied by the background fields coupled to the string. Thus, unlike a point particle which does not imposes any constraint on the external geometry in which it is moving, the consistent quantization of a string gives constraints for the external fields. The background geometry cannot be chosen arbitrarily, but must satisfy the set of equations(also called β-function equations) which guarantee the absence of conformal anomalies. The string effective action used in this paper is the action which reproduces such a set of equations for the background fields, and in particular for the metric.

The derivation of the background equations of motion and of the effective action, from the sigma-model action (A.2), can be performed order by order by using a perturbative expansion in powers of α' (indeed, in the limit $\alpha' \to 0$ the action becomes very large in natural units, so that the quantum corrections are expected to become smaller and smaller). Such a procedure, however, is in general long and complicated, even to lowest order, and a detailed derivation of the background equations is outside the purpose of these lectures. Let me sketch here the procedure for the simplest case in which the only external field coupled to the string is the metric tensor $g_{\mu\nu}$.

In the conformal gauge, the action (A.2) becomes:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\ \partial_i x^\mu \partial^i x^\nu g_{\mu\nu}. \tag{A.11}$$

Let us formally assume a deformation of the number of world sheet dimensions, from 2 to $2+\epsilon$, and perform the conformal transformation: $\eta_{ij} \to \eta_{ij}\exp(\rho)$. Expanding we get, for small ϵ,

$$\begin{aligned} &-\frac{1}{4\pi\alpha'}\int d^{2+\epsilon}\xi\ e^{\rho\epsilon/2}\partial_i x^\mu \partial^i x^\nu g_{\mu\nu} = \\ &-\frac{1}{4\pi\alpha'}\int d^{2+\epsilon}\xi\ \partial_i x^\mu \partial^i x^\nu g_{\mu\nu}\left(1 + \frac{\epsilon}{2}\rho + ...\right). \end{aligned} \tag{A.12}$$

For $\epsilon \to 0$ the ρ dependence disappears, and the classical action is conformally invariant. In order to preserve this invariance also for the quantum theory, at the one loop level, let us treat the sigma model as a quantum field theory for $x^\mu(\sigma,\tau)$, and let us consider the quantum fluctuations $\hat{x}^\mu$ around a given expectation value x_0^μ. For the general reparametrization invariance of the theory we can always choose for x_0 a locally inertial frame, such that $g_{\mu\nu}(x_0) = \eta_{\mu\nu}$. By expanding the metric around x_0, the leading

corrections are of second order in the fluctuations, because in a locally inertial frame the first derivatives of the metric (and then the Cristoffel connection) can always be set to zero (but not the curvature). With an appropriate choice of coordinates, called Riemann normal coordinates, the metric can thus be expanded as:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\nu\alpha\beta}(x_0)\hat{x}^\alpha\hat{x}^\beta + ... \tag{A.13}$$

and the action for the quantum fluctuations becomes, to lowest order in the curvature,

$$S = -\frac{1}{4\pi\alpha'}\int d^{2+\epsilon}\xi\Big[\partial_i\hat{x}^\mu\partial^i\hat{x}_\mu\left(1+\frac{\epsilon}{2}\rho\right) - \frac{1}{3}\partial_i\hat{x}^\mu\partial^i\hat{x}^\nu R_{\mu\nu\alpha\beta}(x_0)\hat{x}^\alpha\hat{x}^\beta\left(1+\frac{\epsilon}{2}\rho\right) + ...\Big]. \tag{A.14}$$

It must be noted that, at the quantum level, the dependence of ρ does not disappear in general from the action in the limit $\epsilon \to 0$, since there are one-loop terms that diverge like ϵ^{-1}, just to cancel the ϵ dependence and to give a contribution proportional to ρ to the effective action. By evaluating, for instance, the two point function for the quantum operator $\hat{x}^\alpha\hat{x}^\beta$, in the coincidence limit $\sigma \to \sigma'$ (the tadpole graph), one obtains [83]

$$\langle\hat{x}^\alpha(\sigma)\hat{x}^\beta(\sigma')\rangle_{\sigma\to\sigma'} \sim \eta^{\alpha\beta}\lim_{\sigma\to\sigma'}\int d^{2+\epsilon}k\frac{e^{ik\cdot(\sigma-\sigma')}}{k^2} \sim \eta^{\alpha\beta}\epsilon^{-1}, \tag{A.15}$$

which gives the one-loop contribution to the action

$$\Delta S \sim \int d^{2+\epsilon}\xi\ \partial_i\hat{x}^\mu\partial^i\hat{x}^\nu R_{\mu\nu}\ \rho. \tag{A.16}$$

This term violates, at one-loop, the conformal invariance, unless we restrict to a background geometry satisfying the condition

$$R_{\mu\nu} = 0, \tag{A.17}$$

which are just the usual Einstein equations in vacuum.

A similar procedure can be applied if the string moves in a richer external background (not only pure gravity). Indeed, pure gravity is not enough, as a consistent quantum theory for closed bosonic strings, for instance, must contain at least three massless state (beside the unphysical tachyon, removed by supersymmetry) in the lowest energy level: the graviton, the scalar dilaton and the pseudo-scalar Kalb-Ramond axion. The sigma model describing the propagation of a string in such a background must thus contain the coupling to the metric, to the dilaton ϕ, and to the two-form $B_{\mu\nu} = -B_{\nu\mu}$:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\ \partial_i x^\mu\partial_j x^\nu\left(\sqrt{-\gamma}\gamma^{ij}g_{\mu\nu} + \epsilon^{ij}B_{\mu\nu}\right) - \frac{1}{4\pi}\int d^2\xi\sqrt{-\gamma}\frac{\phi}{2}R^{(2)}(\gamma), \tag{A.18}$$

where ϵ^{ij} is the two-dimensional Levi-Civita densor density, $\epsilon_{12} = -\epsilon_{21} = 1$, and $R^{(2)}(\gamma)$ is the two-dimensional scalar curvature for the world sheet metric γ. The condition of conformal invariance, at the one-loop level, leads to the equations

$$R_{\mu\nu} + \nabla_\mu\nabla_\nu - \frac{1}{4}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} = 0, \qquad H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\alpha\mu} + \partial_\alpha B_{\mu\nu},$$

$$R + 2\nabla^2\phi - (\nabla\phi)^2 - \frac{1}{12}H^2_{\mu\nu\alpha} = 0,$$
$$\nabla_\mu \left(e^{-\phi} H^{\mu\nu\alpha}\right) = 0, \tag{A.19}$$

which can be obtained by extremizing the effective action

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|}\; e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{1}{12}H^2_{\mu\nu\alpha}\right]. \tag{A.20}$$

(see Appendix C).

It should be noted that the inclusion of the dilaton in the condition of conformal invariance cannot be avoided, since the dilaton coupling in the action (A.18) breaks conformal invariance already at the classical level ($\sqrt{-\gamma}R^{(2)}$ is not invariant under a Weyl rescaling of γ). However, the dilaton term is of order α' with respect to the other terms of the action (for dimensional reasons), so that it is correct to sum up the classical dilaton contribution to the quantum, one-loop effects, as they are all of the same order in α'. Without the dilaton, however, the world sheet curvature density $\sqrt{-\gamma}R^{(2)}$ does not contribute to the string equations of motion, as it is a pure Eulero two-form in two dimensions (just like the Gauss-Bonnet term in four dimensions).

Let me note, finally, that the expansion around x_0 can be continued to higher orders,

$$g(x) = \eta + R\hat{x}\hat{x} + \partial R\hat{x}\hat{x}\hat{x} + R^2\hat{x}\hat{x}\hat{x}\hat{x} + ..., \tag{A.21}$$

thus introducing higher curvature terms, and higher powers of α', in the effective action:

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|}\; e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{\alpha'}{4}R^2_{\mu\nu\alpha\beta} + ...\right]. \tag{A.22}$$

At any given order, unfortunately, there is an intrinsic ambiguity in the action due to the fact that, with an appropriate field redefinition of order α',

$$g_{\mu\nu} \to g_{\mu\nu} + \alpha' \left(R_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi + ...\right)$$
$$\phi \to \phi + \alpha' \left(R + \nabla^2\phi + ...\right), \tag{A.23}$$

we obtain a number of different actions, again of the same order in α' (see for instance [84]). This ambiguity cannot be eliminated until we limit to an effective action truncated to a given finite order.

The higher curvature (or higher derivative) expansion of the effective action is typical of string theory: it is controlled by the fundamental, minimal length parameter $\lambda_s = (2\pi\alpha')^{1/2}$, in such a way that the higher order corrections disappear in the point-particle limit $\lambda_s \to 0$. At any given order in α', however, there is also the more conventional expansion in power of the coupling constant g_s (i.e. the loop expansion of quantum field theory: tree-level $\sim g^{-2}$, one-loop $\sim g^0$, two-loop $\sim g^2$, ...). The important observation is that, in a string theory context, the effective couplig constant is controlled by the dilaton. Consider, for instance, a process of graviton scattering, in four dimensions. Comparing the action (2.4) with the standard, gravitational Einstein action (2.1), it follows that the effective coupling constant, to lowest order, is

$$\sqrt{8\pi G} = \lambda_p = \lambda_s e^{\phi/2} \tag{A.24}$$

(G is the usual Newton constant). Each loop adds an integer power of the square of the dimensionless coupling constant, which is controlled by the dilaton as

$$g_s^2 = (\lambda_p/\lambda_s)^2 = (M_s/M_p)^2 = e^{\phi}. \tag{A.25}$$

We may thus expect, for the loop expansion of the action, the following general scheme:

$$\begin{aligned} S = & -\int e^{-\phi}\sqrt{-g}\left(R + \nabla\phi^2 + \alpha' R^2 + ...\right) && \text{tree level} \\ & -\int \sqrt{-g}\left(R + \nabla\phi^2 + \alpha' R^2 + ...\right) && \text{one} - \text{loop} \\ & -\int e^{+\phi}\sqrt{-g}\left(R + \nabla\phi^2 + \alpha' R^2 + ...\right) && \text{two} - \text{loop} \\ & ..... \end{aligned} \tag{A.26}$$

Unfortunately, each term in the action, at each loop order, is multiplied by a dilaton "form factor" which is different in general for *different fields* and for *different orders*. This difference can lead to an effective violation of the universality of the gravitational interactions [85] in the low-energy, macroscopic regime, and this violation can be reconciled with the present tests of the equivalence principle only if the dilaton is massive enough, to make short enough the range of the non-universal dilatonic interactions.

The tree-level relation (A.25) is valid also for a higher-dimensional effective action, provided e^{ϕ} represents the shifted four-dimensional dilaton which includes the volume of the extra-dimensional, compact internal space, and which controls the grand-unification gauge coupling, α_{GUT}, as [13]

$$\alpha_{GUT} = \exp\langle\phi\rangle = (M_s/M_p)^2 \sim 0.1 - 0.001. \tag{A.27}$$

However, the relation (A.25) is no longer valid, in general, if the gauge interactions are confined in four-dimensions and only gravity propagates in the extra dimensions. In that case the relation depends on the volume of the extra dimensions, whose size may be allowed to be large in Planck units [86]. For internal dimensions of volume V_n the relation becomes, in particular,

$$M_p^2 = M_s^{2+n} V_n e^{-\Phi}, \tag{A.28}$$

where Φ is the dilaton in $d = 3 + n$ dimensions. In this case, the string mass parameter could be much smaller than the value expected from eq. (A.27), provided the internal volume is correspondingly larger.

Appendix B. Duality symmetry

Notations and conventions. In this paper we use the metric signature $(+ - --)$, and we define the Riemann and Ricci tensor as follows:

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} + \Gamma_{\mu\rho}{}^{\beta}\Gamma_{\nu\alpha}{}^{\rho} - (\mu \leftrightarrow \nu),$$
$$R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu}.$$

Consider the gravi-dilaton effective action, in the S-frame, to lowest order in α' and in the quantum loop expansion:

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|}\, e^{-\phi} \left[R + (\nabla\phi)^2\right]. \tag{B.1}$$

For a homogeneous, but anisotropic, Bianchi I type metric background:

$$\phi = \phi(t), \qquad g_{00} = N^2(t), \qquad g_{ij} = -a_i^2(t)\delta_{ij}, \tag{B.2}$$

we have ($H_i = \dot{a}_i/a_i$):

$$\begin{aligned}
&(\nabla\phi)^2 = \frac{\dot{\phi}^2}{N^2}, \qquad \sqrt{-g} = N\prod_{i=1}^{d} a_i, \\
&\Gamma_{00}{}^0 = \frac{\dot{N}}{N} \equiv F, \qquad \Gamma_{0i}{}^j = H_i\delta_i^j, \qquad \Gamma_{ij}{}^0 = \frac{a_i\dot{a}_i}{N^2}\delta_{ij}, \\
&R = \frac{1}{N^2}\left[2F\sum_i H_i - 2\sum_i \dot{H}_i - \sum_i H_i^2 - \left(\sum_i H_i\right)^2\right].
\end{aligned} \tag{B.3}$$

By noting that

$$\begin{aligned}
&\frac{d}{dt}\left[2\frac{e^{-\phi}}{N}\left(\prod_{k=1}^{d} a_k\right)\left(\sum_i H_i\right)\right] \\
&= \frac{e^{-\phi}}{N}\left(\prod_{k=1}^{d} a_k\right)\left[2\sum_i \dot{H}_i - 2F\sum_i H_i - 2\dot{\phi}\sum_i H_i + 2\left(\sum_i H_i\right)^2\right],
\end{aligned} \tag{B.4}$$

the action (B.1), modulo a total derivative, can be rewritten as:

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^dx dt \prod_{i=1}^{d} a_i \frac{e^{-\phi}}{N}\left[\dot{\phi}^2 - \sum_i H_i^2 + \left(\sum_i H_i\right)^2 - 2\dot{\phi}\sum_i H_i\right]. \tag{B.5}$$

We now introduce the so-called shifted dilaton $\overline{\phi}$, defined by

$$e^{-\overline{\phi}} = \int \frac{d^dx}{\lambda_s^d} \prod_{i=1}^{d} a_i e^{-\phi}, \tag{B.6}$$

from which

$$\phi = \overline{\phi} + \sum_i \ln a_i, \qquad \dot{\phi} = \dot{\overline{\phi}} + \sum_i H_i \tag{B.7}$$

(by assuming spatial sections of finite volume, $(\int d^dx\sqrt{|g|})_{t=\text{const}} < \infty$, we have absorbed into ϕ the constant shift $-\ln(\lambda_s^{-d}\int d^dx)$, required to secure the scalar behaviour of $\overline{\phi}$ under coordinate reparametrizations preserving the comoving gauge). The action becomes:

$$S = -\frac{\lambda_s}{2}\int dt \frac{e^{-\overline{\phi}}}{N}\left(\dot{\overline{\phi}}^2 - \sum_i H_i^2\right). \tag{B.8}$$

By inverting one of the d scale factors the corresponding Hubble parameter changes sign,

$$a_i \to \tilde{a}_i = a_i^{-1}, \qquad H_i \to \tilde{H}_i = \frac{\dot{\tilde{a}}_i}{\tilde{a}_i} = a_i\frac{da_i^{-1}}{dt} = -H_i, \tag{B.9}$$

so that the quadratic action (B.8) is clearly invariant under the inversion of any scale factor preserving the shifted dilaton,

$$a_i \to \tilde{a}_i = a_i^{-1}, \qquad \overline{\phi} \to \overline{\phi} \tag{B.10}$$

("scale factor duality", see [9] and the first paper of Ref. [8]).

In order to derive the field equations, it is convenient to use the variables $\beta_i = \ln a_i$, so that $H_i = \dot{\beta}_i$, $\dot{H}_i = \ddot{\beta}_i$, and the action (B.8) is cyclic in β_i. By varying with respect to N, β_i and $\overline{\phi}$, and subsequently fixing the cosmic time gauge $N = 1$, we obtain, respectively,

$$\dot{\overline{\phi}}^2 - \sum_i H_i^2 = 0, \tag{B.11}$$

$$\dot{H}_i - H_i \dot{\overline{\phi}} = 0, \tag{B.12}$$

$$2\ddot{\overline{\phi}} - \dot{\overline{\phi}}^2 - \sum_i H_i^2 = 0. \tag{B.13}$$

This is a system of $(d+2)$ equations for the $(d+1)$ variables $\{a_i, \phi\}$. However, only $(d+1)$ equations are independent (see for instance [21]: eq. (B.11) represents a constraint on the set of initial data).

The above equations are invariant under a time reversal transformation

$$t \to -t, \qquad H \to -H, \qquad \dot{\overline{\phi}} \to -\dot{\overline{\phi}}, \tag{B.14}$$

and also under the duality transformation (B.10). If we invert $k \leq d$ scale factors, $\tilde{a}_1 = a_1^{-1}$, ... $\tilde{a}_k = a_k^{-1}$, the shifted dilaton is preserved, $\overline{\phi} = \tilde{\overline{\phi}}$, provided

$$\overline{\phi} = \phi - \sum_{i=1}^{d} \ln a_i = \tilde{\phi} - \sum_{i=1}^{k} \ln \tilde{a}_i - \sum_{i=k+1}^{d} \ln a_i, \tag{B.15}$$

from which:

$$\tilde{\phi} = \phi - 2\sum_{i=1}^{k} \ln a_i. \tag{B.16}$$

Given an exact solution, represented by the set of variables

$$\{a_1, ...a_d,\ \phi\}, \tag{B.17}$$

the inversion of $k \leq d$ scale factors defines then a new exact solution, represented by the set of variables

$$\{a_1^{-1}, ...a_k^{-1}, a_{k+1}, ...a_d,\ \phi - 2\ln a_1 ... - 2\ln a_k\}. \tag{B.18}$$

By inverting all the scale factors we obtain the transformation

$$\{a_i, \phi\} \to \{a_i^{-1}, \phi - 2\sum_{i=1}^{d} \ln a_i\} \tag{B.19}$$

which, in the isotropic case, corresponds in particular to the duality transformation (2.5).

As a simple example, we consider here the particular isotropic solution

$$a = t^{1/\sqrt{d}}, \qquad \overline{\phi} = -\ln t, \tag{B.20}$$

which satisfies identically the set of equations (B.11) - (B.13). By applying a duality *and* a time-reversal transformation we obtain the four different exact solutions

$$\{a_\pm(t) = t^{\pm 1/\sqrt{d}}, \quad \overline{\phi}(t) = -\ln t\},$$
$$\{a_\pm(-t) = (-t)^{\pm 1/\sqrt{d}}, \quad \overline{\phi}(-t) = -\ln(-t)\}, \tag{B.21}$$

corresponding to the four branches illustrated in Fig. 2, and describing decelerated expansion, $a_+(t)$, decelerated contraction, $a_-(t)$, accelerated contraction, $a_+(-t)$, accelerated expansion, $a_-(-t)$. The solution describes expansion or contraction if the sign of $\dot{a}$ is positive or negative, repectively, and the solution is accelerated or decelerated if $\dot{a}$ and $\ddot{a}$ have the same or the opposite sign, respectively.

It is important to consider also the dilaton behaviour. According to eq. (B.7):

$$\phi_\pm(\pm t) = \overline{\phi}(\pm t) + d \ln a_\pm(\pm t) = \left(\pm\sqrt{d} - 1\right) \ln(\pm t). \tag{B.22}$$

It follows that, in a phase of growing curvature ($t < 0, t \to 0_-$), the dilaton is growing only for an expanding metric, $a_-(-t)$. This means that, in the isotropic case, there are only expanding pre-big bang solution, i.e. solutions evolving from the string perturbative vacuum ($H \to 0, \phi \to -\infty$), and then characterized by a growing string coupling, $\dot{g}_s = (\exp \phi/2)\dot{} > 0$.

In the more general, anisotropic case, and in the presence of contracting dimensions, a growing curvature solution is associated to a growing dilaton only for a large enough number of contracting dimensions. To make this point more precise, consider the particular, exact solution of eqs. (B.11) - (B.13) with d expanding and n contracting dimensions, and scale factors $a(t)$ and $b(t)$, respectively:

$$a = (-t)^{-1/\sqrt{d+n}}, \qquad b = (-t)^{1/\sqrt{d+n}}, \qquad \overline{\phi} = -\ln(-t), \qquad t \to 0_-. \tag{B.23}$$

This gives, for the dilaton,

$$\phi = \overline{\phi} + d \ln a + n \ln b = \frac{n - d - \sqrt{d+n}}{\sqrt{d+n}} \ln(-t), \tag{B.24}$$

so that the dilaton is growing if

$$d + \sqrt{d+n} > n. \tag{B.25}$$

For $n = 6$, in particular, this condition requires $d > 3$. This could represent a potential difficulty for the pre-big bang scenario, which might be solved, however, by quantum cosmology effects [87].

The scale factor duality of the action (B.8) is in general broken by the addition of a non-trivial dilaton potential (unless the potential depends on the dilaton through $\overline{\phi}$, of course). When the antisymmetric tensor $B_{\mu\nu}$ is included in the action, however, the scale factor duality can be lifted to a larger group of global symmetry transformations. To illustrate this important aspect of the string cosmology equations, we will consider here a set of cosmological baground fields $\{\phi, g_{\mu\nu}, B_{\mu\nu}\}$, for which a syncronous frame exists where $g_{00} = 1, g_{0i} = 0, B_{0\mu} = 0$, and all the components ϕ, g_{ij}, B_{ij} do not depend on the spatial coordinates.

Let us write the action

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|}\; e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{1}{12} H^2_{\mu\nu\alpha} \right] \tag{B.26}$$

directly in the syncronous gauge, as we are not interested in the field equations, but only in the symmetries of the action. We set $g_{ij} = -\gamma_{ij}$ and we find, in this gauge,

$$\begin{aligned}
&\Gamma_{ij}{}^{0} = \frac{1}{2}\dot{\gamma}_{ij}, \qquad\qquad \Gamma_{0i}{}^{j} = \frac{1}{2} g^{jk}\dot{g}_{ik} = \frac{1}{2}\left(g^{-1}\dot{g}\right)_i{}^{j} = \left(\gamma^{-1}\dot{\gamma}\right)_i{}^{j} \\
&R_0{}^{0} = -\frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)^2 - \frac{1}{2}\mathrm{Tr}\left(\gamma^{-1}\ddot{\gamma}\right) - \frac{1}{2}\mathrm{Tr}\left(\dot{\gamma}^{-1}\dot{\gamma}\right), \\
&R_i{}^{j} = -\frac{1}{2}\left(\gamma^{-1}\ddot{\gamma}\right)_i{}^{j} - \frac{1}{4}\left(\gamma^{-1}\dot{\gamma}\right)_i{}^{j}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right) + \frac{1}{2}\left(\gamma^{-1}\dot{\gamma}\gamma^{-1}\dot{\gamma}\right)_i{}^{j},
\end{aligned} \tag{B.27}$$

where

$$\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right) = \left(\gamma^{-1}\right)^{ij}\dot{\gamma}_{ji} = g^{ij}\dot{g}_{ji}, \tag{B.28}$$

and so on [note also that $\dot{\gamma}^{-1}$ means $(\gamma^{-1})\dot{}$]. Similarly we find, for the antisymmetric tensor,

$$\begin{aligned}
&H_{0ij} = \dot{B}_{ij}, \qquad\qquad H^{0ij} = g^{ik}g^{jl}\dot{B}_{kl} = \left(\gamma^{-1}\dot{B}\gamma^{-1}\right)^{ij}, \\
&H_{\mu\nu\alpha}H^{\mu\nu\alpha} = 3H_{0ij}H^{0ij} = -3\mathrm{Tr}\left(\gamma^{-1}\dot{B}\right)^2.
\end{aligned} \tag{B.29}$$

Let us introduce the shifted dilaton, by absorbing into ϕ the spatial volume, as before:

$$\sqrt{|\det g_{ij}|}e^{-\phi} = e^{-\overline{\phi}}, \tag{B.30}$$

from which

$$\dot{\overline{\phi}} = \dot{\phi} - \frac{1}{2}\frac{d}{dt}\ln\left(\det\gamma\right) = \dot{\phi} - \frac{1}{2}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right). \tag{B.31}$$

By collecting the various contributions from ϕ, R and H^2, the action (B.26) can be rewritten as:

$$\begin{aligned}
S = -\frac{\lambda_s}{2}\int dt e^{-\overline{\phi}}\Bigg[&\dot{\overline{\phi}}^2 + \frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)^2 - \mathrm{Tr}\left(\gamma^{-1}\ddot{\gamma}\right) \\
&- \frac{1}{2}\mathrm{Tr}\left(\dot{\gamma}^{-1}\dot{\gamma}\right) + \dot{\overline{\phi}}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right) + \frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{B}\right)^2\Bigg].
\end{aligned} \tag{B.32}$$

We can now eliminate the second derivatives, and the mixed term ($\sim \dot{\overline{\phi}}\dot{\gamma}$), by noting that

$$\frac{d}{dt}\left[e^{-\overline{\phi}}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)\right] = e^{-\overline{\phi}}\left[\mathrm{Tr}\left(\gamma^{-1}\ddot{\gamma}\right) + \mathrm{Tr}\left(\dot{\gamma}^{-1}\dot{\gamma}\right) - \dot{\overline{\phi}}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)\right]. \tag{B.33}$$

Finally, by using the identity,

$$\left(\gamma^{-1}\right)\dot{} = -\gamma^{-1}\dot{\gamma}\gamma^{-1} \tag{B.34}$$

(following from $g^{-1}g = \gamma^{-1}\gamma = I$), we can rewrite the action in quadratic form, modulo a total derivative, as

$$S = -\frac{\lambda_s}{2}\int dt e^{-\overline{\phi}}\left[\dot{\overline{\phi}}^2 - \frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{\gamma}\right)^2 + \frac{1}{4}\mathrm{Tr}\left(\gamma^{-1}\dot{B}\right)^2\right]. \tag{B.35}$$

This action can be set into a more compact form by using the $2d \times 2d$ matrix M, defined in terms of the spatial components of the metric and of the antisymmetric tensor,

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix},$$
$$G = g_{ij} \equiv -\gamma_{ij}, \qquad G^{-1} \equiv g^{ij}, \qquad B \equiv B_{ij}, \tag{B.36}$$

and using also the matrix η, representing the invariant metric of the $O(d,d)$ group in the off-diagonal representation,

$$\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{B.37}$$

(I is the unit d-dimensional matrix). By computing $M\eta, \dot{M}\eta$ and $(\dot{M}\eta)^2$ we find, in fact,

$$\mathrm{Tr}\left(\dot{M}\eta\right)^2 = 2\mathrm{Tr}\left[\dot{\gamma}^{-1}\dot{\gamma} + \left(\gamma^{-1}\dot{B}\right)^2\right] = 2\mathrm{Tr}\left[-\left(\gamma^{-1}\dot{\gamma}\right)^2 + \left(\gamma^{-1}\dot{B}\right)^2\right], \tag{B.38}$$

and the action becomes

$$S = -\frac{\lambda_s}{2}\int dt e^{-\overline{\phi}}\left[\dot{\overline{\phi}}^2 + \frac{1}{8}\mathrm{Tr}\left(\dot{M}\eta\right)^2\right]. \tag{B.39}$$

We may note, at this point, that M is a symmetric matrix of the pseudo-orthogonal $O(d,d)$ group. In fact,

$$M^T\eta M = \eta, \qquad M = M^T \tag{B.40}$$

for any B and G. Therefore:

$$M\eta = \eta M^{-1}, \qquad \left(\dot{M}\eta\right)^2 = \eta\left(M^{-1}\right)^{\cdot}M\eta, \tag{B.41}$$

and the action can be finally rewritten as

$$S = -\frac{\lambda_s}{2}\int dt e^{-\overline{\phi}}\left[\dot{\overline{\phi}}^2 + \frac{1}{8}\mathrm{Tr}\ \dot{M}\left(M^{-1}\right)^{\cdot}\right]. \tag{B.42}$$

This form is explicitly invariant under the global $O(d,d)$ transformations (2.10), preserving the shifted dilaton:

$$\overline{\phi} \to \overline{\phi}, \qquad M \to \Lambda^T M\Lambda, \qquad \Lambda^T\eta\Lambda = \eta \tag{B.43}$$

In fact

$$\mathrm{Tr}\ \dot{\tilde{M}}\left(\tilde{M}^{-1}\right)^{\cdot} = \mathrm{Tr}\left[\Lambda^T\dot{M}\Lambda\Lambda^{-1}\left(M^{-1}\right)^{\cdot}\left(\Lambda^T\right)^{-1}\right] = \mathrm{Tr}\ \dot{M}\left(M^{-1}\right)^{\cdot}. \tag{B.44}$$

In the absence of the antysimmetric tensor M is diagonal, and the special $O(d,d)$ transformation with $\Lambda = \eta$ corresponds to an inversion of the metric tensor:

$$M = \mathrm{diag}(G^{-1}, G),$$
$$\tilde{M} = \Lambda^T M\Lambda = \eta M\eta = \mathrm{diag}(G, G^{-1}) \Longrightarrow \tilde{G} = G^{-1}. \tag{B.45}$$

For a diagonal metric $G = a^2 I$, and the invariance under the scale factor duality transformation (2.5) is recovered as a particular case of the global $O(d,d)$ symmetry of the low energy effective action.

Appendix C. The string cosmology equations

In order to derive the cosmological equations let us include in the action, for completeness, the antisymmetric tensor $B_{\mu\nu}$, a dilaton potential $V(\phi)$, and also the possible contribution of other matter sources represented by a Lagrangian density L_m:

$$S = -\frac{1}{2\lambda_s^{d-1}} \int d^{d+1}x\sqrt{|g|}\; e^{-\phi}\left[R + (\nabla\phi)^2 - \frac{1}{12}H^2_{\mu\nu\alpha} + V(\phi)\right] + \int d^{d+1}x\sqrt{|g|}\; L_m. \tag{C.1}$$

In a scalar-tensor model of gravity, expecially in the presence of higher derivative interactions, it is often convenient to write the action in the language of exterior differential forms, as this may simplify the variational procedure (see for instance [88]). Here we will follow however the more traditional approach, by varying the action with respect to $g_{\mu\nu}, \phi$ and $B_{\mu\nu}$. We shall take into account the dynamical stress tensor $T_{\mu\nu}$ of the matter sources (defined in the usual way), as well as the scalar source σ representing a possible direct coupling of the dilaton to the matter fields:

$$\delta_g(\sqrt{-g}L_m) = \frac{1}{2}\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}, \qquad \delta_\phi(\sqrt{-g}L_m) = \sqrt{-g}\sigma\delta\phi. \tag{C.2}$$

We start performing the variation with respect to the metric, using the standard, general relativistic results:

$$\begin{aligned}
&\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu},\\
&\delta\left(\sqrt{-g}R\right) = \sqrt{-g}\left(G_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\nabla^2\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right),
\end{aligned} \tag{C.3}$$

where $G_{\mu\nu}$ is the usual Einstein tensor. It must be noted, however, that the second covariant derivatives of $\delta g^{\mu\nu}$, when integrated by parts, are no longer equivalent to a divergence (and then to a surface integral), because of the dilaton factor $\exp(-\phi)$ in front of the Einstein action, which adds dilatonic gradients to the full variation. By performing a first integration by part, and using the metricity condition $\nabla_\alpha g_{\mu\nu} = 0$, we get in fact:

$$\begin{aligned}
\delta_g S = {}& \frac{1}{2}\int d^{d+1}x\sqrt{|g|}T_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2\lambda_s^{d-1}}\int d^{d+1}x\sqrt{|g|}\; e^{-\phi}\\
&\left[G_{\mu\nu} + \nabla_\alpha\phi g_{\mu\nu}\nabla^\alpha - \nabla_\mu\phi\nabla_\nu + \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 - \frac{1}{2}g_{\mu\nu}V(\phi)\right.\\
&\left.+\frac{1}{2}g_{\mu\nu}\frac{1}{12}H^2_{\alpha\beta\gamma} - \frac{3}{12}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta}\right]\delta g^{\mu\nu}\\
&-\frac{1}{2\lambda_s^{d-1}}\int d^{d+1}x\sqrt{|g|}\;\nabla_\alpha\left[e^{-\phi}g_{\mu\nu}\nabla^\alpha\delta g^{\mu\nu} - e^{-\phi}\nabla_\nu\delta g^{\nu\alpha}\right] = 0.
\end{aligned} \tag{C.4}$$

A second integration by part of $\nabla\delta g^{\mu\nu}$ cancels the bilinear term $\nabla_\mu\phi\nabla_\nu\phi$, and leads to the field equations:

$$\begin{aligned}
&G_{\mu\nu} + \nabla_\mu\nabla_\nu\phi + \frac{1}{2}g_{\mu\nu}\left[(\nabla\phi)^2 - 2\nabla^2\phi - V(\phi) + \frac{1}{12}H^2_{\alpha\beta\gamma}\right] - \frac{1}{4}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta}\\
&\qquad = \frac{1}{2}e^\phi T_{\mu\nu}.
\end{aligned} \tag{C.5}$$

We have chosen units such that $2\lambda_s^{d-1} = 1$, so that e^ϕ represents the $(d+1)$-dimensional gravitational constant (see Appendix A). Also, we have implicitly added to the action the boundary term

$$\frac{1}{2\lambda_s^{d-1}} \int_{\partial\Omega} \sqrt{|g|}\; e^{-\phi} K^\alpha d\Sigma_\alpha, \tag{C.6}$$

whose variation with respect to g exactly cancels the contribution of the total divergence appearing in the last integral of eq. (C.4):

$$\delta_g \int \sqrt{|g|}\; e^{-\phi} K^\alpha d\Sigma_\alpha = \int \sqrt{|g|}\; e^{-\phi} \left(g_{\mu\nu}\nabla^\alpha \delta g^{\mu\nu} - \nabla_\nu \delta g^{\nu\alpha}\right) d\Sigma_\alpha. \tag{C.7}$$

Here K^α is a geometric term representing the so-called extrinsic curvature on the d-dimensional closed hypersurface, of infinitesimal area $d\Sigma_\alpha$, bounding the total spacetime volume over which we are varying the action. Note that the integral (C.6) differs from the usual boundary term, used in general relativity [89] to derive the Einstein equations, only by the presence of the tree-level dilaton coupling $e^{-\phi}$ to the extrinsic curvature.

Let us now perform the variation with respect to the dilaton, again in units $2\lambda_s^{d-1} = 1$. We get the Eulero-Lagrange equations:

$$\begin{aligned} &\partial_\mu \left[-2\sqrt{-g} e^{-\phi}\partial^\mu \phi\right] = e^{-\phi}\sqrt{-g}\left[R + (\nabla\phi)^2 - \frac{1}{12}H^2 + V\right] \\ &- e^{-\phi}\sqrt{-g}V' + \sqrt{-g}\sigma \end{aligned} \tag{C.8}$$

(where $V' = \partial V/\partial\phi$), from which

$$R + 2\nabla^2\phi - (\nabla\phi)^2 + V - V' - \frac{1}{12}H^2 + e^\phi\sigma = 0. \tag{C.9}$$

The variation with respect to $B_{\mu\nu}$,

$$\delta_B \int d^{d+1}x \sqrt{|g|}\; e^{-\phi} \left(\partial_\mu B_{\nu\alpha}\right) H^{\mu\nu\alpha} = 0, \tag{C.10}$$

gives finally

$$\partial_\mu \left(\sqrt{|g|}\; e^{-\phi} H^{\mu\nu\alpha}\right) = 0 = \nabla_\mu \left(e^{-\phi} H^{\mu\nu\alpha}\right). \tag{C.11}$$

Eqs. (C.5, C.9, C.11) are the equations governing the evolution of the string cosmology background, at low energy. Note that eq. (C.5) can also be given in a simplified form: if we eliminate the scalar curvature present inside the Einstein tensor, by using the dilaton equation (C.9), we obtain:

$$R_\mu{}^\nu + \nabla_\mu\nabla^\nu\phi - \frac{1}{2}\delta_\mu^\nu V' - \frac{1}{4}H_{\mu\alpha\beta}H^{\nu\alpha\beta} = \frac{1}{2}e^\phi \left(T_\mu{}^\nu - \delta_\mu^\nu\sigma\right). \tag{C.12}$$

For the purpose of these lectures, it will be enough to derive some simple solution of the string cosmology equations in the absence of the potential ($V = 0$), of the antisymmetric tensor ($B = 0$), and with a perfect fluid, minimally coupled to the dilaton ($\sigma = 0$), as the matter sources. Assuming for the background a Bianchi I type metric, we can work in the syncronous gauge, by setting

$$\begin{aligned} &g_{\mu\nu} = \mathrm{diag}(1, -a_i^2\delta_{ij}), \qquad a_i = a_i(t), \qquad \phi = \phi(t), \\ &T_\mu{}^\nu = \mathrm{diag}(\rho, -p_i^2\delta_i^j), \qquad p_i/\rho = \gamma_i = \mathrm{const}, \qquad \rho = \rho(t). \end{aligned} \tag{C.13}$$

For this background:

$$\Gamma_{0i}{}^{j} = H_i\delta_i^j, \quad \Gamma_{ij}{}^{0} = a_i\dot{a}_i\delta_{ij}, \quad R_0{}^{0} = -\sum_i \left(\dot{H}_i + H_i^2\right),$$

$$R_i{}^{j} = -\dot{H}_i\delta_i^j - H_i\delta_i^j\sum_k H_k, \quad R = -\sum_i \left(2\dot{H}_i + H_i^2\right) - \left(\sum_i H_i\right)^2,$$

$$(\nabla\phi)^2 = \dot{\phi}^2, \quad \nabla^2\phi = \ddot{\phi} + \sum_i H_i\dot{\phi}, \quad \nabla_0\nabla^0\phi = \ddot{\phi},$$

$$\nabla_i\nabla^j\phi = H_i\dot{\phi}\delta_i^j. \tag{C.14}$$

The dilaton eq. (C.9) gives then

$$2\ddot{\phi} + 2\dot{\phi}\sum_i H_i - \dot{\phi}^2 - \sum_i \left(2\dot{H}_i + H_i^2\right) - \left(\sum_i H_i\right)^2 = 0. \tag{C.15}$$

The (00) component of the eq. (C.5) gives

$$\dot{\phi}^2 - 2\dot{\phi}\sum_i H_i - \sum_i H_i^2 + \left(\sum_i H_i\right)^2 = e^{\phi}\rho. \tag{C.16}$$

The diagonal, spatial components (i,i) of eq. (C.5) (the off-diagonal components are trivially satisfied) give

$$\dot{H}_i + H_i\sum_k H_k - H_i\dot{\phi} - \frac{1}{2}\sum_i \left(2\dot{H}_i + H_i^2\right) - \frac{1}{2}\left(\sum_i H_i\right)^2$$
$$-\frac{1}{2}\dot{\phi}^2 + \ddot{\phi} + \dot{\phi}\sum_i H_i = \frac{1}{2}e^{\phi}p_i. \tag{C.17}$$

The last five terms on the left-hand side add to zero because of the dilaton equation (C.15), and the spatial equations reduce to

$$\dot{H}_i - H_i\left(\dot{\phi} - \sum_k H_k\right) = \frac{1}{2}e^{\phi}p_i. \tag{C.18}$$

The above equations are clearly invariant under time-reversal, $t \to -t$. In order to make explicit also their duality invariance, let us introduce again the shifted dilaton (see eq. (B.7)), such that

$$e^{\overline{\phi}} = e^{\phi}/\sqrt{-g}, \qquad \dot{\overline{\phi}} = \dot{\phi} - \sum_i H_i, \tag{C.19}$$

and define

$$\overline{\rho} = \rho\sqrt{-g} = \rho\prod_i a_i, \qquad \overline{p} = p\sqrt{-g} = p\prod_i a_i. \tag{C.20}$$

In terms of these variables, the time and space equations (C.16,C.18), and the dilaton equation (C.15), become, respectively:

$$\dot{\overline{\phi}}^2 - \sum_i H_i^2 = e^{\overline{\phi}}\overline{\rho}, \tag{C.21}$$

$$\dot{H}_i - H_i\dot{\overline{\phi}} = \frac{1}{2}e^{\overline{\phi}}\overline{p}_i, \tag{C.22}$$

$$2\ddot{\overline{\phi}} - \dot{\overline{\phi}}^2 - \sum_i H_i^2 = 0. \tag{C.23}$$

They are explicitly invariant under the scale-factor duality transformation:

$$a_i \to a_i^{-1}, \qquad \overline{\phi} \to \overline{\phi}, \qquad \overline{\rho} \to \overline{\rho}, \qquad \overline{p} \to -\overline{p}. \tag{C.24}$$

which implies, for a perfect fluid source, a "reflection" of the equation of state, $\gamma = p/\rho = \overline{p}/\overline{\rho} \to -\overline{p}/\overline{\rho} = -\gamma$ (see the first paper in Ref. [8]). A general $O(d,d)$ transformation changes however the equation of state in a more drastic way (see [12]), introducing also shear and bulk viscosity.

The above $(d+2)$ equations are a system of independent equations for the $(d+2)$ variables $\{a_i, \phi, \rho\}$. Their combination implies the usual covariant conservation of the energy density. By differentiating eq. (C.21), and using (C.22,C.23) to eliminate $\dot{H}_i$, $\dot{\overline{\phi}}$, respectively, we get in fact

$$\dot{\overline{\rho}} + \sum_i H_i \overline{p}_i = 0, \tag{C.25}$$

which, using the definitions (C.20), is equivalent to

$$\dot{\rho} + \sum_i H_i \left(\rho + p_i\right) = 0. \tag{C.26}$$

In order to obtain exact solutions, it is convenient to include this energy conservation equation in the full system of independent equations.

In these lectures I will present only a particular example of matter-dominated solution by considering a d-dimensional, isotropic background characterized by a power-law evolution,

$$a \sim t^{\alpha}, \qquad \overline{\phi} \sim -\beta \ln t, \qquad p = \gamma\rho. \tag{C.27}$$

We use (C.21,C.23, C.25) as independent equations. The integration of eq. (C.25) gives immediately

$$\overline{\rho} = \rho_0 a^{-d\gamma}; \tag{C.28}$$

eq. (C.21) is then satisfied provided

$$d\gamma\alpha + \beta = 2. \tag{C.29}$$

Finally, eq. (C.23) provides the constraint

$$2\beta - \beta^2 - d\alpha^2 = 0. \tag{C.30}$$

We have then a system of two equations for the two parameters α, β (note that, if α is a solution for a given γ, then also $-\alpha$ is a solution, associated to $-\gamma$). We have in general two solutions. The trivial flat space solution $\beta = 2, \alpha = 0$, corresponds to dust matter ($\gamma = 0$) according to eq. (C.22). For $\gamma \neq 0$ we obtain instead

$$\alpha = \frac{2\gamma}{1 + d\gamma^2}, \qquad \beta = \frac{2}{1 + d\gamma^2}, \tag{C.31}$$

which fixes the time evolution of a and $\overline{\phi}$:

$$a \sim t^{\frac{2\gamma}{1+d\gamma^2}}, \qquad \overline{\phi} = -\frac{2}{1 + d\gamma^2} \ln t, \tag{C.32}$$

and also of the more conventional variables ρ, ϕ:

$$\rho = \overline{\rho} a^{-d} = \rho_0 a^{-d(1+\gamma)}, \qquad \phi = \overline{\phi} + d \ln a = \frac{2(d\gamma - 1)}{1 + d\gamma^2} \ln t. \tag{C.33}$$

This particular solution reproduces the small curvature limit of the general solution with perfect fluid sources (see the last two papers of Ref. [8]), sufficiently far from the singularity. Like in the vacuum solution (B.20) there are four branches, related by time-reversal and by the duality transformation (C.24), and characterized by the scale factors

$$a_{\pm}(\pm t) \sim (\pm t)^{\pm 2\gamma/(1+d\gamma^2)}. \tag{C.34}$$

The duality transformation that preserves $\overline{\phi}$ and $\overline{\rho}$, and inverts the scale factor, in this case is simply represented by the transformation $\gamma \to -\gamma$. Consider for instance the standard radiation-dominated solution, corresponding to $d = 3$, $\gamma = 1/3$, and $t > 0$, and associated to a constant dilaton, according to eq. (C.33). A duality transformation gives a new solution with $\gamma = -1/3$, namely (from (C.32,C.33):

$$a \sim t^{-1/2}, \qquad \rho \sim a^{-2}, \qquad \phi \sim -3 \ln t. \tag{C.35}$$

By performing an additional time reflection we then obtain the pre-big bang solution "dual to radiation", already reported in eq. (2.16).

References

[1] S. Weinberg, *Gravitation and cosmology* (Wiley, New York, 1972).

[2] E. W. Kolb and M. S. Turner, *The Early Universe*, (Addison Wesley, Redwood City, Ca, 1990); A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, Chur, Switzerland, 1990).

[3] A. Vilenkin, Phys. Rev. D **46**, 2355 (1992); A. Borde, A. Vilenkin, Phys. Rev. Lett. **72**, 3305 (1994).

[4] A. Guth, *The inflationary Universe*, (Vintage, London, 1998), p. 271.

[5] A. Vilenkin, Phys. Rev. D **30**, 509 (1984); A. D. Linde, Sov. Phys. JEPT **60**, 211 (1984); Y. Zel'dovich and A. A. Starobinski, Sov. Astron. Lett. **10**, 135 (1984); V. A. Rubakov, Phys. Lett. B **148**, 280 (1984).

[6] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983); S. W. Hawking, Nucl. Phys. B **239**, 257 (1984); S. W. Hawking and D. N. Page, Nucl. Phys. B **264**, 185 (1986).

[7] M. Gasperini, J. Maharana and G. Veneziano, Nucl. Phys. B **472**, 394 (1996); M. Gasperini and G. Veneziano, Gen. Rel. Grav. **28**, 1301 (1996); M. Gasperini, Int. J. Mod. Phys. A **13**, 4779 (1998); M. Cavagliá and C. Ungarelli, Class. Quantum Grav **16**, 1401 (1999).

[8] G. Veneziano, Phys. Lett. B **265**, 287 (1991); M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993); Mod. Phys. Lett. A **8**, 3701 (1993); Phys. Rev. D **50**, 2519 (1994). An updated collections of papers on the pre-big bang scenario is available at `http://www.to.infn.it/~gasperin`.

[9] A. A. Tseytlin, Mod. Phys. Lett. A **6**, 1721 (1991).

[10] A. Sen, Phys. Lett. B **271**, 295 (1991) ; S. F. Hassan and A. Sen, Nucl. Phys. B **375**, 103 (1992).

[11] K. A. Meissner and G. Veneziano, Phys. Lett. B **267**, 33 (1991); Mod. Phys. Lett. A **6**, 3397 (1991).

[12] M. Gasperini and G. Veneziano, Phys. Lett. B **277**, 256 (1992).

[13] V. Kaplunovsky, Phys. Rev. Lett. **55**, 1036 (1985).

[14] R. Brustein and G. Veneziano, Phys. Lett. B **329**, 429 (1994); N. Kaloper, R. Madden and K. A. Olive, Nucl. Phys. B **452**, 677 (1995); R. Easther, K. Maeda and D. Wands, Phys. Rev. D **53**, 4247 (1996).
[15] M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B **272**, 277 (1991).
[16] M. Gasperini, Mod. Phys. Lett. A **14**, 1059 (1999).
[17] D. Shadev, Phys. Lett. B **317**, 155 (1984); R. B. Abbott, S. M. Barr and S. D. Ellis, Phys. Rev. D **30**, 720 (1984); E. W. Kolb, D. Lindley and D. Seckel, Phys. Rev. D **30**, 1205 (1984); F. Lucchin and S. Matarrese, Phys. Lett. B **164**, 282 (1985).
[18] R. Brandenberger and C. Vafa, Nucl. Phys. B **316**, 391 (1989); A. A. Tseytlin and C. Vafa, Nucl. Phys. B **372**, 443 (1992).
[19] M. Gasperini, N. Sánchez and G. Veneziano, Nucl. Phys. B **364**, 365 (1991).
[20] E. R. Caianiello, M. Gasperini and G. Scarpetta, Class. Quantum Grav. **8**, 659 (1991); M. Gasperini, in Proc. of the First Workshop on *"Advances in Theoretical Physics"* (Vietri, October 1990), edited by E. R. Caianiello (World Scientific, Singapore, 1991) p. 77.
[21] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B **494**, 315 (1997).
[22] W. Rindler, Mon. Not. R. Astr. Soc. **116**, 6 (1956).
[23] M. Gasperini, *Inflation and initial conditions in the pre-big bang scenario*, BA-TH/99-332, gr-qc/9902060.
[24] M. S. Turner and E. J. Weinberg, Phys. Rev. D **56**, 4604 (1997).
[25] N. Kaloper, A. Linde and R. Bousso, Phys. Rev. D **59**, 043508 (1999).
[26] M. Maggiore and R. Sturani, Phys. Lett. B **415**, 335 (1997).
[27] A. Buonanno, T. Damour and G. Veneziano, Nucl. Phys. B **543**, 275 (1999).
[28] D. Clancy, J. E. Lidsey and R. Tavakol, Phys. Rev. D **59**, 063511 (1999).
[29] I. Antoniadis, J. Rizos and K. Tamvakis, Nucl. Phys. B **415**, 497 (1994); S. J. Rey, Phys. Rev. Lett. **77**, 1929 (1996); M. Gasperini and G. Veneziano, Phys. Lett. B **387**, 715 (1996); R. Brustein and R. Madden, Phys. Lett. B. **410**, 110 (1997); Phys. Rev. D **57**, 712 (1998); R. Brandenberger, R. Easther and J. Maia, JHEP **9808**, 007 (1998); M. Maggiore and A. Riotto, *D-branes and cosmology*, CERN-TH/98-361, hep-th/9911089; S. Foffa, M. Maggiore and R. Sturani, *Loop corrections and graceful exit in string cosmology*, IFUP-TH 8/99, hep-th/9903008.
[30] M. Gasperini and M. Giovannini, Phys. Lett. B **301**, 334 (1993); Class. Quantum Grav. **10**, L133 (1993); G. Veneziano, *Pre-bangian origin of our entropy and time arrow*, CERN-TH/99-32, hep-th/9902126.
[31] M. Gasperini and M. Giovannini, Phys. Lett. B **282**, 36 (1992); Phys. Rev. D **47**, 1519 (19934).
[32] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B **361**, 45 (1995).
[33] R. Brustein, M. Gasperini and G. Veneziano, Phys. Rev D **55**, 3882 (1997).
[34] See for instance P. Fritschel, in *Second Amaldi Conference on "Gravitational Waves"* (CERN, July 1997), edited by E. Coccia et al. (World Scientific, Singapore, 1998), p. 74.
[35] See for instance A. Brillet, in *Second Amaldi Conference on "Gravitational Waves"* (CERN, July 1997), edited by E. Coccia et al. (World Scientific, Singapore, 1998), p. 86.
[36] See for instance J. Hough, in *Second Amaldi Conference on "Gravitational Waves"* (CERN, July 1997), edited by E. Coccia et al. (World Scientific, Singapore, 1998), p. 97; see also P. Bender, these Proceedings.
[37] R. Durrer, M . Gasperini, M. Sakellariadou and G. Veneziano, Phys. Lett. B **436**, 66 (1998); Phys. Rev. D **59**, 043511 (1999).
[38] M. Gasperini and G. Veneziano, Phys. Rev. D **59**, 043503 (1999).
[39] C. L. Bennet et al., Astrophys. J. **430**, 423 (1994).
[40] E. J. Copeland, R. Easther and D. Wands, Phys. Rev. D **56**, 874 (1997); E. J. Copeland, J. Lidsey and D. Wands, Nucl. Phys. B **506**, 407 (1997); A. Buonanno, K.A. Meissner, C. Ungarelli and G. Veneziano, JHEP01, 004 (1998).
[41] A. Melchiorri, F. Vernizzi, R. Durrer and G. Veneziano, *Cosmic microwave anisotropies and extra dimensions in string cosmology*, astro-ph/9905327.

[42] See for instance the web site: `http://map.gsfc.nasa.gov`.
[43] See for instance the web site: `http://astro.estec.esa.nl/SA-general/Projects/Planck`.
[44] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75**, 3796 (1995).
[45] M. S. Turner and L. M. Widrow, Phys. Rev. D37, 2743 (1988).
[46] M. Gasperini, Phys. Lett. B **327**, 314 (1994); M. Gasperini and G. Veneziano, Phys. Rev. D **50**, 2519 (1994).
[47] T. Damour and A. M. Polyakov, Nucl. Phys. B **423**, 532 (1994).
[48] M. Gasperini, in *Proc. of the 12th It. Conference on "General Relativity and Gravitational Physics"* (Rome, September 1997), edited by M. Bassan et al. (World Scientific, Singapore, 1997), p. 85.
[49] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
[50] J. Hwang, Ap. J. 375, 443 (1991).
[51] R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov and G. Veneziano, Phys. Rev. D **51**, 6744 (1995).
[52] G. F. R. Ellis and S. Bruni, Phys. Rev.D **40**, 1804 (1989).
[53] R. B. Abbott, B. Bednarz and S. D. Ellis, Phys. Rev. D33, 2147 (1986).
[54] M. Gasperini, Phys. Rev. D **56**, 4815 (1997).
[55] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. D **48**, R439 (1993).
[56] L. P. Grishchuk and Y. V. Sidorov, Class. Quantum Grav. **6**, L161 (1989); Phys. Rev. D **42**, 3413 (1990); L. P. Grishchuk, Class. Quantum Grav. **10**, 2449 (1993).
[57] R. Brandenberger, V. Mukhanov and T. Prokopec, Phys. Rev. Lett. **69**, 3606 (1992); M. Gasperini and M. Giovannini, Phys. Lett. B **301**, 334 (1993); Class. Quantum Grav. **10**, L133 (1993).
[58] L. P. Grishchuk, Sov. Phys. JETP **40**, 409 (1975).
[59] M. Gasperini, in *String theory in curved spacetimes*, edited by N. Sánchez (World Scientific, Singapore, 1998), p. 333.
[60] M. Abramowitz and I. A. Stegun, *"Handbook of mathematical functions"* (Dover, New, York, 1972).
[61] V. F. Mukhanov and N. Deruelle, Phys. Rev.D **52**, 5549 (1995).
[62] B. Allen, in *Proc. of the Les Houches School on " Astrophysical sources of gravitational waves"*, edited by J. A. Mark and J. P. Lasota (Cambridge, 1997), p. 373; B. Allen and J. D. Romano, *Detecting a stochastic background of gravitational radiation*, gr-qc/9710117.
[63] B. Allen, E. E. Flanagan and M. A. Papa, *Is the squeezing of relic gravitational waves produced by inflation detectable?*, gr-qc/9906054.
[64] L. P. Grishchuk, *The detectability of relic (squeezed) gravitational waves by laser interferometers*, gr-qc/9801055.
[65] M. Gasperini, in *Proc. of the Second Paris Cosmology Colloquium* (Paris, 1994), edited by H. J. De Vega and N. Sanchez (World Scientific, Singapore, 1995), p. 429; in *"String gravity and physics at the Planck energy scale"*, ed. by N. Sanchez and A. Zichichi (Kluwer Acad. Publ., Dordrecht, 1996), p. 305; in *Second Amaldi Conference on "Gravitational Waves"* (CERN, July 1997), edited by E. Coccia et al. (World Scientific, Singapore, 1998), p. 62.
[66] M. Maggiore, *High energy physics with gravitational wave experiments*, gr-qc/9803028.
[67] L. M. Krauss and M. White, Phys. Rev. Lett. **69**, 869 (1992).
[68] V. Kaspi, J. Taylor and M. Ryba, Ap. J. **428**, 713 (1994).
[69] V. F. Schwarztmann, JETP Letters **9**, 184 (1969).
[70] E. J. Copeland, A. R. Liddle, J. E. Lidsey and D. Wands, Phys. Rev. D **58**, 63508 (1998); Gen. Rel. Grav. **30**, 1711 (1998).
[71] M. Galluccio, F. Litterio and F. Occhionero, Phys. Rev. Lett. **79**, 970 (1997).
[72] A. Buonanno, M. Maggiore and C. Ungarelli, Phys. Rev. D55, 3330 (1997).
[73] R. R. Caldwell and B. Allen, Phys. Rev. D45, 3447 (1992); R. A. Battye and E. P. S. Shellard, Phys. Rev. D53, 1811 (1996).
[74] M. S. Turner and F. Wilczek, Phys. Rev. Lett. 65, 3080 (1990).

[75] S. Y. Khlebnikov and I. I. Tkachev, Phys. Rev. D **56**, 653 (1997); B. Bassett, Phys. Rev. D **56**, 3439 (1997).
[76] P. J. E. Peebles and A. Vilenkin, Phys. Rev. D **59**, 063505 (1999); M. Giovannini, *Production and detection of relic gravitons in quintessential inflation models*, astro-ph/9903004.
[77] P. Astone, in *Second Amaldi Conference on "Gravitational Waves"* (CERN, July 1997), edited by E. Coccia et al. (World Scientific, Singapore, 1998), p. 192.
[78] G. Prodi, talk given at the *Second Int. Conference "Around VIRGO"* (Tirrenia, Pisa, September 1998); see also M. Cerdonio, these Proceedings.
[79] F. Pegoraro, E. Picasso and L. Radicati, J. Phys. A **11**, 1949 (19780.
[80] E. Picasso, private communication.
[81] See for instance J. A. Lobo, in *Second Amaldi Conference on "Gravitational Waves"* (CERN, July 1997), edited by E. Coccia et al. (World Scientific, Singapore, 1998), p. 168.
[82] C. Ungarelli and A. Vecchio, in preparation.
[83] M. B. Green, J. Schwartz and E. Witten, *Superstring theory* (Cambridge University Press, Cambridge, Ma, 1987).
[84] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B **293**, 385 (1987).
[85] T. R. Taylor and G. Veneziano, Phys. Lett. B **213**, 459 (1988).
[86] See for instance I. Antoniadis and B. Pioline, *Large dimensions and string physics at a TeV*, hep-ph/9906480 (CPHT-PC720-0699, March 1999).
[87] A. Buonanno, M. Gasperini, M. Maggiore and C. Ungarelli, Class. Quantum Grav. **14**, L97 (1997).
[88] M. Gasperini and M. Giovannini, Phys. Lett. B **287**, 56 (1992).
[89] G. W. Gibbons and S. Hawking, Phys. Rev. D **15**, 2752 (1977).

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String theory and primordial cosmology

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SUMMARY.

String cosmology aims at providing a reliable description of the very early Universe in the regime where standard-model physics is no longer appropriate, and where we can safely apply the basic ingredients of superstring models such as dilatonic and axionic forces, duality symmetries, winding modes, limiting sizes and curvatures, higher-dimensional interactions among elementary extended object. The sought target is that of resolving (or at least alleviating) the big problems of standard and inflationary cosmology like the spacetime singularity, the physics of the trans-Planckian regime, the initial condition for inflation, and so on.

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Chapter 1

String theory and primordial cosmology

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SUMMARY. String cosmology aims at providing a reliable description of the very early Universe in the regime where standard-model physics is no longer appropriate, and where we can safely apply the basic ingredients of superstring models such as dilatonic and axionic forces, duality symmetries, winding modes, limiting sizes and curvatures, higher-dimensional interactions among elementary extended object. The sought target is that of resolving (or at least alleviating) the big problems of standard and inflationary cosmology like the spacetime singularity, the physics of the trans-Planckian regime, the initial condition for inflation, and so on.

1.1 The standard "Big Bang" cosmology

In the second half of the last century the theoretical and observational study of our Universe, grounded on one hand on the Einstein theory of general relativity, and on the other hand on astronomical observations of every increasing precision, has led to the formulation (and to the subsequent completion) of the so-called *standard cosmological model* (see e.g. [1, 2, 3]).

Such a model – like every physical model – is based on various assumptions. We should mention, in particular:

1. The assumption that the large-scale spacetime geometry can be foliated by a class of three-dimensional space-like hypersurfaces which are exactly homogeneous and isotropic.

2. The assumption that the matter and the radiation filling our Universe behave exactly as a perfect fluid with negligible friction and viscosity terms.

3. The assumption that the radiation is in thermal equilibrium.

4. The assumption that the dominant source of gravity, on cosmological scales, is the so called *dark matter* component of the cosmological fluid (invisible, up to now, to all attempted detection procedures of nongravitational type); and so on.

Using such assumptions, the standard cosmological model has obtained a long and impressive series of successes and experimental confirmations, such as:

1. The geometric interpretation of the apparent recessional velocity of distant light sources, together with a precise theoretical formalization of the empirical Hubble law.

2. The prediction of a relic background of thermal radiation.

3. The explanation of the process of genesis of the light elements and of the other *building blocks* of our present macroscopic world (like the processes of nucleosynhesis and baryogenesis); and so on.

In spite of these important achievements the standard cosmological model was put in trouble when, in the 1980s, the scientific community started to investigate the problem of the origin of the observed galactic structures, and of the small (but finite) inhomogeneity fluctuations presents in the temperature T of the relic background radiation ($\Delta T/T \sim 10^{-5}$). How did originate the temperature inhomogeneities $\Delta T/T$ and, especially, the matter inhomogeneities $\Delta\rho/\rho$ which are at the grounds of the concentration and subsequent growth of the cosmic aggregates (cluster of galaxies, stars and planets) that we presently observe? No temperature fluctuation and density fluctuation should exist, on macroscopic scales, if our Universe would be exactly homogeneous and isotropic as required by the standard cosmological model.

This problem was solved by assuming that the standard cosmological model has to be modified, at some very early epoch, by the introduction of a cosmological phase – called *inflation* – characterized by an accelerated expansion rate [4, 5, 6]. During such a primordial inflationary phase the three-dimensional spatial sections of our Universe underwent a gigantic (almost exponential) growth of proper volume in few units of the Hubbe-time parameter (see see.g. [3, 7, 8]). This process was able to amplify the microscopic quantum fluctuations of the matter fields (and of the geometry), thus producing the macroscopic inhomogeneities required for the formation of the matter structures and of the temperature anisotropies we observe today (see see.g. [8, 9, 10]).

A phase of inflationary evolution like that proposed above, however, cannot be extended back in time to infinity (or, to use the standard terminology, cannot be *past eternal* [11, 12, 13]). If we go back in time to sufficiently earlier epochs we find that the inflationary phase of the standard model has a beginning at

a precise instant of time. Before that time, the Universe was in an extremely hot, dense and curved primordial state – an ultimate concentrate of matter and radiation at extremely high energy and temperature.

This means, in other words, that before starting inflating the Universe was quite close to the so-called *big bang* epoch, namely to the epoch of the huge cosmic explosion which – according to the standard model, even including the inflationary phase – gave rise to the matter and energy species we observe today, and was at the origin of the spacetime itself.

In fact, the big bang epoch of the standard model corresponds (strictly speaking) to a mathematical singularity where the energy density and the spacetime curvature blow up to infinity. We can thus say that to the question *How did the Universe begin?*, the standard cosmological model provides the answer: *The Universe was born from the initial big bang singularity.*

1.1.1 Validity restrictions of the standard cosmological model

It is well known that standard cosmology is based on the Einstein theory of general relativity, which is a *relativistic* theory of gravity, but *not* a *quantum* theory. Hence, like all classical theories, general relativity has a limited validity range. Because of those limits the standard cosmological model cannot be extrapolated to physical regimes where the energy and the spacetime curvature are too high: this prevents taking too seriously the predictions of such a model about the initial singularity.

We should recall, in fact, that a classical model is valid until the corresponding action $S = Et$ is much larger than the elementary *quantum of action* (or Planck's quantum) h. If we take a cosmological patch of the size given by the Hubble radius c/H, we can then estimate the total involved energy E by multiplying the energy density ρ of the gravitational sources by the spatial volume $(c/H)^3$, containing the contribution of all observable matter and radiation at a given time t. The typical cosmological time scale, on the other hand, is provided by the Hubble time H^{-1}, and the energy density ρ is related to the Hubble time by the Einstein equations, which imply (modulo numerical factors of order one) $\rho = c^2H^2/G$, where G is the Newton gravitational constant. By imposing the condition $Et \gg h$ we then find that the standard cosmological model may give a reliable (classical) description of the Universe provided that

$$\frac{c^5}{GH^2} \gg h. \tag{1.1}$$

(This condition, in units $h = c = 1$, can also be rephrased as $H \ll M_{\rm P}$, where $M_{\rm P} = (hc/G)^{1/2}$ is the Planck mass).

The parameters C, G, h appearing in the above equation are constant, while the Hubble parameter H is closely related to the spacetime curvature and is time dependent, $H = H(t)$. According to the standard model, in particular, H grows as we go back in time, and diverges at the time of the big-bang singularity

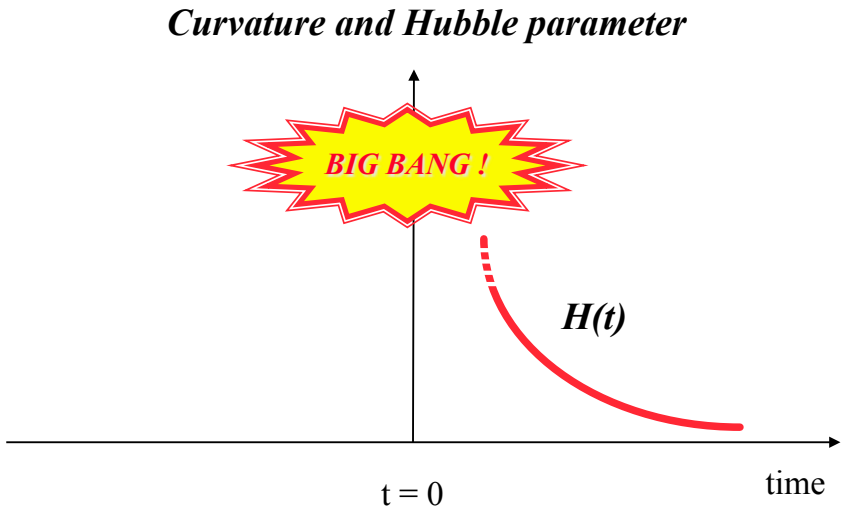

Figure 1.1: *According to the standard cosmological model, the spacetime curvature and the associated Hubble parameter $H(t)$ undergo an unbounded growth as we go back in time, and blow up at the time $t = 0$ of the initial "big bang" singularity.*

(Fig. 1.1). Correspondingly, the ratio c^5/GH^2 decreases and goes to zero at the singularity. Hence, before reaching the big bang epoch we necessarily enter the regime where the condition (1.1) is violated, and the standard cosmological model is no longer valid.

In order to provide a reliable description of the primordial Universe we should thus use a more general approach, based on a theory able to describe gravity also in the quantum regime. A possible candidate for this theory, which is complete, consistent at all energy scales and, besides gravity, also incorporates all fundamental interactions, is the so-called *theory of strings* (see, e.g., [14, 15, 16]).

1.2 String theory

The name of this theory is due to the fact that it proposes a model where the fundamental *building blocks* of our physical description of nature are one-dimensional extended object (elementary "strings", indeed), instead of elementary particles. Such strings can be open (of finite length), or closed, and the spectrum of states associated to their vibration modes can reproduce the particle states of the gravitational interaction and of all the other fundamental (electromagnetic, strong and weak) interactions.

In addition, if the string model is appropriately *supersymmetrized* – namely, if we add to each bosonic degree of freedom a corresponding fermionic partner

– we arrive at the so-called theory of *superstrings*. This model potentially describes not only all interaction fields, but also their elementary sources (quarks and leptons), and thus all possible species and states of matter [14, 15].

But there is more. A basic property of string theory – probably the most revolutionary property, comparing with the other theories – is the property of determining not only the possible form of the interaction terms (which is also done by the usual gauge theories, through the minimal coupling procedure), but also the form of the free-field (kinetic) terms (which in the other theories is always left, to some extent, arbitrary). Indeed, string theory satisfies a new symmetry (called *conformal symmetry*) which rigidly prescribes the allowed free-field dynamics, at any given order of the chosen perturbative expansion [14, 15].

Quantizing a string, and imposing that the conformal symmetry is left unbroken by the quantum corrections (i.e. imposing the absence of *conformal anomalies*), one finds, in fact, that – to lowest order – the electromagnetic field *must* satisfy the Maxwell equations, the gravitational field *must* satisfy the Einstein equations, the spinor fields *must* satisfy the Dirac equations, and so on. All field equations, laboriously discovered in the past centuries through the theoretical elaboration of a large amount of empirical data, can be simply *predicted* by string theory even in the absence of any experimental input!

Finally, as already stressed, string theory is valid for all interactions also in the quantum regime, and can thus be used at arbitrarily high energy scales. In particular, unlike general relativity, can be applied to describe the Universe at epochs arbitrarily near to the big bang epoch. In such a limiting high-energy regime the equations we obtain from string theory are different, in general, from the corresponding field-theory equations, and thus it makes sense to ask the question *What's new from string theory about cosmology?*

In particular, *What's new about the very early epochs at the beginning of the Universe?*

1.3 String cosmology

There are, in particular, two aspects of string theory which can play a relevant role in the formulation of a consistent cosmological scenario.

The first one concerns the so-called *dual symmetry*, typical of one-dimensional extended objects. If such a symmetry is respected (even at the approximate level) by the gravitational dynamics on cosmological scales, then any cosmological phase occurring at $t > 0$, and characterized by a decreasing Hubble parameter H (hence, decreasing curvature), must be associated to a *dual* partner phase, defined at $t < 0$ and characterized by growing H (see [17] for a nontechnical illustration of this duality symmetry). It follows in particular that the present cosmological phase, subsequent to the big bang epoch and well described by the standard model, must be preceded in time by an almost specularly symmetric phase occurring *before the big bang* (Fig. 1.2). Such a duality symmetry should also leave an imprint on the properties of the cosmological perturbations [18].

In Fig. 1.2 both phases are characterized by a curvature (and a Hubble

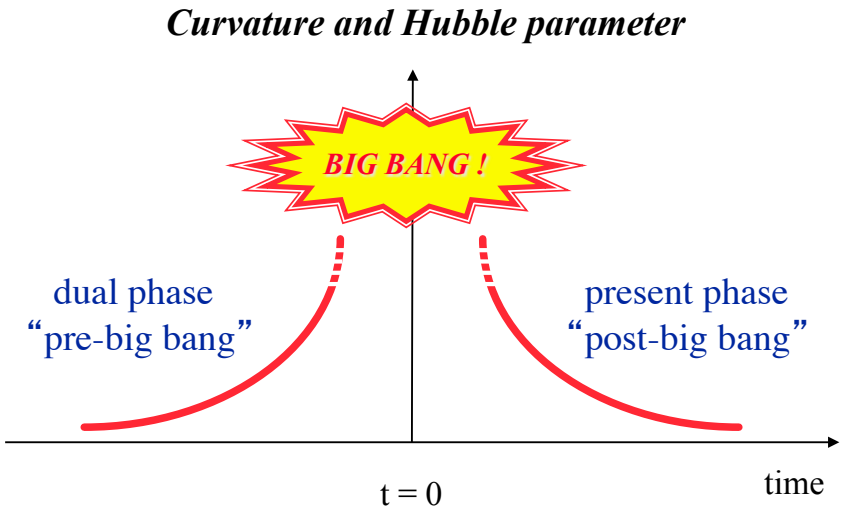

Figure 1.2: *The standard cosmological phase, of "post-big bang" type, is preceded by a (string-theory) dual phase of "pre-big bang" type.*

parameter) which diverges as t goes to zero. If that would be the case, then the two branches of the cosmological evolution would be causally disconnected by a spacetime singularity, with no chances of merging together into a single coherent model of spacetime evolution. It is here, however, that comes into play another crucial aspect of string theory.

String theory is indeed characterized by a fundamental length $\lambda_{\rm s}$, which is a constant parameter of the string action and which controls the typical size of a quantized string. The physical role played by $\lambda_{\rm s}$ is very similar to the role played by the Bohr radius for the atom, which represents the minimal allowed size of the quantum electronic orbitals. The numerical value, however, is quite different: we may expect, in fact, $\lambda_{\rm s} \sim 10^{-33}$ cm (i.e. a value of $\lambda_{\rm s}$ which is about 10 times that of the Planck length $\lambda_{\rm P} = h/M_{\rm P}c^2$), in order that string theory may include a realistic description of all fundamental interactions (different values of λ_s are possible in the presence of large extra dimensions, see below).

Aside from the particular numerical value of $\lambda_{\rm s}$, what is important, in our context, is that proper distances and sizes smaller than $\lambda_{\rm s}$ have no physical meaning in a string-inspired model. It follows that, in a string-cosmology context, the Hubble radius c/H has to be constrained by the condition $c/H \gtrsim \lambda_{\rm s}$. Since the Hubble radius is directly related to the inverse of the spacetime curvature, we can deduce that the curvature cannot blow up to infinity, because of the constraint $H \lesssim c/\lambda_{\rm s}$. Hence, when a given spacetime region has reached the limiting value $H \approx c/\lambda_{\rm s}$, its geometrical state can only evolve in two ways: it can either stabilize at such a maximum value, or start decaying towards lower curvature states after a bounce induced by appropriate "stringy" effects (see

e.g. [19]).

In such a context, the big bang singularity predicted by the standard model and sharply localized at a a given epoch (say, $t = 0$), has thus to be replaced by an extended phase of very high (but finite) maximal curvature: the so-called *string phase* (see e.g. [20, 21]). By combining the existence of the dual symmetry and of a minimal length scale, a string-based model can thus complete the standard cosmological scenario by removing the curvature singularity and extending the physical description of the Universe back in time, beyond the big bang, to infinity. The *big bang era* is still there, but it is deprived of the standard role of initial singularity: it corresponds, instead, to the epoch marking the transition between the growing curvature and the decreasing curvature regime (Fig. 1.3).

Within such a cosmological scenario (first presented in detail in [22]) the initial cosmological state is no longer localized at $t = 0$, but it is moved to the limit $t \to -\infty$, and corresponds to an asymptotic state usually called the *string perturbative vacuum*. Such a new initial state, as illustrated in Fig. 1.3, turns out to be a sort of specularly symmetric version of the final state that would be reached in the asymptotic future by a Universe which keeps expanding for ever according to the standard cosmological dynamics. Namely, a flat, empty and cold initial state, drastically different from the initial hot, explosive state, extremely curved and concentrated, proposed by the standard scenario.

There is, however, a possible asymmetry between the initial and final state of the above string-cosmology model, due to the coupling strength of the fundamental interactions: such a coupling tends to zero as $t \to -\infty$, while it may become very strong in the opposite limit, if not appropriately stabilized (see e.g. [20, 21]). This growth of the coupling can be accompanied, in principle, also by a large amount of entropy production (see e.g. [23]).

1.4 A higher-dimensional Universe

String theory, which is at the grounds of the cosmological scenario described in the previous section, can be consistently formulated only in the context of a higher dimensional spacetime manifold.

In fact, in order to consistently quantize a bosonic string without introducing *ghosts* (states of negative norm), and without violating the Lorentz symmetry, one must introduce a generalized spacetime manifold with 26 dimensions (see e.g. [14, 15]). In this way, however, one obtains a model which has still a pathology, as it contains *tachyons* (states of imaginary mass), which we believe should be absent in any realistic physical model.

In order to eliminates the tachyons, we can generalize the bosonic string model by adding fermion states and considering the so-called *superymmetric* string models, or *superstring* models. In that case, a consistent quantization requires 10 spacetime dimensions, and the number of total dimensions is to be increased up to 11 (with one time-like and 10 space-like dimensions) if we require that the five possible types of superstrings may be connected by duality

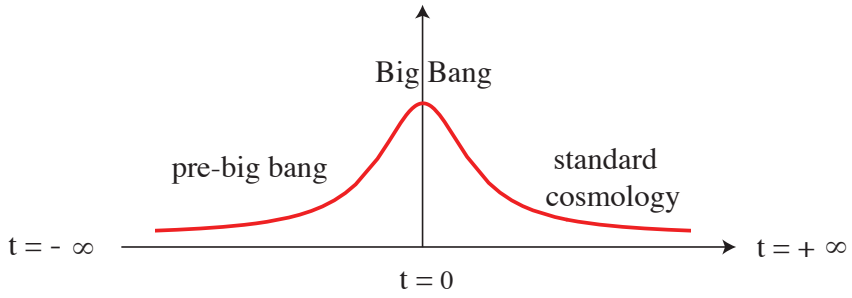

Figure 1.3: *Time evolution of the curvature scale and of the energy density in a typical example of string-cosmology scenario. The big bang epoch does not correspond to a singularity (like in the standard model) but to a phase of maximal, finite curvature. The Universe evolves starting from a flat, cold and empty state called the "string perturbative vacuum", asymptotically localized at* $t = -\infty$.

transformations, and may represent various weak-coupling regimes of a more fundamental theory, called *M-theory* [24] (see also [25]).

Hence, whatever string model is assumed to apply, it is clear that the associated string cosmology scenario must be referred to a higher dimensional Universe. On the other hand, all present phenomenological experience (including the most sensitive high-energy experiments) points at a world with one time-like and *only three* space-like dimensions. We are thus naturally led to the following questions:

If string theory is correct, and the Universe in which we live has a number $d > 3$ *of spatial dimensions, why our experience is only limited to a three-dimensional space? why we cannot detect the additional "extra" dimensions? what happened to those dimensions, if they really exist?*

There are at least two possible answers to the above questions.

There is an *old-fashioned* answer – which, for a long time, has been also the only possible answer to the previous questions – dating back to the so-called Kaluza-Klein model, formulated at the beginning of the last century [26, 27] in the context of a higher dimensional version of general relativity. According to such model, we cannot detect the extra dimensions simply because such dimensions are compactified on length scales of extremely small size (hence, they need extremely high energies in order to be experimentally resolved).

We can take, as a simple example, a long and very thin cylinder. A cylinder is a two-dimensional object but, if it is observed from a distance much larger than its radius, it may appear (in all respects) as being one-dimensional, extended in length but deprived of any sensible thickness. In the same way the spatial extension of our Universe could be largely asymmetric, with three spa-

tial dimensions macroscopically expanded on a large scale, while all the other dimensions *rolled up* in a highly compact way, and confined on a very small length scale – of order (for instance) of λ_s. If we do not have a sufficiently powerful instrument, able to resolve the required (very tiny) distance scales, we will always observe three spatial dimensions only.

Very recently, however, a new possible answer to the dimensionality problem has been suggested by theoretical studies mainly performed in the second half of the 1990s, and closely related to particular string-model configurations, called *branes* [25]. Such a new answer states that we cannot "see" the extra dimensions simply because the fundamental interactions propagate only along three spatial dimensions. All instruments we use to explore the world around us (starting from our eyes up to the more powerful and sophisticated technological tools) have indeed a working mechanism based on the fundamental (electromagnetic, nuclear, and so on) interactions. If such interactions are living only on a restricted subspace of the full spacetime manifold (like, for instance, waves which propagate on the surface of a pond, and not in the direction orthogonal to the pond surface), then the extra dimensions are hidden to our direct experience, even if they are largely (or infinitely) extended.

This second possible answer to the dimensionality problem has suggested new, interesting types of cosmological models, formulated in the context of the so-called *brane-world* scenario (see, e.g., [28]).

1.5 Brane cosmology

According to the so-called brane-world cosmology, our Universe could be a four-dimensional "slice" of a higher-dimensional *bulk* manifold. The elementary charges sourcing the gauge interactions are confined on a three-dimensional hypersurface Σ_3 associated to an object called *Dirichet* 3-*brane* (or D_3-brane), and we cannot detect the external spatial dimensions because the gauge fields of those charges can propagate only on the *world-volume* Σ_4 swept by the time evolution of the brane. (It should be recalled that the description of our Universe as a four-dimensional "domain wall" embedded in a higher-dimensional bulk spacetime was previously suggested, with different motivations, also in [29]).

In a string theory context, however, the confinement mechanism is not equally efficient for all fundamental interactions. Gravity, in particular, is not confined, or is only partially [30] confined, so that it can propagate outside the brane spacetime. This possibility is illustrated in Fig. 1.4, which shows a brane spacetime Σ_4 with two possible sources of interactions. One is a charge, source of the electromagnetic field: the associated electromagnetic waves (or photons) are strcicly confined and can propagate only on Σ_4. The other is a mass, source of the gravitational field: the associated gravitational waves (or gravitons) can leave the brane spacetime Σ_4 and propagate through the external spatial dimensions.

This property of the gravitational field is quite important because, if the higher-dimensional bulk spacetime contains two (or more) fundamental branes,

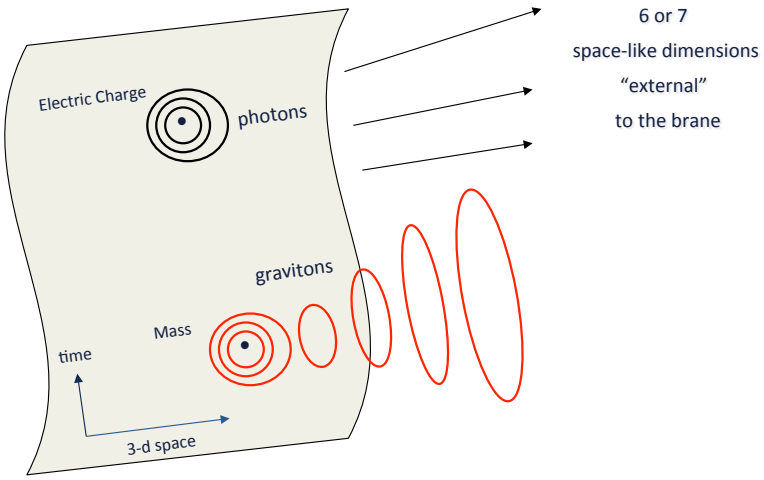

Figure 1.4: *A brane-Universe with one time-like and three space-like dimensions, embedded in an external bulk spacetime characterized by six (according to superstring theory) or seven (according to M-theory) extra spatial dimensions. Electromagnetic forces are confined on the brane spacetime, while gravitational forces propagate also in the directions external to the brane.*

they can interact among themselves gravitationally. And this possibility leads us to an interesting geometric interpretation of the big bang mechanism, namely of the high-energy process which has marked the beginning of the standard cosmological phase, bringing the Universe to the form we are presently observing.

In fact, during the high-curvature phase localized around $t = 0$ (Fig. 1.3), a higher-dimensional Universe tends to be filled by branes which are spontaneously produced in pairs from the high-energy vacuum, and which can gravitationally (and strongly) interact among themselves [21]. According to string theory, on the other hand, the total gravitational force in a higher-dimensional spacetime includes various components: we should mention, in particular, the symmetric-tensor contribution associated to the *graviton*, the scalar contribution associated to the *dilaton*, and the antisymmetric-tensor contribution associated to the *axion* (see, e.g., [20, 21]).

The first two types of forces are always attractive, while the axion force is repulsive between sources of the same sign and attractive between sources of opposite sign (like, for instance, a brane and an antibrane, characterized by opposite axionic charges). It follows, in particular, that if we have two identical branes (or antibranes) in an initial static and symmetric state, then the axion repulsion exactly cancels the attraction due to the graviton and to the dilaton, and the net resulting force is vanishing. If we have instead a brane and an antibrane then the total gravitational force is always non-vanishing and

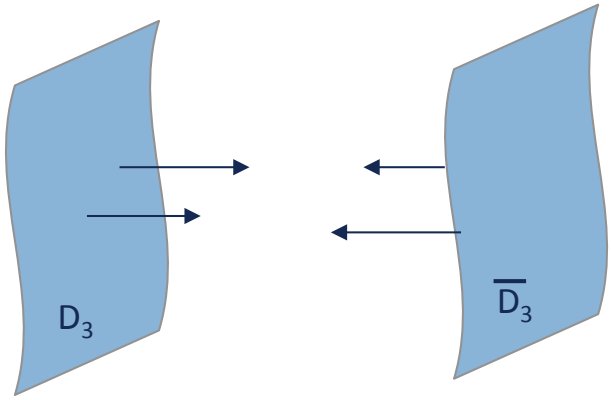

Figure 1.5: *A brane (D_3) and an antibrane ($\overline{D}_3$) tend to collide because the mutual gravitational force they experience in a higher dimensional spacetime is always of attractive type (like the electric force acting between a charged particle and the corresponding antiparticle in the usual three-dimensional space).*

attractive, quite irrespective of their initial configuration.

Because of such attractive force, branes and antibranes, copiously produced during the high-energy pre-big bang phase, tend to collide among themselves (Fig. 1.5): it could be, therefore, that it was the collision of our brane-Universe with an antibrane to simulate the big bang explosion, and trigger the transition from the pre-big bang phase to the phase of standard (post-big bang) evolution. This type of scenario is very similar to the so-called *ekpyrotic* model (first proposed in [31], and later embedded in the context of a more general type of *cyclic* cosmologies, see e.g. [32]), with the only difference that, in the ekpyrotic case, the 3-branes are *domain walls* representing the spacetime boundaries.

1.6 Conclusion

String theory, M-theory, and the related models of brane interactions suggests new and interesting scenarios for the birth of the Universe and its subsequent primordial evolution, not necessarily limited in time by a big bang singularity. They can be tested by present (or near-future) observations concerning the properties – in particular, the "blue" tilt [33] of the spectrum – of the cosmic background of relic gravitational radiation (see, e.g., [34]), the possibility of axion contributions to the CMB anisotropies (see, e.g., [35]), the evolution of the so-called dark energy (or quintessence) field dominating the large-scale dynamics (see, e.g., [36, 37]).

Some of those scenario have been briefly introduced and illustrated in the previous sections. But there are also other, equally interesting scenarios closely related to the previous ones, among which I would like to mention the string-gas [38] and brane-gas [39] cosmologies, based on the repulsive mechanism of winding modes, as well as more general bouncing cosmology models (see, e.g., [40, 41, 42]). Also, models of brane anti-brane inflation [43, 44], where the (time-varying) distance between the two branes plays the role of the inflation field.

All these models have many (and interesting) phenomenological implications, but – as usual in a cosmological context – many studies and many observational data are required before being able of selecting the model most appropriate to our Universe. Thus, we can easily predict that we still have in front of us many years of work and – maybe – of surprising findings.

Bibliography

[1] S. Weinberg: *Gravitation and Cosmology* (Wiley, New York 1972)

[2] S. Dodelson: *Modern Cosmology* (Academic Press, San Diego, CA 2003)

[3] S. Weinberg: *Cosmology* (Oxford University Press, Oxford 2008)

[4] A. Guth: *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*, Phys. Rev. D **23**, 347 (1981)

[5] A. D. Linde: *A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems*, Phys. Lett. B **108**, 389 (1982)

[6] A. Albrecht, P. J. Steinhardt: *Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking*, Phys. Rev. Lett. **48**, 1220 (1982)

[7] E. W. Kolb, M. S. Turner: *The Early Universe* (Addison Wesley, Redwod City, CA 1990)

[8] A.R. Liddle, D.H. Lyth: *Cosmological Inflation and Large-Scale Structure* (Cambridge University Press, Cambridge 2000)

[9] V.F. Mukhanov: *Physical Foundation of Cosmology* (Cambridge University Press, Cambridge 2005)

[10] R. Durrer: *The Cosmic Microwave Background* (Cambridge University Press, Cambridge 2008)

[11] A. Borde, A. Vilenkin: *Eternal inflation and the initial singularity*, Phys. Rev. Lett. **72**, 3305 (1994)

[12] A. Borde, A. Guth, A. Vilenkin: *Inflationary space-times are incomplete in past directions* Phys. Rev. Lett. **90**, 151301 (2003)

[13] A. Mithani, A. Vilenkin: *Did the Universe have a beginning?* (arXiv:1204.4658)

[14] M.B. Green, J. Schwartz, E. Witten: *Superstring Theory* (Cambridge University Press, Cambridge 1987)

[15] J. Polchinski: *String Theory* (Cambridge University Press, Cambridge 1998)

[16] B. Zwiebach: *A First Course in String Theory* (Cambridge University Press, Cambridge 2009)

[17] M. Gasperini: *The Universe before the Big Bang: cosmology and string theory* (Springer-Verlag, Berlin, Heidelberg 2008)

[18] R. Brustein, M. Gasperini, G. Veneziano: *Duality in cosmological perturbation theory*, Phys. Lett. B **431**, 277 (1998)

[19] M. Gasperini, M. Giovannini, G. Veneziano: *Cosmological perturbations across a curvature bounce*, Nucl. Phys. B **694**, 206 (2004)

[20] M. Gasperini: *Elements of String Cosmology* (Cambridge University Press, Cambridge 2007)

[21] M. Gasperini, G. Veneziano: *The Pre-big bang scenario in string cosmology*, Phys. Rep. **373**, 1 (2003)

[22] M. Gasperini, G. Veneziano: *Pre-big bang in string cosmology*, Astropart. Phys. **1**, 317 (1993)

[23] M. Gasperini, M. Giovannini: *Quantum squeezing and cosmological entropy production*, Class. Quant. Grav. **10**, L133 (1993)

[24] E. Witten: *String theory dynamics in various dimensions*, Nucl. Phys. **B443**, 85 (1995)

[25] K. Becker, M. Becker, J. E. Schwarz: *String theory and M-theory: a modern introduction* (Cambridge University Press, Cambridge 2006)

[26] T. Kaluza: *On the Problem of Unity in Physics*, Sitzungsber. Preuss. Akad. Wiss. Berlin, **1921**, 966 (1921)

[27] O. Klein: *Quantum Theory and Five-Dimensional Theory of Relativity*, Z. Phys. **37**, 895 (1926)

[28] D. Langlois: *Brane cosmology: An Introduction*, Prog. Theor. Phys. Suppl. **148**, 181 (2003)

[29] V. A. Rubakov, M. Shaposhnikov: *Do We Live Inside a Domain Wall?*, Phys. Lett. B **125**, 136 (1983)

[30] L. Randall, R. Sundrum: *An Alternative to compactification*, Phys. Rev. Lett. **83**, 4960 (1999)

[31] J. Khoury, B. A. Ovrut, P. J. Steinhard, N. Turok: *The Ekpyrotic universe: Colliding branes and the origin of the hot big bang*, Phys. Rev. D **64**, 123522 (2001)

[32] P. J. Steinhardt, N. Turok: *Cosmic evolution in a cyclic universe*, Phys. Rev. D **65**, 126003 (2002)

[33] R. Brustein, M. Gasperini, M. Giovannini, G. Veneziano: *Relic gravitational waves from string cosmology*, Phys. Lett. B **361**, 45 (1995)

[34] M. Gasperini: *Elementary introduction to pre-big bang cosmology and to the relic graviton background*, in Gravitational Waves, edited by I. Ciufolini, V. Gorini, U. Mischella, P. Frè (IOP Publishing, Bristol 2001), p. 280-337, hep-th/9907067.

[35] R. Durrer, M. Gasperini, M. Sakellariadou, G. Veneziano: *Massless (pseudo)scalar seeds of CMB anisotropy*, Phys. Lett. B **436**, 66 (1998)

[36] M. Gasperini: *Dilatonic interpretation of the quintessence?*, Phys. Rev. D **64**, 043510 (2001)

[37] L. Amendola, M. Gasperini, F. Piazza: *Fitting type Ia supernovae with coupled dark energy*, JCAP **0409**, 014 (2004)

[38] R. Brandenberger, C. Vafa: *Superstrings in the Early Universe*, Nucl. Phys. B **316**, 391 (1989)

[39] S. Alexander, R. Brandenberger, D. Easson: *Brane gases in the early universe*, Phys. Rev. D **62**, 103509 (2000)

[40] T. Biswas, A. Mazumdar, W. Siegel: *Bouncing universes in string-inspired gravity*, JCAP **0603**, 009 (2006)

[41] T. Biswas, T. Koivisto, A. Mazumdar: *Towards a resolution of the cosmological singularity in non-local higher derivative theories of gravity*, JCAP **1011**, 008 (2010)

[42] Y.-F. Cai, D. A. Easson, R. Brandenberger: *Towards a Nonsingular Bouncing Cosmology*, JCAP **1208**, 020 (2012)

[43] C. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh, R. J. Zhang: *The Inflationary brane anti-brane universe*, JHEP **0107**, 047 (2001)

[44] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAlliste, S. P. Trivedi: *Towards inflation in string theory*, JCAP **0310**, 013 (2003)

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Why supersymmetry should be restored at the TeV scale

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Abstract

It is explained why the curvature associated to the vacuum energy density arising from SUSY breaking cannot be completely transferred to the extra spatial dimensions of a bulk space-time manifold, and it is shown – without using hierarchy arguments but only the results of current large-scale observations – why the Tev scale should correspond to the maximal allowed SUSY-breaking scale.

Essay written for the 2008 Awards for Essays on Gravitation
(Gravity Research Foundation, Wellesley Hills, MA, 02481-0004, USA)
and awarded with "Honorable Mention"

The aim of this paper is to argue that we must expect a phenomenological upper bound of about one TeV on the supersymmetry breaking scale characterizing standard-model interactions, provided *i*) the space-time in which we live is the four-dimensional section of a curved higher-dimensional bulk manifold, and *ii*) the vacuum energy density associated to SUSY breaking is gravitating like all other forms of energy. Let us start with two observations which are hardly questionable (at least, at the present stage of understanding of fundamental physics).

The first observation is that supersymmetry is broken, in our four-dimensional Universe, at a scale $M_{\rm SUSY} \gtrsim 10^2$ GeV (as the required superpartners of known particles are not observed, up to this mass/energy scale). This necessarily produces a vacuum energy density $(M_{\rm SUSY}/M_{\rm P})^4 \gtrsim 10^{-64}$, where $M_{\rm P} = (8\pi G)^{-1/2} \sim 10^{18}$ GeV is the reduced Planck mass (as the zero-point energies of boson and fermion fields fail to cancel one another, in general, in the absence of supersymmetry constraints).

The second observation is that such a huge energy density, in spite of its presence, *does not bend* the geometry of the four-dimensional macroscopic space-time that we are presently observing (as the currently observed level of curvature, at large scales, corresponds to a much smaller vacuum energy density, $\rho_V/M_{\rm P}^4 \lesssim 10^{-120}$). Barring the (exotic) possibility that the vacuum energy associated to a SUSY-broken phase is gravitationally neutral, this second observation provides a compelling motivation for the existence of extra spatial dimensions able to absorb the curvature produced by the four-dimensional energy density $M_{\rm SUSY}^4$, and thus characterized by a curvature scale L such that

$$L^{-2} \sim 8\pi G\, M_{\rm SUSY}^4 = \frac{M_{\rm SUSY}^4}{M_{\rm P}^2}. \qquad (1)$$

In that case, our four-dimensional space-time may be seen as a (nearly flat) section of a (highly curved) higher-dimensional bulk manifold – much in the same way as ordinary three-dimensional space, in spite of the energy density it contains, may correspond to a flat section of a curved FRW space-time.

This idea of "off-loading" the gravitational effects of the vacuum energy density along extra spatial dimensions is rather old [1], but has been recently considered with renewed interest [2]-[5] within the so-called "brane-world" scenario, where standard-model (gauge) interactions are confined on the four-dimensional hypersurface swept by the time-evolution of some fundamental three-brane. Putting aside the problems of naturalness, fine-tuning, self-tuning, ... possibly associated to the existing examples of this off-loading mechanism, the crucial point (for the purpose of this paper) concerns the supersymmetry properties that the bulk must satisfy (to a high degree of accuracy) for a successful (i.e. realistic) scenario.

In fact, if bulk supersymmetry is broken at a scale $M_{\rm S}^{\rm bulk}$, it can be shown that our four-dimensional space-time (henceforth brane-world, for short) automatically absorbs from the bulk a vacuum energy density ρ_V, which is determined in general by $M_{\rm S}^{\rm bulk}$ and by the

bulk curvature scale L. Such a contribution arises from the zero-point energies necessarily associated to the quantum fluctuations of the bulk geometry at the brane position [6], which are perceived on the brane as towers of massive gravitational excitations not included into the standard (four-dimensional) field multiplets.

In order to evaluate the energy density transferred in this way from the bulk to the brane we may consider an unperturbed background configuration describing a three-brane embedded in a $D = (4 + n)$-dimensional bulk manifold, whose curvature along the n extra dimensions completely absorbs the huge contribution of brane SUSY breaking, according to Eq. (1). We stress that such a contribution arises from the quantum zero-point energies of all fields localized on the brane, possibly including the massless components of metric fluctuations (representing long-range gravitational interactions) if they are confined on the brane (for instance, through an infinitely attractive potential as in [2]).

The complete spectrum of metric fluctuations also includes, however, a wide sector of tensor/scalar perturbations which can freely propagate along all bulk directions, and whose oscillations evaluated at the brane position can in principle drag up energy from the bulk to the brane. The effective action for such fluctuations, dimensionally reduced (integrating over the directions orthogonal to the brane), canonically normalized, and evaluated at the brane position, describes massive tensor/scalar fields which, once quantized, generate a zero-point energy density [6]:

$$\rho_V \sim L^n \int_0^\lambda \mathrm{d}^n\chi \int_0^\Lambda \mathrm{d}^3k \sqrt{|k|^2 + |\chi|^2}. \tag{2}$$

Here χ_i, $i = 1, \ldots, n$ are the momenta along the n extra dimensions, appearing on the brane as a (continuous) spectrum of mass eigenvalues, and Λ, λ are the (possibly different) cutoff parameters in the momentum spaces corresponding to the spatial dimension internal and external to the brane, respectively. We have supposed, for simplicity, that the n extra dimensions are isotropic (all with the same curvature scale $\sim L$), and non-compact (but "warped" by the bulk curvature), so as to be associated to a continuous spectrum of momenta χ^2.

We can now observe that the vacuum energy (2) of the (bosonic) metric fluctuations, in the case of a perfectly supersymmetric bulk, will be exactly compensated by the opposite contribution of the existing bulk fermionic "superpartners". If, on the contrary, bulk supersymmetry is broken at a scale $M_{\mathrm{S}}^{\mathrm{bulk}}$, then even subtracting the fermionic contribution the net result for the vacuum energy density is in general non-vanishing. Considering, in particular, a model of broken supersymmetry with the same number of boson and fermion degrees of freedom, subtracting from Eq. (2) the associated contribution of the fermionic components of the gravitational supermultiplet, and assuming the so-called "supertrace" cancellation of the mass-squared terms [7] (as in models of spontaneously broken supersymmetry), we may expect from (2) the leading order result [6]

$$\rho_V \sim \left(L\, M_{\mathrm{S}}^{\mathrm{bulk}}\right)^n \left(M_{\mathrm{S}}^{\mathrm{bulk}}\right)^4. \tag{3}$$

We have assumed that $\lambda \sim M_{\rm S}^{\rm bulk}$, since above that scale supersymmetry is restored, and appropriate cancellations are expected to suppress to zero the vacuum energy density sourced by the bulk fields.

It is important to stress that the above result for the vacuum energy density absorbed from the bulk by the brane is valid for $LM_{\rm S}^{\rm bulk} \gtrsim 1$, while it reduces to $\rho_V \sim (M_{\rm S}^{\rm bulk})^4$ when $LM_{\rm S}^{\rm bulk} \lesssim 1$. In other words, the result for ρ_V is *bounded from below* by

$$\rho_V \gtrsim \left(M_{\rm S}^{\rm bulk}\right)^4, \tag{4}$$

quite irrespectively of the continuous or discrete nature of the spectrum of the metric fluctuations (namely, for both compact and non-compact extra dimensions).

In fact, let us first observe that the result (3), obtained in the case of a continuous spectrum, can also be extended to the case of compact extra dimensions (of proper size L) and discrete spectrum, provided the mass-eigenvalue spacing L^{-1} is negligible with respect to the energy scale $M_{\rm S}^{\rm bulk}$ (i.e. for $LM_{\rm S}^{\rm bulk} > 1$). In the opposite case ($LM_{\rm S}^{\rm bulk} < 1$) in which the discrete nature of the spectrum cannot be ignored, instead, the integral $L^n \int {\rm d}^n\chi$ of Eq. (2) is replaced by the (dimensionless) sum operator $\sum_n$ over the eigenvalues χ_n, and result for ρ_V – assuming the convergence of the series of SUSY-breaking corrections at the bulk scale, as before – becomes $\rho_V \sim (M_{\rm S}^{\rm bulk})^4$. This result can also be understood as a consequence of the Casimir effect which, for $L^{-1} > M_{\rm S}^{\rm bulk}$, dominates the vacuum energy density of the compact extra dimensions, leading to an overall bulk energy density $\rho_{\rm bulk} \sim (L^{-1})^n (M_{\rm S}^{\rm bulk})^4$. The energy absorbed by the brane, given by the integration of $\rho_{\rm bulk}$ over the proper volume of the transverse dimensions, is then $\rho_V \sim (M_{\rm S}^{\rm bulk})^4$.

Finally, let us come back to the case of a continuous spectrum of bulk fluctuations, associated to warped, non-compact dimensions with a curvature scale of order L, and leading to the result of Eq. (3). If $LM_{\rm S}^{\rm bulk} > 1$ such a result is always valid, quite independently of the absolute magnitude of the two energy scales L^{-1} and $M_{\rm S}^{\rm bulk}$. If $LM_{\rm S}^{\rm bulk} < 1$ the result (3) is still valid, in principle, but we must take into account that a curvature L^{-1} necessarily induces a bulk supersymmetry breaking characterized by a scale of the same order [8, 9]. Hence, in this case, bulk supersymmetry is automatically broken up to a scale $M_{\rm S}^{\rm bulk} \sim L^{-1}$, and we are led again to $\rho_V \sim L^{-4} \sim (M_{\rm S}^{\rm bulk})^4$.

There are two main conclusions that we can draw from the above discussion. First, when bulk supersymmetry is broken at a scale $M_{\rm S}^{\rm bulk}$, the quantum zero-point energies of the bulk gravitational fluctuations necessarily induce on the brane a vacuum energy density ρ_V which is *at least* of order $(M_{\rm S}^{\rm bulk})^4$. Second, even in the absence of specific sources of bulk SUSY breaking, a curved geometry breaks bulk supersymmetry and thus induces on the brane a vacuum energy density $\rho_V \sim L^{-4}$. On the other hand, the bulk *has* to be curved (according to Eq. (1)) in order to absorb the huge cosmological constant $M_{\rm SUSY}^4$ associated to supersymmetry breaking on the brane. It follows that the process of "off-loading" to the bulk the effects of SUSY breaking cannot have a hundred per cent efficiency: it is

necessarily associated to a gravitational "backreaction" assigning to the brane a minimal level of vacuum energy density which, according to Eq. (1), is given (in Planck units) by

$$\frac{\rho_V}{M_{\rm P}^4} \sim (L\, M_{\rm P})^{-4} \sim \left(\frac{M_{\rm SUSY}}{M_{\rm P}}\right)^8 . \tag{5}$$

Similar results, previously obtained with different arguments [7, 10, 11, 12], can thus find a confirmation which is fully independent of the number and/or compactness of the extra dimensions.

The above result bring us to the final point of this paper. The vacuum energy density that we are presently observing in our Universe is constrained by large-scale measurements (see e.g [13]) to be $\rho_V/M_{\rm P}^4 \lesssim 10^{-120}$. If our four-dimensional macroscopic Universe is a brane-world embedded in a higher-dimensional bulk, then the SUSY breaking of the (standard-model) fields confined in four-dimensions should induce on the brane the minimal level of vacuum energy density (5), and the phenomenological constraints on ρ_V should consequently determine the following upper bound on the allowed SUSY-breaking scale:

$$M_{\rm SUSY} \lesssim 10^{-15} M_{\rm P} \sim 1\,{\rm TeV}. \tag{6}$$

In such a case, supersymmetry effects could become directly observable already within the forthcoming collider experiments (see e.g. [14]), planned to be sensitive to the TeV scale. The non-observation of supersymmetric effects at the TeV scale should be interpreted as experimental evidence against the existence of supersymmetry, and/or against the physical relevance of the scenario of "bulk-diluted" vacuum energy density discussed in this paper. In that case, one should possibly apply to other mechanisms for the cancellation of the vacuum energy density (see e.g. [15]).

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References

[1] V. A Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125**, 139 (1983).

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4960 (1999).

[3] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and R. Sundrum, Phys. Lett. B **480**, 193 (2000); S. Kachru, M. Schulz and E. Silverstein, Phys. Rev. D **62**, 045021 (2000).

[4] C. Csaki, J. Erlich and C. Grojean, Nucl. Phys. B **604**, 312 (2001); Gen. Rel. Grav. **33**, 1921 (2001).

[5] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, Nucl. Phys. B **680** (2004) 389.

[6] M. Gasperini, *Higher-dimensional perturbations of the vacuum energy density*, BA-TH/579-08 (February 2008).

[7] E. Kiritsis, JHEP **9910**, 010 (1999).

[8] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B **436**, 257 (1998).

[9] I. Antoniadis, E. Dudas, and A. Sagnotti, Phys. Lett. B **464**, 38 (1999).

[10] C. Schmidhuber, Nucl. Phys. B **585**, 385 (2000); Nucl. Phys. B **619**, 603 (2001).

[11] T. Banks, Int. J. Mod. Phys. A **16**, 910 (2001); hep-th/0007146.

[12] P. Berglund, T. Hubsc and D. Minic, Phys. Rev. D **67**, 041901 (2003).

[13] Particle Data Group webpage at `http://pdg.lbl.gov/`

[14] J. L. Feng, *Colliders physics and cosmology*, arXiv:0801.1334.

[15] G. L. Alberghi, A. Y. Kamenshchik, A. Tronconi, G. P. Vacca and G. Venturi, *Vacuum energy and standard model physics*, arXiv:0804.4782; A. A. Andrianov, F. Cannata, P. Giacconi, A. Y. Kamenshchik and R. Soldati, Phys. Lett. B **651**, 306 (2007).

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Inhomogeneous Pre-Big Bang String Cosmology

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Abstract

An inhomogeneous version of pre–Big Bang cosmology emerges, within string theory, from quite generic initial conditions, provided they lie deeply inside the weak-coupling, low-curvature regime. Large-scale homogeneity, flatness, and isotropy appear naturally as late-time outcomes of such an evolution.

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1 Introduction

Superstring theory, a supposedly consistent and unified quantum theory of all interactions, suggests, through its peculiar duality symmetries [1], an attractive alternative to standard inflationary cosmology, the so-called pre- Big Bang (PBB) scenario [2] [3] [4].

The basic postulate of PBB cosmology is that the Universe started its evolution from a very perturbative initial state, i.e. from very weak coupling and very small curvatures (in appropriate units, see below). If initial homogeneity is also assumed, this leads automatically to a super-inflationary epoch during which the accelerated expansion of the Universe is driven by the accelerated growth of the effective (gauge and gravity) coupling, i.e. of the dilaton.

This result looks surprising, at first sight, since the equation of state of the dilaton is not of the kind that one usually associates with inflation. Indeed, inflation only occurs if the geometry is looked at in the so-called String(S)-frame, the one appearing in the Nambu-Goto-Polyakov action, while in the Einstein(E)-frame one rather observes an accelerated contraction. The point here is that, precisely at weak coupling and small curvatures, fundamental strings keep their characteristic size constant and sweep geodesic surfaces with respect to the S-frame metric [5] [6]. Thus, in this regime at least, the latter *is* the correct frame for describing how distances among objects vary w.r.t. their intrinsic size, and for determining what is actually meant by small or large curvatures. As long as curvatures remain small (resp. become $O(\frac{1}{\alpha'})$) in the string frame, the low energy effective action of string theory represents an adequate (resp. becomes an inadequate) description of physics.

Going from one "frame" to another, although not a coordinate transformation, is only a matter of field redefinitions. Hence, any physical observable takes the same value in any such "frame". Whenever useful for computations, any other frame can be used, and we shall take advantage of this possibility below. However, the description of physical phenomena is all but intuitive in other frames and, in order to avoid problems of interpretation, it is best to transform back the results to the string frame at the end of the calculations.

The PBB scenario also needs an "exit assumption" from the PBB phase to standard non-inflationary cosmology. Higher derivative (α') corrections are expected to stop the indefinite growth of curvature (see [7] for recent progress on this issue), while loop corrections and/or a non-perturbative potential are usually assumed to stop the indefinite growth of the coupling itself. If such a "graceful exit" turns out to be a property of string theory, one will obtain an

interesting scenario whereby the usually assumed Big Bang initial conditions (a hot, dense and highly-curved state) will be the outcome -rather than the starting point- of inflation. Several interesting phenomenological consequences of the scheme have also been worked out [8].

In spite of the above advantages of the PBB scenario, it is clear that the assumption of homogeneous initial conditions is a difficult one to swallow. After all, inflation is supposed to explain [9] homogeneity(as well as isotropy and flatness)! The purpose of this note is to relax the assumption of initial homogeneity by taking the initial state of the Universe to be in the weak-coupling, weak-curvature regime, but otherwise arbitrary. Generically, PBB behaviour will emerge in suitable regions of space which, eventually, will fill all but an infinitesimal fraction ofspace. Within those regions, the Universe will appear very homogeneous, isotropic and spatially flat.

The plan of the paper is as follows: in Section 2, I give the general form of the equations in the E-frame and describe some general features of their solutions, which will be confirmed by the construction, in Section 3, of the general analytic asymptotic solution near the "singularity". In Section 4, the results of the previous Sections will be reinterpreted in the physical S-frame and semiquantitative local conditions for the onset of PBB behaviour will be given.

Most of our technical results are not really new, only the physical motivation and interpretation are. In particular, our method can be traced back to the work of Belinskii and Kalatnikov [10] which was made recently more systematic through the so-called gradient expansion method either directly in terms of Einstein's equations [11], or within a Hamilton-Jacobi formulation [12].

2 General properties of the solutions in the E-frame

Having made the basic postulate that the Universe started its (pre) history in a weak-coupling, low-curvature state, we will describe its early evolution in terms of the tree-level, low-energy effective action of string theory.

For simplicity we shall consider only the particular case of a critical superstring theory, with vanishing cosmological constant and six frozen internal dimensions. The effective four-

dimensional theory is thus described (in Landau-Lifshitz notations [13]) by the action:

$$\Gamma^S_{eff} = \frac{1}{2}\int dx\sqrt{G}e^{-\phi}\left(R(G) + G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \tag{2.1}$$

where $G_{\mu\nu}$ is the string-frame (S-frame) metric, ϕ is the dilaton and we have also set to zero the antisymmetric tensor field $B_{\mu\nu}$. In the conformally related Einstein frame (E-frame) defined by the new metric $g_{\mu\nu}$ via:

$$g_{\mu\nu} = e^{-\phi}G_{\mu\nu} \tag{2.2}$$

the action becomes:

$$\Gamma^E_{eff} = \frac{1}{2}\int dx\sqrt{g}\left(R(g) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \tag{2.3}$$

and the field equations take the familiar form:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}(\partial\phi)^2 \\ \bigtriangledown^2\phi \equiv g^{\mu\nu}D_\mu\partial_\nu\phi = 0 \end{aligned} \tag{2.4}$$

where we are using units in which $8\pi G_N = 1$.

We also choose to work in the synchronous gauge, i.e. we fix:

$$g_{00} = -1,\ g_{0i} = 0\ (i = 1, 2, 3), \tag{2.5}$$

so that the dynamical variables become the three-metric g_{ij} and the dilaton ϕ.

The Einstein equations for the system can be readily written. They consist of:

- The Hamiltonian constraint

$$2\dot{\phi}^2 + Tr\chi^2 - \chi^2 = 4\mathcal{R} - 2(\bigtriangledown\phi)^2 \tag{2.6}$$

- The three Momentum Constraints

$$\chi_{,i} - \chi^j_{i;j} = -\dot{\phi}\,\bigtriangledown_i\,\phi \tag{2.7}$$

- The dynamical equation for ϕ

$$\ddot{\phi} + \frac{1}{2}\chi\dot{\phi} = \bigtriangledown^2\phi \tag{2.8}$$

- The dynamical equation for g_{ij}

$$\dot{\chi^j_i} + \frac{1}{2}\,\chi\,\chi^j_i = -2\mathcal{R}^j_i + \bigtriangledown_i\phi\,\bigtriangledown^j\,\phi \tag{2.9}$$

where a dot denotes derivative w.r.t. E-frame "cosmic" time τ, covariant derivatives are defined w.r.t. the three-metric g_{ij} (whose three-curvature is denoted by $\mathcal{R}$), and we have defined:

$$\chi_{ij} = \dot{g_{ij}},\ \chi_i^j = \chi_{ik}g^{kj},\ Tr\chi^2 = \chi_i^j\chi_j^i,\ \chi = \chi_i^i = \frac{\dot{g}}{g},\ g = det(g_{ij}). \tag{2.10}$$

It is easy to check that the above equations reduce to the usual cosmological equations once all spatial gradients are set to zero. It is also possible to check that the four constraints, once imposed at an initial time, remain valid at all times thanks to the evolution equations.

The following general inequalities follow as long as the three metric ispositive (semi) definite:

$$Tr\chi^2 \geq \frac{1}{3}\chi^2 \geq 0\ , \tag{2.11}$$

where the first equality is reached in the isotropic case.

Combining the trace of (2.9) and (2.6) we obtain a useful equation which does not involve spatial gradients:

$$\dot{\chi} = -\left(\frac{1}{2}Tr\chi^2 + \dot{\phi}^2\right) \leq 0, \tag{2.12}$$

with the equality sign holding iff $\dot{\phi} = \chi_{ij} = 0$.

Equation (2.12), together with the basic postulate of PBB cosmology, is enough to conclude that, on the $\tau = 0$ hypersurface, we can only have regions with $\chi < 0$. Indeed, regions with $\chi > 0$, given Eq. (2.12), will necessarily have to originate from a large curvature phase in the past. Even regions with $\chi = 0$ are excluded, since, going backward in time, they generically came from a positive χ. Furthermore, the Hamiltonian constraint (2.6) implies that χ can only vanish if

$$\mathcal{R} \geq \frac{1}{4}Tr\chi^2 + \frac{1}{2}\dot{\phi}^2, \tag{2.13}$$

i.e. in regions of sufficiently large positive spatial curvature. In sufficiently homogeneous regions, in which fields vary little over a Hubble radius, we get from (2.11), (2.6) the bounds:

$$-\sqrt{3} \leq \rho \equiv \frac{\chi}{\sqrt{Tr\chi^2}} \leq -1. \tag{2.14}$$

We conclude that generic initial data satisfying the PBB postulate can be presented in terms of quasi-homogeneous regions where $-\sqrt{3} < \rho < -1$, separated by inhomogeneous regions where ρ *can* take any negative value larger than $-\sqrt{3}$. Let us see how each one of these regions evolves in time:

The quasi-homogeneous regions with $\rho < 0$ undergo an accelerated contraction in the Einstein frame and are consistent with the PBB postulate. We shall see below (and in Section 3) more precisely how they evolve in the E-frame while, in Section 4, we will reinterpret those results in the S-frame.

Highly inhomogeneous regions are more difficult to deal with, but are not expected [10] [11] [12] to undergo a marked accelerated (or decelerated) evolution.

It is relatively easy to see that, for quasi-homogeneous regions with $\rho < 0$, the approximation of neglecting spatial derivatives improves with time.Leaving to Section 3 the study of the asymptotic solution, we just note that, neglecting spatial gradients, the trace of (2.9) gives:

$$\chi \sim \frac{2}{\tau - \tau_0} \Rightarrow g \sim (\tau - \tau_0)^2, \tau < \tau_0, \tag{2.15}$$

where, in general, τ_0 can depend on x. Invoking now (2.6) and the definition (2.14) we can determine $\dot{\phi}$ up to a sign ambiguity. However, given the fact that we chose $\chi < 0$, and that we want to start the evolution from weak coupling, we have to choose the sign giving:

$$\dot{\phi} = -\frac{\chi}{\sqrt{2}}\sqrt{1 - \rho^{-2}} \Rightarrow \phi \sim \ -c(x) log(\tau_0 - \tau), \ \ c(x) > 0. \tag{2.16}$$

In order to check that spatial gradients are subleading w.r.t. time derivatives as we approach $\tau = \tau_0$ consider, for instance, the ratio (with no sum over i, j implied):

$$r_{ij} \equiv \frac{g^{ij} \bigtriangledown_i \phi \bigtriangledown_j \phi}{\dot{\phi}^2}. \tag{2.17}$$

As we approach the singularity, using (2.15), (2.16), we find:

$$r_{ij} \sim g \cdot g^{ij} \bigtriangledown_i \phi \bigtriangledown_j \phi \sim \epsilon^{ikn}\epsilon^{jlm} g_{kl} g_{nm} \ log^2(\tau_0 - \tau) \to 0, \tag{2.18}$$

unless the contraction of the three-volume is highly anisotropic.

We thus conclude that, within sufficiently homogeneous and isotropic regions, the approximation of neglecting spatial derivatives becomes increasingly good as we approach the singularity at $\tau = \tau_0$. This conclusion will be fully confirmed in the analytic asymptotic solution presented in Section 3.

Let us now try to understand how things evolve as one goes backward in time, $\tau \to -\infty$. It is not possible to obtain an asymptotic solution in this limit by the method used in Section 3. Nevertheless, we can make the following observations: Eqs. (2.8), (2.12), together imply

that the only early-time fixed points are either at infinity (corresponding to a singularity) or at $\dot{\phi} = \chi_{ij} = \bigtriangledown^2\phi = 0$. This implies however (neglecting boundary terms) $\bigtriangledown_i\phi = 0$ and thus, using Eq. (2.9), $\mathcal{R}_i^j = 0$, which, in three dimensions, is equivalent to flat space. Thus the only non-singular fixed point is the trivial one.

Homogeneous PBB cosmologies are examples of solutions converging towards this trivial fixed point in the far past. Our basic postulate requires that there should be a *finite-measure* basin of attraction (containing the homogeneous cases) towards this fixed point. By using a reliable linear perturbation theory around the homogeneous solutions, it is not hard to show that such a finite basin should exist, although determining its exact extension is highly non-trivial.

Another related issue is whether, within such a basin of attraction, one is really approaching the trivial Minkowski vacuum with vanishing coupling and curvature. The tricky point here is that curvatures and derivatives of ϕ have to go to zero in *string* units, namely they have to become much smaller than e^{ϕ} in the E-frame. Although we have indications that this is indeed the case, we leave a full discussion of this important question to further investigations.

3 General asymptotic solution in the E-frame

In this Section we present the leading term of the general asymptotic solution of Eqs. (2.6)-(2.9) near the singularity hypersurface $\tau = \tau_0(x)$. The construction is quite straightforward (see e.g. ([10])), if one makes a few approximations in solving the equations and then checks, a posteriori, their validity. Finally, a counting of the number of arbitrary functions contained in the solution shows that we are indeed describing the most general solution.

Inserting (2.15) back into Eq.(2.9) we get, for small enough gradients,

$$\dot{\chi}_i^j + \frac{1}{\tau - \tau_0}\chi_i^j = 0, \tau < \tau_0. \tag{3.1}$$

Integrating this equation twice we get:

$$\chi_i^j = \frac{2\lambda_i^j}{\tau - \tau_0},\ g_{ij}(x,\tau) = \left[exp\left(2\lambda(x)log(1 - \frac{\tau}{\tau_0})\right)\right]_i^k\ g_{kj}(x,0) \tag{3.2}$$

where the space-dependent quantities λ_i^k satisfy:

$$\lambda_i^i = 1,\ \frac{1}{3} \leq \lambda_i^k\lambda_k^i \leq 1,\ \lambda_{ij} \equiv \lambda_i^k g_{kj}(x,0) = \lambda_{ji}. \tag{3.3}$$

At this point we use (2.6) and find, asymptotically:

$$\phi(x,\tau) = \phi(x,0) - \sqrt{2}\sqrt{1-\lambda_i^k\lambda_k^i}\; log(1-\frac{\tau}{\tau_0}). \tag{3.4}$$

Rewriting finally Eq.(2.8) in the form:

$$(\sqrt{g}\dot{\phi})^{\cdot} = \sqrt{g}\,\bigtriangledown^2\,\phi, \tag{3.5}$$

we see that it is automatically fulfilled asymptotically since, from (3.2), (3.4), we find $\sqrt{g}\dot{\phi} \sim const(x)$.

A convenient way to present the solution and to check that, indeed, it is general, consists of introducing local coordinates with respect to which the matrix λ and the three-metric are diagonal. Then we can write:

$$g_{ij}(x,\tau) = \sum_a e_i^a(x)e_j^a(x)(1-\frac{\tau}{\tau_0})^{2\lambda_a(x)} \tag{3.6}$$

$$\phi(x,\tau) = \phi(x,0) - \sqrt{2}\sqrt{1-\sum\lambda_a^2}\; log(1-\frac{\tau}{\tau_0}),\ \sum\lambda_a = 1 \tag{3.7}$$

where, at the moment, e_i^a are arbitrary "dreibein" matrices and we note that the parameter ρ of Section 2 is now given by $\rho^{-2} = \sum\lambda_a^2$.

The above solution appears to depend upon thirteen arbitrary functions of space, i.e. nine e_i^a, two of the three λ_a, τ_0, and ϕ_0. However, we have not imposed yet the momentum constraints (2.7). These constraints, being conserved, reduce to their form at $\tau = 0$ and can be used to eliminate three of the nine e_i^a. We are left with ten arbitrary functions. Yet, within the synchronous gauge, we can still perform a four-parameter family of gauge transformations (see, e.g., [13]) allowing us to set $\tau_0(x) = \tau_0$ (independent of x) and to further fix three of the remaining sixindependent quantities e_i^a. We end up with six (physical) arbitrary functions which is, indeed, the correct number [10] for the problem at hand.

4 Physical reinterpretation in the string frame

In order to discuss the physical implications of our results we transform them back to the string frame. Using Eq. (2.2) we obtain

$$G_{ij}(x,\tau) = \sum_a e_i^a(x)e_j^a(x)\cdot e^{\phi(x,0)}(1-\frac{\tau}{\tau_0})^{2\lambda_a-\sqrt{2}\sqrt{1-\sum\lambda_b^2}}\ , \tag{4.1}$$

Since $g_{00} = -1$ has also been changed into a non-trivial G_{00}, westill have to go to the synchronous gauge in the S-frame. This is easily accomplished, to leading order in the gradients, by changing the time variable from τ to t according to:

$$\sqrt{2}(t_0 - t) = \tau_0 \, e^{\phi(x,0)/2}(\sqrt{2} - \sqrt{1 - \sum \lambda_a^2})^{-1}(1 - \tau/\tau_0)^{1 - \frac{1}{\sqrt{2}}\sqrt{1-\sum \lambda_a^2}} \tag{4.2}$$

where we note that, for any choice of the λ_a, $t \to t_0$ as $\tau \to \tau_0$. Finally, after defining approriate S-frame "dreibeins" $E_i^a(x)$, we get:

$$G_{ij}(x,t) = \sum_a E_i^a(x) E_j^a(x)(1 - \frac{t}{t_0})^{2\alpha_a(x)} \tag{4.3}$$

$$\phi(x,t) = \phi(x,0) + \gamma \, log(1 - \frac{t}{t_0}), \tag{4.4}$$

where:

$$\alpha_a = \frac{\lambda_a - \frac{1}{\sqrt{2}}\sqrt{1 - \sum \lambda_b^2}}{1 - \frac{1}{\sqrt{2}}\sqrt{1 - \sum \lambda_b^2}}, \; \gamma = -\frac{\sqrt{2}\sqrt{1 - \sum \lambda_b^2}}{1 - \frac{1}{\sqrt{2}}\sqrt{1 - \sum \lambda_b^2}}. \tag{4.5}$$

The above relations imply:

$$\sum \alpha_a^2 = 1, \; \gamma = -1 + \sum \alpha_a \tag{4.6}$$

so that we find for the ($O(d,d)$-invariant) shifted dilaton [1]:

$$\bar{\phi} \equiv \phi - \frac{1}{2} trlogG \sim -log(1 - \frac{t}{t_0}) + \bar{\phi}(x,0), \tag{4.7}$$

which generalizes the well known S-frame homogeneous cosmology results [2], [3].

Eqs. (4.5) can easily be inverted to express the λ_a's in terms of the α_a's:

$$\lambda_a = \frac{1}{3} + \frac{2}{3}\frac{\alpha_a - \frac{1}{3}\sum \alpha_b}{1 - \frac{1}{3}\sum \alpha_b}. \tag{4.8}$$

Using these equations we can construct, in principle, scale-factor-duality(SFD) related solutions ($\alpha_a \to -\alpha_a$ for some a[2]) even in the inhomogeneous case. Of course, one has to check that the constraints can be simultaneously imposed on each component of the dual pairs.

We are finally in a position to study how the three-volume of a quasi-homogeneous region evolves in the string frame. We find

$$\frac{1}{2}\chi^{(s)} \equiv \frac{1}{2}\frac{\dot{G}}{G} = -\frac{\sum \alpha_a}{t_0 - t} \tag{4.9}$$

which shows that quasi-homogeneous regions undergo *superinflation* in the string frame provided:

$$\sum \alpha_a = \frac{1 - \frac{3}{\sqrt{2}}\sqrt{1 - \sum \lambda_a^2}}{1 - \frac{1}{\sqrt{2}}\sqrt{1 - \sum \lambda_a^2}} < 0. \tag{4.10}$$

This condition is satisfied provided $\rho^{-2} = \sum \lambda_a^2 < 7/9$, which includes all but a small region for the allowed values of this quantity (see (2.14)). Note that the maximal expansion rate is reached for $\sum \alpha_a = -\sqrt{3}$ corresponding to the isotropic case $\lambda_a = 1/3$.

We may also ask about the flatness problem. It is easy to see that the typical quantity measuring flatness, $\frac{\mathcal{R}}{\chi^2}$, contains a logarithmically enhanced term which, for $t \to t_0$, behaves as

$$\frac{\mathcal{R}}{\chi^2} \sim \frac{G^{ij} \bigtriangledown_i \phi \bigtriangledown_j \phi}{\dot{\phi}^2} \sim (t - t_0)^{2-2\mathrm{Max}(\alpha_a)} log^2(t_0 - t), \tag{4.11}$$

and thus goes to zero almost everywhere thanks to (4.6). The non-enhanced terms are more involved, but certainly behave in the same way for sufficiently isotropic situations. Thus, as expected, spatial flatness is generically achieved after a long PBB era.

We conclude that all but the "skin" of the quasi-homogeneous regions with negative ρ inflates when we look at the geometry in the "right" frame and becomes homogeneous and spatially flat. Presumably, after a while, these regions represent by far the largest fraction of the Universe. Furthermore, the isotropic regions dominate all others, providing also a possible "explanation" for the isotropy and flatness of our observable Universe.

5 Conclusions

In this paper we have relaxed the assumption of homogeneity (and spatial flatness) in the initial conditions for string cosmology. The emerging picture, which bears some similarity to Linde's chaotic inflation [14], can be described as follows: the primordial state of the Universe is arbitrarily close to flat space-time but not particularly homogeneous, in the sense that time and space derivatives are both extremely small but comparable. Stochastically, however, sufficiently homogeneous and isotropic conditions will emerge in some regions of space and undergo, in the S-frame, an accelerated expansion yielding homogeneity, flatness and isotropy as one approaches a singularity in the future within a finite proper time. Before such a singularity is reached, the low-energy effective action will cease to be valid and, hopefully,

the higher-derivative corrections present in string theory will lead to a finite-curvature stringy phase [7] rather than to a genuine singularity.

In this connection, it is amusing to speculate that the famous singularity theorems of Hawking and Penrose [15] will find an unexpected (and welcome!) application in string cosmology: since the conditions of the theorems do probably apply to the low-energy effective action, they will imply the necessary growth of curvature up to the scale at which string corrections invalidate the assumptions made in those theorems, i.e. the unavoidable occurrence of a long, dilaton-driven inflationary phase as long as the initial Universe was indeed at very weak coupling and very nearly flat.

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References

[1] G. Veneziano, Phys. Lett. **B265**, 287 (1991);
A.A. Tseytlin, Mod. Phys. Lett **A6** (1991) 1721;
A.A. Tseytlin and C. Vafa, Nucl. Phys. **B372** (1992) 443.
K.A. Meissner and G. Veneziano, Phys. Lett. **B267** (1991) 33;
Mod. Phys. Lett. **A6** (1991) 3397;
A. Sen, Phys. Lett. **B271** (1991) 295;
S.F. Hassan and A. Sen, Nucl. Phys. **B375** (1992) 103;
M. Gasperini and G. Veneziano, Phys. Lett. **B277** (1992) 256.

[2] G. Veneziano, ref.[1]

[3] M. Gasperini and G. Veneziano, Astropart. Phys. **1** (1993) 317;
Mod. Phys. Lett. , **A8** (1993) 3701; Phys. Rev. **D50** (1994) 2519;
R. Brustein and G. Veneziano, Phys. Lett. **B329** (1994) 429;
N. Kaloper, R. Madden and K. A. Olive, Nucl. Phys. **B452** (1995) 677;

Phys. Lett.**B371** (1996) 34;
R. Easther, K. Maeda and D. Wands, Phys. Rev.**53** (1996) 4247;
E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. **D50** (1994) 4868;
ibid. **D51** (1995) 223;
J. D. Barrow and K. E. Kunze, *Inhomogeneous String Cosmologies*, Sussex preprint (hep-th/9701085).

[4] A continuously updated collection of papers on the PBB scenario is available at `<<<<http://www.to.infn.it/teorici/gasperini/`.

[5] N. Sanchez and G. Veneziano, Nucl. Phys. **B333** (1990) 253.

[6] See also: B. A. Campbell, A. Linde and K. A. Olive, Nucl. Phys. **B355** (1991) 146.

[7] M. Gasperini, M. Maggiore and G. Veneziano, *Towards a non-singular pre-big bang cosmology*, CERN-TH/96-267 (hep-th/9611039), to appear in Nucl. Phys. B;
R. Brustein and R. Madden, *Graceful exit and energy conditions in string cosmology*, preprint BGU-PH-97/06 (hep-th/9702043).

[8] R. Brustein et al., Phys. Lett. **B361** (1995) 45;
For a recent comprehensive review, see: M. Gasperini, *Relic gravitons from the Pre-Big-Bang: what we know and what we do not know*, hep-th/9607146;
M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75** (1995) 3796;
E. J. Copeland, R. Easther and D. Wands, *Vacuum fluctuations in axion-dilaton cosmologies*, SUSSEX-TH-97001 (hep-th/9701082).

[9] See, e.g., *Inflationary Cosmology*, L. F. Abbot and So-Young Pi, Eds., World Scientific Publishing Co. (1986).

[10] V. A. Belinskii and I. M. Khalatnikov, Sov. Phys. (JETP)**36** (1973) 591, and references therein.

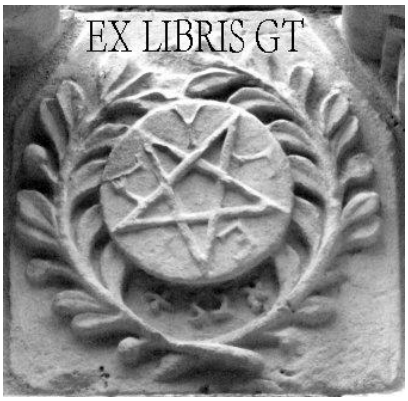
[11] N. Deruelle and D. Langlois, Phys. Rev. **D52** (1995) 2007;
O. Iguchi and H. Ishihara, *Onset of inflation in inhomogeneous cosmology*, Tokyo preprint TIT/HEP-326/COSMO-71 (1997), and references therein.

[12] J. Parry, D. S. Salopek and J. M. Stewart, Phys. Rev. **D49** (1994) 2872, and references therein.

[13] L. Landau and E. M. Lifshitz, *Théorie Classique des Champs* (Ed. Mir, Moscow, 1989).

[14] A. Linde, Phys. Lett. **129B** (1983) 177.

[15] S. W. Hawking and R. Penrose, Proc. Roy. Soc. Lond., **A314** (1970) 529.



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Pre-Big-Bang In String Cosmology

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Abstract

The duality-type symmetries of string cosmology naturally lead us to expect a pre-big-bang phase of accelerated evolution as the dual counterpart of the decelerating expansion era of standard cosmology. Several properties of this scenario are discussed, including the possibility that it avoids the initial singularity and that it provides a large amount of inflation. We also discuss how possible tracks of the pre-big-bang era may be looked for directly in the spectral and "squeezing" properties of relic gravitons and, indirectly, in the distorsion they induce on the cosmic microwave background.

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1. Introduction

The standard cosmological model (SCM) provides the best account so far for the observed properties of our present Universe. Nevertheless, in spite of its success, the SCM cannot be extended backward in time to arbitrarily large curvatures or temperatures without running into problems. We are referring here not only to the well-known phenomenological problems (e.g. horizon, flatness, entropy), but also to what is probably the main conceptual difficulty of the SCM, the initial singularity. The aim of this paper is to present a possible alternative to the initial singularity, which arises naturally in a string cosmology context because of the "large scale-small scale" symmetry (target-space duality) typical of string theory (see Refs. [1–4] for previous pioneer work along this line).

What we shall refer to here as "*string cosmology*" is just the string-theory analogue of the usual Einstein-Friedman (EF) equations of the SCM. The string-cosmology equations are thus obtained by varying the low-energy string theory effective action -describing the massless background fields- supplemented with a phenomenological source term that accounts for possible additional bulk string matter at the classical level. As long as the equation of state of the classical sources [5] is compatible with the properties of string propagation in the massless backgrounds, the resulting system of coupled equations enjoys a much larger symmetry than that of standard cosmology [6].

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This definition of string cosmology is probably insufficient for a truly "stringy" description of the Universe in the high curvature regime (near the Planck scale), where higher-order string corrections cannot be neglected, but it is certainly enough to suggest the alternative to the singularity of the SCM discussed here.

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In the SCM, curvature, source energy density, and temperature all grow as

SCM

we go backward in time: eventually, they blow up at the initial singularity, the so-called big bang. A possible way out of this conclusion is to postulate that the standard picture of space-time as a smooth manifold does not survive near and above the Planck scale, where some drastic modification of the classic geometry is

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to be introduced (see [7] for recent speculations on this subject). There are also, however, more conventional alternatives to the standard singular scenario. The two simplest possibilities are illustrated in **Fig. 1**. In the first case the curvature stops its growth after reaching a maximal value. In the second case, the curvature grows to a maximum value and then starts decreasing again as we go backward in time.

2

The first case corresponds to a de Sitter-like primordial inflationary phase with

unlimited past extension. It is constrained by phenomenological bounds (based, as we shall see in Section 4, on the properties of the relic graviton spectrum), which imply for the final curvature scale a value at least four orders of magnitude below the Planck scale. This may seem unnatural if one believes that the curvature becomes stable just because of quantum gravity effects. Moreover, it has been recently argued [8] that eternal exponential expansion with no beginning is impossible in the context of the standard inflationary scenario. If this is so, the hypothesis of a primordial phase of constant curvature does not help to avoid the problem of the initial singularity.

3 The second possibility describes the alternative to the singularity suggested by the symmetries of string cosmology. In this case, as shown in Section 4, the quoted phenomenological constraints can be evaded and the curvature can reach the Planck scale. In this context the big-bang does no longer corresponds to a singularity, but to an instant of maximal curvature marking the transition from a "string-driven" growing-curvature regime (in the sense explained in Section 2) to the decelerated evolution of the standard scenario. We shall call, in general, *"pre-big-bang"* this new phase characterized by growing curvature and accelerated evolution (as one goes forward in time).

It is important to stress that the growth of curvature, during such a phase, may be due not only (and not necessarily) to the contraction of some internal ???? space, but also to the superinflationary expansion [9] of the three physical spatial dimensions. Thus this scenario is something genuinely different from the old oscillating cosmological model, in which the present expansion is preceded by a contracting phase. Indeed, in this scenario, it is also possible for the Universe to be monotonically expanding (see Section 5). We also stress that this picture is by no means to be regarded as an alternative to the standard (even inflationary) cosmology. It is only a completion of the standard picture, which cannot be extended beyond the Planck era.

This paper is devoted to the discussion of various aspects of this possible alternative to the singularity of the SCM, and is organized as follows:

In Section 2 we shall discuss two string-theoretic motivations for the pre-big-bang scenario: one is based on the solution of the string equations of motion in a background with event horizons [5]; the second rests on a property of the low-energy string effective action, called scale-factor-duality [10–12], a particular case o of a more general $O(d,d)$-covariance of the string cosmology equations [6,13,14].

In Section 3 a simple example of a string-driven pre-big-bang scenario is used

to illustrate the behaviour of the density and of the temperature during the phase of growing curvature, and to discuss its possible relevance as a model of inflation.

If such a scenario is taken seriously, the important question to ask is whether observable tracks of the pre-big-bang era may survive and be available to present observations [cf. the observable tracks of the big bang, the electromagnetic cosmic microwave background (CMB)]. In Section 4 we show that these tracks may exist, and that they are to be looked for in the relic graviton background, either directly or through the distortions they induce on the CMB, perhaps not an impossible dream for the not-too-distant future.

In Section 5 we present solutions to the string cosmology equations which interpolate smoothly between the growing-curvature and the decreasing-curvature phases and in which both the curvature and the effective string coupling (the dilaton) are everywhere bounded. Although these solutions do not yet incorporate a completely self-consistent equation of state for the string sources, they represent good candidates for describing the background dynamics around the maximum curvature regime.

Our main conclusions are summarized in Section 6.

2. String theory motivations for the pre-big-bang scenario

2.1 String propagation in a background with shrinking event horizons

One argument supporting the pre-big-bang scenario is provided by the solution to the string equations of motion in a background manifold with event horizons [5]. Indeed, the curvature growth and the accelerated evolution, typical of a pre-big-bang epoch, are necessarily associated with event horizons whose proper size decreases in time. This has important consequences for objects of finite proper size, such as strings.

In order to present this argument, it is convenient to start by recalling some elementary, kinematical properties of Friedmann-Robertson-Walker spatially flat metrics:

$$g_{\mu\nu} = diag(1, -a^2(t)\delta_{ij}). \tag{2.1}$$

a is the scale factor, and $H = \dot{a}/a$ is the usual Hubble parameter, where a dot denotes differentiation with respect to the cosmic time t.

definitions

Consider the proper distance along a null geodesic

$$d(t) = a(t) \int_{t_1}^{t_2} dt' a^{-1}(t'). \tag{2.2}$$

In the limit $t_2 \to t_{Max}$, where t_{Max} is the maximal future extension of the cosmic time coordinate on the given background manifold, $d(t_1)$ defines the proper size of the *event horizon* at the time t_1 (i.e. the maximal size of the space-time region within which a causal connection can be established). In the limit $t_1 \to t_{min}$, where t_{min} is the maximal past extension of the cosmic time coordinate, $d(t_2)$ defines instead the *particle horizon* at the time t_2 (i.e. the maximal portion of space which can be included inside the past light cone of a given observer).

By means of these definitions, and by using $|H|$ as an indicator of the curvature scale, it is possible to provide a sort of "causal" classification of the isotropic and homogeneous backgrounds based on their asymptotic behaviour, which is reported in **Table I** for the expanding case, and in **Table II** for the contracting one. Note that, in both tables, the linear evolution of the horizon is referred to the cosmic time coordinate; note also the "dual" symmetry which exchanges particle with event horizon when passing from expanding to contracting backgrounds.

1

For the purpose of this paper, three points have to be stressed. The first point is that we are in presence of event horizons only if

$$sign\{\dot{a}\} = sign\{\ddot{a}\}, \tag{2.3}$$

that is only in the case of accelerated evolution. Inflation, on the other hand, is accelerated expansion, and inflation plus the simultaneous shrinking of some internal dimension may be self-sustained (in the absence of dominant vacuum contributions) only in the case of accelerated contraction [15]. Event horizons thus appear naturally during a phase of dimensional decoupling.

2

The second point to be stressed is that, for an accelerated evolution, the proper size of the regions that are initially in causal contact tends to evolve in time, asymptotically, like the scale factor $a(t)$. [This can be easily seen from eq. (2.2) by considering a causally connected region of proper initial size $d(t_1) = t_1$, in a background which is, for example, in accelerated expansion, i.e. $a \sim t^\alpha$, $\alpha > 1$ for $t \to \infty$. One then finds, for $t_2 >> t_1$, that $d(t_2) \simeq d(t_1)a(t_2)/a(t_1)$]. Comparing this behaviour with the time evolution of the event horizon, we may thus conclude that the regions which are initially in causal contact grow faster that the horizon in the case of accelerated expansion, and contract more slowly than the horizon

in the case of accelerated contraction. In both cases, they will tend to cross the horizon. This is not a source of problems for point-like objects, of course, but extended sources may become, in such a situation, larger than the causal horizon itself.

The third important point is that, according to Tables I and II, the curvature is growing only if

$$sign\{\dot{a}\} = sign\{\dot{H}\}. \tag{2.4}$$

A typical pre-big-bang configuration with accelerated evolution and growing curvature is thus necessarily associated with the presence of an event horizon which shrinks linearly with respect to cosmic time: extended objects, in such backgrounds, are doomed to become causally disconnected, quite independently of their proper size.

This effect is peculiar to gravity, and can be easily understood in terms of tidal forces [16]. Consider indeed a free-falling object, of finite size λ, embedded in a cosmological background. Each point of the extended object will fall, in general, along different geodesics. In the free-falling frame of one end of the object, the other end, at a proper distance $|z| = \lambda$, will have a relative acceleration given by the equation of geodesic deviation,

$$|\frac{Du^\mu}{Ds}| = |R^\mu{}_{\nu\alpha\beta} u^\nu u^\alpha z^\beta| = \lambda |\frac{\ddot{a}}{a}|. \tag{2.5}$$

This acceleration defines a local Rindler horizon at a distance

$$d = \lambda^{-1} |\frac{\ddot{a}}{a}|^{-1}, \tag{2.6}$$

which depends only on the background curvature. If the curvature is growing, this distance becomes smaller and smaller, until the two ends become causally disconnected.

In the case of strings, the existence of a causally disconnected asymptotic regime implies that an approximate description of the string motion around the classical path of a point particle is no longer allowed [17]. Suppose indeed that the exact solution of the string equations of motion and constraints in a cosmological background is expanded around a comoving geodesic representing the point-particle motion of the centre of mass of the string [18]. Such an expansion is valid, provided [5,17]

$$|\dot{X}^i| \sim |\frac{\partial X^i}{\partial \sigma}| \tag{2.7}$$

(X^i is any spatial component of the target space coordinates, and σ is the usual world-sheet space variable) and provided the first-order fluctuations, whose Fourier components satisfy the linearized equation

$$\ddot{\chi}^i_n + (\frac{n^2}{\lambda^2} - \frac{\ddot{a}}{a})\chi^i_n = 0, \tag{2.8}$$

stay small [$\chi^i = a(t)X^i$, and $\lambda = m\alpha'$, where m is the string mass and α' the string tension].

If the background curvature is growing, this approximation breaks down asymptotically. Indeed, in the high curvature limit $|\ddot{a}/a| >> n^2\lambda^2$, the asymptotic solution to the fluctuation equation (2.8) can be written in general as

$$X^i = A + B\int \frac{dt}{a^2} \tag{2.9}$$

with A and B independent of time, and one obtains

$$|\dot{X}^i| << |\frac{\partial X^i}{\partial \sigma}| \tag{2.10}$$

in the case of expanding backgrounds, and

$$|\dot{X}^i| >> |\frac{\partial X^i}{\partial \sigma}| \tag{2.11}$$

in the contracting case.

In both cases the condition (2.7) is not satisfied, and the geodesic expansion is no longer valid, asymptotically. In both cases, however, the exact solution can be expanded around a new asymptotic configuration [5,19,20], which is "unstable", in the sense that it is non-oscillating with respect to the world-sheet time.

What is important, in our context, is that this kind of solution, when inserted into the string stress tensor, leads in general to a self-sustained scenario. In other words it leads to an effective equation of state characterizing a source which can drive, by itself, the phase of accelerated evolution and growing curvature. In the perfect fluid approximation and, in particular, for a background with d expanding and n contracting dimensions, such an equation of state takes the form [5]

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$$\rho + dp - nq = 0 \tag{2.12}$$

(ρ is the energy density, p and q are respectively the pressure in the expanding space and in the internal shrinking dimensions).

A kind of explanation of KK shrinking while U expansion

According to this equation of state it is possible to obtain a "string-driven" pre-big-bang phase not only during an anisotropic situation of dimensional decoupling [5], but also in the case of isotropic expansion ($n = 0$) or isotropic contraction ($d = 0$), provided the correct string gravity equations, including the dilaton, are used instead of the Einstein equations [6]. This is to be contrasted with the case of point-like sources, where it is impossible to achieve, asymptotically, the causally disconnected regime characterized by non-oscillating configurations, and it is thus impossible to arrange a self-consistent equation of state able to sustain the pre-big-bang phase.

2.2 Scale factor duality

A second, probably stronger, motivation supporting the pre-big-bang picture follows from a property of the low-energy string effective action, which can be written in D dimensions

$$S = -\frac{1}{16\pi G_D}\int d^D x\sqrt{|g|}e^{-\phi}(R + \partial_\mu\phi\partial^\mu\phi - \frac{H^2}{12} + V) \qquad (2.13)$$

(here $H = dB$ is the antisymmetric tensor field strength and V is a constant which is vanishing in a critical number of space-time dimensions).

The cosmological equations obtained from this action for a homogeneous and spatially flat background are characterized by a symmetry called "scale factor duality" [10]. According to this symmetry, for any given set of solutions $\{a_i(t), \phi(t), 1 = 1, ..., D-1\}$ (a_i are the scale factors of a diagonal, not necessarily isotropic metric in the synchronous frame gauge $g_{00} = 1, g_{0i} = 0$), the configuration obtained through the transformation (for each i)

$$a_i \to \tilde{a}_i = a_i^{-1} \quad , \quad \phi \to \tilde{\phi} = \phi - 2\ln a_i \qquad (2.14)$$

is still a solution of the graviton-dilaton system of equations [10–12].

This transformation is just a particular case of a more general global $O(d,d)$ covariance of the theory [13,14] (d is the number of coordinates upon which the background fields are explicitly independent, in our case the $D-1$ spatial coordinates). In the case of anisotropic backgrounds with generally non-diagonal metrics, this covariance, besides transforming the dilaton field as

$$\phi \to \phi - \ln|\det g_{ij}| \qquad (2.15)$$

also mixes non-trivially the components of the metric and of the antisymmetric tensor as follows:

$$M \to \Omega^T M \Omega, \tag{2.16}$$

where Ω represents a global $O(d,d)$ transformation, and

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{2.17}$$

($G \equiv g_{ij}$ and $B \equiv B_{ij} = -B_{ji}$ are matrix representations of the d by d spatial part of the metric and of the antisymmetric tensor, in the basis in which the $O(d,d)$ metric is in off-diagonal form [6,13]).

In the case of manifolds with spatial sections of finite volume (such as a torus), i.e. $(\int d^d x \sqrt{|G|})_{t=t_1} = const < +\infty$, this $O(d,d)$ covariance can be preserved even if the scalar potential V is taken dilaton-dependent, provided ϕ appears in the potential only through the combination [13]

$$\overline{\phi} = \phi - \ln \sqrt{|G|} \tag{2.18}$$

(we have absorbed into ϕ the constant shift $-\ln \int d^d x$ required to secure the $GL(d)$ coordinate invariance of the corresponding non-local action).

Moreover, the $O(d,d)$ covariance holds even if the equations are supplemented by phenomenological source terms corresponding to bulk string matter [6]. Indeed, their effective equation of state can be expressed in terms of the $2d$-dimensional phase-space variables $Z^A = (P_i, \partial X^i/\partial\sigma)$, representing a solution of the string equations of motion: when the background is changed according to eq. (2.16), the new solution $\tilde{Z} = \Omega^{-1} Z$ generates a new effective equation of state which leaves the overall $O(d,d)$ symmetry unbroken [6].

The importance of scale factor duality, in our context, is that it allows the construction of a mapping relating any decreasing curvature background to a corresponding scenario in which the curvature is increasing.

It is true, indeed, that the Hubble parameter is odd under scale factor duality, $H \to -H$, so that a duality transformation $a \to a^{-1}$ maps an expanding solution of Table I into a contracting one of Table II (and vice versa). However, the cosmological solutions with monotonic evolution of the curvature (see for instance Section 3) are in general defined on a half-line in t, namely $-\infty \leq t \leq t_c$ and $t_c \leq t \leq \infty$, where t_c is some finite value of the cosmic time where a curvature singularity occurs [10]. We are thus allowed, in such a case, to change simultaneously the sign of $\dot{H}$ and $\ddot{a}$ inside a given table (i.e. keeping the sign of $\dot{a}$ constant), by

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performing a duality transformation and by changing, simultaneously, the domain of t from $[t_c, \infty]$ to $[-\infty, -t_c]$, i.e. by performing the inversion $t \to -t$.

This means that, to any given decelerated solution, with monotonically decreasing curvature, typical of the standard post-big-bang cosmology, it is always possible to associate a pre-big-bang solution, with accelerated evolution and increasing curvature, through the transformation

$$a(t) \to a^{-1}(-t). \tag{2.19}$$

dilaton, dilaton, dilaton!!!!

It is important to stress that this mapping property (valid in the cases of both expanding and contracting backgrounds) cannot be achieved in the framework of Einstein's equations, where there is no dilaton. In that case, scale factor duality is broken; one is just left with the more conventional time reversal symmetry $a(t) \to a(-t)$.

Unlike straight scale-factor-duality, the transformation (2.19) has non-trivial fixed points $a(t) = a^{-1}(-t)$ describing models that are monotonically expanding (or contracting), since $H(t) = H(-t)$, and in which the duration of the primordial superinflationary evolution equals that of the dual decelerated phase, as $\dot{H}(t) = -\dot{H}(-t)$. We shall come back on this point in the following section.

3. An example of pre-big-bang dynamics

Consider a spatially flat background configuration, with vanishing antisymmetric tensor and dilaton potential. The dilaton field depends only upon time, and the metric, which is assumed to describe a phase of dimensional decoupling in which d spatial dimensions expand with scale factor $a(t)$, and n dimensions contract with scale factor $b(t)$, is given by

$$g_{\mu\nu} = diag(1, -a^2(t)\delta_{ij}, -b^2(t)\delta_{ab}) \tag{3.1}$$

(conventions: $\mu, \nu = 1, ..., D = d + n$; $i, j = 1, ..., d$; $a, b = 1, ..., n$).

We are working in a synchronous frame in which $g_{00} = 1$, $g_{0i} = 0 = g_{0a}$, so that the time parameter t coincides with the usual cosmic time coordinate. In this gauge, the stress tensor of a comoving source, in the perfect fluid approximation, becomes

$$T_\mu{}^\nu = diag(\rho(t), -p(t)\delta_i^j, -q(t)\delta_a^b), \tag{3.2}$$

where p and q are the pressures in the expanding and contracting space, respectively. It is also convenient to introduce the $O(d+n,d+n)$-invariant expressions for the dilaton and the matter energy density, that in this background take the form

$$\overline{\phi} = \phi - \ln\sqrt{|g|} = \phi - d\ln a - n\ln b \tag{3.3}$$

$$\overline{\rho} = \rho\sqrt{|g|} = \rho a^d b^n \tag{3.4}$$

(we also define, for the pressure, $\overline{p} = p\sqrt{|g|}$).

With these definitions, the equations obtained by varying the effective action (2.13) (with $V = 0$) and the action for the matter sources,

$$2(R_\mu{}^\nu + \bigtriangledown_\mu \bigtriangledown^\nu \phi) - \frac{1}{2}H_{\mu\alpha\beta}H^{\nu\alpha\beta} = 16\pi G_D e^\phi T_\mu{}^\nu \tag{3.5}$$ "einstein"

$$R - (\bigtriangledown_\mu\phi)^2 + 2\bigtriangledown_\mu\bigtriangledown^\mu\phi - \frac{1}{12}H_{\mu\nu\alpha}H^{\mu\nu\alpha} = 0 \tag{3.6}$$ dilaton

$$\partial_\nu(\sqrt{|g|}e^{-\phi}H^{\nu\alpha\beta}) = 0 \tag{3.7}$$ hubble

($\bigtriangledown_\mu$ is the Riemann covariant derivative), can be written in a particularly simple form (when $B = 0$). The dilaton equation (3.6) reduces to

$$\dot{\overline{\phi}}^2 - 2\ddot{\overline{\phi}} + dH^2 + nF^2 = 0, \tag{3.8}$$

where $H = \dot{a}/a$, $F = \dot{b}/b$. Moreover, by combining this equation with the time component of (3.5) we get

$$\dot{\overline{\phi}}^2 - dH^2 - nF^2 = 16\pi G_D\overline{\rho}e^{\overline{\phi}}. \tag{3.9}$$

From the space components of (3.5) we have finally the equations for the external and internal pressure

$$2(\dot{H} - H\dot{\overline{\phi}}) = 16\pi G_D\overline{p}e^{\overline{\phi}} \tag{3.10}$$

$$2(\dot{F} - F\dot{\overline{\phi}}) = 16\pi G_D\overline{q}e^{\overline{\phi}}. \tag{3.11}$$

[Note that the four equations (3.8) to (3.11) are only a particular case of the o $O(d,d)$-covariant string cosmology equations of Ref. [6], for vanishing B,V and a perfect fluid description of the matter sources].

Suppose now that $a(t)$ describes accelerated expansion, $b(t)$ accelerated contraction, and that

$$p = \gamma_1\rho \quad , \quad q = \gamma_2\rho \quad , \quad b = a^{-\epsilon} \tag{3.12}$$

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b = a^-1

with $\gamma_1, \gamma_2, \epsilon$ constants, $\epsilon > 0$. "Stringy" sources, asymptotically consistent with such background [5], must satisfy the equation of state (2.12), which implies

$$d\gamma_1 - n\gamma_2 = -1. \tag{3.13}$$

As discussed in [5], the three possible asymptotic behaviours, $|q| << |p|$,$|q| >> |p|$ and $|q| \sim |p|$ correspond to $\epsilon < 1$, $\epsilon > 1$ and $\epsilon = 1$, respectively. In the first two cases, however, eqs. (3.8) to (3.11) can be consistently solved only for $\rho = p = q = 0$, so that one recovers the vacuum background [10] with a Kasner-like metric solution, and $\overline{\phi} \sim -\ln|t|$.

If we look for solutions with non-vanishing bulk string matter we have thus to consider the third possibility, $b = a^{-1}$. In such a case eqs. (3.10) and (3.11) imply $p = -q$, and the condition (3.13) is non-trivially satisfied by

$$\gamma_1 = -\gamma_2 = \frac{-1}{d+n}. \tag{3.14}$$

We can thus obtain, from eqs. (3.8) to (3.10), the particular exact solution

$$\begin{aligned} a(t) = b^{-1}(t) = (-\frac{t}{t_0})^{-2/(d+n+1)}, \\ \phi = \phi_0 + 2d\ \ln\ a \quad , \quad \rho = \rho_0 a^{n+1-d}, \end{aligned} \tag{3.15}$$

where t_0, ρ_0, ϕ_0 are integration constants, related by

$$\rho_0 e^{\phi_0} = 4\frac{(d+n)(d+n-1)}{(d+n+1)^2}. \tag{3.16}$$

This solution satisfies the properties

$$\begin{aligned} \dot{a} > 0 \quad , \quad \ddot{a} > 0 \quad , \quad \dot{H} > 0 \\ \dot{b} < 0 \quad , \quad \ddot{b} < 0 \quad , \quad \dot{F} < 0 \end{aligned} \tag{3.17}$$

Therefore, in the $t \to 0_-$ limit, it describes a background in which the evolution is accelerated, since $\{\dot{a}, \ddot{a}\}$ and $\{\dot{b}, \ddot{b}\}$ have the same sign, and the curvature is growing, since $\{\dot{a}, \dot{H}\}$ and $\{\dot{b}, \dot{F}\}$ have the same sign (recall Tables I and II). Moreover, the equation of state of the sources,

$$p = -q = -\frac{\rho}{d+n} \tag{3.18}$$

is compatible with the solution of the string equations of motion, so this background is a typical example of *string-driven pre-big-bang* cosmology.

It may be interesting to note that, in the four-dimensional isotropic limit $(n = 0, d = 3)$, the solution (3.15) simply reduces to

$$a = (\frac{-t}{t_0})^{-1/2} \quad , \quad \phi = \phi_0 - 3\ln(-\frac{t}{t_0}) \quad , \quad \rho = -3p = \rho_0(-\frac{t}{t_0}), \tag{3.19}$$

namely to the superinflationary background obtained, via the duality transformation (2.14), from the standard radiation-dominated cosmology (with constant dilaton),

$$a = (\frac{t}{t_0})^{1/2} \quad , \quad \phi = \phi_0 \quad , \quad \rho = 3p = \rho_0(\frac{t}{t_0})^{-2}, \tag{3.20}$$

as already pointed out in [6].

The simple example (3.15) can be used to illustrate two important properties of the pre-big-bang phase: not only the curvature, but also the total effective energy density and the temperature are growing together with the curvature.

For the behaviour of ρ we have indeed, from eq. (3.15):

$$\rho(t) = \rho_0(-\frac{t}{t_0})^{2(d-n-1)/(d+n+1)}. \tag{3.21}$$

Since we are working in a Brans-Dicke frame, we have to include also the time variation of the gravitational constant, $G \sim e^{\phi}$, in the term that plays the role of the total gravitational source. We thus obtain, for the effective energy density,

$$G\rho \sim \rho_0 a^{d+n+1} = \rho_0(-\frac{t}{t_0})^{-2}, \tag{3.22}$$

which is always growing for $t \to 0_-$.

As far as the temperature behaviour is concerned, it must be recalled that, according to the string cosmology equations (3.5) to (3.7), the dilaton gives no contribution to the covariant conservation of the energy of the sources (in spite of its coupling to the tensor $H_{\mu\nu\alpha}$ [6]). Moreover, when $H_{\mu\nu\alpha} = 0$, such a conservation equation defines a regime of adiabatic evolution for the perfect fluid. By using the standard thermodynamical arguments (see for instance [21]) one can thus obtain, for any given equation of state $p = \gamma\rho$, a relation between temperature and scale factor which, in the case of d isotropic spatial dimensions, takes the form

$$a(T) \sim T^{-1/d\gamma}. \tag{3.23}$$

The conservation equation, on the other hand, provides also a relation between scale factor and energy density:

$$\rho(a) \sim a^{-d(1+\gamma)}. \tag{3.24}$$

For a standard radiation-dominated background ($\gamma = 1/d$), such as that of eq. (3.2), one then finds the usual adiabatic decreasing of the temperature:

$$\rho \sim \frac{1}{a^{d+1}} \sim T^{d+1} \quad , \quad T \sim \frac{1}{a}. \tag{3.25}$$

For the dual background (3.19), on the contrary, $\gamma = -1/d$ and the temperature increases together with the scale factor,

$$\rho \sim \frac{1}{a^{d-1}} \sim \frac{1}{T^{d-1}} \quad , \quad T \sim a. \tag{3.26}$$

The same result holds in the more general anisotropic pre-big-bang background of eq. (3.15), with equation of state (3.18). Indeed in such a case the conservation equation reads

$$\dot{\overline{\rho}} = H\overline{\rho} \tag{3.27}$$

and implies

$$\rho(a) \sim a^{n+1-d}. \tag{3.28}$$

In terms of the total proper volume $V = V_d V_n = a^{d-n}$, eq. (3.27) can be rewritten as an adiabaticity condition

$$dS \equiv \frac{1}{T} d(\rho V) + \frac{1}{T}(p\frac{dV_d}{V_d} + q\frac{dV_n}{V_n})V = \frac{1}{T} d(\rho V) + \frac{p'}{T} dV = 0 \tag{3.29}$$

corresponding to the effective pressure

$$p' = p(\frac{d+n}{d-n}) = \frac{\rho}{n-d}. \tag{3.30}$$

The integrability condition $\partial^2 S/\partial V \partial T = \partial^2 S/\partial T \partial V$ then gives

$$\rho(t) \sim T^{n+1-d}, \tag{3.31}$$

which, using eq.(3.28), leads to

$$T \sim a \sim (-t)^{-2/(n+1+d)}. \tag{3.32}$$

Thus we find that, during the pre-big-bang phase, the temperature, also in the anisotropic case, grows proportionally to the expanding scale factor.

Two comments are in order:

The first is that, according to this example, the temperature seems to remain unchanged ($T = \tilde{T}$) when passing from a background a to the dual one $\tilde{a} = a^{-1}$. Indeed, from eqs. (3.25) and (3.26),

$$\tilde{T} \sim \tilde{a} = a^{-1} \sim T. \tag{3.33}$$

This behaviour is to be contrasted with recent results, obtained in a black-hole background, which suggest that the temperature should be inverted under a duality transformation [22].

We must note, however, that the pre-big-bang solution considered here is related to the decelerated post-big-bang phase not only through a duality transformation, but also through a change of the cosmic time range from $[-\infty, 0]$ to $[0, \infty]$. Moreover, what we have considered is the temperature of the sources, and not the horizon temperature as in the black hole case.

Indeed, the appearance of a dynamical event horizon during the pre-big-bang phase is not sufficient to introduce a thermal bath of geometric origin, and to define an intrinsic "background temperature". The reason is that, in contrast with the Rindler or de Sitter case, the Green functions for fields embedded in a manifold with shrinking horizons are not, in general, periodic with respect to the Euclidean time coordinate. In other words, a De Witt detector [23] in the frame of a comoving pre-big-bang observer will measure a particle background, but the response function of the detector is not of the thermal type and cannot define an intrinsic temperature.

Moreover, even introducing (naïvely) a temperature associated with the pre-big-bang event horizon in terms of its local surface gravity, $T_H \sim |t|^{-1}$, it would be impossible in this cosmological context to discuss the transformation of the temperature under duality by using the horizon thermodynamics, because there is no event horizon in the dual, decelerated phase.

It is possible, however, to obtain a thermal spectrum from the geometry, in a cosmological context, by considering the high frequency sector of the particle spectrum produced by a background that evolves smoothly between two asymptotically flat states [23–25]. In this way one can associate a temperature with the geometry, independently from the presence of event horizons, and one can easily show that for a "self-dual" metric [i.e. a metric that satisfies $a^{-1}(t) = a(-t)$, see for instance Section. 5], this "geometric" temperature is inverted under duality, just as in the black hole case.

The demonstration is based on the properties of the Bogoliubov coefficients, $c_+(k)$ and $c_-(k)$, parametrizing for each mode k the transformation between the $|in>$ and $|out>$ vacuum, and normalized in such a way that $|c_-(k)|^2 =< n_k >$ measures the spectral number of produced particles, and $|c_+|^2 = 1 + |c_-|^2$.

Consider indeed the transformation $t \to -t$, connecting an expanding to a contracting background. The $|in>$ and $|out>$ states get interchanged, but the

constant equilibrium temperature T_0 associated with the thermal distribution of the produced particles is preserved, as it depends on the norm of the Bogoliubov coefficients, which is left unchanged

$$|c_+| = | < in, +|out, + > | \quad , \quad |c_-| = | < in, +|out, - > |. \tag{3.34}$$

The local temperature $T(t) = T_0/a(t)$, characteristic of the given geometry, changes however from a regime of adiabatic red-shift to an adiabatic blue-shift, $T(t) \to T(-t) = T_0/a(-t)$. If the metric is self-dual we obtain, in units of T_0,

$$T(a(-t)) = T(a^{-1}(t)) \equiv \tilde{T}(t) = a(t) = \frac{1}{T(a(t))} \tag{3.35}$$

so that the temperature and its dual $\tilde{T}$ satisfy

$$T\tilde{T} = T_0^2 = const, \tag{3.36}$$

in agreement with earlier arguments of string thermodynamics in a cosmological context [2,3].

The second comment regards the growth of the temperature of the source during the pre-big-bang phase [see eqs. (3.25) and (3.26)]. In the regime of radiation-dominated expansion, aT stays constant, and the usual way to get the very large present value of the entropy, starting from reasonable initial conditions, is through the occurrence of some non-adiabatic "reheating" era. In the dual pre-big-bang regime, on the contrary, the temperature of the sources grows with the scale factor just because of the adiabatic evolution. It is thus possible (at least in principle) for the Universe to emerge from the big-bang with hot enough sources and large enough scale factor to solve the entropy and the horizon problems.

Consider, for example, a cosmological evolution which is symmetric around the phase of maximal curvature, in the sense that the post-big-bang decelerated expansion of the three-dimensional space [eq. (3.20)] is preceeded by a corresponding phase of accelerated expansion [eq. (3.19)], which extends in time as long as the dual decelerated phase.

The pre-big-bang superinflation ends at the maximum curvature scale, which is expected to be of the order of the Planck mass, $H_f \sim M_P$. On the other hand, our present curvature scale H_0 is about 60 orders of magnitude below the Planck scale. In the case of a symmetric evolution evolution around the maximum (see e.g. the example presented in Section 5), the pre-big-bang superinflation era must start at a scale $H_i \lesssim H_0 \sim 10^{-60} M_P$, thus providing a total amount of inflation

$$\frac{a_f}{a_i} \sim (\frac{H_f}{H_i})^{1/2} \gtrsim 10^{30}, \tag{3.37}$$

which is just the one required for the solution of the standard cosmological puzzles [26].

Apart from this amusing numerical coincidence, what is important to stress is that in the case of a symmetric temporal extension the pre-big-bang scenario seems capable of providing automatically the required amount of inflation, *for any given value of the final observation time* t_0, thus avoiding the anthropic problem that is in general included in the more conventional, post-big-bang, inflationary models (see [27] for a lucid discussion of the anthropic principle in an inflationary context).

It should be clear, however, that even in this context the large present value of the total entropy would be the consequence of a non-adiabatic conversion of the hot string gas into hot radiation, occurring during the transition to the standard scenario. A possible example of such process is the decay of highly excited string states, created by the background evolution through the mechanism described (for the case of gravitons) in the following section. One should also mention, in this context, the existence of an additional source of radiation entropy due to the presence of non-trivial antisymmetric tensor backgrounds. Indeed, even if the evolution is globally adiabatic, entropy exchanges may occur between B and the fluid part of the source, because of the covariant conservation of the total energy density. As a consequence, the damping of B in time (which is expected to occur if the standard isotropic scenario is to be recovered) should be accompanied by an entropy increased in the radiation fluid, as already stressed in [6].

4. Observable tracks of a pre-big-bang phase

As discussed in the previous sections, there are various reasons for which the occurrence of a period of accelerated evolution and growing curvature, in the past history of our Universe, may seem both plausible (from a string theory point of view) and attractive (for its phenomenological aspects). The important question to ask, therefore, is whether the occurrence of such a phase may be tested experimentally in some way.

An affirmative answer to this question is provided by the study of the cosmic graviton background [28–30]: indeed, gravitons decoupled from matter earlier than any other field (nearly at the Planck scale), so that the shape of the spectrum of the gravitons produced by the transition from the pre-big-bang to the post-big-bang

scenario should survive, nearly unchanged, up to the present time. Given a model of cosmological evolution, in particular, one can compute the expected graviton spectrum, and then constrain the model by using the present observational bounds.

Consider, for example, a generic model of pre-big-bang evolution in which the dilaton is growing, d dimensions expand with scale factor $a(\eta)$ and n dimensions contract with scale factor $b(\eta)$ in such a way that, for $\eta < -\eta_1 < 0$,

$$a \sim (-\eta)^{-\alpha} \quad , \quad b \sim (-\eta)^{\beta} \quad , \quad \phi \sim \gamma \ln a, \tag{4.1}$$

where η is the conformal time coordinate, related to the cosmic time t by $dt/d\eta = a$ (we shall denote, in this section, derivatives with respect to η with a prime). Note that in eq. (4.1) η ranges over negative values, so that α, β and γ are all positive parameters. We shall assume that this phase is followed (at $\eta = -\eta_1$) by the standard radiation-dominated, and matter-dominated (at $\eta = \eta_2$), expansion of three spatial dimensions, with frozen dilaton and radius of the internal space, namely

$$\begin{aligned} &a \sim \eta \quad , \quad b = 1 \quad , \quad \phi = \phi_0 \quad , \quad -\eta_1 < \eta < \eta_2, \\ &a \sim \eta^2 \quad , \quad b = 1 \quad , \quad \phi = \phi_0 \quad , \quad \eta > \eta_2 > 0. \end{aligned} \tag{4.2}$$

The spectrum of the gravitons produced from the vacuum, because of the background variation, is to be computed from the free propagation equation of a metric fluctuation, $h_{\mu\nu} = \delta g_{\mu\nu}$, obtained by perturbing the background equations at fixed sources [31–33] (in our case dilaton and antisymmetric tensor included), $\delta T^{\nu}_{\mu} = \delta\phi = \delta H_{\mu\nu\alpha} = 0$. The contribution to the graviton production from a possible variation of the effective gravitational coupling can be accounted for, in general, by perturbing a Brans-Dicke background [34]. In our case we shall perturb the string gravity equations (3.5) to (3.7) around the cosmological configuration $\{H_{\mu\nu\alpha} = 0, \phi = \phi(t), g_{\mu\nu} = diag(1, g_{ij}, g_{ab})\}$, where $g_{ij} = -a^2\delta_{ij}$ and $g_{ab} = -b^2\delta_{ab}$ are the d- and n-dimensional conformally flat metrics of the expanding and contracting manifolds, respectively.

By imposing the gauge

$$h_{0\mu} = 0 \quad , \quad \bigtriangledown_{\nu} h_{\mu}{}^{\nu} = 0 \quad , \quad g^{\mu\nu} h_{\mu\nu} = 0, \tag{4.3}$$

and by considering a perturbation $h_i{}^j(\vec{x}, t)$ propagating in the expanding "external" space, one then gets the linearized equation [34]

$$g^{\mu\nu} \bigtriangledown_{\mu} \bigtriangledown_{\nu} h_i{}^j - \dot{\phi}\dot{h}_i{}^j = 0, \tag{4.4}$$

where ∇_μ denotes the covariant derivative with respect to the background metric g. By performing a Fourier expansion, and introducing the variable

$$\psi_i{}^j = h_i{}^j b^{n/2} a^{(d-1)/2} e^{-\phi/2}, \tag{4.5}$$

one can conveniently rewrite eq. (4.4), for each mode $\psi_i{}^j(k)$ in terms of the conformal time variable, in the form:

graviton spectrum

$$\psi''(k) + [k^2 - V(\eta)]\psi(k) = 0, \tag{4.6}$$

where

$$\begin{aligned} V(\eta) = {} & \frac{d-2}{2}\frac{a''}{a} + \frac{n}{2}\frac{b''}{b} - \frac{\phi''}{2} + \frac{1}{4}(d-1)(d-3)(\frac{a'}{a})^2 + \frac{n}{4}(n-2)(\frac{b'}{b})^2 \\ & + \frac{1}{4}\phi'^2 + \frac{n}{2}(d-1)\frac{a'}{a}\frac{b'}{b} - \frac{1}{2}(d-1)\frac{a'}{a}\phi' - \frac{n}{2}\frac{b'}{b}\phi' \end{aligned} \tag{4.7}$$

For the phenomenological background (4.1), (4.2) we thus have the effective potential barrier

$$\begin{aligned} V(\eta) &= \frac{1}{4\eta^2}\{[\alpha(d-1-\gamma) - n\beta + 1]^2 - 1\} \quad , \quad \eta < -\eta_1 \\ V(\eta) &= 0 \quad , \quad -\eta_1 < \eta < \eta_2 \\ V(\eta) &= \frac{2}{\eta^2} \quad , \quad \eta > \eta_2, \end{aligned} \tag{4.8}$$

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which clearly displays the usual contribution of the accelerated expansion ($\alpha \neq 0$) [28,29], of the dimensional reduction process ($n\beta \neq 0$) [35,36] and, in addition, of the dilaton variation ($\gamma \neq 0$) [34].

The general solution of eq. (4.6) can be written in terms of the first- and second-kind Hankel functions $H_\nu^{(1)}(k\eta), H_\nu^{(2)}(k\eta)$, of index

$$\nu = \frac{1}{2}[\alpha(d-1-\gamma) - n\beta + 1] \tag{4.9}$$

during the pre-big-bang phase, and of index $|\nu| = 1/2$ and $|\nu| = 3/2$ during the radiation- and matter-dominated phase, respectively. Starting with the initial solution

$$\psi(k) = N\eta^{1/2} H_\nu^{(2)}(k\eta) \quad , \quad \eta < -\eta_1 \tag{4.10}$$

representing positive frequency modes in the $\eta \to -\infty$ limit (and corresponding to the Bunch-Davies "conformal" vacuum [23]), one has in general a linear combination of positive ($H^{(2)}$) and negative ($H^{(1)}$) frequency solutions for $\eta \to \infty$.

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The Bogoliubov coefficients $c_\pm(k)$, describing such mixing, can be obtained by matching the solutions at $\eta = \eta_1$ and $\eta = \eta_2$ ($\eta_2 >> \eta_1$).

For $k\eta_1 < 1$, using the small argument limit of the Hankel functions, and considering only the first background transition, this "sudden" approximation gives

$$|c_-(k)| \simeq (k\eta_1)^{-|\nu+1/2|} \tag{4.11}$$

(we neglect numerical factors of order unity). For $k\eta_1 \gtrsim 1$, i.e. for modes with a comoving frequency k higher than the potential barrier $[V(\eta_1)]^{1/2} \simeq \eta_1^{-1}$, this approximation is no longer valid; then the mixing coefficient c_- is to be computed by replacing the potential step with a smooth transition from $V(\eta_1)$ to 0 (in order to avoid ultraviolet divergences). In this way one finds, however, that for $k\eta_1 > 1$ the mixing is exponentially suppressed [23,25,35], so that it may be neglected for our purposes.

The modes with sufficiently small frequency $k < 1/\eta_2$ are significantly affected also by the second background transition, from the radiation-dominated to the matter-dominated regime [28,37-39]. In the same approximation one finds that, for this frequency sector,

$$|c_-(k)| \simeq (k\eta_1)^{-|\nu+1/2|}(k\eta_2)^{\mp 1}, \tag{4.12}$$

where the sign of the second exponent is -1 ($+1$) if $\nu + 1/2 > 0$ (< 0) [34]. For all the models of dynamical dimensional decoupling proposed up to now (see for instance [5,15,40]), including the pre-big-bang example of the previous section, one has however $\nu + 1/2 > 0$. We shall thus disregard here, for simplicity, the alternative possibility (but we note that for $\nu + 1/2 < 0$ the phenomenological constraints on the graviton spectrum that we shall present below turn out to be somewhat relaxed [34]).

The spectral energy density $\rho(\omega) = \omega(d\rho_g/d\omega)$, which is the variable usually adopted to characterize the energy distribution of the produced gravitons [28,29,39], is simply related to the Bogoliubov coefficient $c_-(k)$ that gives, for each mode, the final number of produced particles:

$$\rho(\omega) \simeq \omega^4 |c_-|^2, \tag{4.13}$$

where $\omega = k/a(t)$ is the proper frequency (again we neglect numerical factors of order unity). On the other hand, $|\eta|^{-1} \simeq |a(\eta)H(\eta)|$; moreover, η_1 corresponds to the beginning of the radiation era, so we can conveniently express the maximum

curvature scale $H_1 \equiv H(\eta_1)$, reached at the end of the pre-big-bang phase, as $H_1^2 \simeq G\rho_\gamma(\eta_1)$, where G is the usual Newton constant, and ρ_γ is the radiation energy density [29]. It follows that the spectral energy density (4.13), at the present observation time t_0, and in units of critical energy density ρ_c, can be finally expressed as

$$\begin{aligned} \Omega(\omega, t_0) \equiv \frac{\rho(\omega)}{\rho_c} &\simeq GH_1^2\Omega_\gamma(t_0)\left(\frac{\omega}{\omega_1}\right)^{4-2|\nu+1/2|} \quad , \qquad \omega_2 < \omega < \omega_1 \\ \Omega(\omega, t_0) \equiv \frac{\rho(\omega)}{\rho_c} &\simeq GH_1^2\Omega_\gamma(t_0)\left(\frac{\omega}{\omega_1}\right)^{4-2|\nu+1/2|}\left(\frac{\omega}{\omega_2}\right)^{-2} \quad , \quad \omega_0 < \omega < \omega_2 \end{aligned} \tag{4.14}$$

Here $\Omega_\gamma(t_0) \sim 10^{-4}$ is the fraction of critical energy density present today in radiation form; $\omega_0 \sim H_0 \sim 10^{-18}$ Hz is the minimal frequency inside the present Hubble radius;

$$\omega_2 = \frac{H_2 a_2}{H_0 a_0}\omega_0 \sim 10^2\omega_0 \tag{4.15}$$

is the frequency corresponding to the radiation $\rightarrow$ matter transition and, finally,

$$\omega_1 = \frac{H_1 a_1}{H_0 a_0}\omega_0 \sim 10^{29}\left(\frac{H_1}{M_P}\right)^{1/2}\omega_0 \tag{4.16}$$

is the maximum cut-off frequency depending on the height of the potential barrier $V(\eta_1)$, i.e. on the big-bang curvature scale H_1.

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see 4.8

The two important parameters of the spectrum (4.14) are the maximum scale H_1 and the power ν which depends on the pre-big-bang kinematics and fixes the frequency behaviour of the energy density of the relic gravitons. In four dimensions ($d = 3, n = 0$), and in the absence of dilaton contributions ($\gamma = 0$), the high-frequency behaviour of the spectrum mimics exactly the behaviour of the curvature scale for $\eta < -\eta_1$, as was stressed in [29,39] (we recall indeed that in such a case $4 - 2|\nu + 1/2| = 2 - 2\alpha$, so that a de Sitter phase at constant curvature, $\alpha = 1$, corresponds to a flat spectrum). If, however, the other contributions to the graviton production are included, then the total spectrum may be flat or decreasing even if the curvature is growing.

What is important to stress is that for a flat or decreasing spectrum the most significant bound is provided by the isotropy of the electromagnetic background radiation [28], $\Omega < \Omega_i$, while if the spectrum is growing fast enough it is only constrained by the critical density bound [34,39], $\Omega < \Omega_c$. The first bound turns out to be most constraining if imposed at the minimum frequency ω_0, the second bound at the maximum frequency ω_1 (there is also a bound obtained from the

pulsar-timing data [28], to be imposed at a frequency $\omega \sim 10^{-8}$ Hz, but it is not significant in our context [34]).

The two constraints

$$\Omega(\omega_0) < \Omega_i \qquad , \qquad \Omega(\omega_1) < \Omega_c \tag{4.17}$$

define an allowed region in the (H_1, ν) plane, which is bounded by the curves [obtained from eq.(4.14)]

$$\begin{aligned} Log(\frac{H_1}{M_P}) &< 2 + \frac{1}{2} Log\Omega_c, \\ Log(\frac{H_1}{M_P}) &< \frac{1}{|\nu + 1/2|}(116 + Log\Omega_i) - 58. \end{aligned} \tag{4.18}$$

The region allowed by the present experimental data ($\Omega_c \sim 1, \Omega_i \sim 10^{-8}$) is shown in **Fig. 2**.

Because of the many approximations made, and of the uncertainties in the experimental data, this figure is expected to give, of course, only a qualitative picture of the possible phenomenological scenario. We can see, nevertheless, that there is a maximum allowed scale of curvature, $H_1 \sim 10^2 M_p$, which can be reached only if the spectrum is growing fast enough, $|\nu + 1/2| \lesssim 1.8$. The general bound on the pre-big-bang kinematics to allow a given final scale $H_1 \leq 10^2 M_P$, in particular, is given by eq. (4.18) as

$$|\nu + \frac{1}{2}| \leq \frac{108}{58 + Log(H_1/M_P)}. \tag{4.19}$$

The Planck scale, in particular, can be reached if $|\nu + 1/2| \lesssim 1.88$. Flat or decreasing spectra, $|\nu + 1/2| \geq 2$, can only be extended to a maximum scale $H_1 \lesssim 10^{-4} M_P$. One thus recovers the well-known bound [41] (already quoted in Section 1) that applies to a four-dimensional primordial phase of the de Sitter type (which has $\nu + 1/2 = 2$, and thus corresponds to a flat spectrum).

This bound is avoided by the pre-big-bang example given in Section .2. In that case

$$\alpha = \beta = \frac{2}{3 + d + n} \qquad , \qquad \gamma = 2d, \tag{4.20}$$

and one gets an increasing-with-energy graviton spectrum with

$$\nu + \frac{1}{2} = \frac{2}{3 + d + n}. \tag{4.21}$$

As a result, the curvature can reach the maximum allowed scale for any number of dimensions.

The situation that we have discussed refers to the case of a "sudden" transition from the growing curvature regime to the decreasing one. The transition, however, could pass through an intermediate de Sitter phase of maximal constant curvature. In such a case, the high-frequency part of the graviton spectrum is flattened, while the lower-frequency part continues to grow. It is still possible to evade the bound $H_1 \lesssim 10^{-4} M_P$, and to consider a de Sitter inflation occurring at the Planck scale, but its duration turns out to be constrained by the present pulsar-timing data, in such a way that $10^{-1} M_P$ is practically the maximum allowed value of H_1 in such a Planck-scale inflation [39].

Note that, in this paper, we are using the data on the isotropy of the electromagnetic CMB only as a bound on the possible presence of a relic graviton background, since we are mainly interested in the limiting value of the curvature for a phase of pre-big-bang evolution. It is also possible, however, to interpret the CMB anisotropy recently detected by COBE [42] as entirely (or at least partially, but significantly) due to a stochastic background of cosmic gravitons [43,34]. By making such an assumption, and by using the graviton spectrum corresponding to our pre-big-bang model (4.21), the COBE result can be read as providing an interesting relation between the maximum scale H_1 and the total number of spatial dimensions $d + n$.

We should mention, finally, that an additional phenomenological signature of a pre-big-bang phase is contained in the squeezing parameter $r(\omega)$ which characterizes the quantum state of the relic cosmic gravitons [44], produced from the vacuum by the pre-big-bang $\rightarrow$ post-big-bang transition.

Such a parameter can be approximately expressed in terms of the Bogoliubov coefficients as [34,44]

$$r(\omega) \simeq \ln |c_-|. \tag{4.22}$$

For the high-frequency part of the spectrum ($\omega > \omega_2$) we thus find, from eq. (4.11),

$$r(\omega) \simeq |\nu + \frac{1}{2}|[25 - \ln(\frac{\omega}{Hz}) + \frac{1}{2} \ln(\frac{H_1}{M_P})]. \tag{4.23}$$

The first term on the right-hand side is expected to be dominant (at least in the presently allowed frequency range for a possible graviton detection); the second term determines the variation with frequency of the squeezing, and the last term provides a correction if the final curvature scale differs from the Planck one. In the

case of a future *direct* observation of the cosmic graviton background, a measure of the parameter $r(\omega)$ would provide information both on the maximum curvature scale H_1 and on the kinematics of a possible pre-big-bang phase.

5. The high curvature regime

In the examples of pre-big-bang scenarios so far considered the growth of the curvature, as well as that of the effective coupling e^{ϕ}, were unbounded [see for instance eq. (3.15)]. According to the phenomenological constraints discussed in the previous section, however, this growth must stop, at the latest around the Planck scale. A realistic model for this scenario should thus describe also a smooth transition from the growing to the decreasing curvature phase, avoiding the singularity both in the curvature and in the dilaton field.

Of course, in the high-curvature regime in which one approaches the Hagedorn and Planck scale, string corrections (of order α' and higher) are expected to modify the naïve low-energy effective action (2.13). Moreover, in such regime a more realistic cosmology is probably obtained by including the contributions of a dilaton potential $V(\phi)$, and also of the antisymmetric tensor field. Indeed, there are regular examples (without sources) in which both a constant dilaton potential [12] and an $O(d,d)$-generated torsion background [45] contribute to violate the strong energy condition and to avoid the singularity.

Let us resort, therefore, to the full $O(d,d)$-covariant equations of string cosmology, obtained from the action (2.13) (supplemented by a phenomenological source term describing bulk string matter). Let us assume, furthermore, that the d-dimensional space has finite volume (even if flat, for simplicity) such as a torus. It is thus possible to make the $O(d,d)$ covariance of the background fields compatible with the $GL(d)$ coordinate invariance, even in the presence of a non-trivial dilaton potential, provided $V = V(\overline{\phi})$, where $\overline{\phi}$ is as defined in eq. (2.18). In such case the field equations, obtained from the total (generally non-local) action, can be written in explicit $O(d,d)$-covariant form as [6]

$$\dot{\overline{\phi}}^2 - 2\ddot{\overline{\phi}} - \frac{1}{8}Tr(\dot{M}\eta)^2 + \frac{\partial V}{\partial\overline{\phi}} - V = 0 \tag{5.1}$$

$$\dot{\overline{\phi}}^2 + \frac{1}{8}Tr(\dot{M}\eta)^2 - V = \overline{\rho}e^{\overline{\phi}} \tag{5.2}$$

$$(e^{-\overline{\phi}}M\eta\dot{M})^{\cdot} = \overline{T} \tag{5.3}$$

where M is the matrix defined in eq. (2.17), $\overline{T}$ is a $2d \times 2d$ matrix representing the spatial part of the string stress tensor (including the possible antisymmetric current density, source of torsion [6]), and η is the $O(d,d)$ metric in off-diagonal form

$$\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{5.4}$$

(Note that, throughout this section, both $\overline{T}$ and $\overline{\rho}$ are expressed in units of $16\pi G_D$, so that they have dimensions L^{-2}.)

These three equations generalize, respectively, the dilaton equation (3.8), and the time and space part of the graviton equations, (3.9) to (3.11), to which they reduce exactly for $V = 0$, vanishing torsion, and diagonal metric background. Their combination provides the useful covariant conservation equation of the source energy density, which can be written in compact form as [6]

$$\dot{\overline{\rho}} + \frac{1}{4} Tr(\overline{T}\eta M \eta \dot{M} \eta) = 0. \tag{5.5}$$

As in ordinary cosmology, this system of equations must be supplemented by an equation of state, which in this context reads as a relation of the form $\overline{T} = \overline{T}(\overline{\rho}, M)$. In order to provide an explicit example of regular pre-big-bang scenarios, we note that the variables of eqs. (5.1) to (5.3) can be separated, and the equations can be solved, for any given $\overline{T}/\overline{\rho}$, provided

$$\frac{\partial V}{\partial \overline{\phi}} = 2V. \tag{5.6}$$

Indeed, using this condition, the combination of eqs. (5.1) and (5.2) gives

$$(e^{-\overline{\phi}})^{\cdot\cdot} = \frac{1}{2}\overline{\rho}. \tag{5.7}$$

Defining a time coordinate ξ such that

$$d\xi = \frac{\overline{\rho}}{\rho_0} dt \tag{5.8}$$

(ρ_0 is a constant, and we shall denote with a prime differentiation with respect to ξ), eqs. (5.7) and (5.3) can then be easily integrated a first time to give

$$\overline{\rho}(e^{-\overline{\phi}})' = \frac{1}{2}\rho_0^2(\xi + \xi_0) \tag{5.9}$$

$$\overline{\rho} M \eta M' = \rho_0^2 \Gamma e^{\overline{\phi}}. \tag{5.10}$$

where

$$\Gamma = \int \frac{\overline{T}}{\overline{\rho}} d\xi \tag{5.11}$$

(ξ_0 is an integration constant).

On the other hand, by using (5.10) to eliminate $M\eta\dot{M}$, and by noting that $\overline{T}/\overline{\rho} = \Gamma'$, we can rewrite the conservation equation as

$$\overline{\rho}' e^{-\overline{\phi}} = -\frac{1}{8}\rho_0^2 Tr(\Gamma\eta\Gamma\eta)'. \tag{5.12}$$

By adding eqs. (5.9) and (5.12), integrating a second time and defining (for later convenience)

$$D(\xi) = 4\beta + (\xi + \xi_0)^2 - \frac{1}{2}Tr(\Gamma\eta)^2 \tag{5.13}$$

(β is an integration constant), we obtain finally

$$\overline{\rho} e^{-\overline{\phi}} = \frac{1}{4} D\rho_0^2, \tag{5.14}$$

which, inserted into (5.9) and (5.10), gives

$$\overline{\phi}' = -\frac{2}{D}(\xi + \xi_0) \tag{5.15}$$

$$M\eta M' = \frac{4\Gamma}{D}. \tag{5.16}$$

We have thus satisfied eq. (5.3), and a combination of eqs. (5.1) and (5.2). We must still impose eq. (5.2) which, using eqs. (5.15) and (5.16) and the identity [6]

$$(M\eta M'\eta)^2 = -(M'\eta)^2 \tag{5.17}$$

reduces to the condition

$$(\xi + \xi_0)^2 - \frac{1}{2}Tr(\Gamma\eta)^2 = D + \frac{VD^2}{4}(\frac{\rho_0}{\overline{\rho}})^2. \tag{5.18}$$

This condition, together with eq. (5.6), can be satisfied in two ways. The first way is trivial, $V = 0$ and $\beta = 0$. The equations (5.15) and (5.16) can then be integrated for any given $\Gamma(\xi)$, but the singularity in this case cannot be avoided, because of the zeros of $D(\xi)$.

The second possibility corresponds to a non-trivial two-loop potential $V(\overline{\phi}) = -V_0 e^{2\overline{\phi}} < 0$, with $\beta = V_0$. In this case too eqs. (5.6) and (5.18) are both satisfied, and we can obtain examples of backgrounds that are exact solutions of the string cosmology equations (5.14)–(5.16) (with $\beta = V_0$), and which describe a

smooth evolution from asymptotically growing to asymptotically decreasing curvature, without singularities. We shall give some examples below.

Let us consider, first of all, a d-dimensional isotropic background, with scale factor a, vanishing torsion ($B = 0$), and sources with a diagonal stress-tensor, so that [6]

$$M\eta M' = 2\frac{a'}{a}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad , \quad \overline{T} = \overline{p}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{5.19}$$

where I is the d-dimensional identity matrix. By defining $\overline{p}/\overline{\rho} = \gamma$ we thus have

$$D = 4V_0 + (\xi + \xi_0)^2 - d(\int \gamma d\xi)^2. \tag{5.20}$$

The integration of the cosmological equations is thus immediate in the case of a perfect fluid source, with $\gamma = const.$ Consider for example the interesting case of the radiation-like solution,

$$\gamma = \frac{1}{d} \quad , \quad \int \gamma d\xi = \frac{1}{d}(\xi + \xi_1) \quad , \quad \xi_1 = const. \, , \tag{5.21}$$

and define, for convenience, the constant parameters

$$\alpha = \frac{d-1}{d} \quad , \quad b = 2(\frac{\xi_0}{\xi_1} - \frac{1}{d}) \quad , \quad c = \frac{4V_0 + \xi_0^2}{\xi_1^2} - \frac{1}{d}. \tag{5.22}$$

By choosing the arbitrary constants ξ_0 and ξ_1 in such a way that

$$\Delta^2 \equiv 4\alpha c - b^2 > 0 \tag{5.23}$$

the integration of eqs. (5.15) and (5.16) then provides the non-singular solution

$$\overline{\phi} = \phi_0 + \ln(\alpha\frac{\xi^2}{\xi_1^2} + b\frac{\xi}{\xi_1} + c)^{-\frac{d}{d-1}} - \frac{4(\xi_1 - \xi_0)}{(d-1)\Delta\xi_1}\arctan(\frac{2\alpha\xi + b\xi_1}{\Delta\xi_1}) \tag{5.24}$$

$$a = a_0(\alpha\frac{\xi^2}{\xi_1^2} + b\frac{\xi}{\xi_1} + c)^{\frac{1}{d-1}}\exp\{\frac{4(\xi_1 - \xi_0)}{(d-1)\Delta\xi_1}\arctan(\frac{2\alpha\xi + b\xi_1}{\Delta\xi_1})\} \tag{5.25}$$

(a_0 and ϕ_0 are dimensionless integration constants). Their combination gives

$$e^{\phi} = a_0^d e^{\phi_0}\exp\{\frac{4(\xi_1 - \xi_0)}{(d-1)\Delta\xi_1}\arctan(\frac{2\alpha\xi + b\xi_1}{\Delta\xi_1})\}, \tag{5.26}$$

which is growing for $\xi_1 > \xi_0$.

Note that the dilaton potential $V(\overline{\phi})$ provides its strongest contribution just in correspondence of the phase of maximum curvature, while it rapidly decays to

running lambda

zero away from this regime. We thushave an example of "running cosmological constant" $\Lambda = V(\overline{\phi})$, which may suggest a new possible approach to the cosmological constant problem based on the damping of Λ obtained from a non-local effective action (see also [46]).

It is also interesting to point out that, in the case of a radiation-like source, the time coordinate ξ coincides with the conformal time η. From eq. (5.14) we indeed obtain, for $\overline{\rho}$

$$\overline{\rho} = \frac{\xi_1^2 \rho_0^2 e^{\phi_0}}{4(\alpha \frac{\xi^2}{\xi_1^2} + b\frac{\xi}{\xi_1} + c)^{\frac{1}{d-1}}} \exp\{-\frac{4(\xi_1 - \xi_0)}{(d-1)\Delta\xi_1} \arctan(\frac{2\alpha\xi + b\xi_1}{\Delta\xi_1})\}, \tag{5.27}$$

so that, from the definition of ξ,

$$\frac{d\xi}{dt} = \frac{\overline{\rho}}{\rho_0} = \frac{1}{4}\xi_1^2 \rho_0 e^{\phi_0}(\frac{a_0}{a}). \tag{5.28}$$

Equations (5.25) and (5.26) describe a radiation-dominated solution, which interpolates without singularities between an initial asymptotic phase of *accelerated contraction*, $a \sim (-t)^{2/(d+1)}$, $\phi = \phi_1 = const.$, *growing curvature*, and a final phase of *decelerated expansion*, $a \sim t^{2/(d+1)}$, $\phi = \phi_2 = const.$, *decreasing curvature*. There are no horizons, and both the curvature and the dilaton field are regular.

In this example the final expanding state is reached starting from an initial contraction. There is, however, also the dual solution $\tilde{\phi}, \tilde{a}$ corresponding to the equation of state

$$\overline{p} = -\frac{\overline{\rho}}{d} \quad , \quad \int \gamma d\tilde{\xi} = -\frac{1}{d}(\tilde{\xi} + \xi_1), \tag{5.29}$$

where $\tilde{\xi}$ is the "dual" time coordinate defined by eq. (5.8) for this new solution. From eqs.(5.14)–(5.16) one obtains, in terms of $\tilde{\xi}$, that the new solution $\{\tilde{\phi}, \tilde{a}\}$ is related to the old one $\{\phi, a\}$ by

$$\begin{aligned} \tilde{\overline{\phi}} &= \overline{\phi}(\tilde{\xi}) \\ (\frac{\tilde{a}}{a_0}) &= a_0 a^{-1}(\tilde{\xi}) \\ \tilde{\overline{\rho}} &= \frac{1}{4}\xi_1^2 \rho_0^2 e^{\phi_0} \frac{a_0}{a(\tilde{\xi})} = \frac{1}{4}\rho_0^2 e^{\phi_0}(\frac{\tilde{a}}{a_0}) \end{aligned} \quad , \tag{5.30}$$

where $d\tilde{\xi} \sim \tilde{a}dt$. In this solution, therefore, the background evolves smoothly from an initial accelerated phase of *superinflationary expansion*, $a \sim (-t)^{-2/(d+1)}$,

with a logarithmically *increasing dilaton*, towards a final asymptotic phase of *decelerated contraction*, $a \sim t^{-2/(d+1)}$, a logarithmically *decreasing dilaton*, and *decreasing curvature*. Again the maximum curvature regime is crossed over without singularities.

These two kinds of solution smoothly connect the pre-big-bang phase to the post-big-bang one by changing the sign of H (from expansion to contraction, and vice-versa). Their existence may suggests a possible scenario in which the transition to the standard cosmology occurs after a period of background oscillations [11,47], namely a series of anisotropic contractions and expansions in which "external" and "internal" coordinates exchange their roles periodically. Such a scenario may also recall the Kasner-like oscillating behaviour, typical in general relativity of a cosmological metric approaching the initial singularity [48], with the only difference that in our case the singularity is smoothed out by a phase of maximal curvature.

This possibility is suggested by solutions obtained for a fixed equation of state. We might expect, however, that the transition from a given kinematical class of background evolution to the dual one might be associated with a corresponding transition between duality-related regimes also in the matter sources. In such a case it becomes possible to go across the phase of maximum curvature, without singularities, even for monotonically expanding (or contracting) "self-dual" solutions.

Consider indeed (always in an isotropic context, with diagonal pressure) a model of sources performing a transition from an equation of state that is typical of string-driven pre-big-bang, $p = -\rho/d$ (see Section 3), to the dual, radiation-dominated regime with $p = \rho/d$. We shall model this transition by

$$\frac{\overline{p}}{\overline{\rho}} = \gamma(\xi) = \frac{\xi}{d\sqrt{\xi^2 + \xi_1^2}} \quad , \quad \int \gamma d\xi = \frac{1}{d}\sqrt{\xi^2 + \xi_1^2}, \tag{5.31}$$

where $|\xi_1|$ is a phenomenological parameter (typically of order M_P^{-1}) that characterizes the time scale of the transition regime. Asymptotically, i.e. for $|\xi| >> |\xi_1|$, one recovers radiation for $\xi > 0$ and the dual state for $\xi < 0$.

By inserting this matter behaviour into the equations (5.14), (5.15) and (5.20), we get a regular solution provided ξ_0 and ξ_1 are chosen such that

$$\alpha c > \xi_0^2, \tag{5.32}$$

where α and c are the constants defined in eq. (5.22). This condition can be satisfied, in particular, by the choice

$$\xi_0 = 0 \quad , \quad 4V_0 = \xi_1^2, \tag{5.24}$$

which nicely simplifies the final expression for the background fields (a more general choice does not change the qualitative behaviour of the solution, which is presented here for illustrative purposes only).

With this choice, the integration of eqs. (5.15) and (5.16) gives

$$e^{\overline{\phi}} = e^{\phi_0}(1+\frac{\xi^2}{\xi_1^2})^{\frac{-d}{d-1}} \tag{5.25}$$

$$a = a_0(\frac{\xi}{\xi_1}+\sqrt{1+\frac{\xi^2}{\xi_1^2}})^{\frac{2}{d-1}} \tag{5.26}$$

where a_0 and ϕ_0 are integration constants. From their combination we have

$$e^{\phi} = a_0^d e^{\phi_0}(1+\frac{\xi}{\sqrt{\xi^2+\xi_1^2}})^{\frac{2d}{d-1}}. \tag{5.27}$$

Moreover, according to eq. (5.14),

$$\frac{\overline{\rho}}{\rho_0} = \frac{d-1}{4d}\xi_1^2\rho_0 e^{\phi_0}(1+\frac{\xi^2}{\xi_1^2})^{-\frac{1}{d-1}} = \frac{d\xi}{dt} \tag{5.28}$$

so that

$$e^{\phi}\frac{\rho}{\rho_0} = \frac{d-1}{4d}\xi_1^2\rho_0 e^{\phi_0}(1+\frac{\xi^2}{\xi_1^2})^{-\frac{d+1}{d-1}} \tag{5.29}$$

and

$$e^{\phi}\frac{p}{\rho_0} = \frac{\xi}{\xi_1}\frac{d-1}{4d}\xi_1^2\rho_0 e^{\phi_0}(1+\frac{\xi^2}{\xi_1^2})^{-\frac{3d+1}{2(d-1)}} \tag{5.30}$$

We note that, for $\xi \to \infty$, eq. (5.26) gives $a(\xi) \sim \xi^{2/(d-1)}$, while for $\xi \to -\infty$, $a(\xi) \sim (-\xi)^{-2/(d-1)}$. From the definition (5.8), it thus follows that, in the asymptotic future $d\xi/dt \sim a^{-1}$, namely ξ tends to coincide with the conformal time, while in the asymptotic past ξ tends to coincide with the "dual" time coordinate, as $d\xi/dt \sim a$.

This solution describes a model that is expanding ($H > 0$), for all t; the Universe, starting from a flat space ($H \to 0$) and weak coupling ($e^{\phi} \to 0$) regime, evolves through a superinflationary phase [$a \sim (-t)^{-2/(d+1)}$] dominated by string-like unstable matter ($p = -\rho/d$), towards a final decelerated [$a \sim t^{2/(d+1)}$], radiation-dominated ($p = \rho/d$) phase with frozen gravitational coupling ($e^{\phi} = const.$). In $d = 3$ spatial dimensions it provides an explicit realization of the model discussed at the end of Section 3.

The curvature is everywhere bounded, growing from $-\infty$ to 0, and decreasing from 0 to ∞. The behaviour of H and $\dot{H}$ is given by

$$H = \frac{a'\overline{\rho}}{a\rho_0} = \frac{\xi_1\rho_0 e^{\phi_0}}{2d}(1+\frac{\xi^2}{\xi_1^2})^{-\frac{d+1}{2(d-1)}} \tag{5.31}$$

$$\dot{H} = H' \frac{\overline{\rho}}{\rho_0} = -\frac{(d+1)\xi\xi_1\rho_0^2 e^{2\phi_0}}{8d^2}(1+\frac{\xi^2}{\xi_1^2})^{-\frac{3d+1}{2(d-1)}}. \tag{5.32}$$

The absence of singularity can be traced back to the fact that this solution is self-dual, in the sense that

$$[\frac{a(-t)}{a_0}]^{-1} = \frac{a(t)}{a_0}. \tag{5.33}$$

In general relativity, such a solution is not allowed, as the field equations are time-symmetric but not invariant under the inversion of the scale factor. The behaviour of $a, e^{\phi}, e^{\phi}\rho, e^{\phi}p, H$ and $\dot{H}$ is plotted in **Fig. 3** for $d = 3$, $\phi_0 = 0$, $a_0 = \rho_0 = \xi_1 = 1$.

An analogous solution obviously exists in which H is always negative (the universe is always isotropically contracting); it is easily obtained in correspondence with the dual equation of state $\gamma(\xi) = -\xi/d\sqrt{\xi^2+\xi_1^2}$. More interesting, in our context, is however the Bianchi I type anisotropic background in which, during the pre-big-bang phase, d dimensions expand with scale factor a, while n dimensions shrink with scale factor $b = a^{-1}$, with an equation of state $p = -q = -\rho/(d+n)$ (the example discussed in Section 3).

In this case, by setting $\gamma = p/\rho$, we have

$$\frac{1}{2}Tr(\Gamma\eta)^2 = (d+n)(\int \gamma d\xi)^2, \tag{5.34}$$

and the equations (5.15) and (5.16) for a and $\overline{\phi}$ reduce to

$$\begin{aligned} \overline{\phi}' &= -\frac{2}{D}(\xi+\xi_0) \\ \frac{a'}{a} &= \frac{2}{D}\int \gamma d\xi = -\frac{b'}{b}, \end{aligned} \tag{5.35}$$

where

$$D = 4V_0 + (\xi+\xi_0)^2 - (d+n)(\int \gamma d\xi)^2. \tag{5.36}$$

In analogy with the previous example, we represent the evolution in time of the sources between the dual asymptotic regimes by

$$\begin{aligned} \frac{p}{\rho} = \gamma(\xi) &= \frac{1}{d+n}\frac{\xi}{\sqrt{\xi^2+\xi_1^2}} = -\frac{q}{\rho} \\ \int \gamma d\xi &= \frac{\sqrt{\xi^2+\xi_1^2}}{d+n} \end{aligned} \tag{5.37}$$

and we choose conveniently the arbitrary parameters in such a way that $\xi_0 = 0$ and $\xi_1^2 = 4V_0$.

The integration of eqs. (5.35) then provides

$$e^{\overline{\phi}} = e^{\phi_0}(1+\frac{\xi^2}{\xi_1^2})^{-\frac{d+n}{d+n-1}} \tag{5.38}$$

$$a = b^{-1} = a_0(\frac{\xi}{\xi_1} + \sqrt{1+\frac{\xi^2}{\xi_1^2}})^{\frac{2}{d+n-1}} \tag{5.39}$$

with a_0 and ϕ_0 arbitrary integration constants. It follows that

$$e^{\phi} = a_0^{d-n} e^{\phi_0}(\frac{\xi}{\xi_1} + \sqrt{1+\frac{\xi^2}{\xi_1^2}})^{\frac{2(d-n)}{d+n+1}}(1+\frac{\xi^2}{\xi_1^2})^{-\frac{d+n}{d+n-1}} \tag{5.40}$$

and that

$$\frac{\overline{\rho}}{\rho_0} = \frac{d\xi}{dt} = \frac{d+n-1}{4(d+n)}\xi_1^2\rho_0 e^{\phi_0}(1+\frac{\xi^2}{\xi_1^2})^{-\frac{1}{d+n-1}}. \tag{5.41}$$

By comparing this last equation with eq. (5.39) we can easily check that this solution, in the $\xi \to -\infty$ limit, exactly describes the regime of pre-big-bang and dynamical dimensional reduction discussed in Section 3, with $a(t) \sim (-t)^{-2/(d+n+1)} = 1/b$. This background, according to the solution (5.39), (5.40), evolves in such a way as to reach a phase of maximal, finite curvature, after which it approaches the dual, decelerated regime in which the internal dimensions are not frozen, but keep contracting like $b(t) = 1/a \sim t^{-2/(d+n+1)}$ for $t \to +\infty$.

It is important to stress that the dilaton coupling, in this case, does not settle down to a finite constant value after the big-bang, but, according to eq.(5.40), tends to decrease during the phase of decreasing curvature. Such a decrease of ϕ is driven by the decelerated shrinking of the internal dimensions which are not frozen, unlike in the previous case.

We have thus provided two examples of self-dual solutions which can model the transition from the growing to the decreasing curvature regime. These examples are not intended to apply too far away from the maximum curvature scale (i.e. in too low a curvature regime where, for example, there are constraints which rule out too fast a variation of the gravitational coupling [49] and of the radius of the internal dimensions [50], such as the ones predicted by our last example). In the low curvature regime $O(d,d)$ symmetry and scale factor duality may indeed be expected to be broken, for instance by a dilaton potential $V(\phi)$. They suggest, however, the interesting possibility of a transition, which may occur even for monotonically evolving scale factors, and which approximates a de Sitter phase in the neighbourhood of the big-bang region ($\dot{H} = 0$).

We note, finally, that in the model of sources that we have used the ratio p/ρ is not a constant, but the pressure is still diagonal. The effective, time-dependent equations of state (5.31) and (5.37) may thus be re-expressed in terms of an effective bulk viscosity, which depends on ρ and which becomes negligibly small in the "in" and "out" asymptotic regimes where the perfect fluid behaviour is recovered. It is thus significative, in this context, that the evolution of a phase of exponential de Sitter expansion down to the decelerated regime, in a model of string-driven inflation [51], can also be interpreted in terms of an effective ρ-dependent, bulk viscosity [52].

We have already seen, on the other hand, that in the pre-big-bang phase the entropy of the sources is constant, as the matter evolution is adiabatic. Near the maximum ($\dot{H} \simeq 0$), however, one can define a horizon entropy [53] which shrinks like the pre-big-bang horizon area. Viscosity, and other dissipative processes, may thus naturally be expected to occur, in that regime, possibly yielding a large entropy increase.

6. Conclusions

In this paper we have described how string theory suggests and supports a picture of the Universe in which the present decelerated expansion is preceeded by a dual phase in which the evolution is accelerated and the curvature is growing (what we have called **"pre-big-bang"**). We stressed that such a scenario is not to be regarded as an alternative to the usual post-Planckian cosmology, whether inflationary or not, but that it represents **a complement** of the standard scenario, which cannot be extended beyond the Planck scale.

In the pre-big-bang phase the growth of curvature may be accompanied (but not necessarily sustained) by the shrinking of some extra "internal" spatial dimensions down to a final compactification scale. This scale, as well as the maximal (big-bang) curvature scale, are both expected to be given by the fundamental length parameter of string theory, i.e. roughly by the Planck scale itself. In the subsequent decelerated phase the internal radius may be frozen, or may keep shrinking, as shown by the examples presented in Section 5.

As we pointed out there, the possibility of avoiding the big-bang singularity appears to be deeply rooted in a typical stringy symmetry, the scale factor duality, which acts on homogeneous cosmological backgrounds. Indeed, the product

of scale-factor-duality and time reversal admits, as non-trivial fixed points, backgrounds which connect smoothly, at $t = 0$, two duality-related regimes, e.g. a superinflationary era and a standard decelerating expansion.

Moreover, the generalization of scale-factor-duality to a full continuous $O(d, d)$ symmetry provides a framework in which entire classes of non-singular cosmological models can be obtained via $O(d, d)$ "boosts" of trivial, or even singular [45] initial conformal backgrounds.

Obviously, the picture presented here is in many respects qualitative and preliminary. Many problems remain to be solved, such as that of formulating a simple, yet consistent equation of state for string sources in curved backgrounds. At a more fundamental level, we might expect that, unless higher-order quantum string effects are consistently taken into account, it will be impossible to avoid a singularity once all physical constraints are imposed.

Nevertheless, we wish to stress that the global picture emerging from our simplified approach is already well defined and allows, in principle, for experimental verifications. We have shown, indeed, that the string cosmology equations and the assumption of a pre-big-bang era provide definite predictions for the spectrum of relic gravitons that should be filling up space not less than the electromagnetic CMB does. Thus such a scenario can be confirmed, disproved, or at least significantly constrained, by some (hopefully near-future) direct or indirect observations and measurements of a gravitational-wave background of cosmological origin. It is perhaps encouraging to recall at this point that the-hot big-bang hypothesis [54] was definitively confirmed through the observation of the relic electromagnetic background radiation [55].

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References

1. E.Alvarez, Phys.Rev.D31(1985)418.
2. Y.Leblanc, Phys.Rev.D38(1988)3087.
3. R.Brandenberger and C.Vafa, Nucl.Phys.B316(1989)391.
4. E.Alvarez and M.A.R.Osorio, Int.J.Theor.Phys.28(1989)949.
5. M.Gasperini, N.Sanchez and G.Veneziano, Int.J.Theor.Phys.A6 (1991)3853; Nucl.Phys.B364(1991)365.
6. M.Gasperini and G.Veneziano, Phys.Lett.B277(1992)256.
7. E.Alvarez, J.Cespedes and E.Verdaguer, Phys. Lett. B289 (1992) 51; A.Ashtekar, "Emergence of discrete structures at the Plank scale", to appear in Proc. of the 6th Int. Workshop on Theor. Phys. "String quantum gravity and physics at the Planck energy scale" (Erice, June 1992).
8. A.Vilenkin, "Did the universe have a beginning?" Caltech preprint CALT-68-1772 (1992).
9. F.Lucchin and S.Matarrese, Phys.Lett.B164(1985)282.
10. G.Veneziano, Phys.Lett.B265(1991)287.
11. A.A.Tseytlin, Mod.Phys.Lett.A6(1991)1721
12. A.A.Tseytlin and C.Vafa, Nucl.Phys.B372(1992)443; A.A.Tseytlin, Class. Quantum Grav.9(1992)979.
13. K.A.Meissner and G.Veneziano, Phys.Lett.B267(1991)33; Mod.Phys.Lett.A6(1991)3397.
14. A.Sen, Phys.Lett.B271(1991)295; S.F.Hassan and A.Sen, Nucl.Phys.B375(1992)103.
15. D.Shadev, Phys.Lett.B137(1984)155; R.B.Abbott, S.M.Barr and S.D.Ellis, Phys.Rev.D30(1984)720; E.W.Kolb, D.Lindley and D.Seckel, Phys.Rev.D30(1984)1205.
16. M.Gasperini, Phys.Lett.B258(1991)70; Gen.Rel.Grav.24(1992)219.
17. N.Sanchez and G.Veneziano, Nucl.Phys.B333(1990)253.
18. H.J.De Vega and N.Sanchez, Phys.Lett.B197(1987)320.
19. Nguyen S.Han and G.Veneziano, Mod.Phys.Lett.6(1991)1993.
20. G.Veneziano, Helvetica Physica Acta 64 (1991) 877.
21. S.Weinberg, Gravitation and cosmology (Wiley, New York, 1971) Chapt.15.
22. A.Giveon, Mod.Phys.Lett.A6(1991)2834.

23. N.D.Birrell and P.C.W.Davies, Quantum fields in curved space (Cambridge Univ. Press, Cambridge, 1982) Chapt.3.
24. L.Parker, Nature 261(1976)20
25. E.Verdaguer, "Gravitational particle creation in the early Universe", Barcelona preprint, UAB-FT-276 (Lectures given at the ERE-91 Meeting, Bilbao, 1991).
26. A.H.Guth, Phys.Rev.D23(1981)347.
27. A.D.Linde, "Inflation and quantum cosmology", preprint CERN-TH.5561/89 (based on lectures given at the Summer School on Particle Physics and Cosmology, Trieste, 1989).
28. L.P.Grishchuk, Sov.Phys.Usp.31(1988)940.
29. L.P.Grishchuk and M.Solokhin, Phys.Rev.D43(1991)2566.
30. K.S.Thorne, in 300 Years of Gravitation", eds. S.W.Hawking and W.Israel (Cambridge Univ.Press, Cambridge, 1988) p.330.
31. E.M.Lifshitz and I.M.Khalatnikov, Adv. Phys.12(1963)208.
32. L.P.Grishchuk, Sov.Phys.JEPT 40(1975)409.
33. L.H.Ford and L.Parker, Phys.Rev.D16(1977)1601.
34. M.Gasperini and M.Giovannini, "Dilaton contributions to the cosmic gravitational wave background", Turin Univ. preprint, DFTT 58/92 (to appear in Phys. Rev. D).
35. J.Garriga and E.Verdaguer, Phys.Rev.D39(1989)1072;
M.Demianski, in Proc. 9th Italian Conference on General Relativity, Capri, 1990, eds. R.Cianci et al. (World Scientific, Singapore, 1991) p.19;
M.Amendola, M.Litterio and F.Occhionero, Phys.Lett.B237(1990)348.
36. M.Gasperini and M.Giovannini, Class. Quant. Grav.9(1992)L137.
37. B.Allen, Phys.Rev.D37(1988)2078.
38. V.Sahni, Phys.Rev.D42(1990)453.
39. M.Gasperini and M.Giovannini, Phys.Lett.B282(1992)36.
40. D.Shadev, Phys.Rev.D39(1989)3155;
R.G.Moorhouse and J.Nixon, Nucl.Phys.B261(1985)172.
41. V.A.Rubakov, M.V.Sazhin and A.V.Veryaskin, Phys.Lett.B115 (1982)189;
R.Fabbri and M.D.Pollock, Phys.Lett.B125(1983)445;
L.F.Abbot and M.B.Wise. Nucl.Phys.B244(1984)54.
42. G.Smoot et al., COBE collaboration preprint 92-04 (to appear in Ap.J.Lett.)
43. L.Krauss and M.White, "Grand unification, gravitational waves and the CMB anisotropy",Yale preprint YCTP-P15-92;

T. Souradeep and V.Sahni, "Density perturbations, gravity waves and the cosmic microwave background", preprint IUCAA (Pune, India), July 1992; F.Lucchin, S.Matarrese and S.Mollerach, "The gravitational wave contribution to CMB anisotropies and the amplitude of mass fluctuations from COBE results", Fermilab-Pub-92/185-A (July 1992).

44. L.P.Grishchuk and Y.V.Sidorov, Class. Quantum Grav.6(1989)L161; Phys.Rev.D42(1990)3413.
45. M.Gasperini, J.Maharana and G.Veneziano, Phys.Lett.B272(1991) 277; M.Gasperini, J.Maharana and G.Veneziano, "Boosting away singularities from conformal string backgrounds", CERN-TH.6634/92; A. Giveon and A. Pasquinucci, " On cosmological string backgrounds with toroidal isometries, Princeton preprint, IASSNS-HEP-92/55.
46. A.A.Tseytlin, Phys.Rev.Lett. 66 (1991) 545.
47. A.A.Tseytlin, "String cosmology and dilaton", to appear in Proc. of the 6th Int. Workshop on Theor. Phys. "String quantum gravity and physics at the Planck energy scale" (Erice, June 1992).
48. V.A.Belinski, E.M.Lifshitz and I.M.Khalatnikov, Adv.Phys.31 (1982)639.
49. R.W.Hellings, "Time variation of the gravitational constant", in Proc. 10th Int. School on Cosmology and Gravitation, Erice 1987, ed. V.N.Melnikov and V. De Sabbata (Kluwer Acad. Pub., Dordrecht) p.215; F.S.Accetta, L.M.Krauss and P.Romanelli, Phys.Lett.B248(1990)146.
50. E.W.Kolb, M.J.Perry and T.P.Walker, Phys.Rev.D33(1986)869; J.D.Barrow, Phys.Rev.D35(1987)1805.
51. N.Turok, Phys.Rev.Lett.60(1987)549.
52. J.D.Barrow, Nucl.Phys.130(1988)743.
53. P.C.W.Davies, Class.Quantum Grav.4(1987)L225.
54. G.Gamow, Phys.Rev.70(1946)572; R.A.Alpher and R.C.Herman, Rev.Mod.Phys.22(1950)153; R.H.Dicke, P.J.E.Peebles, P.G.Roll and D.T.Wilkinson, Ap.J.142(1965)414.
55. A.A.Penzias and R.W.Wilson, Ap.J.142(1965)419.

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Figure captions

Fig. 1
Two possible alternatives to the curvature singularity of the standard cosmological model.

Fig. 2
Present constraints on the maximum allowed curvature scale H_1 (in units of Planck mass), versus the index ν which parametrizes the background kinematics. The allowed region extends down to $|\nu + 1/2| = 0$.

Fig. 3
Time evolution with respect to ξ of the self-dual solution given by eqs. (5.26) and (5.27) around the phase of maximal curvature ($\xi = 0$). a) The scale factor and the dilaton coupling; b) the effective density and pressure; c) the Hubble parameter and its time derivative.

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BACK-REACTION TO DILATON-DRIVEN INFLATION

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Abstract

We compute the leading-order back-reaction to dilaton-driven inflation, due to graviton, dilaton and gauge-boson production. The one-loop effect turns out to be non-vanishing (unlike the case for pure de-Sitter and for power-law inflation), to be of relative order $\ell_P^2 H^2(t)$, and to have the correct sign for favouring the exit to a FRW phase.

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1 Introduction

In recent years, pre-big bang (PBB) cosmology [1] has established itself as a logically consistent alternative to standard slow-roll inflation [2]. While the ultimate verdict on which, if either, of these two versions of cosmology will survive is left to future experiments, at the theoretical level each one has its own advantages and shortcomings.

The trade-off follows from the fundamental physical difference between the two scenarios: in slow-roll inflation the evolution is from larger to smaller curvatures, while in dilaton-driven PBB inflation the opposite is true. Thus, while in the former case Planckian/string-scale curvatures are naturally reached at the very beginning, pushing the problem of initial conditions into the full quantum-gravity domain (see e.g. [3]), in the latter that problem can be tackled within a perfectly known (albeit non-linear) set of equations (see e.g. [4]). On the other hand, since large curvatures naturally occur at the end of dilaton-driven inflation[1], the exit problem becomes harder to analyse than in the conventional scenario.

It is generally believed that the exit problem of string cosmology [5], i.e. achieving a smooth, non-singular transition from inflationary to FRW-type string cosmology solutions, should be solved by a combination of two kinds of effects, corresponding to two different classes of corrections to the low-energy, tree-level effective action. While higher-derivative corrections should be able to bound the growth of curvature [6] (or to allow a reinterpretation of the lowest-order curvature singularity), loop corrections [7], accounting for the back-reaction on the geometry from particle production, should stop the growth of the coupling (of the dilaton) when a certain critical-energy condition [8] is reached. Recent proposals of cosmological entropy bounds [9] appear to support [10] these ideas.

Unfortunately, present techniques are still inadequate for giving a convincing answer to the question of whether any one of these desired effects does actually take place. The question of fully non-perturbative higher-derivative corrections amounts

[1] A phenomenologically interesting consequence of this difference is that, in slow-roll inflation, Planck/string-scale physics is "screened" from observation by the subsequent long inflationary era; in the PBB scenario, it is in principle accessible to observation.

to finding an exact (2D) conformal field theory. However, since supersymmetry is spontaneously broken in the case of cosmology, chances to get such a CFT in closed form are slim.

Similarly, understanding the full effect of back-reaction would require knowledge of the full one-loop effective action up to four derivatives. In spite of brilliant work by Vilkovisky and collaborators [11] on this problem, these are still early times for applying the results so far obtained to our problem. Thus our aim here will be more modest: we will try to understand in which direction does the back-reaction from particle production work in PBB cosmology while it is still a small effect. No attempt will be therefore be made to achieve (a fortiori to describe) a complete exit.

We thus took an easier road, recently opened by Iliopoulos et al. [12] and by Abramo and Woodard [13], [14]. Their method was already checked, whenever possible, against the more intuitive approach of Abramo, Brandenberger and Mukhanov [15]. Apart from our use of a different gauge, known to be more appropriate in string cosmology, our work can be seen as a straightforward application of that method in the context of string cosmology. As a check of gauge invariance, we will also reproduce some results of [13], [14].

Another (apparently strong) limitation of our work is that we consider loop corrections to solutions of the low-energy effective action, i.e. we do not include higher-derivative terms in the latter. Since the back-reaction effect turns out to be of the same order as a four-derivative term, it may look suspicious to neglect such terms in the tree-level action while keeping them in the correction. There is, however, a limit in which such an approximation is justified: it is a large-N limit, where N is the number of species of the produced particles. The one-loop back-reaction is of relative order $g^2 N \ell_s^2 R$ and thus dominates over a tree-level higher-derivative correction $\mathcal{O}(\ell_s^2 R)$ at sufficiently large[2] $g^2 N$. Finally, it can also be shown that higher-loop effects on the metric, the dilaton and the antisymmetric tensor (axion) are parametrically smaller at large $g^2 N$ and small g^2, justifying the one-loop approximation. This observation makes it all the more urgent to devote further effort to a non-perturbative understanding of one-loop back-reaction in cosmology.

[2] N is indeed a large number for a typical string theory gauge group, say $E_8 \otimes E_8$.

The paper is organized as follows: after recalling (Sections 2 and 3) the form of the lowest-order PBB background whose modification we wish to compute, and the general idea of the method, we will introduce (Section 4) our gauge choice and then go, in Section 5, to the heart of the calculation of the back-reaction, first from gravi-dilaton production, and then from gauge-bosons production. In Section 6, after expressing the final result in a convenient form, we will argue that the back-reaction is of the expected order of magnitude and that, at leading order at least, it modifies the original background in the right direction to help with the exit problem. Finally, an explicit check of the gauge independence of our results is presented in an Appendix.

2 PBB Backgrounds and Fluctuations

In units in which $16\pi G = \hbar = c = 1$ the normalized four-dimensional gravi-dilaton effective action in the Einstein frame reads:

$$S = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}(\nabla\varphi)^2\right] \ . \tag{2.1}$$

Even within this minimal set of moduli (corresponding to having frozen the axion as well as internal dimensions), the gravi-dilaton system offers several interesting cosmological solutions, including, of course, the simplest ones, which describe spatially flat, homogeneous and isotropic Universes. In the comoving (cosmic)-time frame the line element takes the form

$$ds^2 = -dt_0^2 + a_0^2(t_0)\, d\vec{x}\cdot d\vec{x}\ , \tag{2.2}$$

where t_0 represents comoving time (we shall often use the suffix 0 to denote tree-level quantities). In the discussion of quantum fluctuations it is often convenient to go over to the so-called conformal frame, in which the metric (2.2) becomes

$$ds^2 = \Omega^2(\eta)\left(-d\eta^2 + d\vec{x}\cdot d\vec{x}\right) \ . \tag{2.3}$$

We recall that comoving and conformal times are related by $dt_0 = \Omega d\eta$. In the conformal frame a typical PBB-type solution appears as follows:

$$\Omega = \left(\frac{\eta}{\eta_i}\right)^{1/2}\ , \qquad \varphi_0 = -\sqrt{3}\ln(-\eta)\ , \qquad \eta_i < \eta < 0\ , \tag{2.4}$$

where, by identifying the (arbitrary) constant η_i with the beginning of dilaton-driven inflation (DDI), we have normalized Ω to 1 at that moment. It is convenient to define, as usual, $\mathcal{H}(\eta) = \Omega'/\Omega = 1/2\eta$, which is related to the physical Hubble parameter H by

$$H_0 = \frac{\dot{a}_0}{a_0} = \frac{\Omega'}{\Omega^2} = H_i \Omega^{-3} = \frac{1}{3t_0}, \qquad t_0 < 0 , \tag{2.5}$$

where $H_i = 1/2\eta_i$ refers to the initial value of the physical Hubble parameter. Note that, at initial time $\eta = \eta_i$, the physical Hubble parameter is given by Ω' since $\Omega_i = 1$.

We remind the reader that the above background corresponds to DDI in the string frame, with a string-frame scale factor blowing up like $(-t_s)^{-1/\sqrt{3}}$ when the comoving string time t_s approaches 0 from negative values ($t_s \to 0^-$). In the Einstein frame, this is seen as a contraction (gravitational collapse, see [4]). What remains frame-independent is the fact that curvature grows in time, rather than decreases as in power-law inflation. This is why the results of [14] cannot be directly used in our context.

Let us now define the fluctuations around the classical backgrounds by:

$$g_{\mu\nu} = \Omega^2 \left(\eta_{\mu\nu} + \psi_{\mu\nu}\right) , \qquad \varphi = \varphi_0 + \phi . \tag{2.6}$$

Our goal here is to compute the back-reaction on the above homogeneous backgrounds (2.4) (i.e. $\psi_{\mu\nu}$ and ϕ) due to quantum fluctuations at leading order. Following [13], [14], we will thus parametrize the correction to the solution (2.4) by

$$\begin{aligned} \langle \psi_{\mu\nu} \rangle &= A(\eta) \bar{\eta}_{\mu\nu} + C(\eta) t_\mu t_\nu \\ \langle \phi \rangle &= D(\eta) \ , \end{aligned} \tag{2.7}$$

where $t_\mu \equiv (-1, 0, 0, 0)$ is a time-like vector and $\bar{\eta}_{\mu\nu} = \eta_{\mu\nu} + t_\mu t_\nu \equiv \mathrm{diag}\,(0, 1, 1, 1)$. In the next section we shall recall the method of Refs. [12], [13], [14] for computing A, C and D. Here we just conclude by recalling that A, C and D have no physical meaning separately. For instance, if A and D are given in terms of C by

$$\begin{aligned} A &= \frac{\mathcal{H}_0}{\Omega}(\eta) \int^\eta d\eta' (\Omega C)(\eta') \\ D &= \frac{1}{2}\frac{\varphi_0'}{\Omega}(\eta) \int^\eta d\eta' (\Omega C)(\eta') \ , \end{aligned} \tag{2.8}$$

then all three can be gauged away by the redefinition of time, $\eta \to \eta + \frac{1}{2}\Omega^{-1} \int^\eta \Omega C$.

3 The Procedure

Since our background is of the power-law (though not of the power-inflation) type, we shall closely follow here the procedure described in [14] for computing A, C, D. We will then stress at which point the two procedures start to diverge. In [14] the authors first compute some amputated one-point functions denoted by α, γ and δ (see below), and then attach external leg propagators in order to reconstruct the desired functions A, C, D. Let us describe the procedure in a different, perhaps simpler, way.

If, generically, we wish to compute a fluctuation Ψ_i over a generic background Ψ_{0i}, then the relevant one-point function is given by the obvious functional integral

$$\langle \Psi_i(x) \rangle = \frac{\int D\Psi_\ell e^{iS[\Psi_{0\ell}+\Psi_\ell]} \Psi_i(x)}{\int D\Psi_\ell e^{iS[\Psi_{0\ell}+\Psi_\ell]}} \ . \tag{3.1}$$

If the action is expanded only up to the quadratic order in Ψ_ℓ, the one-point functions are trivially zero. If, however, the action is expanded up to cubic order, $S[\Psi_{0\ell} + \Psi_\ell] = S[\Psi_{0\ell}] + S_2^{ij}[\Psi_{0\ell}]\Psi_i\Psi_j + S_3^{ijk}[\Psi_{0\ell}]\Psi_i\Psi_j\Psi_k$ (the summation convention is adopted everywhere), and $S_3^{ijk}[\Psi_{0\ell}]$ is treated perturbatively, then there is a non-trivial one-loop correction to the one-point functions (see Fig. 1). This is the familiar story of vacuum-shifting tadpoles in quantum field theory. Analytically the result can be expressed as follows:

$$\langle \Psi_i(x) \rangle = \frac{\int D\Psi_\ell e^{iS[\Psi_{0\ell}]+iS_2^{ij}[\Psi_{0\ell}]\Psi_i\Psi_j} iS_3^{jkn}[\Psi_{0\ell}]\Psi_i(x)\Psi_j\Psi_k\Psi_n + ...}{\int D\Psi_\ell e^{iS[\Psi_{0\ell}]+iS_2^{ij}[\Psi_{0\ell}]\Psi_i\Psi_j}} \ . \tag{3.2}$$

The calculation of the diagrams in Fig. 1 can be performed in two steps. First, we

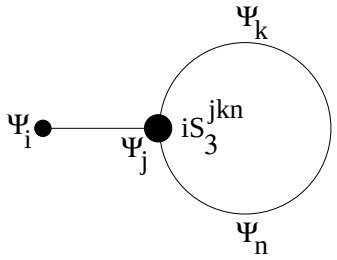

Figure 1: *The one-point functions from tadpoles*

compute the amputated one-point function for all possible three-point vertices, iS_3^{jkn}.

Then the full one-point functions are obtained by attaching the appropriate external lines. Formula (3.2) gives

$$\langle\Psi_i(x)\rangle = \int d^4z \langle\Psi_i(x)\Psi_j(z)\rangle\langle\Psi_k(z)\Psi_n(z)\rangle[iS_3^{jkn}] \ , \tag{3.3}$$

where (dropping indices hereafter), $\langle\Psi(x)\Psi(z)\rangle$ is the standard Feynman propagator for the quadratic action $S_2[\Psi_0]\Psi^2 \equiv \Psi\Delta\Psi$, i.e.

$$\Delta(x)\langle\Psi(x)\Psi(z)\rangle = i\delta^4(x-z) \ . \tag{3.4}$$

Finally, one can express the one-point function as:

$$\langle\Psi(x)\rangle = -\Delta^{-1}(x)\langle\Psi(x)\Psi(x)\rangle[S_3] \ . \tag{3.5}$$

It turns out that the coincident limit propagator $\langle\Psi(x)\Psi(x)\rangle$ is a function of time, x^0, only. Therefore, if one can invert the kinetic operator in the zero-momentum limit, then (3.5) provides our final expression for the one-point function. For the purpose of the first step it is convenient to distinguish three kinds of amputated one-point functions, α, γ and δ, defined respectively in Figs. 2, 3 and 4.

As an illustration of the use of (3.5), we present here the calculation of the one-point function A in the covariant gauge introduced in [12]. Note, from (2.7), that A is nothing but the amplitude $\langle\psi_{11}\rangle$, which can be expanded from the general definition (3.3) and Fig. 1, in terms of the amputated one-point functions α, γ and δ as:

$$A \to \langle\psi_{11}\phi\rangle\cdot\delta + \langle\psi_{11}\psi_{\mu\nu}\rangle t^\mu t^\nu\cdot\gamma + \langle\psi_{11}\psi_{\mu\nu}\rangle\bar{\eta}^{\mu\nu}\cdot\alpha \ . \tag{3.6}$$

As in (3.5), the external line propagators can be replaced by the inverse of the kinetic operators in the zero-momentum limit. In the covariant gauge of Iliopoulos et al. the Lagrangian can be diagonalized by transforming the variables into suitable linear combinations. The propagators of the original variables can be read off from those of the diagonalized variables by applying the inverse transformation. For a standard power-law-inflation background, $\Omega \sim t^s$ with $s > 1$, one easily finds

$$\begin{aligned}
\langle\psi_{11}\phi\rangle &= -\frac{\sqrt{s}}{s+1}i\Delta_A + \frac{\sqrt{s}}{s+1}i\Delta_C \\
\langle\psi_{11}\psi_{\mu\nu}\rangle t^\mu t^\nu &= \frac{1}{s+1}i\Delta_A + \frac{s}{s+1}i\Delta_C \\
\langle\psi_{11}\psi_{\mu\nu}\rangle\bar{\eta}^{\mu\nu} &= \left(-4+\frac{3}{s+1}\right)i\Delta_A + \frac{3s}{s+1}i\Delta_C
\end{aligned} \tag{3.7}$$

where $i\Delta_A$ and $i\Delta_C$ are particular propagators of the diagonalizing variables. They can be replaced by the inverses of the corresponding kinetic operators in the zero-momentum limit, $1/D_A$ and $1/D_C$. With this replacement, when (3.7) is plugged back into the expression for the one-point function A in (3.6), we immediately reach the desired expression for A:

$$A = \frac{1}{D_A}\left[-4\alpha + \frac{1}{s+1}(3\alpha+\gamma) - \frac{\sqrt{s}}{s+1}\delta\right] + \frac{1}{D_C}\left[\frac{s}{s+1}(3\alpha+\gamma) + \frac{\sqrt{s}}{s+1}\delta\right] \ . \quad (3.8)$$

This equation is the same as was given in [14]. The above outline shows how simply the relevant formulae can be obtained once the general result (3.5) has been derived. In the Iliopoulos et al. gauge, which does need ghosts, there are three different types of loops, shown in Figs. 2, 3, and 4, where the dotted line describes the graviton, the dashed line the ghost, and the solid line the dilaton.

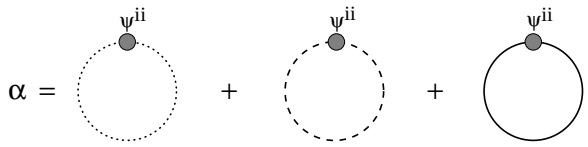

Figure 2: *The α-type amputated one-point function*

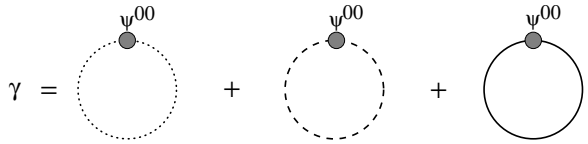

Figure 3: *The γ-type amputated one-point function*

Let us now briefly compare this approach to the one of Ref. [15], whose authors have used a more intuitive method by computing the energy-momentum tensor of the quantum fluctuations and their back-reaction over the original backgrounds directly. The quantum fluctuations are obtained by solving the linear order perturbation equations and the constraints. In our approach, as in [13], one uses instead the constraints

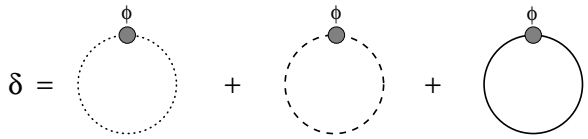

Figure 4: *The δ-type amputated one-point function*

and the gauge-fixing conditions to obtain the Feynman propagators entering the loops. In [15] the back-reaction was computed through systematic corrections of Einstein's equations, the first non-vanishing correction being quadratic in the Planck length ℓ_P, the small expansion parameter characterizing quantum fluctuations to leading order. The counterpart to this in our approach is that the same parameter, ℓ_P^2, controls the loop expansion in quantum gravity. Incidentally, this last observation neatly solves an apparent paradox: classical gravitational waves always carry positive energy while, as we shall see, the back-reaction from the quantum production of gravitational waves from the vacuum turns out to have, at least in some cases, negative energy. The point is that, in the case of the amplification of vacuum fluctuations, there is no parameter allowing a separation of real-graviton production from other virtual one-loop effects whose sign is undetermined. This a-priori non-determination of the sign of the back-reaction is self-evident in the effective action approach [11].

We will follow the same procedure as outlined above in a different gauge where physical variables are more explicit. The usefulness of this gauge was first pointed out in [16] and its significance has been explored in detail in [17], in the context of linear order perturbations to PBB backgrounds. In the next section we summarize results in the off-diagonal gauge relevant for the background (2.4). In the Appendix we will comment on the gauge dependence of the results by comparing them to those obtained in other gauges, e.g. in the covariant gauge of [12].

4 Propagators in the off-diagonal gauge

As explained in the previous section, in order to analyse the perturbations we need to expand (2.1) to quadratic and cubic orders in the fluctuations. We find:

$$S = \int d^4x\sqrt{-g}\Big[g^{\alpha\beta}g^{\rho\sigma}g^{\mu\nu}\Big\{\frac{1}{2}\psi_{\alpha\rho,\mu}\psi_{\nu\sigma,\beta} - \frac{1}{2}\psi_{\alpha\beta,\rho}\psi_{\sigma\mu,\nu} + \frac{1}{4}\psi_{\alpha\beta,\rho}\psi_{\mu\nu,\sigma} \\ -\frac{1}{4}\psi_{\alpha\rho,\mu}\psi_{\beta\sigma,\nu}\Big\}\Omega^4 - \frac{1}{2}g^{\rho\sigma}g^{\mu\nu}\psi_{\rho\sigma,\mu}\psi_{\nu}{}^{\alpha}(\Omega^2)_{,\alpha} - \varphi_0'\partial_\nu\phi\, g^{0\nu} - \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\, g^{\mu\nu}\Big] \quad (4.1)$$

Here the 'pseudo-graviton' ($\psi_{\mu\nu}$) indices are raised and lowered by $\eta_{\mu\nu}$. To complete the expansion of the $g_{\mu\nu}$ fields in terms of the background and the pseudo-graviton, we recall the expansions

$$\begin{aligned} g^{\mu\nu} &= \Omega^{-2}(\eta^{\mu\nu} - \psi^{\mu\nu} + \psi^{\mu\sigma}\psi^{\nu}_{\sigma} + ...) \\ \sqrt{-g} &= \Omega^4\left[1 + \frac{1}{2}\psi + (\frac{1}{8}\psi^2 - \frac{1}{4}\psi^{\alpha\beta}\psi_{\alpha\beta}) + ...\right] , \end{aligned} \quad (4.2)$$

where $\psi = \psi^{\mu}_{\mu}$.

Following [18] we split the fluctuations into tensor and scalar parts by

$$\psi_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix} + \begin{pmatrix} 2\Phi & -\partial_i B \\ -\partial_i B & 2(\Psi\delta_{ij} - \partial_i\partial_j E) \end{pmatrix} , \quad (4.3)$$

where h_{ij} represents the (transverse and traceless) tensor part with $h_{ii} = 0$ and $\partial_i h_{ij} = 0$. The trace ψ is given by $\psi = -2\Phi + 6\Psi - 2\nabla^2 E$, where $\nabla^2 \equiv \partial_i\partial_i$. The action (2.1), when expanded up to quadratic order in the variables (4.3), shows that B and Φ behave like Lagrange multipliers, giving rise to the following constraints (a prime refers to $\partial/\partial\eta$):

$$\begin{aligned} B &: \quad \Psi' + \mathcal{H}\Phi + \frac{1}{4}\varphi_0'\phi = 0 \\ \Phi &: \quad 4\nabla^2\Psi - 12\mathcal{H}\Psi' + \varphi_0'\phi' - 4\mathcal{H}\nabla^2(B - E') = 0 . \end{aligned} \quad (4.4)$$

Making use of the first constraint in (4.4), it is possible to diagonalize the quadratic part of the action in our background as

$$S_2 = \frac{1}{2}\int d^4x\left[\Big(v'v' - \partial_i v\partial_i v + \frac{\Omega''}{\Omega}v^2\Big) + \frac{1}{2}\Big(\zeta_{ij}'\zeta_{ij}' - \partial_k\zeta_{ij}\partial_k\zeta_{ij} + \frac{\Omega''}{\Omega}\zeta_{ij}\zeta_{ij}\Big)\right] , \quad (4.5)$$

where v and ζ_{ij} are the canonical variables, given in terms of the original fields as

$$v = \Omega(\phi - \frac{\varphi_0'}{\mathcal{H}}\Psi) = \Omega(\phi + 2\sqrt{3}\Psi)\,, \qquad \zeta_{ij} = \Omega h_{ij}\ . \tag{4.6}$$

Owing to two local gauge symmetries that preserve the form (4.3), two out of the four functions can be fixed through two independent gauge choices. The off-diagonal gauge corresponds to choosing $\Psi = E = 0$, which simplifies the first canonical variable to $v = \Omega\phi$. The rest of the variables become related to this canonical variable in simple ways because of other relations following from the constraints (4.4), $\Phi = \sqrt{3}\phi/2$ and $\nabla^2 B = -\sqrt{3}\phi'/2$.

The process of obtaining the propagator for the canonical variables is simplified by dragging the scale factor, Ω, from the canonical variables to the quadratic differential operator. In the off-diagonal gauge this amounts to

$$S_2 = \frac{1}{2}\int d^4x \left[\phi\Delta\phi + \frac{1}{2}h_{ij}\Delta h_{ij}\right]\,, \qquad \Delta = \Omega\left[-\partial_\eta^2 + \nabla^2 + \frac{\Omega''}{\Omega}\right]\Omega. \tag{4.7}$$

The differential operator Δ, in the zero-momentum limit, is easily inverted:

$$\Delta_0^{-1} f(\eta) \equiv -\int_{\eta_i}^{\eta} d\eta_1 \Omega^{-2}(\eta_1)\int_{\eta_i}^{\eta_1} d\eta_2 f(\eta_2)\,, \quad \Delta_0 = \Omega\left[-\partial_\eta^2 + \frac{\Omega''}{\Omega}\right]\Omega, \tag{4.8}$$

as has been advertised in [14]. This choice of the inverse operator also enforces the boundary conditions

$$\Delta_0^{-1} f(\eta_i) = (\Delta_0^{-1} f)'(\eta_i) = 0, \tag{4.9}$$

a property we require for the corrections A, C and D.

From the quadratic action (4.7) it is easy to read out the Feynman propagators for the physical modes

$$\begin{aligned} \langle\phi(x)\phi(y)\rangle &= i\Delta^{-1}(x,y) \\ \langle h_{ij}(x)h_{kl}(y)\rangle &= 2T_{ijkl}\left[i\Delta^{-1}(x,y)\right]\,, \end{aligned} \tag{4.10}$$

where T_{ijkl} is an appropriately normalized tensor enforcing the transverse and traceless conditions on gravitons, i.e. $\partial_i T_{ijkl} = \partial_j T_{ijkl} = \partial_k T_{ijkl} = \partial_l T_{ijkl} = 0$ and $T_{iikl} = T_{ijkk} = 0$. For our purpose we are interested in computing the propagator

$\Delta^{-1}(x,y)$ in the coincident limit $x \to y$. To do this we go to the Fourier-transformed variables

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\phi_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}\,, \qquad h_{ij}(x) = \int \frac{d^3k}{(2\pi)^3}h_{\mathbf{k}ij}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}\quad, \tag{4.11}$$

where both the mode functions $\phi_{\mathbf{k}}$ and $h_{\mathbf{k}ij}$ can be obtained by solving the differential equations $\Delta f_{\mathbf{k}}(\eta) = 0$. The solutions can be expressed in terms of Hankel functions of the second kind. Notice that, $v = \Omega\phi$ being the canonical variable, one should normalize its modes in the far past, $\eta \to -\infty$, as $1/\sqrt{|\mathbf{k}|}$. By the large-argument limit of the Hankel function, this completely fixes the modes:

$$f_{\mathbf{k}}(\eta) = \frac{1}{2\Omega}\sqrt{\pi\eta}H_0^{(1)}(|\mathbf{k}\eta|)\,. \tag{4.12}$$

Consequently the propagator in the coincident-point limit takes the form

$$\begin{aligned}
\lim_{x\to y}\langle\phi(x)\phi(y)\rangle &= \frac{|\eta|}{8\pi\Omega^2}\int dk k^2 H_0^{(1)*}(k|\eta|)H_0^{(1)}(k|\eta|) \\
\lim_{x\to y}\langle h_{ij}(x)h_{kl}(y)\rangle &= \frac{|\eta|}{4\pi\Omega^2}\int dk k^2(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}-\delta_{ik}\frac{k_jk_l}{k^2}-\delta_{jl}\frac{k_ik_k}{k^2} \\
&\quad -\delta_{il}\frac{k_jk_k}{k^2}-\delta_{jk}\frac{k_ik_l}{k^2}+\delta_{kl}\frac{k_ik_j}{k^2}+\delta_{ij}\frac{k_kk_l}{k^2}+\frac{k_ik_jk_kk_l}{k^4}) \\
&\quad \times H_0^{(1)*}(k|\eta|)H_0^{(1)}(k|\eta|)\,.
\end{aligned} \tag{4.13}$$

Here we have replaced the transverse, traceless T_{ijkl} tensor by its Fourier transform. The integral in (4.13) is in general divergent in the ultraviolet. It should be kept in mind, however, that the present approach should be complemented by a momentum cutoff as a result of the ultraviolet sickness of gravity. We set the cutoff, at any given time, at $k_{max} \sim \Omega|H|$, which corresponds to a wavelength of the order of the Hubble radius. This is also the scale at which the mode functions switch from free plane waves to 'frozen out' modes. For modes larger than this wavelength the bosonic contribution dominates its fermionic counterpart (we assume some underlying supersymmetry) as a result of Bose-condensation. This is the effect we are after. For shorter-wavelength modes new degrees of freedom should appear, which would ultimately provide better ultraviolet behaviour, as in superstring theory. We also impose an infrared cutoff coming from a finite duration of dilaton-driven inflation, $k_{min} \sim |H_i|$, corresponding to the maximal Hubble radius. In conclusion we restrict momenta in all loops to be in

the range: $|H_i| \leq k \leq \Omega|H| = |H_i|/\Omega^2$. At this upper-momentum/short-wavelength cutoff, $\Omega|H| \sim 1/|\eta|$, i.e. our cutoff corresponds to $|k\eta| \sim 1$. Note that, in the off-diagonal gauge, as a result of the constraints, all fields can be expressed in terms of ϕ. Thus only one type of amputated loops, the δ-type, effectively arises on the right-hand side of (3.5).

5 Computation of the one-point functions

In the first two subsections we shall compute the amputated one-point functions due to graviton, dilaton, and gauge-boson loops. In the last subsection the external zero-momentum propagators will be glued in to reconstruct the full one-point functions.

5.1 Graviton and dilaton loops

To get the three-point vertices we use our expansion of the action (4.1) to cubic order in the fluctuations, S_3. The results are summarized in tables 1 and 2, showing all the relevant three-point vertices. In table 3 we list all the coincident loops appearing in δ and their respective contributions to the loop integrals ($x = |k\eta|$). There is another way of doing the same computation, which involves, in general, four types of amputated loops in the off-diagonal gauge (Fig. 5).

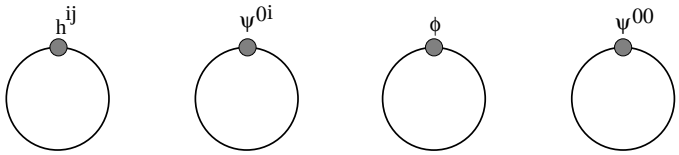

Figure 5: *All possible types of amputated one-point functions in the off-diagonal gauge. Here the line represents all possible coincident propagators.*

The first type of loops, having h_{ij} in the external leg, turn out to be zero, since the loop (see both tables 1 and 2), in the coincident limit, always picks up a factor of δ_{ij}, which gives vanishing contributions because of the tracelessness of h_{ij}. The second type of loops, having Ψ_{0i} in the external leg, also gives vanishing contributions, but for a different reason. Note that $\Psi_{0i} = \partial_i B$ can be integrated by parts to act on

the loop. The loops, however, being functions only of time in the coincident limit (4.13) give zero when hit by a space derivative. Thus only the third and fourth types of loops survive in the off-diagonal gauge, the δ-type and γ-type loops as defined in Figs. 3 and 4.

1	$\frac{1}{2}H_i\partial_2^{\alpha_3}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)t^{\beta_3}$
2	$-H_i\partial_2^{\alpha_3}\eta^{\alpha_1\alpha_2}\eta^{\beta_2\beta_1}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)t^{\beta_3}$
3	$-H_i\partial_2^{\beta_1}\eta^{\alpha_2\beta_2}\eta^{\beta_3\alpha_1}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)t^{\alpha_3}$
4	$\frac{1}{4}\Omega^2\partial_2^{\beta_3}\partial_3^{\alpha_2}\eta^{\alpha_1\beta_1}\eta^{\beta_2\alpha_3}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
5	$-\Omega^2\partial_2^{\beta_3}\partial_3^{\alpha_1}\eta^{\beta_1\alpha_2}\eta^{\beta_2\alpha_3}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
6	$\frac{1}{2}\Omega^2\partial_2^{\beta_3}\partial_3^{\alpha_2}\eta^{\beta_1\alpha_3}\eta^{\beta_2\alpha_1}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
7	$\frac{1}{4}\Omega^2\partial_2^{\alpha_3}\partial_3^{\beta_3}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
8	$\frac{1}{2}\Omega^2\partial_2^{\alpha_3}\partial_3^{\beta_3}\eta^{\alpha_1\alpha_2}\eta^{\beta_2\beta_1}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
9	$\frac{1}{2}\Omega^2\partial_2^{\alpha_1}\partial_3^{\beta_3}\eta^{\alpha_2\beta_2}\eta^{\beta_1\alpha_3}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
10	$\frac{1}{2}\Omega^2\partial_2^{\alpha_1}\partial_3^{\beta_2}\eta^{\alpha_3\beta_3}\eta^{\beta_1\alpha_2}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
11	$\frac{1}{8}\Omega^2\partial_2^{\alpha_4}\partial_3^{\alpha_5}\eta_{\alpha_4\alpha_5}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\eta^{\alpha_3\beta_3}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
12	$\frac{1}{2}\Omega^2\partial_2^{\alpha_4}\partial_3^{\alpha_5}\eta_{\alpha_4\alpha_5}\eta^{\alpha_1\alpha_2}\eta^{\alpha_3\beta_3}\eta^{\beta_2\beta_1}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
13	$\frac{1}{4}\Omega^2\partial_2^{\alpha_1}\partial_3^{\beta_1}\eta^{\alpha_2\beta_2}\eta^{\alpha_3\beta_3}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
14	$\frac{1}{8}\Omega^2\partial_2^{\alpha_4}\partial_3^{\alpha_5}\eta_{\alpha_4\alpha_5}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\alpha_3}\eta^{\beta_3\beta_2}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
15	$\frac{1}{2}\Omega^2\partial_2^{\alpha_4}\partial_3^{\alpha_5}\eta_{\alpha_4\alpha_5}\eta^{\alpha_1\alpha_2}\eta^{\beta_2\alpha_3}\eta^{\beta_3\beta_1}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$
16	$\frac{1}{4}\Omega^2\partial_2^{\alpha_1}\partial_3^{\beta_1}\eta^{\alpha_2\alpha_3}\eta^{\beta_3\beta_2}\psi_{\alpha_1\beta_1}(1)\psi_{\alpha_2\beta_2}(2)\psi_{\alpha_3\beta_3}(3)$

Table 1: *List of all $\psi_{\alpha_1\beta_1}\psi_{\alpha_2\beta_2}\psi_{\alpha_3\beta_3}$ vertices. While choosing an external leg one should permute $1,2,3$ in each vertex.*

1	$-\frac{\sqrt{3}}{4}H_i\partial_1^{\alpha_6}\eta^{\alpha_4\alpha_5}\eta^{\beta_4\beta_5}\phi(1)\psi_{\alpha_4\alpha_5}(2)\psi_{\beta_4\beta_5}(3)t_{\alpha_6}$
2	$\frac{\sqrt{3}}{2}H_i\partial_1^{\alpha_4}\eta^{\alpha_5\beta_5}\eta^{\alpha_6\beta_6}\phi(1)\psi_{\alpha_5\alpha_6}(2)\psi_{\beta_5\beta_6}(3)t_{\alpha_4}$
3	$\sqrt{3}H_i\partial_1^{\alpha_5}\eta^{\alpha_7\alpha_8}\phi(1)\psi_{\alpha_4\alpha_5}(2)\psi_{\alpha_7\alpha_8}(3)t^{\alpha_4}$
4	$-2\sqrt{3}H_i\partial_1^{\alpha_5}\eta^{\alpha_6\alpha_7}\phi(1)\psi_{\alpha_4\alpha_6}(2)\psi_{\alpha_5\alpha_7}(3)t^{\alpha_4}$
5	$-\frac{1}{4}\Omega^2\partial_1^{\alpha_4}\partial_2^{\alpha_5}\eta_{\alpha_4\alpha_5}\eta^{\alpha_6\alpha_7}\phi(1)\phi(2)\psi_{\alpha_6\alpha_7}(3)$
6	$\frac{1}{2}\Omega^2\partial_1^{\alpha_4}\partial_2^{\alpha_5}\phi(1)\phi(2)\psi_{\alpha_4\alpha_5}(3)$

Table 2: *All possible $\phi\psi_{\alpha_1\beta_1}\psi_{\alpha_2\beta_2}$ and $\phi\phi\psi_{\alpha_1\beta_1}$ vertices. Again one should permute appropriate indices when considering each vertex.*

$\langle\phi,\phi'\rangle$	$\int dx x^3 H_0^{(1)*}(x)H_0'^{(1)}(x)/16H_i\pi\lvert\eta\rvert^4$
$\langle\phi,\phi''\rangle$	$-\int dx x^4 H_0^{(1)*}(x)H_0''^{(1)}(x)/16H_i\pi\lvert\eta\rvert^5$
$\langle\partial_i\phi,\partial_i\phi\rangle$	$-\int dx x^4 H_0^{(1)*}(x)H_0^{(1)}(x)/16H_i\pi\lvert\eta\rvert^5$
$\langle\partial_i\phi,\psi_{0i}\rangle$	$-\sqrt{3}\int dx x^3 H_0^{(1)*}(x)H_0'^{(1)}(x)/32H_i\pi\lvert\eta\rvert^4$
$\langle\partial_i\phi,\psi'_{0i}\rangle$	$\sqrt{3}\int dx x^4 H_0^{(1)*}(x)H_0''^{(1)}(x)/32H_i\pi\lvert\eta\rvert^5$
$\langle\partial_i\phi',\psi_{0i}\rangle$	$\sqrt{3}\int dx x^4 H_0'^{(1)*}(x)H_0'^{(1)}(x)/32H_i\pi\lvert\eta\rvert^5$
$\langle\psi_{0i},\psi'_{0i}\rangle$	$-3\int dx x^3 H_0'^{(1)*}(x)H_0''^{(1)}(x)/64H_i\pi\lvert\eta\rvert^4$
$\langle\partial_i\psi_{0j},\partial_j\psi_{0i}\rangle$	$-3\int dx x^4 H_0'^{(1)*}(x)H_0'^{(1)}(x)/64H_i\pi\lvert\eta\rvert^5$
$\langle\partial_i\psi_{0j},\partial_i\psi_{0j}\rangle$	$-3\int dx x^4 H_0'^{(1)*}(x)H_0'^{(1)}(x)/64H_i\pi\lvert\eta\rvert^5$
$\langle\psi_{ij},\psi'_{ij}\rangle$	$3\int dx x^3 H_0^{(1)*}(x)H_0'^{(1)}(x)/8H_i\pi\lvert\eta\rvert^4$
$\langle\partial_i\psi_{jk},\partial_j\psi_{ik}\rangle$	0
$\langle\partial_i\psi_{jk},\partial_i\psi_{jk}\rangle$	$-3\int dx x^4 H_0^{(1)*}(x)H_0^{(1)}(x)/8H_i\pi\lvert\eta\rvert^5$
$\langle\phi',\phi'\rangle$	$-\int dx x^4 H_0'^{(1)*}(x)H_0'^{(1)}(x)/16H_i\pi\lvert\eta\rvert^5$
$\langle\psi'_{ij},\psi'_{ij}\rangle$	$-3\int dx x^4 H_0'^{(1)*}(x)H_0'^{(1)}(x)/8H_i\pi\lvert\eta\rvert^5$

Table 3: *All possible coincident loop propagators appearing for the off-diagonal gauge. The prime in $H_0'^{(2)}$ refers to $\partial/\partial x$, otherwise to $\partial/\partial\eta$.*

Another important feature of the off-diagonal gauge is the absence of the Faddeev–Popov ghosts. This can be readily seen from the gauge transformations of the gauge-fixing functions Ψ and E. Their gauge transformations have been given in [18]:

$$\Psi \to \Psi + \mathcal{H}\xi(x)\,, \qquad E \to E - \overline{\xi}(x)\,, \tag{5.1}$$

where ξ and $\overline{\xi}$ are the two gauge parameters associated with the coordinate transformations: $\eta \to \eta - \xi$ and $x^i \to x^i - \partial_i\overline{\xi}$. Clearly, the Faddeev–Popov determinant is field-independent and hence the ghosts are decoupled, unlike what happens in the covariant gauge of [12].

As an example, we present a sample calculation, say the three-point vertex #5 of table 2. For illustrative purposes, it is easier to make use of the constraints at the beginning (in contrast, in the computer code its more convenient to impose the constraints later). As we have argued, this does not make a difference, which is at the same time an important consistency check for our method. Making use of the constraints in the off-diagonal gauge presented in the previous section, the $\mathrm{tr}(\psi_{\mu\nu})$,

present in the vertex #5, can be replaced by the scalar fluctuation ϕ. As we said, in this case there is only one type of amputated loop $\delta_{\rm eff}$, that can be read off from the vertex #5:

$$\delta_{\rm eff} = \frac{\sqrt{3}}{4}[-\Omega^2\langle\phi'\phi'\rangle + 4H_i\langle\phi\phi'\rangle + \Omega^2\langle\phi\phi\rangle'' + \Omega^2\langle\partial_i\phi\partial_i\phi\rangle] \,. \tag{5.2}$$

Now, one should look at table 3, where various propagators have been listed in this gauge and pick up the leading-order terms from (5.2) as we approach the limit, $\eta \to 0^-$. Recall that $\Omega^2 \sim |\eta|$, and hence all the terms in (5.2) are of the same order, $1/\eta^4$.

5.2 Gauge-boson loops

Let us now study the back-reaction produced by gauge fields to the background (2.4). There are two distinct types of gauge fields appearing in the low-energy string effective action — the NS-type and the R-type. The NS-type couple to all the fundamental moduli while the R-type couple only to the metric. Let us first concentrate on the NS-type gauge bosons. In the canonical frame the action is:

$$S^{\rm em} = \frac{1}{4}\int d^4x e^{-\varphi}\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \,, \tag{5.3}$$

which supplements the graviton–dilaton action (2.1) in the presence of the NS-type fields. Since the gauge-boson action (5.3) is already quadratic in the gauge fields, these decouple, to linear order, from other fluctuations. Also, interactions stemming from a possible non-Abelian gauge structure can be ignored. We shall work in the Coulomb gauge where the temporal component of the gauge field is zero and the spatial components satisfy $\partial_i A_i = 0$. Going over to a canonical gauge-field variable, the action can be written in the form

$$S_2^{\rm em} = \frac{1}{2}\int d^4x \zeta_i \overline{\Delta}\zeta_i \,, \qquad \overline{\Delta} = -\partial_\eta^2 + \nabla^2 - e^{\varphi_0/2}(e^{-\varphi_0/2})'' \,, \tag{5.4}$$

where the canonical field is given by $\zeta_i = e^{-\varphi_0/2}A_i$. Fourier expanding as before, the mode functions satisfy the homogeneous equation

$$\zeta_i = \int \frac{d^3k}{(2\pi)^3}\zeta_{\mathbf{k}i}(\eta)e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}} \,, \quad \overline{\Delta}\zeta_i = 0 \,, \quad \mathbf{k}^i\zeta_{\mathbf{k}i} = 0 \,, \tag{5.5}$$

the solution of which can be written down [19] in terms of the Hankel functions of the second kind. Following the same normalization arguments as presented before, the Coulomb-gauge Feynman propagator for the canonical fields can be expressed in the form

$$\langle\zeta_i(x)\zeta_j(y)\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{4}\Big(\delta_{ij} - \frac{k^ik^j}{k^2}\Big)\left[\sqrt{\pi\eta}H_\lambda^{(1)}(kx^0)\right]\left[\sqrt{\pi\eta}H_\lambda^{(1)*}(ky^0)\right]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \quad (5.6)$$

where the index $\lambda = (\sqrt{3}-1)/2$. The cubic vertices have been listed in table 4 and the coincident loops are presented in table 5.

For the R-type gauge bosons, the coupling to the dilaton being absent in the effective action (5.3), the mode functions satisfy the free wave equations: $\zeta''_{\mathbf{k}i} + k^2\zeta_{\mathbf{k}i} = 0$. Clearly, these modes are not parametrically amplified; and hence their contributions to the back-reaction are negligible.

The relevant diagrams for gauge bosons are the same as in Fig. 5 except that here the solid line represents the gauge loop. Notice, from table 4, that all the three-point vertices are quadratic in the gauge fields and at the linear order there is no coupling between the gauge fields and the gravitational perturbations. Among the loops, see again Fig. 5, the ones that have h_{ij} and ψ_{0i} as external legs are zero for the same reason as presented before. The rest of the diagrams are of the γ or δ type. As an example, let us take the vertex #6 of table 4. It contributes to $\delta_{\rm eff}$ as

$$\delta_{\rm eff} = \frac{1}{2}\exp(-\varphi_0)[-\langle A_i'A_i'\rangle + \langle\partial_kA_i\partial_kA_i\rangle] \,. \quad (5.7)$$

The exponential factor is common to all the vertices and is $\sim |\eta|^{\sqrt{3}}$, which cancels the inverse factor in the second coincident-limit propagator, as can be seen from table 5. It requires some more effort to see that this is also happening for the first entry of table 5, but again, even for gauge bosons, the leading effect in the amputated one-point functions is $\sim 1/\eta^4$.

1	$-\frac{1}{4}\exp(-\varphi_0)\partial_2^{\alpha_5}\partial_3^{\beta_6}\eta_{\alpha_5\beta_6}\eta^{\alpha_4\beta_4}\eta^{\beta_5\alpha_6}A_{\beta_5}(2)A_{\alpha_6}(3)\psi_{\alpha_4\beta_4}(1)$
2	$\frac{1}{4}\exp(-\varphi_0)\partial_2^{\alpha_5}\partial_3^{\beta_5}\eta^{\alpha_4\beta_4}A_{\beta_5}(2)A_{\alpha_5}(3)\psi_{\alpha_4\beta_4}(1)$
3	$\frac{1}{2}\exp(-\varphi_0)\partial_2^{\alpha_5}\partial_3^{\beta_5}\eta^{\alpha_6\beta_6}A_{\alpha_6}(2)A_{\beta_6}(3)\psi_{\alpha_5\beta_5}(1)$
4	$\frac{1}{2}\exp(-\varphi_0)\partial_2^{\beta_5}\partial_3^{\alpha_5}\eta_{\alpha_5\beta_5}\eta^{\alpha_4\alpha_6}\eta^{\beta_4\beta_6}A_{\alpha_6}(2)A_{\beta_6}(3)\psi_{\alpha_4\beta_4}(1)$
5	$-\exp(-\varphi_0)\partial_2^{\alpha_5}\partial_3^{\beta_4}\eta^{\alpha_4\beta_5}A_{\beta_5}(2)A_{\alpha_5}(3)\psi_{\alpha_4\beta_4}(1)$
6	$\frac{1}{2}\exp(-\varphi_0)\partial_2^{\alpha_5}\partial_3^{\beta_6}\eta_{\alpha_5\beta_6}\eta^{\beta_5\alpha_6}\phi(1)A_{\beta_5}(2)A_{\alpha_6}(3)$
7	$-\frac{1}{2}\exp(-\varphi_0)\partial_2^{\alpha_5}\partial_3^{\beta_5}\phi(1)A_{\beta_5}(2)A_{\alpha_5}(3)$

Table 4: *All possible $\phi A_{\alpha_1}A_{\beta_1}$ and $\psi_{\alpha_1\beta_1}A_{\alpha_2}A_{\beta_2}$ vertices. Again one should permute indices when considering each vertex.*

$\langle A'_{i_1}, A'_{i_1}\rangle$	$\int dx x^2[\sqrt{-\eta}(\eta/\eta_i)^{-\sqrt{3}/2}H_\lambda^{(1)*}(\lvert k\eta\rvert)]' \cdot$ $[\sqrt{-\eta}(\eta/\eta_i)^{-\sqrt{3}/2}H_\lambda^{(1)}(\lvert k\eta\rvert)]'/8\pi\lvert\eta\rvert^3$
$\langle \partial_{i_1}A_{i_2}, \partial_{i_1}A_{i_2}\rangle$	$\int dx x^4(\eta/\eta_i)^{-\sqrt{3}}H_\lambda^{(1)*}(x)H_\lambda^{(1)}(x)/8\pi\lvert\eta^4\rvert$

Table 5: *All coincident photon loops. All primes refer to $\partial/\partial\eta$.*

5.3 Adding external propagators

Following (3.5) we have first computed the amputated one-point functions in the off-diagonal gauge. As it turned out, after summing over all the vertices in tables 1 and 2, the leading-order terms, as $\eta \to 0$, have the time dependence $1/\eta^4$. We now attach the external line, i.e. the inverse of the zero-momentum kinetic operator, Δ_0. From (4.8) we note that our amputated quantities have to go through a double η-integral and multiplication by Ω^{-2}. Thus the leading-order time dependence of the one-point functions is simply $1/\eta^3$. Going through the procedure more carefully, we see that, apart for some numerical factors (which are different for different one-point functions), the leading-order contribution to all one-point functions is of order $1/H_i\eta^3$. This result suggests that, in the context of PBB cosmology, the dominant modes are those near the upper momentum cut-off $k \sim \eta^{-1}$. As we have already argued, contributions from modes beyond this cut-off, $|k\eta| \sim 1$, have to be combined with those from other fields, e.g. from fermions. The resulting combined effect will be typical of a short-distance renormalization of the bare parameters of the Lagrangian, which should be mild in a supersymmetric theory. Apart from this, the physical

effect from short distances should be parametrically smaller than the effect we have computed. Considering (2.4), (2.5), and the formula $H_i = 1/2\eta_i$, the contribution we have found can be simply expressed, in our units, as:

$$(A,\, C,\, D) \sim \frac{1}{H_i \eta^3} \sim \frac{H_i^2}{\Omega^6} \equiv H_0^2 \,. \tag{5.8}$$

What about the ratio between the three one-point functions in the off-diagonal gauge? Our answer turns out to be simply:

$$A = 0\,, \qquad C = \sqrt{3} D\,, \qquad D = -\Delta_0^{-1} \delta_{\rm eff} \,. \tag{5.9}$$

We should point out that, while (5.9) appears to parametrize only certain forms of metric fluctuations, other contributions, such as $\langle \psi_{0i} \rangle$ or $\langle \psi_{ij} \rangle$, $i \neq j$, have been shown to be exactly zero. Also, the vanishing of α easily follows from the property $h_{ii} = 0$ and from the coincident-point limit of the loop (4.13).

To reach (5.9) one could follow another path, which parallels more closely [13] and [14]. One can take into account all types of loops –corresponding to all the variables appearing in the fluctuations (2.6)– and use the constraints at the end. As a matter of fact, this also provides a good consistency check of the method. Here again, from very general considerations (cf. the three-point vertices in tables 1, 2 and 4), loops with ψ_{0i} and h_{ij} as external legs vanish, and only δ-type and the γ-type amputated functions survive. Thus the results (5.9) follow in a different way:

$$A = 0\,, \qquad C = \sqrt{3} D\,, \qquad D = -\Delta_0^{-1} \left(\delta + \sqrt{3} \gamma \right) \quad , \tag{5.10}$$

where (5.10) is identical to (5.9) since the constraints provide the relation $\delta_{\rm eff} = \delta + \sqrt{3}\gamma$.

We are now ready to discuss the physical implications of our results for the PBB scenario.

6 Discussion

The physical effects due to back-reaction are best analysed, in a PBB context, by going over to the string frame. In order to prepare for that, let us write down the

final outcome of our calculation in the off-diagonal gauge as

$$\begin{aligned} \langle ds^2 \rangle &= \Omega^2[-(1-C)d\eta^2 + d\vec{x}\cdot d\vec{x}] = -dt^2 + a^2(t) d\vec{x}\cdot d\vec{x} \\ \langle \varphi \rangle &= \varphi_0 + D \ , \end{aligned} \tag{6.1}$$

where C and D are as given explicitly in (5.10). Since we limit ourselves to times for which $C \ll 1$, the new comoving time t is related to the background comoving time by

$$dt \simeq \Omega \left(1 - \frac{1}{2}C\right) d\eta = \left(1 - \frac{1}{2}C\right) dt_0 \ . \tag{6.2}$$

Expressing the dilaton and the Hubble parameter H associated with the scale factor a in the new comoving time, and after using the relation (5.9) between C and D we find

$$\begin{aligned} \langle \varphi \rangle &\simeq -\frac{2}{\sqrt{3}} \Big[\ln(-t) - \sqrt{3}D\Big] \\ H &= \frac{d}{dt}\langle \ln a \rangle \simeq \frac{1}{3t}\Big[1 + \sqrt{3}D\Big] \ . \end{aligned} \tag{6.3}$$

In order to go over to the string frame, we introduce the corresponding comoving time $dt_s = \exp(\langle\varphi\rangle/2)dt$ and the scale factor $a_s = \exp(\langle\varphi\rangle/2)a$. After a straightforward calculation the resulting string-frame Hubble parameter and dilaton can be written as

$$\begin{aligned} H_s &= \frac{d}{dt_s}\langle \ln a_s \rangle \simeq -\frac{1}{\sqrt{3}t_s}\left[1 + \frac{7\sqrt{3}-3}{2}D\right] \ , \\ \langle \varphi \rangle &\simeq -(\sqrt{3}+1)\Big[\ln(-t_s) - (2\sqrt{3}-3)D\Big] \ . \end{aligned} \tag{6.4}$$

It turns out that, in the Einstein frame, $D(t)$ is a positive number times the square of the Hubble parameter. For the minimal gravi-dilaton system, we find:

$$D(t) = 0.0047 H_0^2(t) \ . \tag{6.5}$$

Although the coefficient of H_0^2 looks small, we have to recall that, in our units ($16\pi G = \hbar = c = 1$), Planck's time is given by $t_P^2 = 1/16\pi$. In other words, the (relative, dimensionless) correction to the tree-level background is:

$$D = 0.24\, t_P^2 H_0^2 \ , \tag{6.6}$$

i.e. does not contain any particularly small number.

When one goes to the string frame, in which the string length ℓ_s is a constant, the same result as (6.6) for D follows, provided one interprets t_P as the effective, time-dependent, Planck time $t_P = \ell_s e^{\phi/2}$ and H_0 is expressed in terms of the string-frame Hubble parameter and of $\dot{\phi}$ (now the dot refers to derivative w.r.t. cosmic string time) by the standard relation $H_0 = H_s - \dot{\phi}/2$.

When the NS-gauge-boson loops are included the numbers in (6.5) change but the conclusions do not. The contribution of a single gauge boson to D turns out to be

$$D = 0.00016 H_0^2 = 0.008\ t_P^2 H_0^2\ , \tag{6.7}$$

i.e. of the same sign but about 30 times smaller than that of the gravi-dilaton system (6.5). The result (6.7) should be multiplied by the number of NS gauge bosons, which, in typical superstring theories, could be in the range of hundreds.

In both cases the one-loop correction adds to the rate of inflation and to the rate of growth of the dilaton. At first sight, this seems to go against an eventual exit from inflation. However, what is crucial for the exit is not so much the *absolute* rates of growth but rather the *relative* rate of dilaton and scale-factor growth.

To be more precise, let us compute directly the effect of back-reaction on the string-frame quantity $\dot{\overline{\varphi}}$, where $\overline{\varphi} = \varphi - 3\ln a_s$ is the duality-invariant shifted dilaton and, as before, the dot refers to the time derivative in the string frame. From Eq. (6.4) we easily compute:

$$\dot{\overline{\varphi}} = -\frac{1}{t_s}\left[1 - \frac{9-3\sqrt{3}}{2} D\right]\ . \tag{6.8}$$

The positive sign of D is forcing $\dot{\overline{\varphi}}$ to decrease (relative to the tree-level value) as the result of a competition between a growing dilaton and a growing Hubble parameter in the string frame. It is thus obvious that the ratio $\dot{\overline{\varphi}}/H_s$ will change from its tree-level value of $\sqrt{3}$ towards smaller values. As shown in Fig. 6, this is precisely as required by a graceful exit [5], i.e. the phase-space trajectory in the $\dot{\overline{\varphi}}$-H plane should turn anticlockwise. Incidentally, this is also what the Hubble entropy bound of [9] requires [10].

Another way the effect (6.8) can be understood is by computing the effective energy and pressure densities produced by the quantum fluctuations and by check-

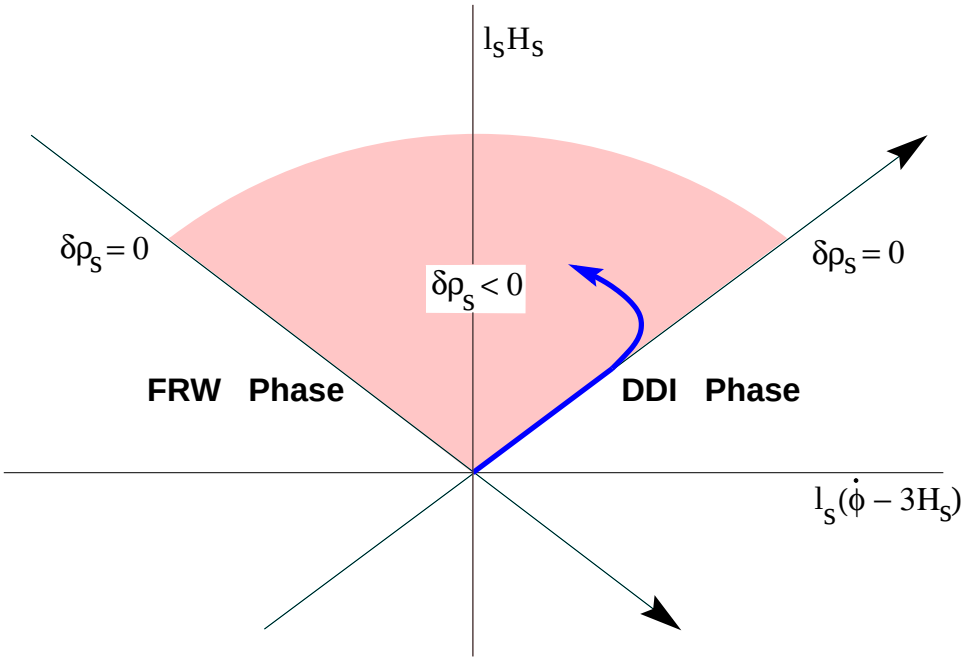

Figure 6: *The phase diagram of PBB cosmology. The curve shows the direction in which the back-reaction we computed is modifying the lowest order solution (straight line) helping an eventual flow from DDI to a FRW-phase.*

ing whether the condition $\delta\rho_s + \delta p_s < 0$ is obeyed, where $\delta\rho_s$ and δp_s are the energy and pressure densities in the quantum fluctuations. Let us study this first in the Einstein frame. For an additional ideal fluid of energy-momentum tensor $T^\mu_\nu = \mathrm{diag}(-\delta\rho, \delta p, \delta p, \delta p)$ added to the dilaton the energy density and pressure density appear in the Einstein–Friedmann equations as

$$\delta\rho = 6H^2 - \frac{1}{2}\dot{\varphi}^2\,, \quad -\delta p = 4\dot{H} + 6H^2 + \frac{1}{2}\dot{\varphi}^2\,. \tag{6.9}$$

When both $\delta\rho$ and δp vanish, we recover, of course, the tree-level solutions. Inserting instead the corrected metric and dilaton backgrounds, we easily obtain:

$$\delta\rho = -\frac{4}{\sqrt{3}}\frac{D}{t^2}\,, \quad \delta p = 0\,. \tag{6.10}$$

Since D is positive, see (6.5) and (6.6), the energy density is negative in the Einstein frame. The value $\delta p = 0$ is special to the background (2.4). We have checked that, for other power-law backgrounds (e.g. for radiation-like solutions) δp does not vanish.

To translate these results to the string frame, we note that the corresponding energy and pressure densities are given by

$$\begin{aligned} e^{\varphi}\delta\rho_s &= e^{-\varphi}\delta\rho = -3H_s^2 + \dot{\overline{\varphi}}^2 \\ e^{\varphi}\delta p_s &= e^{-\varphi}\delta p = 2\dot{H}_s - 3H_s^2 - 2H_s\dot{\overline{\varphi}} - \dot{\overline{\varphi}}^2 + 2\ddot{\overline{\varphi}}\,, \end{aligned} \tag{6.11}$$

so that, again, $\delta\rho_s < 0$ and $\delta p_s = 0$. Hence, the condition $\delta\rho_s + \delta p_s < 0$ is clearly met, see Fig. 6.

We conclude that, from all points of view, the computed back-reaction goes in the right direction to favour a change to the FRW branch and to avoid violation of the entropy bounds.

Appendix

Here we compare our results with similar calculations done in other gauges, such as the Iliopoulos et al. gauge, which has been extensively used in the recent literature to compute back-reaction to chaotic [13] and power-law inflation [14]. Specifically, in the latter case, the authors found no strong effect (i.e. growing with time) at one loop. It might not be so clear to the reader why the one-loop effect is so dominant in PBB inflation, which is also essentially power law. To clarify the situation it is important to recall the basic distinction between PBB and standard power-law inflation. In the PBB case the Hubble parameter grows with time, as $t \to 0^-$, and as a result higher frequency modes leave the horizon at higher values of the Hubble parameter, see Fig. 7. The loop integrals are thus dominated by the latest value of the Hubble parameter, in contrast with the case of standard power law, where they are dominated by its initial value. It is also important to keep in mind the two different time limits: in the case of standard power law, the limit is $t \to \infty$, whereas, in PBB it is, $t \to 0^-$. This makes a crucial difference when deciding which are the dominant or subdominant effects.

As a check of gauge invariance, we feel it is worthwhile to redo the standard power-law calculation in the off-diagonal gauge. It will provide a direct check of the results obtained in the gauge used in [14], as well as help to better understand the differences between the two cases in the same gauge. Let us recall some of the basic

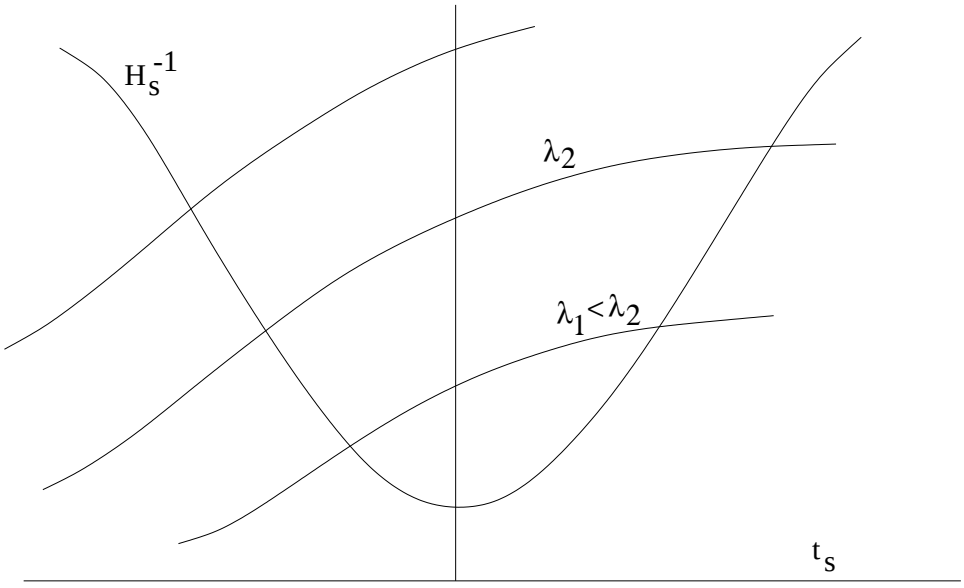

Figure 7: *Horizon crossings of various scales in PBB cosmology.*

formulas for standard power-law inflation. The scale factor, Hubble radius, etc., are given by ($s > 1$):

$$\ln a_0 = s \ln t_0 = -\frac{s}{s-1} \ln \eta \,, \quad H_0 = H_i a_0^{-1/s} = \frac{s}{t_0} \,, \quad H_i = -\frac{s}{s-1} \frac{1}{\eta_i} \tag{6.12}$$

and the dilaton is given by $\varphi_0 = 2\sqrt{s} \ln t_0$. For an arbitrary value of s, (6.12) describes a background solution only in the presence of a potential

$$V(\varphi) = 2H_i^2 \left(3 - \frac{1}{s}\right) \exp\left[-\varphi/\sqrt{s}\right] . \tag{6.13}$$

Note that for the special value $s = 1/3$, corresponding to the PBB case, the potential vanishes. To proceed as before, in the off-diagonal gauge, we have to first find the quadratic constraints through which the metric perturbations become related to the canonical variable ϕ. One finds

$$\Phi = -\frac{1}{2\sqrt{s}} \, \phi \,, \qquad \nabla^2 B = \frac{1}{2\sqrt{s}} \, \phi' \,. \tag{6.14}$$

The rest of the analysis is almost the same as before, except one should take into account the extra three-point vertices coming from the potential (6.13). Using the

constraints and the gauge choice, the one-point functions turn out to have the following forms

$$A = 0\,, \quad C = -\frac{1}{\sqrt{s}} D\,, \quad D = -\Delta_0^{-1}\left(\delta - \frac{1}{\sqrt{s}}\gamma\right) \tag{6.15}$$

where δ and γ represent the same amputated diagrams as before. After summing over the vertices, it turns out that the leading effect in the one-point functions is of the form[3] $D \sim \ln t_0$. If we proceed, as in the case of PBB, to study the contribution of C and D to φ and $H_{\rm one-loop}$, we find that the leading logarithms disappear, irrespective of the numerical factor in front of $\ln t_0$. The reason for the cancellation is twofold: firstly, the relation (6.15) between C and D insures that the changes in φ and H are proportional. Secondly, it is easy to show that for $C \sim \ln t_0$ the change in H is subleading since $\dot{C}t/C \sim (1/\ln t) \to 0$.

Finally let us briefly point out another technical difference between the PBB and the standard power-law cases. In the PBB case the constraints relating Φ and B to ϕ have signs opposite to those in (6.14) implying that C and D have the same sign, see (5.9). This is why their effects on φ and $H_{\rm one-loop}$ add up, in contrast to the case of standard power law, where they compete with each other. Furthermore, the leading behaviour is no longer logarithmic, explaining why a large effect survives only in the PBB case.

Acknowledgements

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[3]For instance, for the particular case $s = 2$, we found $D = -\frac{9\sqrt{2}}{512}\ln t_0$.

References

[1] G. Veneziano, Phys. Lett. B265, 287 (1991); M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993); Phys. Rev. D 50, 2519 (1994); G. Veneziano, hep-th/9802057 Proc. of the 35th Course of the Erice School of Subnuclear Physics, "Highlights: 50 Years Later" (Erice, August 1997), ed. A. Zichichi; G. Veneziano, hep-th/9902097 Proc. of the 1998 Erice School on Subnuclear Physics, "From the Planck length to the Hubble Radius" (Erice, September 1998), ed. A. Zichichi; For an updated list of references visit the web page `http://www.to.infn.it/~gasperin/`.

[2] A. Guth, Phys. Rev. Lett. 49, 1110 (1982); A. Linde, Phys. Lett. 129B, 177 (1983); A. Linde, Phys. Lett. B175, 395 (1986); K.A. Olive, Phys. Rep. 190, 307 (1990).

[3] S.W. Hawking and N. Turok, Phys. Lett. B425, 25 (1998); gr-qc/9802062; Phys. Lett. B432, 271 (1998); A. Linde, Phys. Rev. D58, 083514 (1998); A. Vilenkin, Phys. Rev. D57, 7069 (1998); Phys. Rev. D58, 067301 (1998).

[4] G. Veneziano, Phys. Lett. B406, 297 (1997); M.S. Turner and E.J. Weinberg, Phys. Rev. D56, 4604 (1997); A. Buonanno, K. Meissner, C. Ungarelli and G. Veneziano, Phys. Rev. D57, 2543 (1998); J. Maharana, E. Onofri and G. Veneziano, JHEP 04, 004 (1998); D. Clancy, J.E. Lidsey and R. Tavakol, Phys. Rev. D58, 44017 (1998); N. Kaloper, A. Linde and R. Bousso, Phys. Rev. D59, 043508 (1999); A. Buonanno, T. Damour and G. Veneziano, Nucl. Phys. B543, 275 (1999).

[5] R. Brustein and G. Veneziano, Phys. Lett. B329, 429 (1994); N. Kaloper, R. Madden and K.A. Olive, Nucl. Phys. B452, 677 (1995); M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B494, 315 (1997); R. Brustein and R. Madden, Phys. Rev. D57, 712 (1998).

[6] M. Gasperini and G. Veneziano, Gen. Rel. Grav. 28, 1301 (1996); M. Maggiore, Nucl. Phys. B525, 413 (1998); R. Brustein and R. Madden, JHEP 9907, 006 (1999).

[7] R. Easther and K. Maeda, Phys. Rev. D54, 7252 (1996); S. Foffa, M. Maggiore and R. Sturani, Nucl. Phys. B552, 395 (1999).

[8] G. Veneziano, hep-th/9802057, see [1]; A. Buonanno, K. Meissner, C. Ungarelli and G. Veneziano, JHEP 9801, 004 (1998).

[9] R. Easther and D.A. Lowe, Phys. Rev. Lett. 82, 4967 (1999); G. Veneziano, Phys. Lett. B454, 22 (1999); hep-th/9907012; D. Bak and S.J. Rey, hep-th/9902173; R. Brustein, gr-qc/9904061; N. Kaloper and A. Linde, hep-th/9904120.

[10] G. Veneziano in [9]; R. Brustein in [9]; R. Brustein, S. Foffa and R. Sturani, hep-th/9907032.

[11] See e.g. A.O. Barvinsky and G.A. Vilkovisky, Phys. Lett. B131, 313 (1983); Nucl. Phys. B333, 471 (1990); Nucl. Phys. B439, 561 (1995).

[12] J. Iliopoulos, T.N. Tomaras, N.C. Tsamis and R.P. Woodard, Nucl. Phys. B534, 419 (1998).

[13] L.R.W. Abramo and R.P. Woodard, Phys. Rev. D60, 044010 (1999).

[14] L.R.W. Abramo and R.P. Woodard, Phys. Rev. D60, 044011 (1999).

[15] L.R.W. Abramo, R.H. Brandenberger and V.F. Mukhanov, Phys. Rev. Lett. 78, 1624 (1997); Phys. Rev. D56, 3248 (1997).

[16] J. Hwang, Astrophys. J. 375, 443 (1991).

[17] R. Brustein, M. Gasperini, M. Giovannini, V.F. Mukhanov and G. Veneziano, Phys. Rev. D51, 6744 (1995).

[18] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, Phys. Rep. 215, 203 (1992).

[19] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. 75, 3796 (1995); D. Lemoine and M. Lemoine, Phys. Rev. D52, 1955 (1995); A. Buonanno, K. Meissner, C. Ungarelli and G. Veneziano, JHEP 9801, 004 (1998); R. Brustein and M. Hadad, Phys. Rev. D57, 725 (1998).

GETTING D=3 EXPANDING DIMENSIONS
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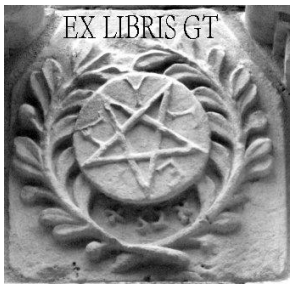
MANIFESTLY $O(d,d)$ INVARIANT APPROACH TO SPACE-TIME DEPENDENT STRING VACUA



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Abstract

An $O(d,d)$ symmetry of the manifold of string vacua that do not depend on d (out of D) space-time coordinates has been recently identified. Here we write down, for $d = D-1$, the low energy equations of motion and their general solution in a manifestly $O(d,d)$-invariant form, pointing out an amusing similarity with the renormalization group framework. Previously considered cosmological and black hole solutions are recovered as particular examples.

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conformal bg=str vacuum?????

1. INTRODUCTION

Narain O(d,d)

Narain's original idea [1], [2] of an $O(d,d)$ symmetry acting on the manifold of static, d-dimensional string compactifications, has been recently generalized [3] to the case of massless backgrounds (metric, torsion and dilaton) which simply do not depend upon a particular set of (possibly non-compact) string coordinates $X^a (a = 1, 2, ...d)$.

A σ-model argument for the validity of such a symmetry at all orders in α' (and possibly at higher genus as well) runs as follows [3]: Consider a conformal background (a string vacuum) of the kind specified above: the associated nilpotent BRST operator Q will depend trivially (i.e. quadratically at most) on the $2d$ phase-space variables $Z = (P_a, X'^a)$. Perform now [3] a global, canonical $O(d,d)$ transformation on the Zs. Since this transformation preserves commutation relations and Wick contractions, the new BRST operator will also be nilpotent. However, as it turns out, the change in Z can be traded for a change in the backgrounds, implying that also the transformed backgrounds define a (generally inequivalent) conformal theory. $O(d,d)$ is thus a symmetry of this particular class of string vacua.

Depending on the theory at hand, a subgroup of $O(d,d)$, which we shall call the —gauge group— $\mathcal{G}$, leaves the theory unchanged. Inequivalent vacua are thus points in the coset $O(d,d)/\mathcal{G}$. While for the static, fully compact case one has [1], [2] the d^2-dimensional space given by $\mathcal{G} = O(d) \bigotimes O(d)$, for the completely non-compact (but time-dependent) case the coset has dimensionality $d(d-1)/2$ [3]. It has been argued by Sen [4] that, in this latter case, our coset is equivalent to $O(d) \bigotimes O(d)/O(d)$ and that the $O(d) \bigotimes O(d)$ symmetry persists to all orders in α'. For the sake of generality we shall base our discussion here on the full original $O(d,d)$ symmetry.

Discrete subgroups of $O(d,d)$ (somewhat improperly called —duality transformations—) had been previously discussed both in the context of cosmological solutions [5], [6] and in that of $2D$ black-holes [7], [8], [9]. Finally, the complete $O(d,d)$ group has been employed to generate, from some known solutions, radically new candidate conformal backgrounds both in the black-hole [10] and in the cosmological [11] case. An interesting aspect of $O(d,d)$, made apparent by these constructions, is that it can transform gauge-equivalent backgrounds into gauge-inequivalent ones, as evidentiated by the presence of a non trivial torsion field B in the transformed solutions.

B

In this paper we shall continue our work [3] by rewriting, for $d = D-1$, the field equations in a manifestly $O(d,d)$-invariant form.This allows us to give the general solution of the (lowest order) equations in terms of quadratures. Actually, equations (and solutions) take the form of renormalization-group equations (and solutions) in real cosmic time both for the coupling constant of string theory and for the matrix descibing the metric and torsion backgrounds. This analogy is used in order to obtain a general classification of the solutions. Finally, we shall recover several known backgrounds as particular cases of our general solution.

2. $O(d,d)$ SYMMETRIC EQUATIONS OF MOTION AND THEIR GENERAL SOLUTION

In this Section we shall derive manifestly $O(d,d)$-invariant equations of motion for the low energy effective action of string gravity coupled to the dilaton and the antisymmetric tensor:

$$S = \int d^D x \, \sqrt{-G} \, e^{-\phi} \left[\Lambda - R - G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] \tag{1}$$

Here Λ is the cosmological constant (proportional to D-10), ϕ is the (Fradkin-Tseytlin) dilaton [12] $G_{\mu\nu}$ is the σ-model metric and $H_{\mu\nu\rho}$ is expressed, as usual, in terms of $B_{\mu\nu}$ via :

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \text{cyclic} \tag{2}$$

If G and B are only functions of time, invariance under general coordinate transformations and under $B_{\mu\nu} \to B_{\mu\nu} + + \partial_{[\mu} \Lambda_{\nu]}$ allows always to bring them to the form:

$$G = \begin{pmatrix} -1 & 0 \\ 0 & G(t) \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & B(t) \end{pmatrix} \tag{3}$$

Hereafter we shall use the symbols G and B to denote the $d \times d$ matrices appearing in (3).

As shown in [3], by introducing a $2d \times 2d$ matrix M and a field Φ defined as

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{4}$$

$$\Phi = \phi - \ln \sqrt{det \; G}. \tag{5}$$

the action (1) takes the form:

action

$$S = \int dt\ e^{-\Phi} \left\{ \Lambda + (\dot{\Phi})^2 + \frac{1}{8}\ \mathrm{Tr}\left[\dot{M}\eta\ \dot{M}\eta\right] \right\} \tag{6}$$

where η defines an $O(d,d)$ metric in off-diagonal form i.e.

$$\eta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \tag{7}$$

The action (6) is manifestly invariant under the a global $O(d,d)$ group acting as :

$$\Phi \to \Phi, \quad M \to \Omega^T M \Omega \tag{8}$$

$$\Omega^T \eta \Omega = \eta \tag{9}$$

For $d < D-1$ and backgrounds G and B which are not block-diagonal, the σ-model argument given in Sect.1 still suggests an $O(d,d)$ symmetry with the off-diagonal blocks of M transforming s by the obvious left or right multiplication with Ω.

EOMs

We will now obtain manifestly $O(d,d)$-invariant equations of motion from the action (6). The first equation follows from reintroducing G_{00} in the action and from setting to zero the corresponding variation. This gives directly the "Zero Energy" condition

conds

$$(\dot{\Phi})^2 + \frac{1}{8}\ \mathrm{Tr}\left[\dot{M}\eta\ \dot{M}\eta\right] - V = 0 \tag{10}$$

where we allow now for a more general potential $V(\Phi)$ rather than just a constant. If we assume that the potential does not break the symmetry explicitly then it can depend on M only through a function of the invariants $\mathrm{Tr}(M\eta)^p, p = 1, 2, ...$ However, for p odd these traces vanish and for p even they are equal to $2d$; hence the potential can depend only on Φ.

The variation of the action with respect to Φ yields:

$$(\dot{\Phi})^2 - 2\ddot{\Phi} - \frac{1}{8}\ \mathrm{Tr}\left[\dot{M}\eta\ \dot{M}\eta\right] + \frac{\partial V}{\partial \Phi} - V = 0 \tag{11}$$

EOM 1

The variation of the action with respect to M has to be done carefully, since M is subject to several constraints. The simplest way to proceed is to write

$$\delta M = \Omega^T M \Omega - M \tag{12}$$

where $\Omega = 1 + \epsilon$ belongs to $O(d,d)$. Expanding the variation of the action to terms linear in ϵ and using the fact that $\epsilon\eta$ is antisymmetric we get the equation:

$$\partial_t(M\eta\dot{M}) = \dot{\Phi}(M\eta\dot{M}) \tag{13}$$

which can be immediately integrated once to give:

$$\mathrm{e}^{-\Phi}(M\eta\dot{M}) = \mathrm{const} = A. \tag{14}$$

From its definition the constant matrix A satisfies

$$A^T = -A, \qquad M\eta A = -A\eta M. \tag{15}$$

It is obvious that eqs. (10), (11) and (13) are invariant under the full $O(d,d)$ group defined by (8) and (9).

Substituting (14) into (10) we obtain the first order equation for Φ:

$$(\dot{\Phi})^2 = \frac{\exp(2\Phi)}{8}\ \mathrm{Tr}(A\eta)^2 + V(\Phi) \tag{16}$$

which can be solved by quadratures:

$$t = \int_{\Phi_0}^{\Phi} dy(\frac{\exp(2y)}{8}\ \mathrm{Tr}(A\eta)^2 + V(y))^{-1/2} \tag{17}$$

It can be easily shown that the solution of (16) automatically satisfies (11) , provided that $\dot{\Phi} \neq 0$. The solution (17) can be used next to define a "dilaton time" τ:

$$\tau = \int_{t_0}^{t} \mathrm{e}^{\Phi}\ dt'. \tag{18}$$

In terms of τ the general solution of (14) simply reads :

$$M(t) = \exp(-A\eta\tau)M(t_0). \tag{19}$$

EOM 2

Hence the whole solution can be explicitly given in terms of quadratures, which is a welcome surprise for a complicated system such as the one described by (1).

To end this Section we would like to point out an amusing similarity between equations (16), (18) and (19) and the renormalization group (RG) equations. Introducing the "reduced" coupling constant α by:

$$\alpha = \mathrm{e}^{\Phi} \tag{20}$$

what is that devil's
cooking with so strange results??

we can rewrite (16) as a RG-equation for the running of α:

"RG" eqs

$$\frac{d\alpha}{dt} = \beta(\alpha) \tag{21}$$

where the β-function is given by:

$$\beta(\alpha) = \pm\alpha\sqrt{V(\alpha) + \frac{\alpha^2}{8}\mathrm{Tr}(A\eta)^2} \tag{22}$$

Once eq. (22) is solved, eq. (18) is the usual definition of the "RG-time" in terms of which moments of structure functions evolve to leading order exponentially (generally with a matrix-valued exponent because of operator mixing), i.e. precisely as in eq. (19). The standard RG analysis of UV or IR fixed points can be applied to our system choosing, for definiteness, the plus sign in eq. (22) (this corresponds, physically, to choosing solutions which evolve from weak coupling in the far past). This analysis leads to the following classification scheme for the general solution:

a) If $\beta(\alpha)$ has no other zero but the trivial one then we have two possibilities:

a_1)

$$\int_{\alpha_0}^{\infty} \frac{dx}{\beta(x)} < \infty \tag{23}$$

There is a singularity in Φ at a finite $t = t_c$. There can also be a (coordinate or curvature) singularity if

$$\int_{\alpha_0}^{\infty} \frac{dxx}{\beta(x)} = \infty \tag{24}$$

In this case the typical cosmology near t_c is superinflationary (Cf. [5]).

a_2)

$$\int_{\alpha_0}^{\infty} \frac{dx}{\beta(x)} = \infty \tag{25}$$

In this case no singularity can develop at any finite t.

b) If $\beta(\alpha)$ has a zero at some $\alpha^* > 0$ then one has again two possibilities:

b_1)

$$\int_{\alpha_0}^{\alpha^*} \frac{dx}{\beta(x)} < \infty \tag{26}$$

No singularity develops at any finite t and the coupling constant, after growing to α^*, goes back to zero after an infinite time.

b_2)

$$\int_{\alpha_0}^{\alpha^*} \frac{dx}{\beta(x)} = \infty \tag{27}$$

In this case the theory has a genuine late-time fixed point $\alpha = \alpha^*$. At late times a regime of exponential inflation ($M(t) \sim \exp(\alpha^* A\eta t)$) sets in.

3. COSMOLOGICAL SOLUTIONS AND 2D BLACK HOLES AS PARTICULAR CASES

In this Section we will solve the equations (16) and (19) for some special potentials. The solutions we find include time dependent cosmological solutions of [5] and 2D black holes [7],[8], as well as some recently studied "boosted" versions of both [11], [10], but the family of solutions of (16) and (19) is much larger.

We start with the case of vanishing potential $V = 0$ (critical dimensions). In this case (16) can be immediately solved to give

$$e^{\Phi} = \alpha = C/(T-t); \qquad C = \sqrt{8/\mathrm{Tr}(A\eta)^2} \tag{28}$$

From (18) we get

$$\tau = C \ln \frac{T-t}{T-t_0} \tag{29}$$

so that the solution for M reads:

$$M(t) = \exp\left(CA\eta \ln \frac{T-t}{T-t_0}\right) \tag{30}$$

where we assumed that $M(t_0) = \mathbf{1}$.

To be more explicit let us assume some specific form for A:

$$A = \begin{pmatrix} 0 & -A_d \\ A_d & 0 \end{pmatrix} \tag{31}$$

where $A_d = diag(a_1, .., a_d)$. In this case (30) reads:

$$M(t) = \begin{pmatrix} diag\left[\left(\frac{T-t}{T-t_0}\right)^{-2\alpha_1}, ..\right] & 0 \\ 0 & diag\left[\left(\frac{T-t}{T-t_0}\right)^{2\alpha_1}, ..\right] \end{pmatrix} \tag{32}$$

where $\alpha_i = a_i/\sqrt{\Sigma a_i^2}$. These solutions are exactly the ones discussed in [5], [6](see also [13]). For $\alpha_i = \alpha\delta_{i1}$ we recover the special case of the completely flat "Milne" metric

[11], while, by inverting and boosting the corresponding A, we arrive at the one-parameter family of non-trivial backgrounds discussed in [11].

We now want to discuss the solutions in presence of the potential $V = \Lambda$ =const. The solution of (16) can be immediately given:

$$e^{\Phi} = \alpha = C\sqrt{\Lambda}/\sinh(\sqrt{\Lambda}(T-t)); \qquad C = \sqrt{8/\mathrm{Tr}(A\eta)^2} \tag{33}$$

From (18) we get

$$\tau = C \ln \frac{\tanh(\sqrt{\Lambda}(T-t)/2)}{\tanh(\sqrt{\Lambda}(T-t_0)/2)} \tag{34}$$

so that the solution for M reads:

$$M(t) = \exp\left(CA\eta \ln \frac{\tanh(\sqrt{\Lambda}(T-t)/2)}{\tanh(\sqrt{\Lambda}(T-t_0)/2)} \right) \tag{35}$$

where we again assumed that $M(t_0) = \mathbf{1}$.

For $d = 1$ we take

$$A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \qquad a > 0 \tag{36}$$

and assume that $M(-\infty) = \mathbf{1}$. Then

$$M(t) = \begin{pmatrix} \tanh^{-2}(\sqrt{\Lambda}(T-t)/2) & 0 \\ 0 & \tanh^{2}(\sqrt{\Lambda}(T-t)/2) \end{pmatrix} \tag{37}$$

$$e^{\Phi} = \frac{2\sqrt{\Lambda}}{a\sinh(\sqrt{\Lambda}(T-t))} \tag{38}$$

After the identification [3]:

$$t \leftrightarrow ix, \ T = 0, \ \Lambda = -4 \tag{39}$$

we recover the $k = 9/4$ black hole solution of [7], [8]. Acting on the solution with η we recover the "dual" 2D black hole of ref.[9].

For $d > 1$ we can use the same A (with zeroes in the additional entries) and reproduce the $k \neq 9/4$ black holes of [7], while, by boosting any of these solutions one recovers the "twisted" black holes of [10].

We wish to conclude by mentioning one amusing solution that exists for $d = 9$ (usual uncompactified superstring). As A we take

$$A = \begin{pmatrix} 0 & diag(-a_1, .., -a_9) \\ diag(a_1, .., a_9) & 0 \end{pmatrix} \tag{40}$$

Then M is equal to

$$M = \begin{pmatrix} diag\left(\tanh^{-2\alpha_1}(\sqrt{\Lambda}(T-t)/2), ..\right) & 0 \\ 0 & diag\left(\tanh^{2\alpha_1}(\sqrt{\Lambda}(T-t)/2), ..\right) \end{pmatrix} \tag{41}$$

where $\alpha_i = a_i/\sqrt{\Sigma a_i^2}$. The scalar curvature and dilaton are given by:

$$R = -\frac{\Lambda}{\cosh^2(\sqrt{\Lambda}(T-t))/2)}\left[\sum \alpha_i - \frac{(\Sigma\alpha_i - 1)^2}{4\sinh^2(\sqrt{\Lambda}(T-t)/2))}\right] \tag{42}$$

$$e^{\phi} = \frac{(\tanh(\sqrt{\Lambda}(T-t)/2))^{\Sigma\alpha_i - 1}}{a\ \cosh^2(\sqrt{\Lambda}(T-t)/2))}. \tag{43}$$

Hence we have a singularity for $t \to T$ unless

$$\sum \alpha_i = 1. \tag{44}$$

Assuming that all $|\alpha_i|$ are equal, the only solution for α_i is (-1/3,-1/3,-1/3,+1/3,..,+1/3) so that three dimensions expand and six contract!

GREAT GODS!!!!!!!!!!!!!!!!!!!!!!!!!!

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References

[1] K. S. Narain, Phys. Lett. B169 (1986) 41.

[2] K. S. Narain, M. H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.

[3] K.A. Meissner and G. Veneziano, Symmetries of Cosmological Superstring Vacua, CERN-TH.6138/91, to appear in Phys.Lett.B.

[4] A. Sen, $O(d) \bigotimes O(d)$ Symmetry of the Space of Cosmological Solutions in String Theory, Scale Factor Duality, and Two Dimensional Black Holes, Tata Institute preprint, TIFR/TH/91-35.

[5] G.Veneziano, Scale Factor Duality for Classical and Quantum Strings, CERN-TH-6077/91.

[6] A. A. Tseytlin, Duality and Dilaton, Mod. Phys. Lett. 6A (1991) 1721; Space-Time Duality, Dilaton and String Cosmology, to appear in the Proceedings of the First International A. D. Sakharov's Conference in Physics, Moscow, May 1991.

[7] E.Witten, On String Theory and Black Holes, IASSNS-HEP-91/12.

[8] G. Mandal, A.M. Sengupta and S.R. Wadia, Classical Solutions of 2-Dimensional String Theory, IASSNS-HEP-91/10; see also, K. Bardakci, A. Forge and E. Rabinovici, Nucl. Phys. B344 (1990) 344; S. Elitzur, A. Forge and E. Rabinovici, Nucl. Phys. B359 (1991) 581; I. Bars and D. Nemeschansky, Nucl. Phys. B348 (1991) 89.

[9] A. Giveon, Target Space Duality and Stringy Black Holes, LBL-30671; E. Smith and J. Polchinski, preprint, UTTG-07-91; R. Dijkgraaf, H. Verlinde and E. Verlinde, preprint LBL-30747 (1991).

[10] A. Sen, Twisted Black p-brane Solutions in String Theory, Tata Institute preprint, TIFR/TH/91-37.

[11] M. Gasperini, J. Maharana and G. Veneziano, From Trivial to Non-Trivial Conformal String Backgrounds via $O(d,d)$ Transformations,preprint CERN-TH-6214/91

[12] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. 158B (1985) 316; E. Witten, Phys. Lett. 149B (1984) 351.

[13] M. Mueller, Nucl. Phys. B337 (1990) 37.

From SFD to O(d,d)

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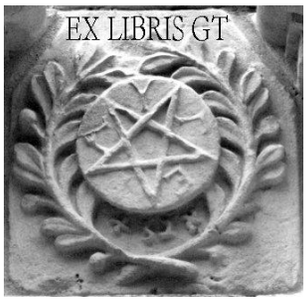
SYMMETRIES OF COSMOLOGICAL SUPERSTRING VACUA



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Abstract

The concept of discrete scale-factor-duality transformations is extended to a full, continuous $O(d,d)$ group. The action of the group transforms "cosmological" solutions of the low-energy, string-modified Einstein equations (including non-trivial dilaton and anti-symmetric tensor fields) into other solutions, thus representing the natural generalization of Narain's construction to time-dependent, not-necessarily-compact manifolds. Possible connections with $2D$ black holes are also pointed out.

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1. INTRODUCTION

In this short note we shall develop and extend the concept [1] of scale factor duality (SFD) as a symmetry group of classical string motions, of the low-energy string effective action and of the corresponding equations of motion.

Let us recall a few characteristics of SFD which make it quite distinct from the more conventional R-duality [2] [3]:

1. SFD does not rely on compactification of target space.

2. Even for a compact target space, SFD relates a physically expanding to a physically contracting dimension. By contrast, time-dependent R-duality relates, e.g. an expanding circle to one which apparently contracts, but actually describes, in dual terms, the same physical expansion.

3. While R-duality could presumably be a true symmetry, i.e. one relating two theories with the same spectrum and S-matrix, SFD is more likely [1] to connect different time-dependent vacua of string theory. Thus SFD appears rather as a generalization of Narain's construction [4] [5] of inequivalent static compactifications to a time-dependent and not necessarily compact target space.

Our results will make the connection to Narain's work even more compelling. Indeed, we shall be able to show that the low-energy effective action possesses, for time-dependent metric, torsion, and dilaton background fields ($G_{\mu\nu}, B_{\mu\nu}, \phi$; $\mu, \nu = 0, 1, 2, ...d$), a full continuous $O(d,d)$ symmetry group whose action transforms "cosmological" solutions of the equations of motion into other (generally inequivalent) solutions.

In Sect. 2 we shall give a simple derivation of the $O(d,d)$ symmetry of the low-energy effective action, for time-dependent $G_{\mu\nu}, B_{\mu\nu}$ and ϕ and obtain the corresponding equations of motion. In Sect. 3 we elucidate the meaning of the symmetry as well as its relation to Narain's work and we comment, finally, on the possible relevance of our results for two-dimensional black holes.

and the field Φ defined as:

$$\Phi = \phi - \ln \sqrt{det\ G}. \tag{7}$$

In this notation the action (1) takes its final form:

$$S = \int dt\ e^{-\Phi} \left\{ \Lambda + (\dot{\Phi})^2 + \frac{1}{8}\ \mathrm{Tr}\left[\dot{M}\eta\ \dot{M}\eta\right] \right\}, \tag{8}$$

where

$$\eta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \tag{9}$$

is an $O(d,d)$ metric in non-diagonal form and $\mathbf{1}$ stands for the d-dimensional unit matrix.

The action (8) is manifestly invariant under the global $O(d,d)$ group acting as:

$$\Phi \to \Phi, \quad M \to \Omega M \Omega^T \tag{10}$$

$$\Omega^T \eta \Omega = \eta. \tag{11}$$

The occurence of $O(d,d)$ as a way to relate different string vacua immediately suggests a relation with Narain's work [4]. The discussion of this relation is deferred to Sect. 3.

Let us recover here instead the particular cases discussed in ref. [1]. For $B = 0$ and a diagonal G, the Z_2^d group of ref. [1] is simply given in terms of a generic d-dimensional projector Π by the choice:

$$\Omega = \begin{pmatrix} \Pi & \mathbf{1} - \Pi \\ \mathbf{1} - \Pi & \Pi \end{pmatrix}, \quad \Pi^2 = \Pi. \tag{12}$$

Notice that, for a non-diagonal G, this subgroup of $O(d,d)$ generates in general a non-vanishing "torsion" B.

We close this section by writing down the equations of motion that follow from the original action once the fields are restricted to the type specified in eq. (3). Generally speaking, by just varying the reduced action (8), one might lose some of the equations. In our case, one can check that the only "missing" condition is the one that follows from reintroducing G_{00} in the action and from setting to zero the corresponding variation. This gives directly the "zero energy" condition

$$(\dot{\Phi})^2 + \frac{1}{8}\ \mathrm{Tr}\left[\dot{M}\eta\ \dot{M}\eta\right] - V = 0, \tag{13}$$

and is equivalent to a change in M:

$$M \to M = \Omega^T M \Omega. \tag{20}$$

This is precisely the change of M given in (10)(see e.g. [1] , [10] for a discussion of the need to transform the dilaton ϕ too). Indeed, any M can be obtained by the action of the group acting, e.g. on $M = \mathbf{1}$. M itself belongs to $O(d,d)$, but, unlike the most general Ω, M is symmetric. The subgroup of $O(d,d)$ that leaves a given M invariant is an $O(d) \times O(d)$, exactly as in Narain's case [4] where, for constant G and B, inequivalent theories (different Ms) are labelled by points in the coset $O(d,d)/(O(d) \times O(d))$.

For time-dependent solutions the situation is somewhat different since one also has to specify the first derivative of M at $t = 0$. A counting of parameters now shows that $O(d,d)$ is not large enough to connect all possible solutions. $M(0)$ and $\dot{M}(0)$ contain indeed $2d^2$ parameters, while an $O(d,d)$ transformation contains $2d^2 - d$ parameters. That means that there are still d invariants to be specified, in order to characterize a class of solutions. It is not hard to prove that a convenient set of invariants is provided by

$$I_p = \mathrm{Tr}(\eta \dot{M})^{2p}, \qquad p = 1, ..., d. \tag{21}$$

Ms which correspond to different invariants should obviously yield inequivalent solutions but one could ask whether M's which are related by $O(d,d)$ rotations correspond to the same theory as naively suggested by the fact that they are are related through the "canonical" transformation (18).

The same question could be asked, of course, in the constant-background case. Consider, for instance, the case of a single compactified dimension, a circle of radius R, and $B = 0$. The general "canonical" transformation would seem to be:

$$X' \to aP \ , \qquad P \to a^{-1} X' \tag{22}$$

which would lead to the (incorrect) symmetry under $G \to a^{-2} G^{-1}$. The problem with such a transformation is that it does not preserve the spectrum of (the zero modes of) X' and P (it is not unitary) for arbitrary values of a. Only if we take $a = R^2$, is the spectrum left unchanged and we can really perform the necessary change of variable in the functional integral. We thus end up with just the usual, discrete Z_2 symmetry [2],[7]:

$$G \to R^{-4} G^{-1}. \tag{23}$$

References

[1] G.Veneziano, Scale Factor Duality for Classical and Quantum Strings, CERN-TH-6077/91 and references therein.

[2] K. Kikkawa and M. Yamasaki, Phys. Lett. B149 (1984) 357.

[3] N. Sakai and I. Senda, Progr. Theor. Phys. 75 (1986) 692.

[4] K. S. Narain, Phys. Lett. B169 (1986) 41.

[5] K. S. Narain, M. H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.

[6] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. 158B (1985) 316; E. Witten, Phys. Lett. 149B (1984) 351.

[7] A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B322 (1989) 167.

[8] A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 669.

[9] K.A.Meissner and G.Veneziano, to appear.

[10] A. A. Tseytlin, Duality and the Dilaton, Johns Hopkins preprint JHU-TIPAC-91008 (1991).

[11] E.Witten, On String Theory and Black Holes, IASSNS-HEP-91/12.

[12] G.Mandal, A.M.Sengupta, S.R.Wadia, Classical Solutions of 2-Dimensional String Theory, IASSNS-HEP-91/10.

[13] A.Giveon, Target Space Duality and Stringy Black Holes, LBL-30671.

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Dilaton Couplings at Large Distances

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Abstract

The large-distance behaviour of dilaton couplings is studied within the framework of four-dimensional heterotic superstring theories. String loop corrections violate universality of dilaton and graviton couplings, increasing dilaton couplings at large distances. This implies a serious problem for string theories, unless a non-zero dilaton mass is generated.

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The presence of dilatons – massless, totally gauge-neutral scalars – seems to be inevitable in string theories. One generally expects that dilatons interact with matter no stronger than gravity, i.e. with couplings of order $\mathcal{O}(\kappa \equiv \sqrt{4\pi G_N})$; the number of dilatons, as well as detailed forms of their couplings, depend on particular models under consideration.

The physical existence of dilatons is very much constrained by the results of experiments designed to study deviations from Einstein's theory of general relativity. Typically, such experiments provide various bounds on the dilaton couplings, depending on the dilaton mass.[1]

The determination of the dilaton mass presents a very difficult problem, related to ultra-violet string physics, supersymmetry breaking etc. On the other hand, as argued in this letter, some general aspects of dilaton interactions, like their large-distance behaviour, do *not* depend on the ultra-violet physics and can be studied by using an effective field theory approximation. To be specific, we consider here only the minimal case of one dilaton field, within the framework of so-called four-dimensional heterotic superstring theories [2], however our analysis could be readily extended to more complicated dilaton systems.

In four-dimensional heterotic superstring theories, the dilaton field ϕ (more precisely, its exponential) belongs to the chiral supermultiplet S, whose interactions with matter and gauge fields are completely determined by the Kähler potential $J = -\log(S+\bar{S})+\ldots$ [3]. Let (A,ψ) and (B,ξ) denote two chiral matter superfields of mass $m \ll 1/\kappa$, in the representations r and $\bar{r}$ of some generic gauge group, respectively.[2] The interaction Lagrangian for this dilaton - gauge - matter system is given by [3]:

$$\begin{aligned}
\mathcal{L}_{\rm int} \;=\; & -\kappa\phi[m\psi\xi + m\bar{\psi}\bar{\xi} + 2m^2 A^\dagger A + 2m^2 B B^\dagger] \\
& -i\sqrt{2}g\kappa\phi[A^\dagger T^{(a)}\psi\lambda^{(a)} - B T^{(a)}\bar{\xi}\bar{\lambda}^{(a)} - \bar{\lambda}^{(a)}\bar{\psi}T^{(a)}A + \lambda^{(a)}\xi T^{(a)}B^\dagger] \\
& -g^2\kappa\phi[A^\dagger T^{(a)}A - B T^{(a)}B^\dagger]^2 \;+\; \frac{1}{2}\kappa\phi F^{(a)}_{\mu\nu}F^{(a)\mu\nu} \;+\; \mathcal{O}(\kappa^2)\,, \qquad (1)
\end{aligned}$$

where g denotes the gauge coupling constant and $T^{(a)}$ the gauge group generators in the representation r. In eq.(1) we used the notation of ref.[5].

The dilaton interactions of eq.(1) exhibit the property that one would generally expect

[1] See e.g. ref.[1].

[2] Such small masses can be generated in some phenomenologically interesting superstring models by a string analogue of the Higgs mechanism [4].

for any massless neutral scalar: at the classical level, the zero-momentum coupling to a particle on its mass-shell is proportional to its rest mass. Since the dilaton under consideration originated from string theory, it is not surprising that its coupling constant is equal to the graviton coupling constant κ. Such a "strongly" coupled massless dilaton would lead to unacceptable deviations from Einstein's theory of general relativity; for instance, non-relativistic measurements of the gravitational constant would yield a result twice as large as the observation of light deflection from the sun. In the case of such a strong coupling, laboratory, geophysical and astronomical data exclude a dilaton mass lower than 10^{-4} eV [6]. Could radiative corrections change this conclusion?

Instead of using the fully-fledged string apparatus in the calculation of radiative corrections to the dilaton couplings, we shall first follow the effective field theory approach of ref.[7], to estimate the size of string-loop effects; to be specific, we discuss the one-loop effects, however our arguments apply to an arbitrary number of loops as well.

In order to determine the one-loop dilaton couplings to the order $\mathcal{O}(\kappa)$, we evaluate the kinetic energy corrections and the one-particle irreducible dilaton vertices. The kinetic energy corrections should be absorbed into rescalings of fields and masses, so that the one-loop effective Lagrangian, expressed in terms of these rescaled fields, acquires the canonical form of the kinetic energy terms. Finally, the dilaton vertices should be expressed in terms of these rescaled fields and masses.

Since dimension d operators in the Lagrangian are weighted by the factor κ^{d-4}, which is of the order of the respective power of the ultra-violet string momentum cut-off $1/\alpha' = g^2/2\kappa^2$, only the loops of light (i.e. of masses $m \ll 1/\kappa$) string excitations may give rise to large corrections, by yielding logarithms which become singular in the infra-red limit of vanishing masses or momenta. Moreover, a simple power-counting argument shows that the loops involving gravitational interactions are free of infra-red singularities to the order $\mathcal{O}(\kappa)$.[3] The only terms that require special attention are the matter mass terms with "unnaturally" small $m \ll 1/\kappa$, however in the supersymmetric theory under consideration their smallness is protected to all orders by chiral symmetries that are restored in the limit of $m \to 0$; some logarithmic corrections are expected, though. We thus conclude that only the loops involving light particles and gauge interactions may give rise to non-trivial radiative corrections, provided that some infra-red singularities are present. This justifies the use of quantum field theory, instead of a fully-fledged string apparatus, in the calculation of these large contributions.

[3]Soft graviton singularities [8] are absent to this order.

As far as the technical aspect of this calculation is concerned, the simplest way to extract the leading logarithms from the Feynman diagrams is to use dimensional regularisation of ultra-violet divergences, by continuing momentum integrals to $D = 4 - 2\epsilon$ dimensions.[4] We subtract the ultra-violet ϵ^{-1} poles, and take the limit $\epsilon \to 0$. At this point, the scale μ of the leading logarithmic terms, of order $g^2 \log(\frac{m}{\mu})^2/16\pi^2 \sim \mathcal{O}(1)$, should be identified with the string ultra-violet cut-off: $\mu \sim 1/\kappa$. One important comment is in order here. Radiative corrections induce some interactions which are absent at the tree-level, for instance $\phi \partial_\mu A^\dagger \partial^\mu A$ etc. These terms, as well as corrections to the terms already present in eq.(1), are individually gauge-dependent, however as expected, this gauge-dependence cancels out in on-mass-shell amplitudes, i.e. after using equations of motion.

As already mentioned in the introduction, we are mainly interested in the large distance behaviour of the dilaton couplings, therefore we restrict our attention to dilaton couplings at zero momentum transfer. We obtain the following form of the effective on-mass-shell scattering Lagrangian, quadratic in the matter fields:

$$\mathcal{L}^{\rm eff}_{\rm quadr} = -K\kappa\phi[m_R\psi\xi + m_R\bar{\psi}\bar{\xi} + 2m_R^2 A^\dagger A + 2m_R^2 BB^\dagger] \; + \; \mathcal{O}(\kappa^2)\,, \tag{2}$$

where now all matter fields correspond to rescaled canonical fields, with canonical kinetic terms and inertial mass:

$$m_R^{1-\rm loop} = m[1 - \frac{g^2 C(r)}{8\pi^2}\log(\kappa^2 m^2)]\,. \tag{3}$$

In eq.(3), $C(r)$ denotes the quadratic Casimir operator for the representation r. The factor K of eq.(2), which parametrises deviations of the dilaton coupling from the graviton coupling constant, is given by:

$$K^{1-\rm loop} = 1 - \frac{g^2 C(r)}{4\pi^2}\log(\kappa^2 m^2)\,. \tag{4}$$

In order to extend our result, eqs.(2)-(4), beyond the one-loop level, and to perform summation of the leading logarithmic terms, it is convenient to make use of the sigma-model approach to string theory. Let $\tilde{G}_{\mu\nu}$ and $\tilde{\phi}$ denote the background metric and the background dilaton field, respectively, of the non-linear string sigma-model. A generic string h-loop contribution to the effective Lagrangian is of the form [9]:

$$\alpha' \Delta\mathcal{L}^{\rm eff} = \sqrt{-\tilde{G}}\,\tilde{O} e^{2(h-1)\tilde{\phi}} g^{2h-2} Z_O^{(h)}\,, \tag{5}$$

[4]No divergences other than logarithmic are encountered in the two- and three-point Green's functions under consideration; these are exactly the infra-red singular logarithms we want to extract.

where $\tilde{O}$ is an operator without the constant $\tilde{\phi}$ mode. In four dimensions, the sigma-model metric and dilaton are related to the physical metric $G_{\mu\nu}$ and dilaton ϕ (of the S-matrix approach) [9] by the following equations:

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} e^{2\kappa\phi} , \qquad \tilde{\phi} = \kappa\phi . \tag{6}$$

For an operator $\tilde{O}$ of conformal weight w, we obtain the following coupling of the zero-momentum physical dilaton:

$$\alpha' \Delta\mathcal{L}^{\rm eff} = \sqrt{-G}\, O e^{2(1+w)\kappa\phi} g^{2h-2} Z_O^{(h)} e^{2h\kappa\phi} . \tag{7}$$

Note that the h-loop dilaton coupling to the operator O is completely determined by the h-loop contribution $Z_O^{(h)}$ to the operator O.

Since the four-dimensional string theory under consideration is supersymmetric, the higher-loop corrections to the factor K, eq.(2), can be determined by studying the dilaton coupling to the scalar field A. The effective Lagrangian for this dilaton - scalar system is given by:

$$\begin{aligned}
\alpha'\mathcal{L}^{\rm eff}(A,\phi) &= -\partial_\mu A^\dagger \partial^\mu A [\sum_{h=0}^{\infty} g^{2h-2} Z_k^{(h)} e^{2h\kappa\phi}] - m^2 A^\dagger A e^{2\kappa\phi} [\sum_{h=0}^{\infty} g^{2h-2} Z_m^{(h)} e^{2h\kappa\phi}] \\
&= -\partial_\mu A^\dagger \partial^\mu A [\sum_{h=0}^{\infty} g^{2h-2} Z_k^{(h)}] - m^2 A^\dagger A [\sum_{h=0}^{\infty} g^{2h-2} Z_m^{(h)}] \\
&\quad - 2\kappa\phi \{ \partial_\mu A^\dagger \partial^\mu A [\sum_{h=0}^{\infty} h g^{2h-2} Z_k^{(h)}] + m^2 A^\dagger A [\sum_{h=0}^{\infty} (h+1) g^{2h-2} Z_m^{(h)}] \} \\
&\quad + \mathcal{O}(\kappa^2) .
\end{aligned} \tag{8}$$

We now perform the rescaling of the scalar field A, to bring its kinetic energy to the canonical form. In terms of this rescaled field, the Lagrangian of eq.(8) is given by:

$$\begin{aligned}
\mathcal{L}^{\rm eff}(A,\phi) &= -\partial_\mu A^\dagger \partial^\mu A - m_R^2 A^\dagger A \\
&\quad - 2\kappa\phi \{ \partial_\mu A^\dagger \partial^\mu A \, (g^2 \frac{\partial}{\partial g^2} \log Z_k) + m_R^2 A^\dagger A \, (1 + g^2 \frac{\partial}{\partial g^2} \log Z_m) \} ,
\end{aligned} \tag{9}$$

where the inertial mass is given by:

$$m_R^2 = m^2 \frac{Z_m}{Z_k} , \tag{10}$$

$$Z_k = \sum_{h=0}^{\infty} g^{2h} Z_k^{(h)} \quad , \quad Z_m = \sum_{h=0}^{\infty} g^{2h} Z_m^{(h)} . \tag{11}$$

After using the equations of motion for the scalar field A, we obtain the following result for the factor K, defined in eq.(2):

$$K \;=\; 1 + g^2 \frac{\partial}{\partial g^2} \log(\frac{Z_m}{Z_k}) \;=\; 1 + g^2 \frac{\partial}{\partial g^2} \log m_R^2 \,. \tag{12}$$

The leading logarithmic terms in the mass rescaling factor $Z_m Z_k^{-1}$, eq.(10), can be summed up to all orders in the gauge coupling constant by using the standard renormalisation group techniques. From the one-loop result of eq.(3), we obtain:

$$\frac{Z_m}{Z_k} = \left[\frac{g^2(m^2)}{g^2}\right]^{4C(r)/\beta_0} , \tag{13}$$

where the running coupling constant is defined by the usual leading logarithmic relation:

$$\frac{1}{g^2(m^2)} = \frac{1}{g^2} + \frac{\beta_0}{16\pi^2} \log(\kappa^2 m^2) \,. \tag{14}$$

By substituting eq.(13) into eq.(12), we obtain the following result for the factor K in the leading logarithmic approximation:

$$K^{\rm llog} = 1 - \frac{g^2(m_R^2)C(r)}{4\pi^2} \log(\kappa^2 m_R^2) \,. \tag{15}$$

Thus in the leading logarithmic approximation, the inclusion of higher-loop effects amounts to the replacement of the coupling constant g^2 in the one-loop result of eq.(4) by the running coupling constant corresponding to the energy scale set by the rest mass of particles interacting with the dilaton. Note that the dilaton coupling is always *larger* than the graviton coupling constant, for any gauge group or representation r.

There is another way of deriving eq.(12), which sheds some light on the dilaton interactions in confining theories, for instance on the dilaton - nucleon interactions. Eq.(5) shows that to all orders in the string loop expansion parameter g (gauge coupling constant):

$$\mathcal{L}^{\rm eff}(\tilde{\phi}, \tilde{G}_{\mu\nu}) \;=\; \mathcal{L}^{\rm eff}(\tilde{\phi} + \log g, \tilde{G}_{\mu\nu}) \;=\; \mathcal{L}^{\rm eff}(\kappa\phi, G_{\mu\nu} e^{2\kappa\phi}) \,, \tag{16}$$

so that:

$$\frac{\partial}{\partial \tilde{\phi}} \mathcal{L}^{\rm eff} \;=\; 2g^2 \frac{\partial}{\partial g^2} \mathcal{L}^{\rm eff} \,. \tag{17}$$

The coupling of the zero-momentum physical dilaton ϕ is given by:

$$\frac{\partial}{\partial \phi} \mathcal{L}^{\rm eff}(\phi = 0) \;=\; 2\kappa(-G^{\mu\nu} \frac{\partial}{\partial G^{\mu\nu}} + g^2 \frac{\partial}{\partial g^2}) \mathcal{L}^{\rm eff}(\phi = 0) \,. \tag{18}$$

For a canonically normalised field A, the most general quadratic form consistent with general covariance, is given by:

$$\mathcal{L}_{\rm quadr}(\phi=0, G_{\mu\nu}, A) \;=\; -\sqrt{-G}(G^{\mu\nu}\partial_\mu A^\dagger \partial_\nu A + m_R^2 A^\dagger A)\,. \tag{19}$$

After applying eq.(18) and using the equations of motion, we obtain:

$$\mathcal{L}^{\rm eff}_{\rm quadr}(\phi, A) \;=\; -2\kappa(m_R^2 + g^2\frac{\partial m_R^2}{\partial g^2})\phi A^\dagger A \;=\; -2\kappa(1+g^2\frac{\partial}{\partial g^2}\log m_R^2)m_R^2\phi A^\dagger A\,, \tag{20}$$

in agreement with the result of eq.(12) for the factor K. The same result can be derived explicitly for fermions, without using supersymmetry. Of course, in a supersymmetric theory the factor K is equal for fermions and bosons of equal masses.

It is clear from our derivation that eq.(20) follows from: i) general covariance and ii) particular dependence of the effective action on the background dilaton field: $\mathcal{L}^{\rm eff}(\tilde\phi) = \mathcal{L}^{\rm eff}(\tilde\phi + \log g)$. It is generally expected that non-perturbative effects do not spoil general covariance. Also, it seems reasonable to assume that low-energy phenomena do not affect the Planck-scale relation between the coupling constant and the dilaton background. In other words, we expect that, even in a confining theory like QCD, the dilaton couples to the rest mass m_R of composite particles, e.g. nucleons, with the factor K given in eq.(12). In a confining theory, the typical particle mass is of order of the strong interaction scale $\Lambda \sim \kappa^{-1}\exp(-8\pi^2/\beta_0 g^2)$ [7], therefore we expect:

$$K \;\approx\; 1 + \frac{16\pi^2}{\beta_0 g^2} \;\gg\; 1\,. \tag{21}$$

Similar considerations can be repeated for the graviton couplings, with the conclusions that: i) the graviton coupling κ remains unaffected by radiative corrections, ii) the inertial mass m_R is equal to the gravitational mass, i.e. the mass that couples to spin-two gravitons. The equivalence principle is *not* violated in the graviton sector, as expected from general covariance.

We conclude that the universality of dilaton and graviton couplings does not hold beyond the tree-level; as seen from eqs.(15) and (21), at low energies the dilatons couple stronger than the gravitons. The effective non-relativistic coupling, induced by the superposition of one-dilaton and one-graviton exchanges is equal to $(1+K)\kappa$. In the presence of dilaton interactions, the equivalence principle *is* violated: non-relativistic matter interacts with couplings different from massless gauge bosons that couple to gravitons only,

with coupling equal to κ. Moreover, since the dilaton coupling contains the factor K, eq.(12), which depends on the rest mass, even the weak equivalence principle is violated.

In summary, we studied radiative effects in dilaton interactions with the conclusions that at large distances dilatons couple stronger than gravitons, and violate the equivalence principle. We confirm that without addressing the dilaton mass generation problem, it is not possible to avoid serious phenomenological problems in string theories with massless dilatons.

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References

[1] A. De Rújula, Phys. Lett. B180 (1986) 213;
J. Ellis, N.C. Tsamis, and M. Voloshin, Phys. Lett. B194 (1987) 291.

[2] K.S. Narain, Phys. Lett. B169 (1986) 41;
H. Kawai, D.C. Lewellen and S.-H.H. Tye, Phys. Rev. Lett. 57 (1986) 1832, Nucl. Phys. B288 (1987) 1;
I. Antoniadis, C. Bachas and C. Kounnas, Nucl. Phys. B289 (1987) 87;
W. Lerche, D. Lüst and A.N. Schellekens, Nucl. Phys. B287 (1987) 477.

[3] S. Ferrara, L. Girardello, C. Kounnas and M. Porrati, Phys. Lett. B192 (1987) 368, B194 (1987) 358;
I. Antoniadis, J. Ellis, E. Floratos, D.V. Nanopoulos and T. Tomaras, Phys. Lett. B191 (1987) 96.

[4] I. Antoniadis, C. Bachas and C. Kounnas, Phys. Lett. B200 (1988) 297.

[5] J. Wess and J. Bagger, Supersymmetry and Supergravity (Princeton University Press, 1983).

[6] A. De Rújula, private communication (1988); see also ref.[1] and:
I.I. Shapiro et al., J. Geophys. Res. 82 (1977) 4329.

[7] G. Veneziano, "Topics in String Theory", lectures given at Beijing and Kanpur, CERN–TH.5019/88 (1988);
T.R. Taylor and G. Veneziano, "Strings and $D = 4$", CERN–TH.5085/88 (1988).

[8] S. Weinberg, Phys. Rev. 140 (1965) B516.

[9] E. Fradkin and A. Tseytlin, Phys. Lett. B160 (1985) 69;
C. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. B262 (1985) 593.

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Quantum hair and the string-black hole correspondence

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Abstract

We consider a thought experiment in which an energetic massless string probes a "stringhole" (a heavy string lying on the correspondence curve between strings and black holes) at large enough impact parameter for the regime to be under theoretical control. The corresponding, explicitly unitary, S-matrix turns out to be perturbatively sensitive to the microstate of the stringhole: in particular, at leading order in l_s/b, it depends on a projection of the stringhole's Lorentz-contracted quadrupole moment. The string-black hole correspondence is therefore violated if one assumes quantum hair to be exponentially suppressed as a function of black-hole entropy. Implications for the information paradox are briefly discussed.

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1 Introduction

The issue of a possible loss of quantum coherence in processes in which a black hole is produced and then evaporates has been the subject of much debate since Hawking's claim [1] that black holes should emit an exactly thermal spectrum of light quanta (see e.g. [2] for a recent review). Progress from string theory on the microscopic understanding of black-hole entropy [3] and on the AdS-CFT correspondence [4], has lent strong support to the belief that no loss of information/quantum-coherence should occur. However, even in the AdS/CFT case, understanding how unitarity on the CFT side teaches about information recovery on the gravity side remains unclear (see [5], [6]).

Another "ab initio" approach to the same problematics consists of the study of trans-planckian-energy collisions of massless strings as a function of center of mass energy (or of the associated gravitational radius R), of impact parameter b, and of the string-length scale l_s, with the relative ratios of these scales defining different regimes for the process [7].

In this framework it has been possible to recover, within a unitarity-preserving S-matrix, both General Relativity expectations and string-size related modifications of it [8], albeit in regimes in which no-black hole formation is expected according to closed-trapped surface criteria[9]. Dealing with the complementary regime (corresponding to $R \gg b, l_s$) has met with more limited success, although some progress has been made in understanding how the threshold of black-hole production can be approached from below [10]. An approximation to deal with the full-collapse regime, proposed a few years ago in [11], appears to predict correctly the existence and rough values of some critical ratios for the onset of collapse, but, unfortunately, has failed so far to provide a unitary description of the process beyond such critical points [12].

Given the above difficulties, the attention has been shifted to a supposedly easier problem [13], that of the scattering of a closed light string off a stack of N $D-p$-branes at small string coupling and large N. Here the equivalent of the black-hole formation regime is the one in which the closed string is absorbed by the brane system and its energy is dissipated in open string excitations of the stack itself. In spite of some progress [13] [14], understanding how information about the initial state gets encoded in the final one is still far from settled.

One problem is that information, if it's to be eventually recovered, has to start coming out, at the latest, by the so-called Page time [15], corresponding roughly to the time by which the evaporating black hole has lost half of its entropy S. In order for this to be possible, the rate of information retrieval cannot be too small, e.g. cannot be of order $\exp(-S)$, at least not after the Page time. Information retrieval should instead be easy if "quantum hair" is inversely proportional to S, as recently proposed in a toy model identifying black holes with a self-sustained critical Bose-Einstein condensate of $N \sim S$ gravitons [16]. Similar claims have been made in [17] on the basis of general uncertainty-principle considerations applied to the geometry itself. Indeed, once an effective classical geometry with an information-free horizon is assumed (even an effective one that corrects the classical horizon), continuous

information loss looks inevitable[2].

In this paper we will address this kind of questions using the correspondence between strings and black holes [18, 19, 20] that occurs when the mass of the former is tuned to the value $M_{SH} = M_s g_s^{-2}$, giving a Schwarzschild radius $R = O(l_s)$. By going to small enough string coupling we can make the entropy of such "string-holes" (SH) arbitrarily large:

$$S_{SH} = \left(\frac{l_s}{l_P}\right)^{D-2} = \left(\frac{M_P}{M_s}\right)^{D-2} = g_s^{-2} \gg 1\,. \tag{1}$$

It is particularly appealing that, for SHs, the question of the size of quantum hair becomes one about whether it is perturbative or not in the string coupling constant. In our case, the role of the parameter N of [16] is played by the string coupling which, for a given string mass, is tuned to a critical value. Unfortunately, and unlike in the simple model of [16], we are presently unable to perform a reliable calculation when g_s and/or M are parametrically larger than their critical values.

Furthermore, in order be able to claim that strings of mass $M_{SH} = M_s g_s^{-2}$ can also be seen as black holes, we have to impose that they are compact enough not to exceed in size their own Schwarzschild radius $R = O(l_s)$, and to check that this restriction does not invalidate the entropy estimate (1). This question was addressed in [20] (see also [19]), where it was argued that the entropy of string states of mass $M \leq M_{SH} = g_s^{-2} M_s$ is shared among states of different size r according to:

$$S(M,r) \sim \frac{M}{M_s}\left(1 - c_1\frac{l_s^2}{r^2}\right)\left(1 - c_2\frac{r^2}{(\alpha' M)^2}\right)\left(1 + c_3\left(\frac{R}{r}\right)^{D-3}\right), \tag{2}$$

where c_i are positive constants of $O(1)$. For $M \ll M_{SH}$ the last terms is negligible and the first two factors give a maximal entropy for $r \sim l_s\sqrt{\frac{M}{M_s}}$, the random-walk value. However, there is still an entropy $O(M/M_s)$ in "compact" strings and, furthermore, as one approaches $M = M_{SH}$, the third term in (2) helps favoring such strings.

Another way of reaching a similar result consists in counting string states at level $N = \alpha' M^2$ produced by oscillators of index larger than K. A simple argument, based on evaluating the corresponding partition function, shows that the entropy of such states is still $O(\sqrt{N})$ if $K \sim \sqrt{N}$. They will generally correspond to occupation numbers $\leq O(1)$ for $O(\sqrt{N})$ oscillators (providing the right value for their mass) and will have a size of order:

$$r^2 \sim l_s^2 \sum_{n>\sqrt{N}} \frac{1}{n}\langle a_n^\dagger a_n\rangle \sim l_s^2\,. \tag{3}$$

This is the kind of states we shall focus our attention on. We recall that, not only entropy, but, qualitatively, many other properties of strings and black holes (decay rates, evaporation time etc.) match on the correspondence line [19, 20].

The idea, therefore, is to consider a thought experiment in which a massless string probes a stringhole target, a process somewhere in between those discussed in [7] (where both projectile and target are massless) and [13] (where the target is infinitely heavy). Studying such a process at sufficiently large impact parameters for the approximations to be under control turns out to be sufficient to reveal whether the quantum hair of such SHs is perturbative or not in $1/S \sim g_s^2$. This appears to be the string-theory counterpart to checking (albeit only at a specific point) whether quantum hair is perturbative in $1/N$ in the approach of [16].

2 A thought experiment revealing quantum hair

We work in flat 10-dimensional spacetime with $(10 - D)$ dimensions compactified at the string-length scale so that the effective large-distance physics lives in D spacetime dimensions. We are also assuming to be working at very small string coupling g_s so that, as already indicated in (1), there is a large hierarchy between the string and Planck mass scales.

Consider now a process in which a massless "probe" string collides with a well-defined heavy (and for the moment generic) "target" string of mass $M \gg M_P \gg M_s$. Let us also take a high-energy limit in which the energy E of the probe string in the rest frame of the heavy one is much larger than M_s and yet much smaller than M,

$$M_s M \ll s - M^2 = -2p \cdot P = 2EM \ll M^2 \,, \tag{4}$$

so that the light string does indeed act (almost) as a probe and yet we can apply a high-energy limit in which graviton exchange dominates.

Following the logic of [7] (see also [21], [13]) we can argue that, at large-enough impact parameter b, the elastic scattering amplitude is given by the semiclassical eikonal formula:

$$S(E, M, b) \sim \exp(i\frac{\mathcal{A}_{cl}}{\hbar}) = \exp\left(i\frac{4GEM}{\hbar}c_D b^{4-D}\right) \equiv e^{2i\delta(E,M,b)} \; ; \; c_D = \Omega_{D-4}^{-1} \equiv \frac{\Gamma(\frac{D-4}{2})}{2\pi^{\frac{D-4}{2}}}. \tag{5}$$

As a consistency check, we note that, when one goes back from b to q-space (or deflection angle θ), one recovers, at the saddle point of the b-integral, the classical Einstein relation (generalized to arbitrary D) between deflection angle, mass, and impact parameter:

$$\theta = \frac{8\pi GM}{\Omega_{D-2} b^{D-3}} \sim \left(\frac{R}{b}\right)^{D-3} \ll 1 \;\; ; \;\; (GM)^{\frac{1}{D-3}} \sim R \ll b \,, \tag{6}$$

where R is the Schwarzschild radius of the heavy string. Obviously, the above formula satisfies the "no-hair" theorem, in the sense that it is sensitive to the mass of the heavy string state but not to its microscopic quantum numbers.

Diagrammatically, the result (5) comes from exponentiating the exchange of a single graviton between the light and the heavy string. Both (5) (and (6)) are indeed only valid at sufficiently large impact parameter (small deflection angle) and suffer from corrections of

higher order in R/b (θ). These will reconstruct, for instance, the deflection formula in the full Schwarzschild (or Kerr if we consider a target with spin) metric. As shown long ago by Duff [22], they correspond, diagrammatically, to exponentiating connected graviton-tree (fan) diagrams in which a single vertex (the trunk of the tree) is attached to the probe string while all the branches terminate on the heavy one, giving the appropriate powers of R and b. These classical correction still satisfy the no-hair condition as well as elastic unitarity. On the other hand if instead"hairy" corrections to $\delta(E, M, b)$ are exponentially suppressed for large black holes, we would expect them to show up in the form:

$$\delta(E, M, b) \to \delta(E, M, b)(1 + \text{classical corrections} + e^{-cS}\hat{Q}) \,, \tag{7}$$

where c is some constant and $\hat{Q}$ represents, schematically, a quantum-hair operator taking different expectation values depending on the black hole microstate. We will check below whether an ansatz like (7) is satisfied for the particular stringhole states introduced in the previous Section.

To address this question recall that, as discussed in [7] and [13] in two different contexts, there are also "string corrections" to the leading eikonal form. These are related to the fact that strings are extended objects and therefore suffer tidal forces when moving in a non trivial geometry[23][1]. Fortunately, at least at small scattering angle, such corrections are fully under control and lead to a unitary S-matrix. Unitarity is now satisfied in a less trivial way: different channels couple, elastic unitarity is violated, but one still obtains a fully unitary S-matrix in the Hilbert space of two arbitrary string states. The question is whether this non-trivial S-matrix contains information about the actual state of the heavy string, and at which level.

Building on the work of [7] and [13] we can be confident that the tidal excitation of both the light and the heavy string are captured, at leading order in θ, by the replacement:

$$\delta(E, M, b) \to \hat{\delta}(E, M, b) = \langle \delta(b + \hat{X}_H - \hat{X}_L) \rangle = 2GEM\hbar^{-1} c_D \langle (b + \hat{X}_H - \hat{X}_L)^{4-D} \rangle \,. \tag{8}$$

Here $\hat{X}_H$ and $\hat{X}_L$ represent the heavy and light string position operators, stripped of their zero modes (which give b), evaluated at $\tau = 0$, and averaged over σ. These operations, together with a normal-ordering prescription, are indicated in (8) by the brackets, i.e.

$$\langle (b + \hat{X}_H - \hat{X}_L)^{4-D} \rangle \equiv \frac{1}{4\pi^2} \int_0^{2\pi} d\sigma_L \int_0^{2\pi} d\sigma_H : \Big(b + \hat{X}_H(\sigma_H, 0) - \hat{X}_L(\sigma_L, 0) \Big)^{4-D} : \,. \tag{9}$$

In words, the classical phase shift is replaced by the average of a *quantum* phase shift in which the impact parameter is affected by a quantum uncertainty encoded in the string position operators.

For what concerns the excitation of the light string, further justification of the above formula comes from the study of string-brane collision discussed in [13], specialized to the

[1] Although all calculations are performed in flat spacetime the effects of an effective non-trivial geometry emerge from the calculation.

case of a stack of 0-branes. For the excitation of the heavy string we can instead appeal to the quantization of the heavy string in the shock-wave metric produced by the light one [24].

Following [7], we now expand (8) to quadratic order in the $\hat{X}$ (the linear order clearly averages out to zero) to get the leading correction in an expansion in $(l_s/b)^2$:

$$2(\hat{\delta}-\delta) = \frac{2\pi GEM(D-2)}{\hbar\Omega_{D-2}b^{D-2}}\langle Q_H^{ij}+Q_L^{ij}\rangle \hat{b}_i\hat{b}_j\,. \tag{10}$$

Here Q_H^{ij} is the $(D-2)$-dimensional (i.e. Lorentz-contracted in the direction of the incoming momentum) quadrupole operator for the heavy string[2].

$$Q_H^{ij} = \hat{X}_H^i\hat{X}_H^j - \frac{\delta_{ij}}{D-2}\sum_{i=1}^{D-2}\hat{X}_H^i\hat{X}_H^i\,, \tag{11}$$

and is projected along the unit vector $\hat{b}$ in the direction of the impact parameter. This projection can also be written in the form:

$$Q_H^{ij}\hat{b}_i\hat{b}_j = \hat{X}_H^i\hat{X}_H^j\left(\hat{b}_i\hat{b}_j - \frac{\delta_{ij}}{D-2}\right) \equiv \Pi_{ij}\hat{X}_H^i\hat{X}_H^j\,. \tag{12}$$

As indicated in (10), we get a similar term for the probe string. At this order in l_s/b the S-matrix thus factorizes in the form:

$$S(E,M,b) = \exp(2i\delta)\ \Sigma_L\ \Sigma_H\ ;\ \Sigma_{L,H} = \exp\left(i(D-2)\Delta\ \tilde{Q}_{L,H}^{ij}\ \hat{b}_i\hat{b}_j\right)\,, \tag{13}$$

where we have defined the dimensionless quantities[3]:

$$\Delta = \frac{2\pi GEMl_s^2}{\hbar\Omega_{D-2}b^{D-2}}\quad ;\quad \tilde{Q}^{ij} = l_s^{-2}Q^{ij}\,, \tag{14}$$

the latter being the quadrupole measured in string-length units. Since the quadrupole operators are hermitian (see also below), each factor appearing in (13) corresponds to a unitary operator. The first two factors are independent of the particular state chosen for the heavy string. Let us therefore concentrate our attention on Σ_H (dropping for simplicity the H suffix). The operator appearing at the exponent in Σ can be easily written down:

$$\tilde{Q}^{ij}\ \hat{b}_i\hat{b}_j = \Pi_{ij}\ \sum_{n=1}^{\infty}\frac{1}{n}\left(a_n^{\dagger i}a_n^j + \tilde{a}_n^{\dagger i}\tilde{a}_n^j + a_n^i\tilde{a}_n^j + a_n^{\dagger i}\tilde{a}_n^{\dagger j}\right)\,. \tag{15}$$

Its diagonal matrix elements are sensitive to the (projected, transverse) quadrupole of the heavy string, while the transitions to other states, induced by terms with two creation or

[2]I am grateful to T. Damour for this interesting remark.

[3]The value of Δ, when compared to unity, determines [7] whether the probe string gets excited or not by tidal forces. However, once more, these effects will *not* depend on the particular state of the target string.

two annihilation operators, correspond to a quadrupole-like excitation of the original string itself. This is hardly surprising in view of the intimate relation between tidal forces and quadrupole moments (see e.g. [25]), and simply appears as a generalization of known facts to an ultra-relativistic situation involving strings (our quadrupole, in particular, is a purely geometrical object).

In order to have an estimate of quantum hair we need to normal order the whole exponential operator occurring in Σ. Following again [7], we find:

$$\begin{aligned}
\Sigma_H &= \Sigma^{(univ)}\ \Sigma^{(hair)}\ ;\ \Sigma^{(univ)} = \Gamma(1+i\Delta)^{D-3}\ \Gamma(1-i(D-3)\Delta) \\
\Sigma^{(hair)} &= :\exp\left(\sum_{n=1}^{\infty}(a_n^{\dagger i}+\tilde{a}_n^i)(a_n^j+\tilde{a}_n^{\dagger j})\left[C_n(\Delta)(\delta_{ij}-\hat{b}_i\hat{b}_j)+\tilde{C}_n(\Delta)\hat{b}_i\hat{b}_j\right]\right): \\
C_n(\Delta) &= -\frac{i\Delta}{n+i\Delta}\ ;\ \tilde{C}_n(\Delta) = C_n(-(D-3)\Delta)\,. \qquad (16)
\end{aligned}$$

At this point the explicit calculation of the S-matrix is simple, in particular between coherent states. $\Sigma^{(univ)}$, being a c-number, does not depend on the internal quantum numbers of the heavy string and, together with similar factors coming from the light string, provides absorption and further contributions to the phase shifts, but no hair. Instead, the operator $\Sigma^{(hair)}$ generates matrix elements that feel the nature of the microstate in which the heavy string actually is. Note that normal ordering has slightly upset the exact quadrupole structure appearing in (11), (12) (which is however recovered for $n \gg \Delta$).

Let us now specify further the process described in the previous section in order to make contact with black-hole physics. To this purpose we shall identify the heavy string with a "stringhole" state described in Sec. 1. The reason for choosing that precise (within factors $O(1)$) value of M is twofold. Choosing M in the range $M_P \ll M \ll M_{SH}$ leads to reliable results, but the string, in this case, is below the correspondence curve, its size is larger than its Schwarzschild radius and therefore is not a collapsed object [18]. On the other hand, various approximations that can be justified for strings of mass up to M_{SH} cease to be valid for strings with $M >> M_{SH}$, i.e. strings that would simulate "large" black holes in string-length units.

Let us first evaluate the quantity Δ in (14) for the SH case. Up to numerical factors:

$$\Delta = \frac{GEMl_s^2}{\hbar\, b^{D-2}} \to \frac{El_s}{\hbar}\left(\frac{l_s}{b}\right)^{D-2} \sim \frac{E}{M_s}\theta^{\frac{D-2}{D-3}}\,. \qquad (17)$$

Given our bounds (4) on E we find:

$$\theta^{\frac{D-2}{D-3}} \ll \Delta \ll g_s^{-2}\theta^{\frac{D-2}{D-3}}\,. \qquad (18)$$

Obviously, even keeping $\theta \ll 1$, but finite and g_s-independent, we can make $\Delta \gg 1$ (yet $\ll g_s^{-2}$) for sufficiently small g_s and with E in a parametrically large region.

In order to estimate the size of quantum hair we note that the coefficients $C_n(\Delta)$ appearing in (16) become $O(1)$ at $n < \Delta$ or of order Δ/n at $n > \Delta$. As already discussed,

typical SHs will have most of the non vanishing occupation numbers of $O(1)$ in oscillators with $n \sim \sqrt{N} \sim g_s^{-2}$. In that case $C_n \sim \Delta/n$ and eq. (16) simplifies further:

$$\Sigma^{(hair)} =: \exp\left(-i(D-2)\Delta \sum_{n=1}^{\infty} \frac{1}{n}(a_n^{\dagger i} + \tilde{a}_n^i)(a_n^j + \tilde{a}_n^{\dagger j})\Pi_{ij}\right) : \ . \tag{19}$$

The basic observation is that the operator appearing in the exponent of (19) is completely unrelated to the one giving the mass of the SH[4] and therefore will distinguish degenerate microstates. It contains positive definite diagonal terms that correspond to the transverse, projected quadrupole of the SH. The non-positive definite terms, corresponding to inelastic transitions, are also microstate-dependent through a similar quadrupole operator. There is also a state-dependent absorption from the real part of C_n. This is suppressed by an extra factor $\Delta/n \sim g_s^2\Delta$ and is not controlled by the quadrupole.

The dominant terms sum up to something $O(1)$ but can still take different values of that same order within the whole SH ensemble. Hence, an experiment measuring the phase of the S-matrix should be able to reduce our ignorance on the state of the SH by a factor $O(2)$[5]. Interpreting this as a reduction on the total number of states e^S, it will correspond to a decrease of $O(1)$ in entropy, meaning that the whole information can be recovered after $O(S)$ experiments. The minimal duration of each experiment being $O(l_s)$, namely the light-crossing time for a SH, the total time needed to recover the information will be of order of the Page/evaporation time $SR \sim g_s^{-2} l_s$.

In our approximation quantum hair also appears to be suppressed with respect to the no-hair terms by a power of the scattering angle. While this is still sufficient for our qualitative discussion, we think that our results should qualitatively extend to scattering angles of O(1). Checking this is not easy since, precisely for a SH target, string-size and classical corrections kick in simultaneously as we increase the scattering angle (although the different θ-dependence should help separate the two kinds of corrections). If so, the quantum hair revealed by our scattering process (with higher multipoles appearing besides Q_{ij}) will indeed approach $O(1)$ for a probe-energy of order of the Hawking/Hagedorn temperature of the SH. Such impact parameters and energies are precisely those typical of Hawking's radiation. Actually, as well known in particle physics (see e.g. [26]), a decay amplitude is usually to be corrected by a "final-state interaction" which basically amounts to multiplying the naive decay amplitude by a factor $S^{1/2}(E,b) \sim \exp(i\delta(E,b))$, where the typical values of E and b will be M_s and l_s respectively. In other words, such a quantum hair may directly leave its imprint in the decay of a SH. Admittedly, all these are hand-waving arguments that should be analyzed more carefully.

In any case, it appears that the quantum-hair amplitude is *not* suppressed, relative to no-hair terms, by $\exp(-S_{SH}) \sim \exp(-g_s^{-2})$ but rather, at most, by a small inverse power of

[4]It is also clearly non-degenerate with the spin of the SH.

[5]If instead we wish to distinguish SH states with differences $O(1)$ in the occupation numbers the sensitivity of (19) will have a suppression factor of $O(\Delta/n) \sim g_s^2\Delta$.

$S \sim g^{-2}$, i.e. is a perturbative effect in the string coupling. Note, however, that a generic individual element of the S-matrix is always suppressed by an $\exp(-\Delta)$ "non-perturbative" factor, which gets compensated by the exponentially large number of final states contributing to inclusive-enough cross sections. Indeed, given that Σ_H is unitary, it is easy to lose its sensitivity to quantum hair if traces over the heavy initial and/or final SH states are taken. Summing individual transition probabilities over final SH states corresponds to considering an inclusive cross section, while tracing/averaging over the initial SH state corresponds to an initial mixed state. In both cases it is quite clear that unitarity of Σ_H washes out all the leading-order SH hair discussed so far[6]. Only subleading terms and/or appropriate interference experiments will be able to leave information about the state of the SH on the probe-string. Whether this is in principle sufficient to retrieve enough information on the SH is not completely obvious.

In conclusion, the results we have presented point in the direction of some perturbative quantum hair being revealed in our thought experiment, very likely something of order $1/S \sim g_s^2$ for a probe of $E \sim M_s$ during a collision (horizon-crossing) time $O(l_s)$. At least naively, this would allow to retrieve the full information about the microstate of the string hole within its evaporation time of order $g_s^{-2} l_s$. This result can be related to the fact that, for a stringhole, the concept of an infomation-free horizon does not make sense (the horizon being as large as the string itself) and, in this sense, it is similar to what is believed to occur for the so-called fuzzballs states of string theory (see [28] and references therein). It would be interesting to see whether thought experiments of the kind discussed here using fuzzballs could reveal a similar amount of quantum hair. Of course the issue of whether or not spacetime around the horizon can be considered to be empty is also very relevant in the recent firewalls debate [29].

Can we reconcile our finding with an exponentially small amount of quantum hair for large black holes (i.e. for black holes much heavier that M_{SH}, for which our simple analysis fails to provide a reliable answer)? Clearly an expression like that of (7) is in contradiction with our findings but one could instead imagine an ansatz like:

$$\delta \to \delta(E, M, b)(1 + \text{classical corrections} + e^{-c\frac{S}{S_{SH}}}\hat{Q}) \,, \tag{20}$$

which would only give an exponential suppression for black holes that are much heavier that those on the correspondence line. Indeed, a single string may fail to represent black holes above the correspondence curve (seen in that case as a critical line separating two phases), in which case $\Sigma^{(hair)}$ could change quite abruptly above the phase transition.

Another possible objection to drawing strong conclusions from our results lies in the possibility[7] that SHs do *not* represent typical black holes but only a tiny fraction of them. In that case, their long hair will make them atypical "hippie-like" black holes within a vast majority of "bald" ones.

[6] At order $l_s^4 b^{-4}$ the eikonal operator will give terms proportional to $X_L^2 X_H^2$ that destroy factorization.

[7] This possible loophole was suggested by M. Porrati.

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References

[1] S. W. Hawking, *Nature* **248** (1974) 30; *Commun. Math. Phys.* **43** (1975) 199.

[2] S. Mathur, *The information paradox : A pedagogical introduction*, arXiv:0909.1038.

[3] A. Strominger and C. Vafa, *Phys.Lett.* **B379** (1996) 99.

[4] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Phys. Lett.* **428** (1998) 105; E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253.

[5] J. M. Maldacena, *J. High Energy Phys.* 0304 (2003) 02; *JHEP* **0304** (2003) 021.

[6] J.L.F. Barbon and E. Rabinovici, *JHEP* **0311** (2003) 047.

[7] D. Amati, M. Ciafaloni and G. Veneziano, *Phys. Lett.* **B197** (1987) 81; *Int. J. Mod. Phys.* **A3** (1988) 1615.

[8] D. Amati, M. Ciafaloni, and G. Veneziano, *Phys. Lett.* **B216** (1989) 41.

[9] D. M. Eardley and S. B. Giddings, *Phys. Rev.* **D66** (2002) 044011; E. Kohlprath and G. Veneziano,*JHEP* **06** (2002) 057; H. Yoshino and Y. Nambu, *Phys. Rev.* **D67** (2003) 024009; S. B. Giddings and V. S. Rychkov, *Phys. Rev.* **D70** (2004) 104026.

[10] G. Veneziano, *JHEP* **11** (2004) 001.

[11] D. Amati, M. Ciafaloni, and G. Veneziano, *JHEP* **02** (2008) 049; G. Marchesini and E. Onofri, *JHEP* **06** (2008) 104; G. Veneziano and J. Wosiek, *JHEP* **09** (2008) 023; G. Veneziano and J. Wosiek, *JHEP* **09** (2008) 024.

[12] M. Ciafaloni and D. Colferai, *JHEP* **11** (2008) 047; *JHEP* **12** (2009) 062.

[13] G. D'Appollonio, P. Di Vecchia, R. Russo and G. Veneziano, *JHEP* **1011** (2010) 100.

[14] W. Black and C. Monni, *Nucl. Phys.* **B859** (2012) 299-320; M. Bianchi and P. Teresi, *Scattering higher spins off D-branes* [arXiv: 1108.1071].

[15] D. N. Page, *Phys. Rev. Lett.* **71** (1993) 3743.

[16] G. Dvali and C. Gomez, *Black Hole's* $1/N$ *Hair*, arXiv:1203.6575; *Black Hole Macro-Quantumness*, to appear, and references therein.

[17] R. Brustein, *Origin of the blackhole information paradox*, arXiv:1209.2686.

[18] G. Veneziano, *Europhys. Lett.* **2** (1986) 133; Proc. Meeting on "Hot hadronic matter: theory and experiments", Divonne, June 1994, eds. J. Letessier, H. Gutbrod and J. Rafelsky, NATO-ASI Series B: Physics (Plenum Press, New York, 1995), **346**, p. 63; L. Susskind, *Some speculations about black hole entropy in string theory*, [hep-th/9309145]; E. Halyo, A. Rajaraman and L. Susskind, *Phys. Lett.* **B392** (1997) 319; E.Halyo, B. Kol, A. Rajaraman and L. Susskind, *Phys. Lett.* **B 401** (1997) 15.

[19] G.T. Horowitz and J. Polchinski, *Phys. Rev.* **D55** (1997) 6189; ibid. **57** (1998) 2557;

[20] T. Damour and G. Veneziano, *Nucl. Phys.* **B568** (2000) 93.

[21] M. Fabbrichesi and R. Iengo, *Phys. Lett.* **B264** (1991) 319.

[22] M. J. Duff , *Phys.Rev.* **D7** (1973) 2317.

[23] S. B. Giddings, *Phys.Rev.* **D74** (2006) 106005.

[24] H. J. de Vega and N. G. Sanchez, *Nucl. Phys.* **B317** (1989) 706–730; G. T. Horowitz and A. R. Steif, *Phys. Rev.* **D42** (1990) 1950–1959; S. B. Giddings, D. J. Gross, and A. Maharana, *Phys. Rev.* **D77** (2008) 046001.

[25] T. Damour and A. Nagar, *Phys. Rev.* **D80** (2009) 084035.

[26] M. Suzuki and L. Wolfenstein, *Phys. Rev.* **D60** (1999) 074019.

[27] P. C. Aichelburg and R. U. Sexl, *Gen. Rel. Grav.* **2** (1971) 303–312; see, e.g. E. Kohlprath and G. Veneziano, *JHEP* **06** (2002) 057 for its generalization.

[28] S. Mathur, *The fuzzball proposal for black holes: an elementary review*, Fortsch. Phys. **53** (2005) 793.

[29] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, *Complementarity or firewalls?*, arXiv: 1207.3123 [hep-th].

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INFLATING, WARMING UP, AND PROBING THE PRE-BANGIAN UNIVERSE

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ABSTRACT

Classical and quantum gravitational instabilities, can, respectively, inflate and warm up a primordial Universe satisfying a superstring-motivated principle of "Asymptotic Past Triviality". A physically viable big bang is thus generated without invoking either large fine-tunings or a long period of post-big bang inflation. Properties of the pre-bangian Universe can be probed through its observable relics, which include: i) a (possibly observable) stochastic gravitational-wave background; ii) a (possible) new mechanism for seeding the galactic magnetic fields; iii) a (possible) new source of large-scale structure and CMB anisotropy.

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1 INTRODUCTION

I would like to begin this talk by asking a very simple question: Did the Universe start "small"? The naive answer is: Yes, of course! However, a serious answer can only be given after defining the two keywords in the question: What do we mean by "start"? and What is "small" relative to? In order to be on the safe side, let us take the "initial" time to be a bit larger than Planck's time, $t_P \sim 10^{-43}$ s. Then, in standard Friedmann–Robertson–Walker (FRW) cosmology, the initial size of the (presently observable) Universe was about 10^{-2} cm. This is of course tiny w.r.t. its present size ($\sim 10^{28}$ cm), yet huge w.r.t. the horizon at that time, i.e. w.r.t. $l_P = ct_P \sim 10^{-33}$ cm. In other words, a few Planck times after the big bang, our observable Universe consisted of $(10^{30})^3 = 10^{90}$ Planckian-size, causally disconnected regions.

More precisely, soon after $t = t_P$, the Universe was characterized by a huge hierarchy between its Hubble radius and inverse temperature on one side, and its spatial-curvature radius and homogeneity scale on the other. The relative factor of (at least) 10^{30} appears as an incredible amount of fine-tuning on the initial state of the Universe, corresponding to a huge asymmetry between time and space derivatives. Was this asymmetry really there? And if so, can it be explained in any more natural way?

It is well known that a generic way to wash out inhomogeneities and spatial curvature consists in introducing, in the history of the Universe, a long period of accelerated expansion, called inflation [1]. This still leaves two alternative solutions: either the Universe was generic at the big bang and became flat and smooth because of a long *post*-bangian inflationary phase; or it was already flat and smooth at the big bang as the result of a long *pre*-bangian inflationary phase.

Assuming, dogmatically, that the Universe (and time itself) started at the big bang, leaves only the first alternative. However, that solution has its own problems, in particular those of fine-tuned initial conditions and inflaton potentials. Besides, it is quite difficult to base standard inflation in the only known candidate theory of quantum gravity, superstring theory. Rather, as we shall argue, superstring theory gives strong hints in favour of the second (pre-big bang) possibility through two of its very basic properties, the first in relation to its short-distance behaviour, the second from its modifications of General Relativity even at large distance. Let us briefly comment on both.

2 (Super)String inspiration

2.1 Short Distance

Since the classical (Nambu–Goto) action of a string is proportional to the area A of the surface it sweeps, its quantization must introduce a quantum of length λ_s through:

$$S/\hbar = A/\lambda_s^2 \, . \tag{1}$$

This fundamental length, replacing Planck's constant in quantum string theory [2], plays the role of a minimal observable length, of an ultraviolet cut-off. Thus, in string theory, physical quantities are expected to be bound by appropriate powers of λ_s, e.g.

$$\begin{aligned} H^2 \sim R \sim G\rho < \lambda_s^{-2} \\ k_B T/\hbar < c\lambda_s^{-1} \\ R_{comp} > \lambda_s \ . \end{aligned} \tag{2}$$

In other words, in quantum string theory (QST), relativistic quantum mechanics should solve the singularity problems in much the same way as non-relativistic quantum mechanics solved the singularity problem of the hydrogen atom by putting the electron and the proton a finite distance apart. By the same token, QST gives us a rationale for asking daring questions such as: What was there before the big bang? Certainly, in no other present theory, can such a question be meaningfully asked.

2.2 Large Distance

Even at large distance (low-energy, small curvatures), superstring theory does not automatically give Einstein's General Relativity. Rather, it leads to a scalar-tensor theory of the JBD variety. The new scalar particle/field ϕ, the so-called dilaton, is unavoidable in string theory, and gets reinterpreted as the radius of a new dimension of space in so-called M-theory [3]. By supersymmetry, the dilaton is massless to all orders in perturbation theory, i.e. as long as supersymmetry remains unbroken. This raises the question: Is the dilaton a problem or an opportunity? My answer is that it is possibly both; and while we can try to avoid its potential dangers, we may try to use some of its properties to our advantage ... Let me discuss how.

In string theory ϕ controls the strength of all forces, gravitational and gauge alike. One finds, typically:

$$l_P^2/\lambda_s^2 \sim \alpha_{gauge} \sim e^{\phi} \ , \tag{3}$$

showing the basic unification of all forces in string theory and the fact that, in our conventions, the weak-coupling region coincides with $\phi \ll -1$. In order not to contradict precision tests of the Equivalence Principle and of the constancy of the gauge and gravitational couplings in the recent past (possibly meaning several million years!) we require [4] the dilaton to have a mass and to be frozen at the bottom of its own potential *today*. This does not exclude, however, the possibility of the dilaton having evolved cosmologically (after all the metric did!) within the weak coupling region where it was practically massless. The amazing (yet simple) observation [5] is that, by so doing, the dilaton may have inflated the Universe!

A simplified argument, which, although not completely accurate, captures the essential physical point, consists in writing the ($k = 0$) Friedmann equation:

$$3H^2 = 8\pi G\rho \ , \tag{4}$$

and in noticing that a growing dilaton (meaning through (3) a growing G) can drive the growth of H even if the energy density of standard matter decreases in an expanding Universe. This

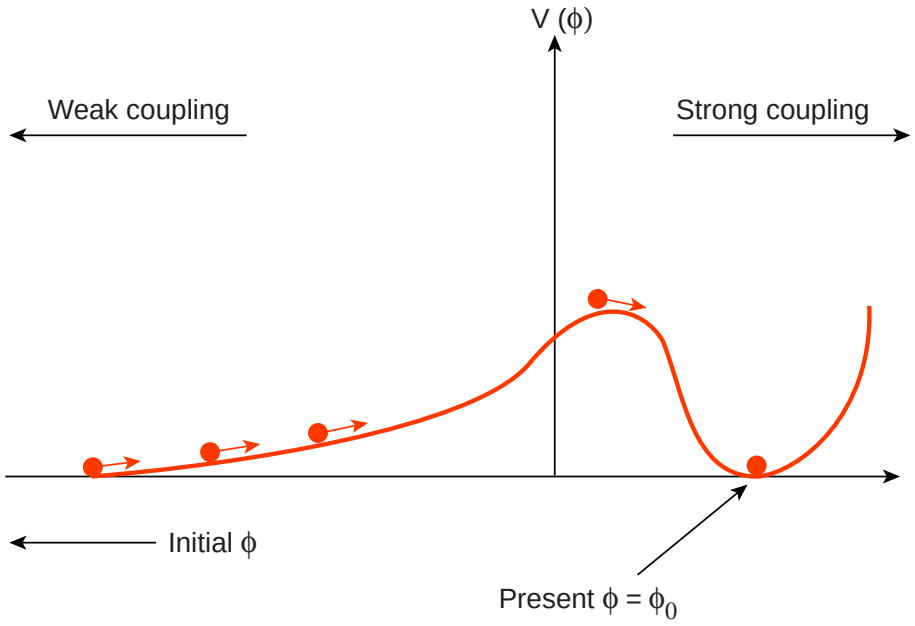

Figure 1:

new kind of inflation (characterized by growing H and ϕ) has been termed dilaton-driven inflation (DDI). The basic idea of pre-big bang cosmology [5, 6, 7] is thus illustrated in Fig. 1: the dilaton started at very large negative values (where it is massless), ran over a potential hill, and finally reached, sometime in our recent past, its final destination at the bottom of its potential ($\phi = \phi_0$). Incidentally, as shown in Fig. 1, the dilaton of string theory can easily roll-up —rather than down— potential hills, as a consequence of its non-standard coupling to gravity.

DDI is not just possible. It exists as a class of (lowest-order) cosmological solutions thanks to the duality symmetries of string cosmology. Under a prototype example of these symmetries, the so-called scale-factor duality [5], a FRW cosmology evolving (at lowest order in derivatives) from a singularity in the past is mapped into a DDI cosmology going towards a singularity in the future. Of course, the lowest order approximation breaks down before either singularity is reached. A (stringy) moment away from their respective singularities, these two branches can easily be joined smoothly to give a single non-singular cosmology, at least mathematically. Leaving aside this issue for the moment (see Section V for more discussion), let us go back to DDI. Since such a phase is characterized by growing coupling and curvature, it must itself have originated from a regime in which both quantities were very small. We take this as the main lesson/hint to be learned from low-energy string theory by raising it to the level of a new cosmological principle [8] of "Asymptotic Past Triviality".

3 Asymptotic Past Triviality

The concept of Asymptotic Past Triviality (APT) is quite similar to that of "Asymptotic Flatness", familiar from General Relativity [9]. The main differences consist in making only assumptions concerning the asymptotic past (rather than future or space-like infinity) and in the additional presence of the dilaton. It seems physically (and philosophically) satisfactory to identify the beginning with simplicity (see e.g. entropy-related arguments concerning the arrow of time). What could be simpler than a trivial, empty and flat Universe? Nothing of course! The problem is that such a Universe, besides being uninteresting, is also non-generic. By contrast, asymptotically flat/trivial Universes are initially simple, yet generic in a precise mathematical sense. Their definition involves exactly the right number of arbitrary "integration constants" (here full functions of three variables) to describe a general solution (one with some general, qualitative features, though). This is why, by its very construction, this cosmology cannot be easily dismissed as being fine-tuned.

It is useful to represent the situation in a Carter–Penrose diagram, as in Fig. 2. Here past infinity consists of two pieces: time-like past infinity, which is shrunk to a point I_-, and past null-infinity, $\mathcal{I}_-$ represented by a line at 45 degrees. Note that this region of the diagram is "non-physical" in FRW cosmology, since it lies behind (i.e. before) the big bang singularity (also shown in the diagram). Instead, we shall be giving initial data infinitesimally close to I_- and $\mathcal{I}_-$, and ask whether they will evolve in such a way as to generate a physically interesting big bang-like state at some later time. Generating so much from so little looks a bit like a miracle. However, we will argue that it is precisely what should be expected, owing to well-known classical and quantum gravitational instabilities.

4 Inflation as a classical gravitational instability

The assumption of APT entitles us to treat the early history of the Universe through the classical field equations of the low-energy (because of the small curvature) tree-level (because of the weak coupling) effective action of string theory. For simplicity, we will illustrate here the simplest case of the gravi-dilaton system already compactified to four space-time dimensions. Other fields and extra dimensions will be mentioned below, when we discuss observable consequences. The (string frame) effective action then reads:

$$\Gamma_{eff} = \frac{1}{2\lambda_s^2} \int d^4x \sqrt{-g}\; e^{-\phi}\; (\mathcal{R} + \partial_\mu \phi \partial^\mu \phi)\; . \tag{5}$$

In this frame, the string-length parameter λ_s is a constant and the same is true of the curvature scale at which we have to supplement eq. (5) with corrections. Similarly, string masses, when measured with the string metric, are fixed, while test strings sweep geodesic surfaces with respect to that metric. For all these reasons, even if we will allow metric redefinitions in order to simplify our calculations, we shall eventually turn back to the string frame for the physical interpretation of the results. We stress, however, that, while our intuition is not frame independent, physically measurable quantities are.

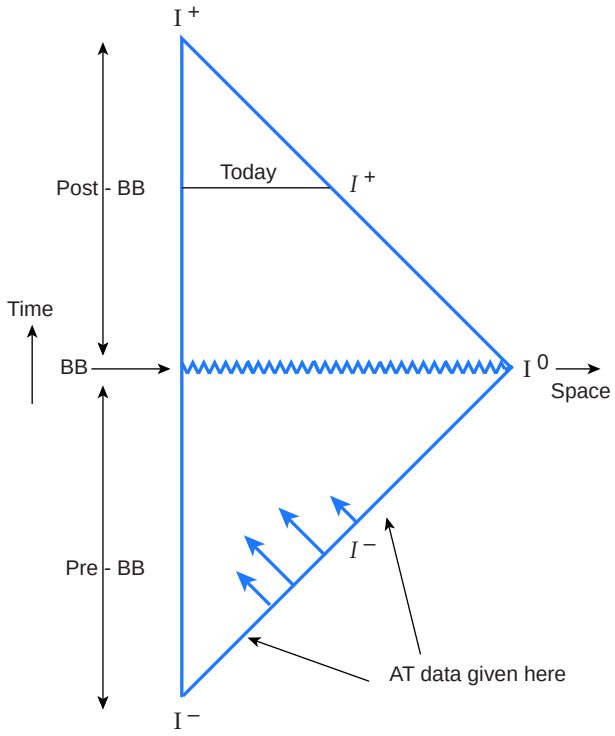

Figure 2:

Even assuming APT, the problem of determining the properties of a generic solution to the field equations implied by (5) is a formidable one. Very luckily, however, we are able to map our problem into one that has been much investigated, both analytically and numerically, in the literature. This is done by going to the so-called "Einstein frame". For our purposes, it simply amounts to the field redefinition

$$g_{\mu\nu} = g^{(E)}_{\mu\nu} e^{\phi - \phi_0} \ , \tag{6}$$

in terms of which (5) becomes:

$$\Gamma_{eff} = \frac{1}{2l_P^2} \int d^4x \sqrt{-g^{(E)}} \ \left(\mathcal{R}^{(E)} - \tfrac{1}{2} g^{\mu\nu}_{(E)} \partial_\mu \phi \partial_\nu \phi \right) \ , \tag{7}$$

where ϕ_0 ($l_P = \lambda_s e^{\phi_0/2}$) is the present value of the dilaton (of Planck's length).

Our problem is thus reduced to that of studying a massless scalar field minimally coupled to gravity. Such a system has been considered by many authors, in particular by Christodoulou [10], precisely in the regime of interest to us. In line with the APT postulate, in the analogue gravitational collapse problem, one assumes very "weak" initial data with the aim of finding under which conditions gravitational collapse later occurs. Gravitational collapse means that the (Einstein) metric (and the volume of 3-space) shrinks to zero at a space-like singularity. However, typically, the dilaton blows up at that same singularity. Given the relation (6) between the Einstein and the (physical) string metric, we can easily imagine that the latter blows up near the singularity as implied by DDI.

How generically does this happen? In this connection it is crucial to recall the singularity theorems of Hawking and Penrose [11], which state that, under some general assumptions, singularities are inescapable in GR. One can look at the validity of those assumptions in the case at hand and finds that all but one are automatically satisfied. The only condition to be imposed is the existence of a closed trapped surface (a closed surface from where future light cones lie entirely in the region inside the surface). Rigorous results [10] show that this condition cannot be waived: sufficiently weak initial data do not lead to closed trapped surfaces, to collapse, or to singularities. Sufficiently strong initial data do. But where is the border-line? This is not known in general, but precise criteria do exist for particularly symmetric space-times, e.g. for those endowed with spherical symmetry. However, no matter what the general collapse/singularity criterion will eventually turn out to be, we do know that:

- it cannot depend on an over-all additive constant in ϕ;
- it cannot depend on an over-all multiplicative factor in $g_{\mu\nu}$.

This is a simple consequence of the invariance (up to an over-all factor) of the effective action (7) under shifts of the dilaton and rescaling of the metric (these properties depend crucially on the validity of the tree-level low-energy approximation and on the absence of a cosmological constant).

We conclude that, generically, some regions of space will undergo gravitational collapse, will form horizons and singularities therein, but nothing, at the level of our approximations, will be able to fix either the size of the horizon or the value of ϕ at the onset of collapse. When this is translated into the string frame, one is describing, in the region of space-time within the horizon, a period of DDI in which both the initial value of the Hubble parameter and that of ϕ are left arbitrary. These two initial parameters are very important, since they determine the range of validity of our description. In fact, since both curvature and coupling increase during DDI, at some point the low-energy and/or tree-level description is bound to break down. The smaller the initial Hubble parameter (i.e. the larger the initial horizon size) and the smaller the initial coupling, the longer we can follow DDI through the effective action equations and the larger the number of reliable e-folds that we shall gain.

This does answer, in my opinion, the objections raised recently [12] to the PBB scenario according to which it is fine-tuned. The situation here actually resembles that of chaotic inflation [13]. Given some generic (though APT) initial data, we should ask which is the distribution of sizes of the collapsing regions and of couplings therein. Then, only the "tails" of these distributions, i.e. those corresponding to sufficiently large, and sufficiently weakly coupled regions, will produce Universes like ours, the rest will not. The question of how likely a "good" big bang is to take place is not very well posed and can be greatly affected by anthropic considerations.

In conclusion, we may summarize recent progress on the problem of initial conditions by saying that [8]: **Dilaton-driven inflation in string cosmology is as generic as gravitational collapse in General Relativity.** At the same time, having a sufficiently long period of DDI amounts to setting upper limits on two arbitrary moduli of the classical solutions.

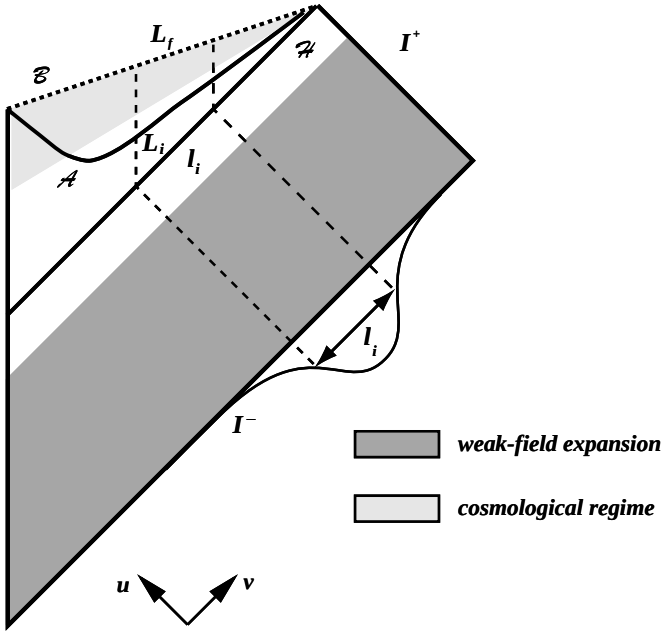

Figure 3:

Our scenario is illustrated in Figs. 3 and 4, both taken from Ref.[8]. In Fig. 3, I show, for the spherically symmetric case, a Carter–Penrose diagram in which generic (but asymptotically trivial) dilatonic waves are given around time-like (I^-) and null ($\mathcal{I}^-$) past-infinity. In the shaded region near $I^-, \mathcal{I}^-$, a weak-field solution holds. However, if a collapse criterion is met, an apparent horizon, inside which a cosmological (generally inhomogeneous) PBB-like solution takes over, forms at some later time. The future singularity of the PBB solution at $t = 0$ is identified with the space-like singularity of the black hole at $r = 0$ (remember that r is a time-like coordinate inside the horizon). Figure 4 gives a $(2 + 1)$-dimensional sketch of a possible PBB Universe: an original "sea" of dilatonic and gravity waves leads to collapsing regions of different initial size, possibly to a scale-invariant distribution of them. Each one of these collapses is reinterpreted, in the string frame, as the process by which a baby Universe is born after a period of PBB inflationary "pregnancy", with the size of each baby Universe determined by the duration of its pregnancy, i.e. by the initial size of the corresponding collapsing region. Regions initially larger than 10^{-13} cm can generate Universes like ours, smaller ones cannot.

A basic difference between the large numbers needed in (non-inflationary) FRW cosmology and the large numbers needed in PBB cosmology should be stressed at this point. In the former, the ratio of two classical scales, e.g. of total curvature to its spatial component, which is expected to be $O(1)$, has to be taken as large as 10^{60}. In the latter, the above ratio is initially $O(1)$ in the collapsing/inflating region, and ends up being very large in that region thanks to DDI. However, the common order of magnitude of these two classical quantities is a free parameter, and is taken to be much larger than a classically irrelevant quantum scale.

We can visualize analogies and differences between standard and pre-big bang inflation by comparing Figs. 5a and 5b. In these, we sketch the evolution of the Hubble radius and

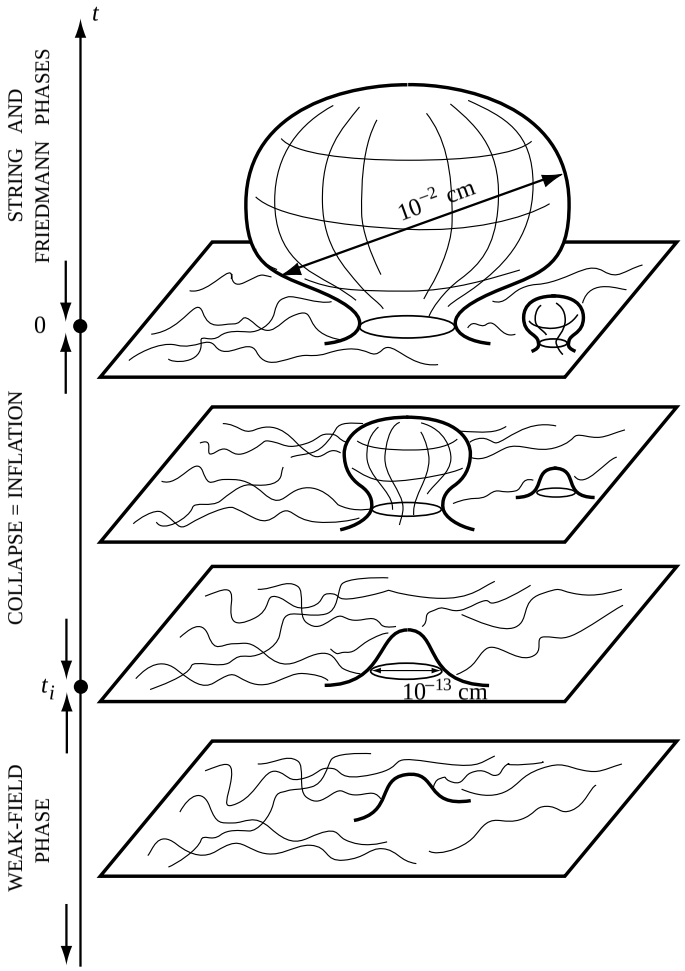
t
STRING AND FRIEDMANN PHASES
0
COLLAPSE = INFLATION
t_i
WEAK-FIELD PHASE
10^{-2} cm
10^{-13} cm

Figure 4:

of a fixed comoving scale (here the one corresponding to the part of the Universe presently observable to us) as a function of time in the two scenarios. The common feature is that the fixed comoving scale was "inside the horizon" for some time during inflation, and possibly very deeply inside at its onset. Also, in both cases, the Hubble radius at the beginning of inflation had to be large in Planck units and the scale of homogeneity had to be at least as large. The difference between the two scenarios is just in the behaviour of the Hubble radius during inflation: increasing in standard inflation (a), decreasing in string cosmology (b). This is what makes PBB's "wine glass" more elegant, and stable! Thus, while standard inflation is still facing the initial-singularity question and needs a non-adiabatic phenomenon to reheat the Universe (a kind of small bang), PBB cosmology faces the singularity problem later, combining it to the exit and heating problems (discussed in Sections V and VIB, respectively).

In the end, what saves PBB cosmology from fine-tuning is (not surprisingly!) supersymmetry. This is what protects us from the appearance of a cosmological constant in the weak-coupling regime. Even a relatively small cosmological constant would invalidate our scale-invariance arguments and force DDI to be very short [5]. Thus, amusingly, while an effective cosmological constant is at the basis of standard (post-big bang) inflation, its absence in the weak coupling region is at the basis of PBB inflation. This may allow us to speculate that the absence (or extreme smallness) of the present cosmological constant may be related to a mysterious degeneracy between the perturbative and the non-perturbative vacuum of superstring theory.

5 The exit problem/conjecture

We have argued that, generically, DDI, when studied at lowest order in derivatives and coupling, evolves towards a singularity of the big bang type. Similarly, at the same level of approximation, the non-inflationary solutions emerge from a singularity. Matching these two branches in a smooth, non-singular way has become known as the (graceful) exit problem in string cosmology [14]. It is, undoubtedly, the most important theoretical problem facing the whole PBB scenario.

There has been quite some progress recently on the exit problem. However, for lack of space, I shall refer the reader to the literature [14] for details. Generically speaking, toy examples have shown that DDI can flow, thanks to higher-curvature corrections, into a de-Sitter-like phase, i.e. into a phase of constant H (curvature) and constant $\dot{\phi}$. This phase is expected to last until loop corrections become important (see next section) and give rise to a transition to a radiation-dominated phase. If these toy models serve as an indication, the full exit problem can only be achieved at large coupling and curvature, a situation that should be described by the newly invented M-theory [3].

It was recently pointed out [15] that the reverse order of events is also possible. The coupling may become large *before* the curvature. In this case, at least for some time, the low-energy limit of M-theory should be adequate: this limit is known [3] to give $D = 11$ supergravity and is therefore amenable to reliable study. It is likely, though not yet clear, that, also in this case,

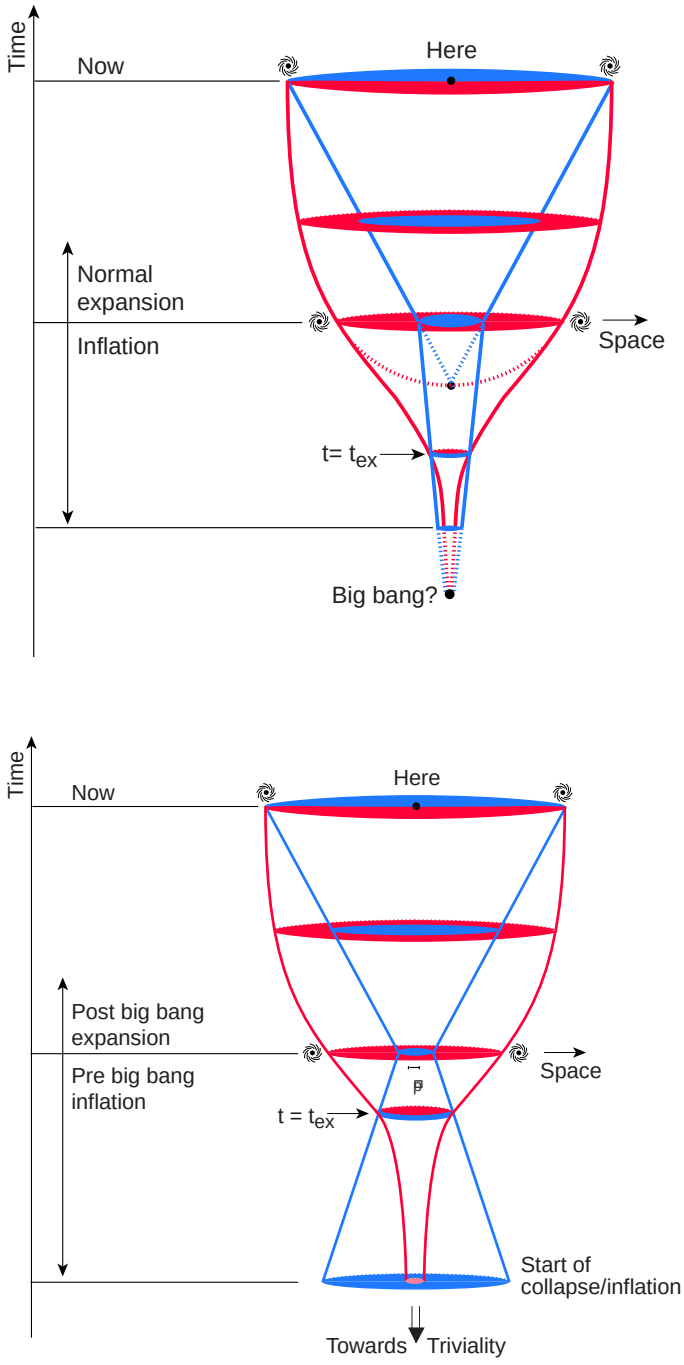
Time
Now
Here
Normal
expansion
Inflation
Space
t= t_ex
Big bang?
Time
Now
Here
Post big bang
expansion
Pre big bang
inflation
Space
t = t_ex
Start of
collapse/inflation
Towards
Triviality

Figure 5:

strong curvatures will have to be reached before the exit can be completed. In the following, we will assume that:

- the big bang singularity is avoided thanks to the softness of string theory;
- full exit to radiation occurs at strong coupling and curvature, according to a criterion given in Section VIB.

6 Observable relics and heating the pre-bang Universe

6.1 PBB relics

Since there are already several review papers on this subject (e.g. [16]), I will limit myself to mentioning the most recent developments, after recalling the basic physical mechanism underlying particle production in cosmology [17]. A cosmological (i.e. time-dependent) background coupled to a given type of (small) inhomogeneous perturbation Ψ enters the effective low-energy action in the form:

$$I = \frac{1}{2}\int d\eta\; d^3x\; S(\eta)\left[\Psi'^2 - (\nabla\Psi)^2\right]. \tag{8}$$

Here η is the conformal-time coordinate, and a prime denotes $\partial/\partial\eta$. The function $S(\eta)$ (sometimes called the "pump" field) is, for any given Ψ, a given function of the scale factor $a(\eta)$, and of other scalar fields (four-dimensional dilaton $\phi(\eta)$, moduli $b_i(\eta)$, etc.), which may appear non-trivially in the background.

While it is clear that a constant pump field S can be reabsorbed in a rescaling of Ψ, and is thus ineffective, a time-dependent S couples non-trivially to the fluctuation and leads to the production of pairs of quanta (with equal and opposite momenta). One can easily determine the pump fields for each one of the most interesting perturbations. The result is:

$$\begin{aligned} \text{Gravity waves, dilaton} &: S = a^2 e^{-\phi} \\ \text{Heterotic gauge bosons} &: S = e^{-\phi} \\ \text{Kalb} - \text{Ramond, axions} &: S = a^{-2} e^{-\phi}\,. \end{aligned} \tag{9}$$

A distinctive property of string cosmology is that the dilaton ϕ appears in some very specific way in the pump fields. The consequences of this are very interesting:

- For gravitational waves and dilatons, the effect of ϕ is to slow down the behaviour of a (remember that both a and ϕ grow in the pre-big bang phase). This is the reason why those spectra are quite steep [18] and give small contributions at large scales. Thus one of the most robust predictions of PBB cosmology is a small tensor component in

the CMB anisotropy[1]. The reverse is also true: at short scales, the expected yield in a stochastic background of gravitational waves is much larger than in standard inflationary cosmology. This is easily understood: in standard inflation the GW spectrum is either flat or slowly decreasing (as a function of frequency). Since COBE data [19] set a limit on the GW contribution at large scales, this bound holds a fortiori at shorter scales, as those of interest for direct GW detection. Thus, in standard inflation, one expects

$$\Omega_{GW} < 10^{-14} \, . \tag{10}$$

Since the GW spectra of PBB cosmology are "blue", the bound by COBE is automatically satisfied, with no implication on the GW yield at interesting frequencies. Values of Ω_{GW} in the range of 10^{-6}–10^{-7} are possible in some regions of parameter space, which, according to some estimates of sensitivities [20], could be inside detection capabilities in the near future.

- For gauge bosons there is no amplification of vacuum fluctuations in standard cosmology, since a conformally flat metric (of the type forced upon by inflation) decouples from the electromagnetic (EM) field precisely in $D = 3 + 1$ dimensions. As a very general remark, apart from pathological solutions, the only background field which, through its cosmological variation, can amplify EM (more generally gauge-field) quantum fluctuations is the effective gauge coupling itself [21]. By its very nature, in the pre-big bang scenario the effective gauge coupling inflates together with space during the PBB phase. It is thus automatic that any efficient PBB inflation brings together a huge variation of the effective gauge coupling and thus a very large amplification of the primordial EM fluctuations [22, 23, 24]. This can possibly provide the long-sought origin for the primordial seeds of the observed galactic magnetic fields. Notice, however, that, unlike GW, EM perturbations interact quite considerably with the hot plasma of the early (post-big bang) Universe. Thus, converting the primordial seeds into those that may have existed at the proto-galaxy formation epoch is by no means a trivial exercise. Work is in progress to try to adapt existing codes [25] to the evolution of our primordial seeds.

- Finally, for Kalb–Ramond fields and axions, a and ϕ work in the same direction and spectra can be large even at large scales [26]. An interesting fact is that, unlike the GW spectrum, that of axions is very sensitive to the cosmological behaviour of internal dimensions during the DDI epoch. On one side, this makes the model less predictive. On the other, it tells us that axions represent a window over the multidimensional cosmology expected generically from string theories, which must live in more that four dimensions. Curiously enough, the axion spectrum becomes exactly HZ (i.e. scale-invariant) when all the nine spatial dimensions of superstring theory evolve in a rather symmetric way [23]. In situations near this particularly symmetric one, axions are able to provide a new mechanism for generating large-scale CMB anisotropy and LSS.

[1]This, however, refers just to first-order tensor perturbations; the mechanism —described below— of seeding CMB anisotropy through axions would also give a tensor (and a vector) contribution whose relative magnitude is being computed.

A recent calculation [27] of the effect gives, for massless axions,

$$l(l+1)C_l \sim O(1)\left(\frac{H_{max}}{M_P}\right)^4 (\eta_0 k_{max})^{-2\alpha}\frac{\Gamma(l+\alpha)}{\Gamma(l-\alpha)}\ , \tag{11}$$

where C_l are the usual coefficients of the multipole expansion of $\Delta T/T$

$$\langle \Delta T/T(\vec{n})\ \ \Delta T/T(\vec{n}')\rangle \ = \ \sum_l (2l+1)C_l P_l(\cos\theta)\ , \tag{12}$$

and the parameters H_{max}, k_{max}, α are defined by the primordial axion energy spectrum in critical units as:

$$\Omega_{ax}(k) = \left(\frac{H_{max}}{M_P}\right)^2 (k/k_{max})^\alpha\ . \tag{13}$$

In string theory, as repeatedly mentioned, we expect $H_{max}/M_P \sim M_s/M_P \sim 1/10$ and $\eta_0 k_{max} \sim 10^{30}$, while the exponent α depends on the explicit PBB background with the above-mentioned HZ case corresponding to $\alpha = 0$. The standard tilt parameter $n = n_s$ (s for scalar) is given by $n = 1 + 2\alpha$ and is found, by COBE, to lie between 0.9 and 1.5, corresponding to $0 < \alpha < 0.25$ (a negative α leads to some theoretical problems). With these inputs we can see that the correct normalization ($C_2 \sim 10^{-10}$) is reached for $\alpha \sim 0.2$, which is just in the middle of the allowed range. In other words, unlike in standard inflation, we cannot predict the tilt, but when this is given, we can predict (again unlike in standard inflation) the normalization.

Our model, being of the isocurvature type, bears some resemblance to the one recently advocated by Peebles [28] and, like his, is expected to contain some calculable amount of non-Gaussianity, which is being calculated and will be checked by the future satellite measurements (MAP, PLANCK).

- Many other perturbations, which arise in generic compactifications of superstrings, have also been studied, and lead to interesting spectra. For lack of time, I will refer to the existing literature [23, 24].

6.2 Heat and entropy as a quantum gravitational instability

Before closing this section, I wish to recall how one sees the very origin of the hot big bang in this scenario. One can easily estimate the total energy stored in the quantum fluctuations, which were amplified by the pre-big bang backgrounds. The result is, roughly,

$$\rho_{quantum} \sim N_{eff}\ H_{max}^4\ , \tag{14}$$

where N_{eff} is the effective number of species that are amplified and H_{max} is the maximal curvature scale reached around $t = 0$. We have already argued that $H_{max} \sim M_s = \lambda_s^{-1}$, and we know that, in heterotic string theory, N_{eff} is in the hundreds. Yet this rather huge energy density is very far from critical, as long as the dilaton is still in the weak-coupling region,

justifying our neglect of back-reaction effects. It is very tempting to assume [23] that, precisely when the dilaton reaches a value such that $\rho_{quantum}$ is critical, the Universe will enter the radiation-dominated phase. This PBBB (PBB bootstrap) constraint gives, typically:

$$e^{\phi_{exit}} \sim 1/N_{eff} \ , \tag{15}$$

i.e. a value for the dilaton close to its present value.

The entropy in these quantum fluctuations can also be estimated following some general results [29]. The result for the density of entropy S is, as expected

$$S \sim N_{eff} H_{max}^3 \ . \tag{16}$$

It is easy to check that, at the assumed time of exit given by (15), this entropy saturates a recently proposed holography bound [30]. This also turns out to be a physically acceptable value for the entropy of the Universe just after the big bang: a large entropy on the one hand (about 10^{90}); a small entropy for the total mass and size of the observable Universe on the other, as often pointed out by Penrose [31]. Thus, PBB cosmology neatly explains why the Universe, at the big bang, looks so fine-tuned (without being so) and provides a natural arrow of time in the direction of higher entropy.

7 Conclusions

- Pre-big bang (PBB) cosmology is a "top–down" rather than a "bottom–up" approach to cosmology. This should not be forgotten when testing its predictions.
- It does not need to invent an inflaton, or to fine-tune its potential; inflation is "natural" thanks to the duality symmetries of string cosmology.
- It makes use of a classical gravitational instability to inflate the Universe, and of a quantum instability to warm it up.
- The problem of initial conditions "decouples" from the singularity problem; it is classical, scale-free, and unambiguously defined. Issues of fine tuning can be addressed and, I believe, answered.
- The spectrum of large-scale perturbations has become more promising through the invisible axion of string theory, while the possibility of explaining the seeds of galactic magnetic fields remains a unique prediction of the model.
- The main conceptual (technical?) problem remains that of providing a fully convincing mechanism for (and a detailed description of) the pre-to-post-big bang transition. It is very likely that such a mechanism will involve both high curvatures and large coupling and should therefore be discussed in the (yet to be fully constructed) M-theory [3]. New ideas borrowed from such theory and from D-branes [32, 15] could help in this respect.

- Once/if this problem will be solved, predictions will become more precise and robust, but, even now, with some mild assumptions, several tests are (or will soon become) possible, e.g.
  - the tensor contribution to $\Delta T/T$ should be very small (see, however, footnote Section VI);
  - some non-Gaussianity in $\Delta T/T$ correlations is expected, and calculable.
  - the axion-seed mechanism should lead to a characteristic acoustic-peak structure, which is being calculated;
  - it should be possible to convert the predicted seed magnetic fields into observables by using some reliable code for their late evolution;
  - a characteristic spectrum of stochastic gravitational waves is expected to surround us and could be large enough to be measurable within a decade or so.

References

[1] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990); A.D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, New York, 1990).

[2] G. Veneziano, Europhys. Lett. **2** (1986) 133; *The Challenging Questions*, Erice, 1989, ed. A. Zichichi (Plenum Press, New York, 1990), p. 199.

[3] See, e.g., E. Witten, Nucl. Phys. **B443** (1995) 85;
P. Horawa and E. Witten, Nucl. Phys. **B460** (1996) 506.

[4] T.R. Taylor and G. Veneziano, Phys. Lett. **B213** (1988) 459.

[5] G. Veneziano, Phys. Lett. **B265** (1991) 287.

[6] M. Gasperini and G. Veneziano, Astropart. Phys. **1** (1993) 317, Mod. Phys. Lett. **A8** (1993) 3701, Phys. Rev. **D50** (1994) 2519.

[7] An updated collection of papers on the PBB scenario is available at http://www.to.infn.it/~gasperin/.

[8] A. Buonanno, T. Damour and G. Veneziano, *Pre-big bang bubbles from the gravitational instability of generic string vacua*, hep-th/9806230; see also, G. Veneziano, Phys. Lett. **B406** (1997) 297; A. Buonanno, K.A. Meissner, C. Ungarelli and G. Veneziano, Phys. Rev. **D57** (1998) 2543, and references therein.

[9] R. Penrose, *Structure of space-time*, in *Battelle Rencontres*, ed. C. Dewitt and J.A. Wheeler, Benjamin, New York, 1968.

[10] D. Christodoulou, Commun. Pure Appl. Math. **56** (1993) 1131, and references therein.

[11] R. Penrose, Phys. Rev. Lett. **14** (1965) 57; S. W. Hawking and R. Penrose, Proc. Roy. Soc. Lond. **A314** (1970) 529.

[12] M. Turner and E. Weinberg, Phys. Rev. **D56** (1997) 4604; N. Kaloper, A. Linde and R. Bousso, *Pre-big bang requires the Universe to be exponentially large from the very beginning*, hep-th/9801073.

[13] A. Linde, Phys. Lett. **129B** (1983) 177.

[14] R. Brustein and G. Veneziano, Phys. Lett. **B329** (1994) 429; N. Kaloper, R. Madden and K.A. Olive, Nucl. Phys. **B452** (1995) 677, Phys. Lett. **B371** (1996) 34; R. Easther, K. Maeda and D. Wands, Phys. Rev. **D53** (1996) 4247; M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. **B494** (1997) 315; R. Brustein and R. Madden, Phys. Lett. **B410** (1997) 110, Phys. Rev. **D57** (1998) 712.

[15] M. Maggiore and A. Riotto, *D-branes and Cosmology*, hep-th/9811089;
see also T. Banks, W. Fishler and L. Motl, *Duality versus Singularities*, hep-th/9811194.

[16] G. Veneziano, in *String Gravity and Physics at the Planck Energy Scale*, Erice, 1995, eds. N. Sanchez and A. Zichichi (Kluver Academic Publishers, Boston, 1996), p. 285; M. Gasperini, ibid., p. 305.

[17] See, e.g., V. F. Mukhanov, A. H. Feldman and R. H. Brandenberger, Phys. Rep. **215** (1992) 203.

[18] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. **B361** (1995) 45; R. Brustein et al., Phys. Rev. **D51** (1995) 6744.

[19] G. F. Smoot et al., Ap. J. **396** (1992) L1; C. L. Bennet et al., Ap. J. **430** (1994) 423.

[20] P. Astone et al., Phys. Lett. **B385** (1996) 421; B. Allen and R. Brustein, Phys. Rev. **D55** (1997) 970.

[21] B. Ratra, Astrophys. J. Lett. **391** (1992) L1.

[22] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75** (1995) 3796;
D. Lemoine and M. Lemoine, Phys. Rev. **D52** (1995) 1955.

[23] A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, JHEP **1** (1998) 4;

[24] R. Brustein and M. Hadad, Phys. Rev. **D57** (1998) 725.

[25] R. M. Kulsrud, R. Cen, J. P. Ostriker and D. Ryu, Ap. J. **480** (1997) 481.

[26] E.J. Copeland, R. Easther and D. Wands, Phys. Rev. D56 (1997) 874; E.J. Copeland, J.E. Lidsey and D. Wands, Nucl. Phys. **B506** (1997) 407.

[27] R. Durrer, M. Gasperini, M. Sakellariadou and G. Veneziano, Phys. Lett. **B436** (1998) 66, Phys. Rev. D59 (1999) 043511; M. Gasperini and G. Veneziano, Phys. Rev. D59 (1999) 043503.

[28] P. J. E. Peebles, *An isocurvature CDM cosmogony. I and II*, astro-ph/9805194, and astro-ph/9805212.

[29] M. Gasperini and M. Giovannini, Phys. Lett. **B301** (1993) 334; Class. Quant. Grav. **10** (1993) L133; R. Brandenberger, V. Mukhanov and T. Prokopec, Phys. Rev. Lett. **69** (1992) 3606; Phys. Rev. **D48** (1993) 2443.

[30] W. Fischler and L. Susskind, *Holography and Cosmology*, hep-th/9806039; see also D. Bak and S.-J. Rey, *Holographic principle and string cosmology*, hep-th/9811008; A. K. Biswas, J. Maharana and R.K. Pradhan, *The holography principle and pre-big bang cosmology*, hep-th/9811051.

[31] see, e.g., R. Penrose, *The Emperor's new mind*, (Oxford University Press, New York, 1989), Chapter 7.

[32] A. Lukas, B.A. Ovrut and D. Waldram, Phys. Lett. **B393** (1997) 65; Nucl. Phys. **B495** (1997) 365; F. Larsen and F. Wilczek, Phys. Rev. **D55** (1997) 4591; N. Kaloper, Phys. Rev. **D55** (1997) 3394; H. Lu, S. Mukherji and C.N. Pope, Phys. Rev. **D55** (1997) 7926; R. Poppe and S. Schwager, Phys. Lett. **B393** (1997) 51; A. Lukas and B. A. Ovrut, Phys. Lett. **B437** (1998) 291; N. Kaloper, I. Kogan and K. A. Olive, Phys. Rev. **D57** (1998) 7340.

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A SIMPLE/SHORT INTRODUCTION TO PRE-BIG-BANG PHYSICS/COSMOLOGY

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ABSTRACT

A simple, non-technical introduction to the pre-big bang scenario is given, emphasizing physical motivations, considerations, and consequences over formalism.

1. Introduction

It is commonly believed (see e.g.[1]) that the Universe – and time itself – started some 15 billion years ago from some kind of primordial explosion, the famous Big Bang. Indeed, the experimental observations of the red-shift and of the Cosmic Microwave Background (CMB) lead us quite unequivocally to the conclusion that, as we trace back our history, we encounter epochs of increasingly high temperature, energy density, and curvature. However, as we arrive close to the singularity, our classical equations are known to break down. The earliest time we can think about classically is certainly larger than the so-called Planck time, $t_P = \sqrt{G_N \hbar} \sim 10^{-43}$s ($c = 1$ throughout). Hence, the honest answer to the question: Did the Universe and time have a beginning? is: We do not know, since the answer lies in the unexplored domain of quantum gravity.

Besides the initial singularity problem – and in spite of its successes – the hot big bang model also has considerable phenomenological problems. Amusingly, these too can be traced back to the nature of the very early state of the Universe. Let us briefly recall why.

General Relativity, together with the cosmological principle (i.e. the assumption of a homogeneous, isotropic Universe over large scales), allows us to describe the geometry of space-time through the Friedmann-Robertson-Walker (FRW) metric (see e.g.[2]):

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] , \; d\Omega^2 = d\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 , \qquad (1)$$

where, as usual, the discrete variable $k = 0, 1, -1$ distinguishes the cases of a flat, closed, or open Universe, respectively. In the presence of some matter (fluid), described by an energy density ρ and pressure density p, the evolution of the Universe

is controlled by the Einstein-Friedmann equations

$$\begin{aligned} H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \\ \dot{H} + H^2 = \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) \ , \end{aligned} \tag{2}$$

which, together, imply the energy conservation equation:

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + p) \ . \tag{3}$$

Given a model for the sources, or, more specifically, given a relation between ρ and p (so-called equation of state), one can easily solve these equations and find the usual FRW cosmological solutions. For the matter- and radiation-dominated cases, respectively,

$$\begin{aligned} p = 0 \ &\Rightarrow \ a(t) \sim t^{2/3}, \ \rho \sim a^{-3} \\ p = \rho/3 \ &\Rightarrow \ a(t) \sim t^{1/2}, \ \rho \sim a^{-4} \end{aligned} \tag{4}$$

where, for simplicity, the spatial-curvature term $\frac{k}{a^2}$ was neglected.

The so-called horizon problem arises as follows. The observable part of our Universe, our present horizon, is given by the distance that light has travelled since the big bang, or about 10^{28} cm. At earlier times, the horizon was much smaller. For a hypothetical observer looking at the sky a few Planck times after the Big Bang, the horizon was not much bigger that a few Planck lengths, say about 10^{-32} cm. Instead, as easily seen from (4), the portion of space that corresponds to our present horizon was, for that same hypothetical observer, some 30 orders of magnitude larger than the Planck length, or about 1 mm. In other words, at that time, what has nowadays become our observable Universe consisted of $(10^{30})^3 = 10^{90}$ Planckian-size, causally disconnected regions. There is no reason to expect that conditions in all those regions were initially the same, since there was never any thermal contact between them. Yet, today, all those 10^{90} regions make up our observable Universe and all appear to resemble one another (to within one part in 10^5).

Who prepared such a smooth and very unlikely initial state? Perhaps God, who picked up a very special point in the huge space of all possible initial configurations (see, in this connection, a nice picture in Roger Penrose's book[3]). If, instead, we do not accept God's fine-tuning, or, in more scientific terms, we do not want to attribute homogeneity to some unknown Planckian physics, two logical possibilities are left to our choice:

- Time did have its beginning at the big bang, when initial conditions were rather random, but a period of superluminal expansion (inflation) brought all those 10^{90} patches in causal contact sometimes between the big bang and the present time. This is the standard *post-big bang* inflation paradigm (see e.g.[4]).

- Time did *not* have its beginning with the big bang and some pre-big bang physics cooked up a "good" big bang from a more generic (less fine-tuned) initial state. This is the attitude one takes in the so-called *pre-big bang* scenario.

One can similarly argue that there are two ways of solving the second major problem of standard cosmology, the so-called flatness problem. Today, space is, to a very good approximation, Euclidean. If it does have any spatial curvature (represented by the $\frac{k}{a^2}$ term in the cosmological equations), this is of $O(H^2)$, i.e. extremely small. Given the solutions (4) it is easy to check that, in order to have such conditions today, one has to start, at the Planck time, with an extremely flat Universe, i.e. with a curvature radius a some 30 orders of magnitude larger that the characteristic length scale at the time, $H^{-1} \sim l_P$. Again, two possibilities come to mind: either the Universe was not particularly flat at the beginning – and subsequent inflation stretched out spatial curvature – or some pre-big bang physics prepared a nice spatially-flat big bang.

Conventional inflation again chooses the first alternative. To succeed, it needs a weakly-coupled scalar field, the so-called inflaton, which, some time after the big bang, finds itself *homogeneously* displaced from the minimum of its potential, and slowly rolls towards it. While doing so, if certain conditions are met, the effective equation of state is $p \sim -\rho$ and the Universe expands quasi-exponentially. One needs this period of exponential growth to last for a long enough time for all our accessible Universe to come in causal contact. This can be achieved at the price of fine-tuning certain masses or couplings.

My main reservations towards this solution (see[5] for a different criticism) are that no one has a convincing model for what the inflaton ought to be and, even more seriously, that it is not easy to justify the initial conditions that can provide a sufficiently long inflationary phase. One is back somehow to the starting point, since the conditions at the onset of inflation have to come from a previous phase, and this inevitably leads us to giving initial conditions in the mysterious Planckian era. Quantum cosmology has then been invoked; however, I am also sceptical about present quantum cosmology arguments[6] "predicting" inflation since they are based on the so-called minisuperspace truncation of the Wheeler–DeWitt equation. Such an approach is only justified for a fairly homogeneous initial universe, which is just what we do *not* wish to assume.

How come string theory prefers the second way out? In order to explain this I have to open a parenthesis and tell you about some striking properties of quantum (super)strings.

2. Three properties of (super)strings

I will concentrate on just three properties of strings which are crucial to the understanding of their possible cosmological implications:

- 1. There is a fundamental length scale in string theory, providing a characteristic size for strings and thus an ultraviolet cutoff. Thanks to this property, superstring theory can be taken seriously as a candidate finite quantum theory of gravity (and of the other interactions as well). The fundamental length scale λ_s is given in terms of the string tension T by the formula

$$\lambda_s = \sqrt{\hbar/T} \ . \tag{5}$$

  Actually λ_s, rather than T, is the fundamental parameter of the theory, providing a meaning for what short and large distances mean. When fields vary little over a string length λ_s one recovers a field-theoretic description given by a local Lagrangian with the smallest number of derivatives.

- 2. Couplings are not God-given constants; they are VEVs which, hopefully, become dynamically determined. In particular, a scalar field, the so-called dilaton ϕ, controls all sorts of couplings, gravitational and gauge alike. Since, in our normalizations,

$$T \cdot G_N = l_P^2/\lambda_s^2 \sim \alpha_{gauge} \sim e^{\phi} \ , \tag{6}$$

  we see that the weak coupling region is $\phi \ll -1$. By contrast, at present, $e^{\phi} \sim 1/25$, implying $\lambda_s \sim 10 l_P \sim 10^{-32}$ cm. In the weak coupling region, perturbative superstring theory is an adequate description of physics, implying that the dilaton itself behaves like a massless particle. As such, the dilaton can easily evolve cosmologically while it is deeply inside the perturbative region. Precision tests of the equivalence principle imply, however, that the dilaton has a mass, i.e. that near its present value $\phi = O(-1)$, its potential has a minimum with finite curvature.

  If both the coupling and derivatives are small, physics is adequately described by the tree-level low-energy effective action of string theory, which reads:

$$\begin{aligned} \Gamma_{eff} &= \frac{1}{2}\int d^4x\sqrt{-g}\ e^{-\phi}\left[\lambda_s^{-2}(\mathcal{R} + \partial_\mu\phi\partial^\mu\phi + H^2_{\mu\nu\rho}) + F^2_{\mu\nu} + \text{higher derivatives}\right] \\ &+ \left[\text{higher orders in } e^{\phi}\right] \ , \end{aligned} \tag{7}$$

  where we have included the contributions of the Kalb-Ramond antisymmetric tensor field through its field strength $H_{\mu\nu\rho}$. Note the two kinds of corrections alluded to in (7). They intervene, respectively, whenever space-time derivatives (i.e. energies) or the string coupling e^{ϕ} become appreciable. Equation (7) will be our starting point to describe pre-big bang cosmology.

- 3. Cosmological field equations exhibit new stringy symmetries whose most interesting representative (scale-factor duality or SFD) acts as follows:

$$a(t) \to a^{-1}(-t) \ \ , \ \ \phi(t) \to \phi(-t) - 2d\ \log a(-t) \ \ , d = 3 \ . \tag{8}$$

  The interest of this duality transformation lies in the fact that it maps ordinary FRW cosmologies with $H > 0$, $\dot{H} < 0$, and $\dot{\phi} = 0$ at $t > 0$ into inflationary

cosmologies with $H > 0$, $\dot{H} > 0$, and $\dot{\phi} > 0$ at $t < 0$. Actually, the dual cosmologies are of the so-called super-inflation (or pole-inflation) type, i.e. have a growing – rather than a constant – Hubble parameter. Since many of the distinctive consequences of PBB cosmology originate from this peculiar feature, let me explain in simplified terms where it comes from Eqs. (2) imply that, for an expanding Universe, ρ and H^2, being proportional (take for simplicity the case of $k = 0$), decrease together in time. This is true if G_N is constant. In string theory, where G_N is controlled by the dilaton through Eq. (6), it is perfectly possible to have a growing H while ρ decreases, provided ϕ is also growing.

The suggestion from string theory now becomes almost a compelling one: Can one put together a standard cosmology at $t > 0$ and a dual cosmology at $t < 0$ to generate a single scenario containing dilaton-driven pre-big bang inflation and FRW post-big bang behaviour? Since, in such a construct, the Hubble parameter grows for $t < 0$ and decreases for $t > 0$, it should reach its maximum at $t = 0$, instant therefore identified with the occurrence of the Big Bang of standard cosmology. The problem is that this maximum is actually infinite if one works in the context of the low-energy effective theory, i.e. if all corrections in (7) are neglected. The pre- (post-) big bang phases have singularities in the future(past), probably a consequence of the validity, in that approximation, of the assumption leading to the Hawking-Penrose singularity theorems.[7] However, that approximation breaks down as soon as the Hubble parameter reaches values $O(\lambda_s^{-1})$ leading us to expect that the maximal value of H, reached at $t = 0$, should be actually of the order of the fundamental length of string theory.

It is easy, actually, to write down mathematical expressions that interpolate smoothly between the inflationary and the FRW branches. For instance, the ansatz

$$\begin{aligned} \hat{a}(t) &= \left(\frac{t + \sqrt{t^2 + \lambda_s^2}}{\lambda_s}\right)^{1/2} \\ \hat{\phi}(t) &= \phi_0 + \frac{3}{2}\,\log\left(1 + \frac{t}{\sqrt{t^2 + \lambda_s^2}}\right) \end{aligned} \tag{9}$$

is easily checked to approach a standard radiation-dominated cosmology with constant dilaton at $t \gg \lambda_s$ and to a dual dilaton-driven inflationary cosmology at $t \ll -\lambda_s$. The question is: Does anything like this come from the true (i.e. high-curvature and/or loop-corrected) field equations? Leaving this hard question to the final section, we turn instead to the formulation of the basic pre-big bang postulate.

3. The pre-big bang postulate

Clearly, if we want to use a dual cosmology for the prehistory of the Universe, given the positive signs of $\dot{H}$ and $\dot{\phi}$, we have to start from (very?) small initial values

for H and e^{ϕ}. Although not strictly necessary, we will also impose, for the sake of simplicity, an almost empty initial Universe. This leads us to the following basic postulate of PBB cosmology:

The Universe started its evolution from the most simple initial state conceivable in string theory, its perturbative vacuum. This corresponds to an (almost)

EMPTY, COLD, FLAT, FREE

Universe as opposed to the standard

DENSE, HOT, HIGHLY-CURVED

initial state of conventional cosmology.

For this assumption to make sense I will have to argue that the new initial conditions are able to provide, at later times, a hot big bang with the desired characteristics thanks to a long pre-big bang inflationary phase. This looks a priori a very hard task, but I will explain below how it can possibly happen. Before discussing this let me illustrate, in two figures, the qualitative differences between the standard (non-inflationary) model, the standard inflationary scenario, and ours.

In Fig. 1 I am plotting, against cosmic time, the behaviour of the Hubble parameter H measured in Planck units $M_P \sim 10^{19}$ GeV. Note that, while in the first model H is a concave function of time, in the second it has a long flat plateau where $H/M_P \leq 10^{-5}$ (this constraint comes from COBE's data, see e.g.[4]). Finally, in the proposed scenario, H/M_P grows all the way to a maximal value $O(10^{-1})$, but quickly becomes very small at both large positive and large negative t.

In Fig. 2 we can see the consequences of this behaviour of H on the kinematics of horizon crossing. During inflation, increasingly small scales are pushed out of the horizon by the accelerated expansion of the Universe. However, while in standard inflation (Fig. 2a) larger scales cross the horizon at slightly larger values of H, in the pre-big bang scenario (Fig. 2b) it is the other way around: the larger the scale, the smaller the value of H at horizon crossing. As we shall see in Sec. 4, the value of H/M_P at horizon crossing is the determining quantity for evaluating the present magnitude of quantum fluctuations at different length scales. Hence the above kinematics of horizon crossing will have an important bearing on the spectrum of quantum fluctuations.

Before turning to that, we should discuss how dilaton-driven inflation sets in during the pre-big bang phase.

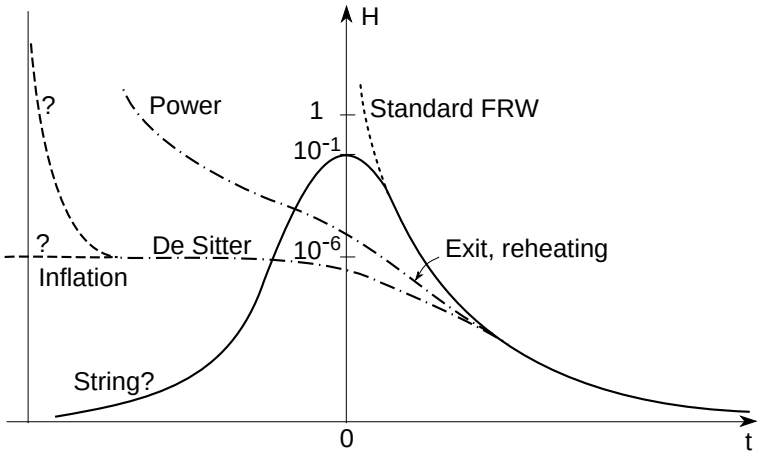

Figure 1

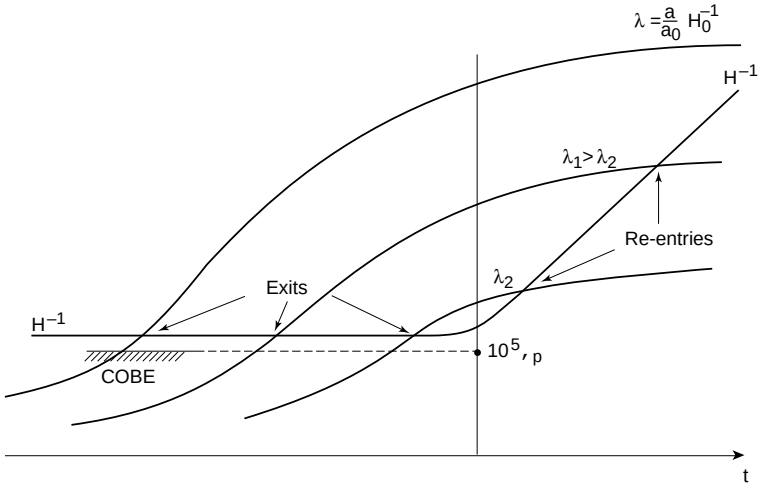

Figure 2a

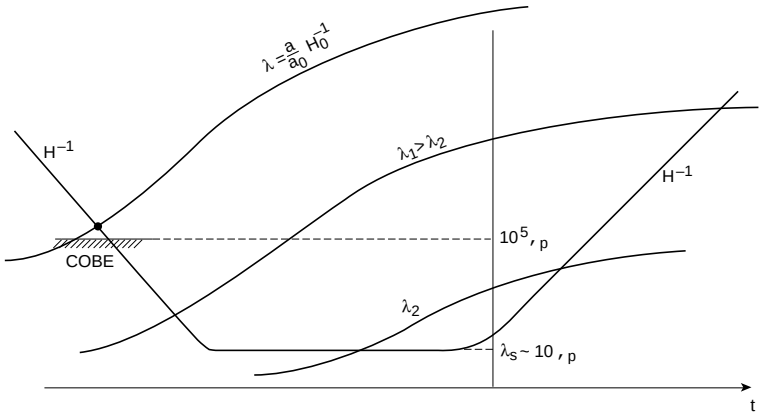

Figure 2b

4. Pre-big bang inflation as a classical instability

The key to understanding how our apparently innocent initial conditions can give inflation is in the last attribute we assumed for the primordial state: the Universe starts deeply inside the perturbative region, i.e. at very small coupling. In terms of the dilaton this means that ϕ started very large and negative. This entitles us to treat the early history of the Universe classically. Since we have also assumed that it was almost flat, we are also entitled to use the low-energy approximation to string theory. All this can be summarized by saying that we can describe the evolution of the Universe through the classical field equations of the low-energy tree-level effective action (7); in the simplest case used here for illustration, this reduces to:

$$\Gamma_{eff} = \frac{1}{2\lambda_s^2} \int d^4x \sqrt{-g}\ e^{-\phi} (\mathcal{R} + \partial_\mu \phi \partial^\mu \phi)\ . \tag{10}$$

It has been known for some time (see[12] for a review) that, if homogeneity and spatial flatness are assumed, then inflationary behaviour automatically follows from the stated initial conditions. Indeed homogeneous, spatially flat solutions fall in *four* categories, of which only *one* satisfies the pre-big bang postulate. The other three exhibit either strong coupling or strong curvature (or both) in the far past.

However, assuming homogeneity from the start is not very satisfactory. If we want to solve in a natural way the homogeneity and flatness problems, we have to start with generic (i.e. not particularly fine-tuned) initial conditions near the perturbative vacuum. During the past year this problem has been tackled by several groups, with somewhat controversial conclusions. Let me try to explain how I see the present situation.

Assume that, at some remote time much before the Big Bang, the Universe was not particularly homogeneous, in the sense that spatial gradients and time derivatives were both of the same order. Assume also, in accordance with the PBB postulate, that both kinds of derivatives were tiny in string units. It can be shown[13] that these initial conditions can lead to a chaotic version of PBB inflation since, as the system evolves, certain patches develop where time derivatives slightly dominate over spatial gradients. Provided this situation is met when the kinetic energy in the dilaton is a non-negligible fraction of the critical density, dilaton-driven inflation sets in,[13,14] blowing up the patch and making it homogeneous, isotropic and spatially flat. The evolution can be studied by analytical methods (gradient expansion[15]) since the approximation of neglecting spatial gradients w.r.t. time derivatives becomes increasingly accurate within the inflating patch.

The controversial issue is that, in order to have sufficient inflation in the patch, dilaton-driven inflation has to last sufficiently long. Its duration is not infinite since it is limited, in the past, by the conditions I just described and, in the future, by the time at which, inevitably, curvatures become of string-size and we can no longer trust the low-energy approximation. Thus, as it was actually noticed from the very beginning,[9] a successful PBB scenario does require very perturbative initial conditions, so that it takes a long time (during which the Universe inflates) to reach the BB singularity.

A particular case of this "fine-tuning" was discussed recently by M. Turner and E. Weinberg.[16] They consider a homogeneous, but not spatially flat Universe and notice that the duration of PBB inflation is limited in the past by the initial value of the spatial curvature. This has to be taken very small in string units if sufficient inflation is to be achieved.

The (almost philosophical) issue is whether this is or is not fine-tuning. String theory has a single length parameter, λ_s, but, fortunately, it has massless states and low-energy vacua (such as Minkowsky space-time) whose characteristic scale is much larger than λ_s. Hence I see nothing wrong in starting the evolution of the Universe in a state of low-energy, small curvatures, and small coupling. Actually, I find it very amusing that a classical instability pushes the Universe from low energy (curvature) and small coupling towards high energy (curvature) and large coupling.

Another result, which has emerged very recently,[13,14] is the behaviour of pre-big bang cosmology in the asymptotic past. If we evolve the system from the initial conditions I described above towards the past, we seem to find two possible behaviours. Either we reach a singularity at some finite cosmic time in the past, or we flow smoothly into a trivial space-time. The first alternative, which looks generic for positive spatial curvature (e.g. a $k = +1$ Universe with $\Omega > 1$) has to be excluded since it contradicts our basic postulate. The second alternative, which looks generic for negative spatial curvature (e.g. a $k = -1$ Universe with $\Omega < 1$), is perfectly consistent with our philosophy and leads to an interesting conjecture[14] for the whole history of time that I will describe below. It would be very interesting if pre-big bang cosmology did predict that the Universe is open, something that appears to be definitely favoured at present (see e.g.[17]) by direct measurements of the red-shift-to-distance relation and by models of large-scale-structure formation.

The complete history at which we arrive can be best drawn on a diagram (Fig. 3) reminiscent of (but actually quite different from) a Carter-Penrose diagram,[18] which artificially squashes space-time at infinity. By a simple change of the radial coordinate we can rewrite Eq. (1) as

$$ds^2 = -dt^2 + a(t)^2 \left[dR^2 + r(R)^2 d\Omega^2\right] , \tag{11}$$

and then define the coordinates x, y in Fig. 3 in terms of t and of the proper distance aR by:

$$\tan(y \pm x) = t \pm aR . \tag{12}$$

As $t \to -\infty$, the Universe approaches an exact vacuum of string theory. However, if equal-time hypersurfaces are taken to be those with a roughly constant energy density, we find that triviality is approached "a la Milne" i.e. the metric becomes

$$ds^2 = -dt^2 + t^2 \left[dr^2/(1 + r^2) + r^2 d\Omega^2\right] , \tag{13}$$

while the dilaton approaches an arbitrary constant. For negative time this is just a negative curvature FRW Universe, which contracts linearly in time, $a(t) = -t$. For positive t it is the linearly expanding Universe towards which we are evolving today if $\Omega < 1$ (for a discussion of this late-time behaviour see, for instance, [19]). In the Milne

Universe the evolution of the scale factor is driven by the negative spatial curvature ($-k\ a^{-2} \to t^{-2}$). Also shown in Fig. 3 are some time-like geodesics corresponding to a fixed comoving distance R from the origin.

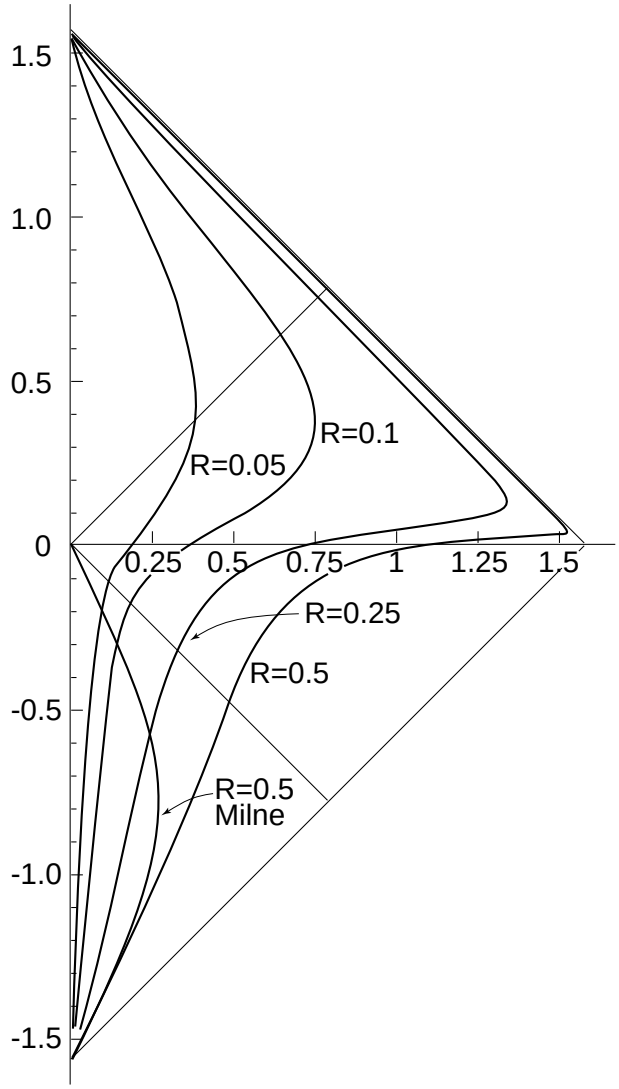

Figure 3

The generic regular solution thus approaches Milne as $t \to -\infty$ but, at any finite large $(-t)$, also contains small dilatonic (and gravitational-wave) perturbations giving $0 < \Omega \ll 1$. As $t \to -\infty$, $\Omega \to 0$. As time goes forward, instead, Ω tends to grow until, at some critical time $-T_0$, Ω becomes $O(1)$, in some region of space. From that moment on, in that "lucky" patch, the metric starts to deviate from Milne (this is shown, in particular, for the $R = 0.5$ geodesic) and dilaton-driven inflation sets in, pushing Ω extremely close to 1 in that patch.

The rest of the story goes as follows: thanks to the string UV cutoff, when the curvature becomes $O(\lambda_s^{-2})$, and/or the coupling becomes $O(1)$, a stringy mechanism

prevents reaching the singularity, and a smooth transition to standard hot big bang FRW cosmology follows. An interesting quantum mechanism, described in Sec. 5, is able to provide radiation, temperature and entropy. However, by then, the inflated patch is both homogeneous and spatially flat: we have been able to produce a "good" big bang! Post-big bang evolution is from now on standard, with one qualification. Although we have achieved $\Omega = 1$ through inflation, we had to start from an open Universe and thus $\Omega = 1 - \epsilon$, $\epsilon \ll 1$. Inevitably, FRW evolution will make Ω deviate more and more from 1 until, once more, the Universe will go back to a linearly expanding Milne-like Universe. It would be wrong to think, however, that the Universe will just follow the time-reverse of its original life, firstly because the final coupling is much larger than the initial one, and, secondly, because entropy has kept increasing all the time: presumably, in this scenario, the very final stage of the Universe will consist of an ever increasingly dilute gas of slowly evaporating black holes...

5. Quantum Mechanical Heating of the Universe and Observable PBB relics

Since there are already several review lectures on this subject (e.g.[12,20]), I will limit myself to mention the most recent developments simply after recalling the basic physical mechanism underlying particle production in cosmology.[21] A cosmological (i.e. time-dependent) background coupled to a given type of (small) inhomogeneous perturbation Ψ enters the effective low-energy action in the form:

$$I = \frac{1}{2} \int d\eta \; d^3x \; S(\eta) \left[\Psi'^2 - (\nabla \Psi)^2 \right] . \tag{14}$$

Here η is the conformal time coordinate, and a prime denotes $\partial/\partial\eta$. The function $S(\eta)$ (sometimes called the "pump" field) is, for any given Ψ, a given function of the scale factor, $a(\eta)$, and of other scalar fields (four-dimensional dilaton $\phi(\eta)$, moduli $b_i(\eta)$, etc.) which may appear non-trivially in the background.

While it is clear that a constant pump field S can be reabsorbed in a rescaling of Ψ, and is thus ineffective, a time-dependent S couples non-trivially to the fluctuation and leads to the production of pairs of quanta (with opposite momenta). Looking back at Eq. (7), one can easily determine the pump fields for each one of the most interesting perturbations. The result is:

$$\begin{aligned} \text{Gravity waves, dilaton} &: S = a^2 e^{-\phi} \\ \text{Heterotic gauge bosons} &: S = e^{-\phi} \\ \text{Kalb} - \text{Ramond, axions} &: S = a^{-2} e^{-\phi} \end{aligned} \tag{15}$$

A distinctive property of string cosmology is that the dilaton ϕ appears in some very specific way in the pump fields. The consequences of this are very interesting:

- For gravitational waves and dilatons the effect of ϕ is to slow down the behaviour of a (remember that both a and ϕ grow in the pre-big bang phase). This is the

reason why those spectra are quite steep[22] and give small contributions at large scales.

- For (heterotic) gauge bosons there is no amplification of vacuum fluctuations in standard cosmology, while, in string cosmology, all the "work" is done by the dilaton. In the pre-big bang scenario, the coupling must grow by as large a factor as the one by which the Universe has inflated. This implies a very large amplification of the primordial quantum fluctuation,[23] possibly explaining the long-sought origin of seeds for the galactic magnetic fields.

- Finally, for Kalb-Ramond fields and axions, a and ϕ work in the same direction and spectra can be large even at large scales.[24] Note, incidentally, that the power of a in S is determined by the rank of the corresponding tensor. It is well known, however, that the Kalb-Ramond field can be reduced to a (pseudo)scalar field, the axion, through a duality transformation. This turns out to change S into S^{-1}, i.e. the pump field for the axion is actually $a^2 e^{\phi}$. An interesting duality of cosmological perturbations, reminiscent of electric-magnetic (or strong-weak) duality, can be argued[25] to guarantee the equivalence of the Kalb-Ramond and axion spectra.

- Many other fluctuations, which arise in generic compactifications of superstrings, have also been studied and lead to interesting spectra. For lack of time, I will refer to the existing literature.[26,27]

The possible flatness of axionic spectra in pre-big bang cosmology leads to hopes that, in such a scenario, there is a natural way to generate an interesting spectrum of large-scale fluctuations, one of the much advertised properties of the standard inflationary scenario. Work is still in progress to establish whether this hope is indeed realized.

Before closing this section, I wish to recall how one sees the very origin of the hot big bang in this scenario. One can easily estimate the total energy stored in the quantum fluctuations which were amplified by the pre-big bang backgrounds. The result is, roughly,

$$\rho_{quantum} \sim N_{eff}\ H_{max}^4\ , \tag{16}$$

where N_{eff} is the effective number of species that are amplified and H_{max} is the maximal curvature scale reached around $t = 0$ (this formula has to be modified in case some spectra show negative slopes). We have already argued that $H_{max} \sim \lambda_s^{-1}$ and we know that, in heterotic string theory, N_{eff} is in the hundreds. Yet this rather huge energy density is very far from critical as long as the dilaton is still in the weak-coupling region, justifying our neglect of back-reaction effects. It is very tempting to assume that, precisely when the dilaton reaches a value such that $\rho_{quantum}$ is critical, the Universe will enter the radiation-dominated phase. This too is, at present, the object of active investigation.

6. Conclusion

Pre-big bang cosmology appears to have survived its first 6-7 years of life. Interest in (criticism of) it is clearly growing. It is perhaps time to make a balance sheet.

Conceptual (technical?) and phenomenological problems include:

- Graceful exit from dilaton-driven inflation to FRW cosmology is not fully understood, in spite of recent progress.[28] Possibly, new ideas borrowed from M-theory and D-branes could help in this respect.[29]
- A scale-invariant spectrum of large-scale perturbations does not look automatic, although, for the first time, thanks to the flat axion spectra, it does not look impossible either.

Attractive features include:

- No need to "invent" an inflaton, or to fine-tune potentials.
- Inflation is "natural" thanks to the duality symmetries of string cosmology.
- The initial conditions problem is decoupled from the singularity problem: a solution to the former is already shaping up and looks exciting.
- A classical gravitational instability finds a welcome use in providing inflation; a quantum instability (pair creation) is able to heat up an initially cold Universe and generate a standard hot big bang with the additional features of homogeneity, flatness and isotropy.
- Last but not least: one is dealing with a highly constrained, predictive scheme which can be tested/falsified by low-energy experiments thanks to the fact that a huge red-shift has brought the scale of Planckian physics down to that of human beings:

$$(l_P/H_0)^{1/2} \sim 1 \text{ mm} \tag{17}$$

References

1. S. Hawking in *Proceedings of the Texas/ESO-CERN Symposium on Relativistic Astrophysics, Cosmology, and Fundamental Physics*, Brighton 1990, eds. J.D. Barrow, L. Mestel and P.A. Thomas, *Ann. NY Acad. Sci.* **647**, 315 (1991).
2. S. Weinberg, *Gravitation and Cosmology*, (John Wiley and Sons, Inc., New York, 1972).
3. R. Penrose, *The Emperor's New Mind*, (Oxford University Press, New York, 1989), Fig. (7.19).
4. E. W. Kolb and M. S. Turner, *The Early Universe*, (Addison-Wesley, Redwood City, Ca, 1990).

5. R. Penrose, *Difficulties with inflationary Cosmology* in *Proc. 14th Texas Symposium on Relativistic Astrophysics*, ed. E. J. Fenyves, *Ann. NY Acad. Sci.* **571**, 249 (1989).
6. See e.g. *Euclidean Quantum Gravity*, eds. G. W. Gibbons and S. W. Hawking, (World Scientific,Singapore, 1993), Part III.
7. S. W. Hawking and R. Penrose, *Proc. Roy. Soc.* A **314**, 529 (1970).
8. G. Veneziano, *Quantum Strings and the Constants of Nature*, in *The Challenging Questions*, Erice, 1989, (ed. A. Zichichi, Plenum Press, New York, 1990).
9. G. Veneziano, *Phys. Lett.* B **265**, 287 (1991).
10. M. Gasperini and G. Veneziano, *Astropart. Phys.* **1**, 317 (1993); *Mod. Phys. Lett.* A **8**, 3701 (1993); *Phys. Rev.* D **50**, 2519 (1994).
11. An updated collection of papers on the PBB scenario is available at `http://www.to.infn.it/~gasperin/`.
12. G. Veneziano, *Status of String Cosmology: Basic Concepts and Main Consequences*, in *String Gravity and Physics at the Planck Energy Scale*, Erice, 1995, eds. N. Sanchez and A. Zichichi, (Kluver Academic Publishers, Dordrecht, 1996), p. 285.
13. G. Veneziano, *Phys. Lett.* B **406** , 297 (1997); see also: A. Feinstein, R. Lazkoz and M.A. Vazquez-Mozo, *Closed Inhomogeneous String Cosmologies*, hep-th/9704173;
J. D. Barrow and M. P. Dabrowski, *Is there Chaos in Low-energy String Cosmology?*, hep-th/9711049; K. Saygily, *Hamilton-Jacobi Approach to Pre-Big Bang Cosmology at Long Wavelengths*, hep-th/9710070.
14. A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, *Phys. Rev.* D **57**, 2543 (1998); see also: J. D. Barrow and K. E. Kunze, *Spherical Curvature Inhomogeneities in String Cosmology*, hep-th/9710018.
15. V. A. Belinskii and I.M. Khalatnikov, *Sov. Phys. (JETP)* **36**, 591 (1973);
N. Deruelle and D. Langlois, *Phys. Rev.* D **52**, 2007 (1995);
J. Parry, D. S. Salopek and J. M. Stewart, *Phys. Rev.* D **49**, 2872 (1994).
16. M. Turner and E. Weinberg, *Phys. Rev.* D **56**, 4604 (1997).
17. J. Glanz, *New Light on the Fate of the Universe*, *Science* **278**, 799 (1997) and references therein.
18. See e.g. Ref. 6.
19. Ya. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics*, Vol II (Structure and Evolution of the Universe), Section 2.4.
20. G. Veneziano, Ref. 12; M. Gasperini, *Status of String Cosmology: Phenomenological Aspects*, ibid., p. 305; *Relic gravitons from the pre-big bang: what we know and what we do not know*, hep-th/9607146.
21. See, e.g., V. F. Mukhanov, A. H. Feldman and R. H. Brandenberger, *Phys. Rep.* **215**, 203 (1992).
22. R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Lett.* B **361**, 45 (1995); R. Brustein *et al. Phys. Rev* D **51**, 6744 (1995).
23. M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Rev. Lett.* **75**, 3796 (1995); D. Lemoine and M. Lemoine, *Phys. Rev.* D **52**, 1955 (1995).
24. E.J. Copeland, R. Easther and D. Wands, *Phys. Rev.* D **56**, 874 (1997);
E.J. Copeland, J.E. Lidsey and D. Wands, *S-duality invariant perturbations*

in string cosmology, SUSSEX-TH-97-007 (hep-th/9705050).
25. R. Brustein, M. Gasperini and G. Veneziano, *Duality in Cosmological Perturbation Theory*, CERN-TH/98-53, to appear on hep-th.
26. R. Brustein and M. Hadad, *Phys. Rev.* D **57**, 725 (1998).
27. A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, *Quantum Inhomogeneities in String Cosmology*, *JHEP* **01**, 004 (1998).
28. M. Gasperini, M. Maggiore and G. Veneziano, *Nucl. Phys.* B **494**, 315 (1997); R. Brustein and R. Madden, *Phys. Lett.* B **410**, 110 (1997); *Phys. Rev.* D **57**, 712 (1998).
29. See, e.g. R. Poppe and S. Schwager, *Phys. Lett.* B **393**, 51 (1997);
A. Lukas, B.A. Ovrut and D. Waldram, *Phys. Lett.* B **393**, 65(1997); *String and M-theory cosmological solutions with Ramond forms*, UPR-723T (hep-th/9610238);
N. Kaloper, *Phys. Rev.* D **55**, 3394 (1997);
H. Lu, S. Mukherji and C.N. Pope, *Phys. Rev.* D **55**, 7926 (1997);
N. Kaloper, I. Kogan and K. A. Olive, *Cos(M)ological Solutions in M- and String Theory*, hep-th/9711027.

Note Added

Since I gave this lecture, two relevant papers have appeared:

1) N. Kaloper, A. Linde and R. Bousso (hep-th/9801073) have added further points to the criticism of the PBB scenario expressed in Ref. 16.

2) A numerical study of the spherically symmetric case by J. Maharana, E. Onofri and G. Veneziano (gr-qc/9802001) appears to support the idea discussed in Section 4 that PBB behaviour emerges generically from initial conditions sufficiently close to Milne's trivial vacuum.

These two papers confirm that much more work is still needed to clarify all the relevant issues raised by the new cosmological setup I discussed here.

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AN ENLARGED UNCERTAINTY PRINCIPLE FROM GEDANKEN STRING COLLISIONS?*)

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ABSTRACT: After recalling recent results on Planckian energy superstring collisions, we present evidence for a minimal scale that can be probed by such gedanken experiments. This scale is nothing but the Planck constant itself of string theory.

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1. INTRODUCTION

I shall start this talk by summarizing the progress made over the last couple of years on the structure of superstring collisions at or above Planckian energies.

I have reported elsewhere [1] on how these calculable gedanken collisions elucidate the way in which string theory recovers classical general relativity (CGR) and predicts finite, calculable quantum corrections to it. In this optics, Planckian energies are very useful since, after all, energy is the source of gravitational interactions and Planckian energies imply large deviations from flat space-time.

On the whole, these results reinforce the belief that superstring theory represents a serious candidate for a finite, theoretically consistent quantum theory of gravity. We shall call it QSG, for Quantum String Gravity.

The focus of our attention here will be a different one: we shall exploit "gedanken" collisions more in the traditional way in which gedanken experiments have been used in the early days of Quantum Mechanics. We would like to find out, in particular, whether or not, in QSG, we have to give up other concepts - besides those of absolute time and simultaneous measurability of position and momentum- simply because there is no (even gedanken) experiment, that can define those concepts in an operative way.

The reason why we might expect some strengthening (enlargement) of the uncertainty principle (UP) in string theory is that it contains a scale parameter, a fundamental length λ_s. We have argued sometime ago [2] that λ_s is the only fundamental

constant appearing in string theory besides the speed of light c (to be set equal to 1 in the following).

λ_s plays, at one time, the role of Planck's constant $\hbar$ and that of the UV cutoff. The reason for this "dual" role of λ_s is that strings acquire, upon quantization, a characteristic size $O(\lambda_s)$. The string size induces a UV cut off which is thus related to λ_s.

The lack of any need for introducing other constants besides c and λ_s in string theory is made almost obvious if energies are measured in units of length (through E--> α'E): after all, because of the string tension, length is energy and energy is length in classical string theory!

However, since geometric string units are not of general use, I shall stick to conventional units of energy. I shall also leave factors of α' and h explicit in all formulae in order to distinguish easily classical from quantum effects. Thus, in the following:

$$c=1 \ , \ 1/2\pi \ \alpha' = T = \Delta E / \Delta x = \text{classical tension}, \ \hbar \neq 1$$

with the string length parameter λ_s and UV momentum cutoff Δp given and related to each other by:

$$\Delta x = \Delta p/T = (2\hbar \ \alpha')^{1/2} = \lambda_s$$

The fact that both Δx and Δp scale like $\hbar^{1/2}$ is a well-known feature of the harmonic oscillator, the basic structure underlying strings.

Our aim here is to attempt the exploration of scales smaller than λ_s. In order to have a chance to succeed we certainly need enormous energies and momentum transfers. The question is whether or not this is also sufficient.

I shall immediately argue that, irrespectively of strings, in a theory containing CGR in the large distance limit, fixed angle scattering at ever increasing (and over-Planckian) energies tests ever increasing scales, against UP-based intuition.

The reason behind this statement is simple: let us fix the c.m. scattering angle θ ($\theta << 1$ but fixed as E goes to infinity) and consider, for instance, the case of c.m. energy $E= M_P/\theta$ i.e. $\Delta p = \theta\ E= M_P$. The UP would say that such an experiment is testing a scale:

$$\Delta x = \hbar/\Delta p = \hbar/ M_P = l_P = (G\hbar)^{1/2}$$

where we have introduced Newton's constant G and Planck's length l_P.

The point, however, is that the range of gravitational interactions at this energy is

$$R_S = 2GE = l_P / \theta$$

The two gravitons deflect each other by an angle θ if they impinge on one another with a transverse distance (impact parameter) given by a simple generalization of Einstein's 1916 formula:

$$\Delta x = R_S/\theta = l_P /\theta^2 >> l_P.$$

Hence, while we naively expect to test $(\Delta x)_{UP} = l_P$, in reality we test the (much) larger scale $(\Delta x)_E = l_P/\theta^2$. The general situation is shown in fig1, where, besides the UP-forbidden region (vertical shading), we have drawn the short distance region (oblique shading) screened by CGR. This kind of GCR effects will play the role of a "backgroud noise" for our gedanken experiments aimed at exploring very tiny scales.

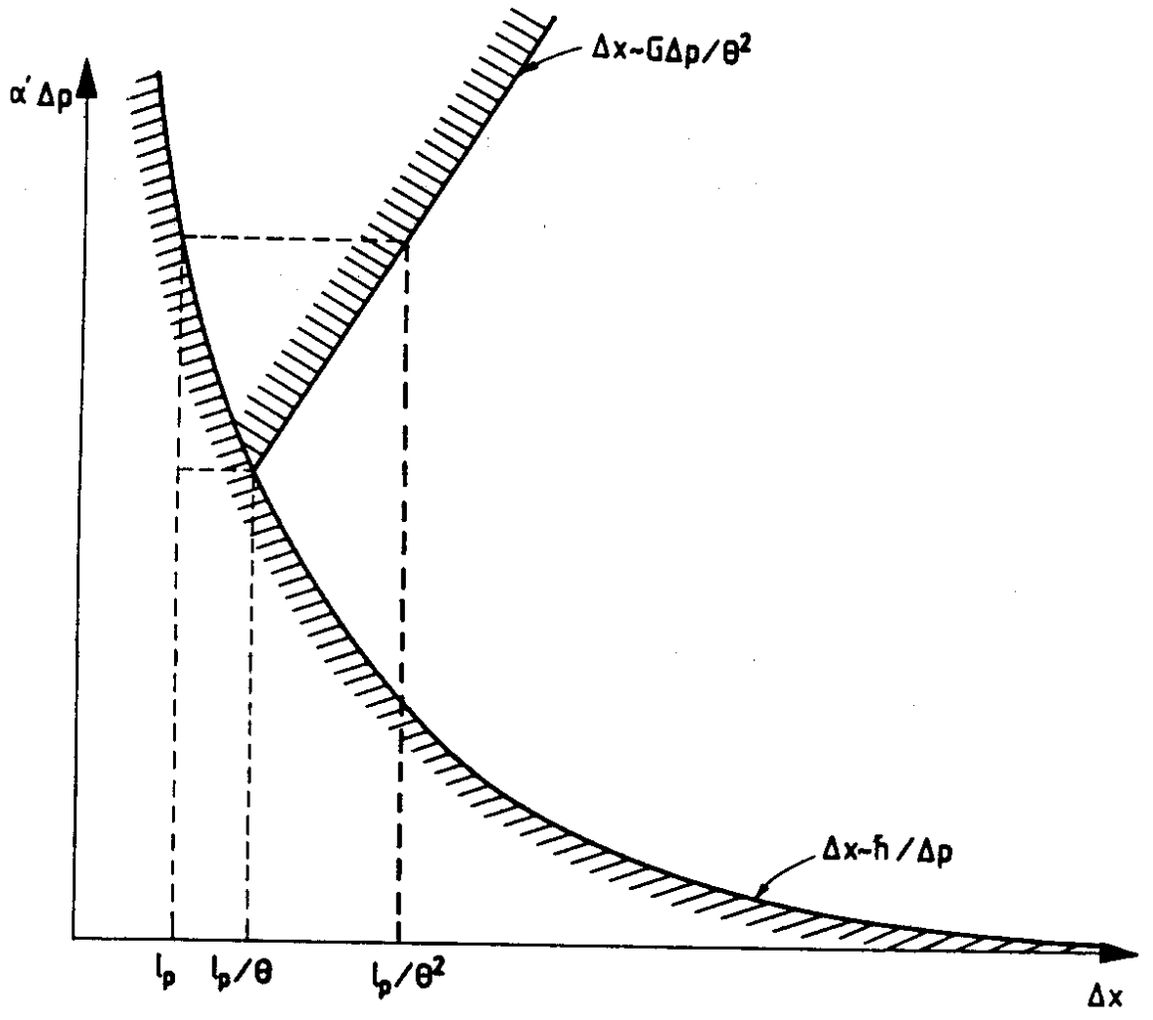

Fig. 1. The uncertainty principle Δx, $\alpha'\Delta p$ plane. The vertically shaded region corresponds to the usual UP while the obliquely shaded region is the one screened by CGR effects.

2. CHECKING THE "CORRESPONDENCE PRINCIPLE"

Before going beyond conventional physics we ought to ask ourselves: is string theory capable of reproducing the CGR effects mentioned above?

The answer would seem to be an obvious yes, since we believe string theory to contain CGR, but, actually, things are not so trivial. Indeed, it is possible to show that at no fixed order of the string loop expansion (i.e. at no fixed genus) string theory is able to reproduce the CGR expectation. This statement uses the well-known fact that dual tree amplitudes decrease at fixed angle and high energy as:

$$A = \exp(-\hbar^{-1}\alpha' s f(\theta)) \quad , s = E^2$$

thus predicting zero cross section in the kinematical region which we expect dominated by a large (semi) classical process.

The above result was recently generalized by Gross and Mende [3] to arbitrary fixed genus with the simple result that the exponential cutoff at genus g is milder by a factor (1+g). Gross and Mende also "explain" this result by the fact that, at very large energies, the functional integral over the string coordinate $X^\mu(z)$ is dominated by a "classical" path which turns out to be imaginary:

$$X^\mu(z)_{class.} = i\,\alpha' \Sigma_i\, p_i^\mu \log(z-z_i)/(1+g).$$

When inserted back in the action this gives the quoted result.

Our interpretation of this result is that, because of its softness, string theory is unable to provide a large Δp in one single exchange (the opposite of QCD so to say). Thus, at any fixed genus, fixed-angle, asymptotic-energy scattering occurs only as a typical quantum tunnelling phenomenon and, as such, is

depressed by an $\exp(-S_E/\hbar)$ factor.

The hint on how to get out of this impasse is already contained in the fixed genus calculation which clearly shows that the higher the genus the less the damping. Is it then possible that an all genus resummation can give back CGR?

This is indeed what happens [4-5] as it is best seen by working in impact parameter b (the variable related to $q = \Delta p$ by a Fourier transform). It turns out that it is in a region of s and b, rather than in a region of s and q, that we can justify our approximate resummation of all genus contributions. When we transform back to q, we need to integrate over b but, in many interesting cases, we shall find a contribution from a saddle point in the region of b which is under control. It is almost unconceivable that such contribution will be cancelled by regions of b on which we have less control.

In practice, the calculation is done by using (old fashioned) Regge-Gribov techniques. The results have been double checked at the one [6] and more recently at the two-loop [7] level by explicit calculation on the appropriate Riemann surfaces.

The specific process considered is graviton-graviton collisions in type 2 superstring theory with D uncompactified dimensions (out of 10). One finds, for $b >> (2\hbar\ \alpha')^{1/2} = \lambda_s$, $b >> (GE)^{1/D-3} = R_s$:

$$S(b,s) = \exp\ (2i\hbar^{-1}\ \delta(b,s))$$

where

$$\delta(b,s) \cong Gs\ b^{4-D}\ (4-D)^{-1} \text{ --->} Gs\ \ln b \qquad (\text{in } D=4).$$

Inserting the above expressions into the Fourier transform we find that, at a given q, the integral over b is dominated by a saddle occurring at:

$$q = -2\, d\delta(b,s)/db \Longrightarrow \quad b^{D-3} = R_s^{D-3} / \theta \qquad (8GE/\theta \text{ in } D=4),$$

rather than at $b=\hbar/q$. The moral to be drawn is that the infinite-genus resummation has created a real saddle point which was absent at any fixed genus. This is possible because, at a genus which grows with E, the enormous momentum transfer q can be shared among so many exchanged gravitons that each one of them carries a small Δp. We shall come back to this important point in Sect. 4.

For the time being we simply stress that our deflection formula is the same as the classical expression for the deflection of a graviton moving in the metric produced by the other. Such metrics had been originally constructed by Aichelbourg and Sexl (AS) [8] and have been recently generalized [9].

Actually, 't Hooft and, more recently, Jackson have obtained [10] the same result for the S-matrix starting from the assumption that, in first approximation, the scattering process can be assimilated to propagation in a fixed external metric.

This is not only a non-trivial confirmation of the validity of our approximate resummation method; it also shows how string theory is able to reproduce CGR results without any a priori assumpion: an effective metric is generated in the collision process which, in itself, takes place in flat space-time.

Incidentally, resummation over genus is also crucial for reestablishing unitarity bounds which are badly violated at tree (or fixed genus) level [11]. For impact parameters larger than R_s unitarity can be explicitly verified. As it turns out [4], the above elastic eikonal formula gets replaced, below some intermediate scale b_d, by an operator eikonal formula describing elastic as well as diffractively excited channels in a way consistent with unitarity. Problems with unitarity only reappear as one approaches $b= R_s$, as we shall discuss below.

3. GETTING RID OF BACKGROUND NOISE

At this point we may claim to have established a sort of "correspondence principle" at large scales. This fact gives us some confidence in trying to move to shorter distances. How far down can we test the theory? We would like string theory to provide the answer, but first he have to overcome the background noise due to CGR.

As is well known to experimentalists, in order to eliminate a background we must first compute it! Unfortunately, the background becomes large and complicated as we approach short distances, meaning here b-->R_S. The situation however is not completely hopeless since, as we move down in b, we find two very different situations according to whether R_S is larger or smaller than λ_S. Let us discuss both cases in succession:

1) $R_S > \lambda_S$.

This is the limit of very large energies at fixed G/α' and is, at present, the difficult case from the computational point of view. One can easily identify corrections $O(R_S/b)$ which we expect to be essential to save unitarity again.

Resumming all corrections of the type $(R_S/b)^n$ looks like a formidable task a priori. One simplification could be, however, that these classical corrections (not involving $\hbar$) correspond to string loops which have been pinched in such a way as to leave a tree diagram if the fast external lines are removed. It is not impossible that such trees can be resummed through the classical solution of a kind of Reggeon field theory, as done in the old days [12] for the problem of hadron-nucleus or nucleus-nucleus collisions. This possibility is now being investigated.

Its outcome, though not particularly relevant for the questions we focus on here, could be extremely rewarding: at $b < R_S$ the whole two graviton system has center of mass energy M and angular momentum J satisfying :

$$J < GM^2$$

which is the condition [13] for having a collapsed object, a sort of Kerr black hole. Thus, one expects the initial state to be completely absorbed, as $b \to R_S$, and the final state to consist of soft radiation (rather than of energetic particles) with a thermal spectrum characterized by Hawking's temperature (modulo small string corrections).

It is obvious, however, that from the point of view of our goal here, short distances remain screened behind the horizon in this unfavourable case!

2) $R_S < \lambda_S$.

This case, corresponding to large strings in Planck units, allows for a "window", a finite energy interval (see fig.2), in which we may be able to explore scales smaller than λ_S.

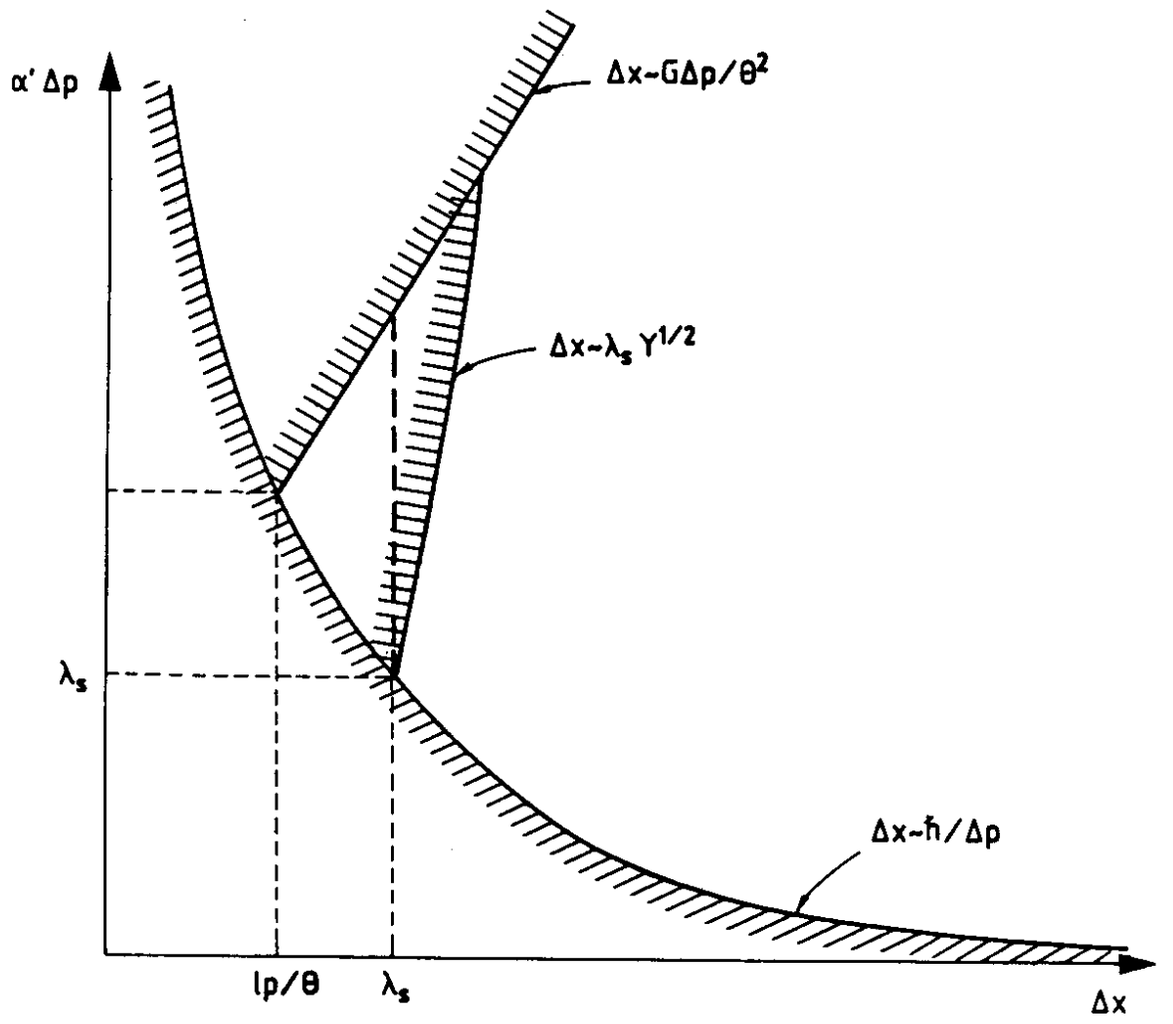

Fig 2. Same plane as in fig.1, where the region screened by quantum- string-size effects is shaded horizontally. Also shown as a dashed line is the energy "window" discussed in sect.3.

Actually, such simplifications occur in this regime that we are able to go even <u>below</u> R_S. Indeed:

1. Higher order classical effects become small in this

case since they are actually $O(R_S/(b+\lambda_S))$ which is always less than one.

2. S(b,s) can be estimated (including string corrections) since δ takes a simple form at small b (it is regular, unlike what happens in field theory).

The net result is that no sign of a singularity is found as one approaches $b=R_S$. This means, presumably, that there is no meaning attached to a black hole whose horizon is smaller than the string scale itself. We conclude that, by a judicious choice of the kinematical regime, we have been able to get rid of the background noise due to CGR effects!

4. A MINIMAL EXPLORABLE LENGTH?

Let us now discuss in more detail the favourable case $R_S < \lambda_S$. The deflection angle can be computed [4] and is found to be that of a test particle moving in the generalized AS metric [9] which corresponds to a massless beam of energy profile [1]:

$$E(r<b) = E\ \Gamma^{-1}(D/2-1)\ \Gamma(D/2-1,\ b^2/\lambda_S^2\, Y)$$

where $\Gamma(a,b)$ is the incomplete Γ-function.

This represents a beam which is practically homogeneous below $r_0=\lambda_S Y^{1/2}$ and exponentially falling above it. As intuitively obvious, the deflection angle reaches a maximum precisely at $b= r_0$ with $\Theta_{max} = (R_S/r_0)^{D-3}$.

This gives us hope [14] to be able to explore short distances by fixed angle experiments and we can try two situations:

2a) $\Theta < \Theta_{max}$. There are two real saddles in this case, one at $b=b_> > r_0$ and one at $b_< < r_0$. Unfortunately, it is the former

saddle that gives the dominant contribution and thus, at these angles, we only explore distances larger than λ_s.

2b) $\Theta > \Theta_{max}$. There is no classically allowed process in this case and scattering proceeds via quantum tunnelling, as in the fixed genus case. The saddle is now imaginary and is located at:

$$b_s = 2i\,\lambda_s\, Y^{1/2}\ (\ln\ \Theta/\Theta_{max} - i\pi/2)^{1/2}$$

Thus, again, $b_s > \lambda_s$. Actually, the results of ref. [14] can be rewritten in terms of the average genus <g> contributing to the process in the form:

$$b_s = i\,\alpha'\,q\,Y\,(1+\langle g\rangle)^{-1}\ ,$$

where

$$(1+\langle g\rangle) = \alpha'\,q\,Y^{1/2}/\lambda_s$$

is the average genus contributing to the process.

The result $b_s = i\alpha' q\ (1+\langle g\rangle)^{-1}$ bears a close resemblance to the Gross-Mende fixed genus result [3]. Notice, however, that, unlike them, we are able to resum over g, to estimate <g>, and to get a definite prediction for b_s.

We can finally conclude that, unfortunately, we have been unable to test distances smaller than λ_s by fixed angle, high energy collisions! The picture which has finally emerged is summarized in fig.(2) again. Obviously the string-size effects can shield the dangerous, Planck length region in Δx.

Let us now interpret and discuss our findings. For this purpose it is useful to rewrite our main results in a somewhat different form which, although slightly approximate, is able to describe all the various regimes at once and is quite suggestive.

This is:

$$\Delta x > \not{h} / \Delta p_S + \alpha' \Delta p_S$$

where

$$\Delta p_S = \Delta p / \langle 1+g \rangle$$

is the momentum transfer per loop. These equations, besides giving immediately a lower bound on Δx of order λ_S, show an amusing "transmutation" of quantum effects into classical ones and viceversa. Indeed, for small Δp_S, the first term dominates and appears to be quantum in nature. However, one finds easily from the results of Sect. 2 that in this case:

$$1+ \langle g \rangle = G\, s/\not{h}$$

so that

$$\not{h} / \Delta p_S = G\, E / \theta \; = b_{class.}$$

which is the classical impact parameter yielding Δp.

Conversely, for large Δp_S, the second term dominates and would appear to be classical (no apparent $\not{h}$ there). After insertion of

$$\langle g \rangle \; = \alpha' q\, Y^{1/2} / \lambda_S \langle g \rangle,$$

however, this contribution to Δx becomes controlled by λ_S, the Planck constant of string theory.

The guess made by Gross [15] (and independently by the author [16]) that $\Delta x > \alpha' \Delta p$ thus turns out not to be confirmed by the detailed analysis of ref[14]. Obviously, unless one has a way to estimate $\langle g \rangle$, any dependence of Δx from Δp is possible depending on the way $\langle g \rangle$ grows with energy!

Having failed to test distances shorter than λ_s by fixed angle collisions, we may ask if there could be other ways to approach these scales. One obvious way would be to work directly at fixed small b (for $R_s < \lambda_s$). We have seen that this is possible and that we found nothing special occurring at these scales.

One possible attitude would be to say that short distances are not just unreachable, they are simply irrelevant. This is quite possible: in string theory large virtual momenta are so cut off that observable do not seem to depend on the short distance degrees of freedom.

As I have explained elsewhere [1], it is possible that an enlarged equivalence principle makes physics depend only upon background fields which are averaged over volumes of order of the string size λ_s itself.

This would be in agreement with independent arguments [17] based on the high temperature behaviour of free energy in string theory. Finally, the existence of a minimal length in string theory is also suggested by the so-called duality of closed strings [18], a symmetry with respect to which a distance of order λ_s corresponds to the fixed point.

5. CONCLUSIONS

We have presented some evidence that, in string theory, fixed angle, high energy collisions fail to test distances smaller than the fundamental scale parameter of string theory: λ_s. Increasing simply the energy does not pay off beyond a certain limit, because of a sort of classical gravitational instability not dissimilar from the Jeans instability discussed in ref [17]. The best compromise seems to be reached by considering large, but not asymptotic energies, a "window" which exists for sufficiently small Newton constant (in α' units). Even then, Planckian scales remain unaccessible.

Can all this be of any relevance for physics? Well, our gedanken collisions may have not always been ... gedanken. In the very early universe, very close to the initial singularity, large curvatures, large densities and large temperatures are very likely to have existed. In that regime, the universe was very quantum... and perhaps very stringy, its own size having possibly been as large as a few times λ_s.

It is precisely in that era that CGR predicts a singularity and that ordinary quantum gravity is unable to make reliable predictions, since semiclassical methods are bound to be inadequate. It is there that QSG may emerge as the only tool available today for providing definite answers. Obviously, what I have described today is only a little taste of the wonderful things that may be waiting for us at the Planck scale!

REFERENCES

1. G. Veneziano, DST Workshop on Superstring Theory (H. S. Mani and R. Ramachandran Eds.) World Scientific Publ. Co. (1988) page1; Talk given at the 5th Marcel Grossmann meeting (Perth Aug. 1988), preprint BU-HEP-88-47 (1988).

2. G. Veneziano, Europhysics Letters, 2 (1986) 133.

3. D. J. Gross and P. F. Mende, Phys. Lett. 197B (1987) 129; Nucl. Phys. B303 (1988) 407.

4. D. Amati, M. Ciafaloni and G. Veneziano, Phys Lett. 197B (1987) 81; Int. Journ. Mod. Phys., 3A (1988) 1615.

5. I. Muzinich and M. Soldate, Phys. Rev. D37 (1988) 353.

6. B. Sundborg, Nucl. Phys. B306 (1988) 545.

7. E. Gava, R. Jengo and C. J. Zhu, Sissa preprint (1989). M. Ademollo, A. Bellini and M. Ciafaloni, Univ. of Florence preprint (1989).

8. P. C. Aichelbourg and R. V. Sexl, Gen. Rel. Grav. 2 (1971) 303.

9. T. Dray and G. 't Hooft, Nucl. Phys. B253 (1985) 173. V. Ferrari, P. Pendenza and G. Veneziano, Gen Rel. Grav. 20 (1988)1185.

10. G. 't Hooft, Phys Lett. 198B (1987) 61. J. D. Jackson, preprint CERN TH-5150 (1988).

11. M. Soldate, Phys. Lett. 186B (1987) 321.

12. A. Schwimmer, Nucl. Phys. B94 (1975) 445.

13. S. W. Hawking, Phys. Rev. D13 (1976) 191.

14. D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B216 (1989) 41.

15. D. J. Gross, Proc. Int. Conf. Munich (Aug. 1988), preprint PUPT-1108 (1988).

16. G. Veneziano, Invited talk at the annual meeting of the Italian Phys. Soc. (Naples, Oct. 1987); see also ref. 2.

17. J. J. Atick and E. Witten, preprint IASSNS-HEP-88/14 (1988).

18. see, for instance, K. Kikkawa and M. Yamasaki, Phys. Lett. 149B (1984) 357;
N. Sakai and I. Senda, Prog. Theor. Phys. 75 (1986) 692;
V. P. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. B287 (1987) 402;
A. Giveon, E. Rabinovici and G. Veneziano, preprint CERN-TH.5106 (1988) to appear in Nucl. Phys. B;
J.Lauer, J. Mas and H. P. Nilles, preprint MPI-PAE/PTh 19/89(1989).

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Dilaton Cosmology and Phenomenology

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Summary. This paper is dedicated to Gabriele Veneziano on his 65*th* birthday. Most of the results reported here are known results, due to Gabriele, or obtained in collaboration with him, or inspired by our joint work on string cosmology. A few new results are also presented concerning the duality invariance of a non-local dilaton coupling to the matter sources, and its possible cosmological applications in the context of the dark-energy scenario.

Contribution to the book:

"String theory and fundamental interactions:
celebrating Gabriele Veneziano on his 65th birthday",
eds. M. Gasperini and J. Maharana
(Lecture Notes in Physics, Springer-Verlag, Berlin/Heidelberg, 2007)

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Dilaton cosmology and phenomenology

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Summary. This paper is dedicated to Gabriele Veneziano on his $65th$ birthday. Most of the results reported here are known results, due to Gabriele, or obtained in collaboration with him, or inspired by our joint work on string cosmology. A few new results are also presented concerning the duality invariance of a non-local dilaton coupling to the matter sources, and its possible cosmological applications in the context of the dark-energy scenario.

Foreword and introduction

My collaboration with Gabriele Veneziano has continued, almost uninterruptedly, for more than fifteen years (even now we are preparing a joint contribution to the book *"Beyond the Big Bang"*, which will be published by Springer, as this book). Our first meeting dates back to 1989, when Gabriele came to the University of Turin to give a series of talks and seminars. At that time I was working there as a young researcher at the Department of Theoretical Physics, and I remember that Sergio Fubini (professor at the same Department) introduced me to Gabriele before a seminar. After the seminar we went to my office, and we started talking about cosmology, big bang, inflation, and strings. Gabriele was able to make me feel at ease, in spite of the fact that I was a bit embarrassed, being face to face with such a world-renown scientist like him: before that meeting, indeed, I knew him only for having seen his name quoted in many books and articles as one of the founders of string theory. I could not imagine that I was about to embark on the most stimulating and important adventure of my scientific life.

After that meeting we started collaborating on string cosmology, and I visited very often the Theory Division (now "TH Unit") at CERN, living in Geneva also for long periods. This has given me the opportunity to appreciate Gabriele not only as a scientist – whose inventiveness, originality, profundity of thought will not be stressed here, because they are well-known to the physicist community – but also for his human qualities. His tireless enthusiasm for

physics, his generosity in sharing knowledge, his intellectual honesty, always make working with him a rewarding and enjoyable experience. I have countless memories of days spent discussing and working out calculations on the blackboard of his office (see Fig. 1), with short "coffee breaks" every now and them, talking about physics even during lunch and dinner. Countless are the things I have learned from him, not only from a scientific but also from a human point of view. I will be grateful to him forever.

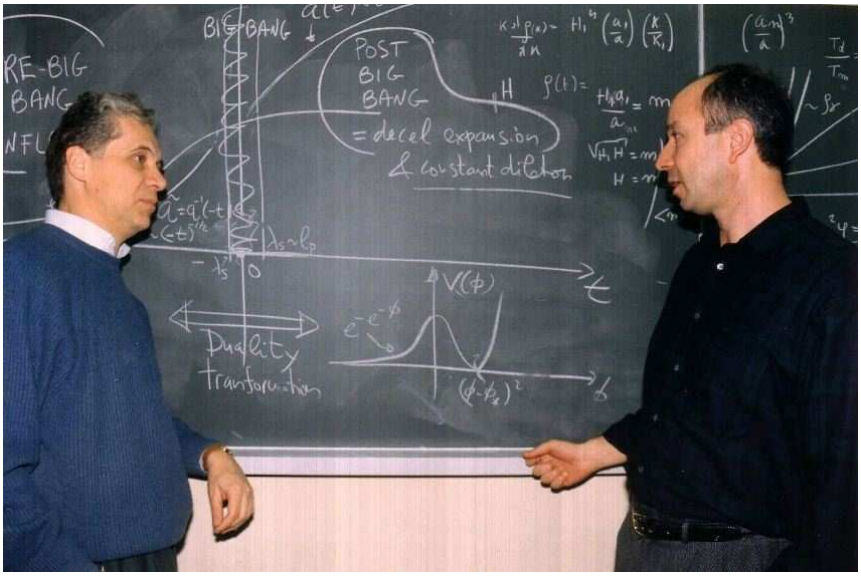

Fig. 1. Gabriele Veneziano (left) and the author (right), talking about dilatons at CERN (January 1994).

Choosing among the lines of research developed in collaboration with Gabriele, I will concentrate the contribution to this book on the possible role played by the dilaton in a cosmological context, with particular attention to the phenomenological aspects of dilaton cosmology. The dilaton is a fundamental scalar field appearing in all models of superstrings, dilaton cosmology is probably the most natural and typical form of "string cosmology", and a direct/indirect confirmation (or disproof) of its predictions could give us important experimental information on string theory in general.

The present contribution contains three lectures. The first lecture (Sect. 1) is devoted to the presentation of a primordial cosmological scenario in which the background evolution is dominated by the dilaton, and the Universe is driven through an accelerated phase representing the "dual" counterpart of the standard, decelerated evolution. The second lecture (Sect. 2) will discuss the possibility that a cosmic background of relic dilaton radiation could have

survived until present, and could be detectable by the gravitational antennas that are presently operating (or planned for a near-future operation). Finally, the third lecture (Sect. 3) will suggest a possible "dilatonic" origin of the dark-energy fluid dominating the cosmic acceleration recently observed on large scales, stressing the main differences from other, more conventional models of scalar "quintessence".

Notations and conventions

Unless otherwise stated, the following conventions are used throughout this paper: Greek indices run from 0 to d, Latin indices from 1 to d, where $d = D-1$ is the number of spatial dimensions of the D-dimensional space-time manifold. The metric signature is:

$$g_{\mu\nu} = \mathrm{diag}(+,-,-,-,\cdots).$$

The Riemann tensor and its contractions are defined by:

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} + \Gamma_{\mu\rho}{}^{\beta}\Gamma_{\nu\alpha}{}^{\rho} - (\mu \leftrightarrow \nu),$$
$$R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu}, \qquad R = R_{\mu}{}^{\mu}, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

The conventions for the covariant derivative are:

$$\nabla_\mu V_\alpha = \partial_\mu V_\alpha - \Gamma_{\mu\alpha}{}^{\beta} V_\beta, \qquad \nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma_{\mu\beta}{}^{\alpha} V^\beta.$$

Finally, we often use the convenient notation:

$$(\nabla\phi)^2 \equiv \nabla_\mu \phi \nabla^\mu \phi, \qquad \nabla^2 \phi \equiv \nabla_\mu \nabla^\mu \phi.$$

1 Dilaton-dominated inflation: the pre-big bang scenario

If we apply to the "specialized" literature for a description of the birth and of the first moments of our Universe, we may read, in the (probably) most ancient and authoritative book, that

> *"In the beginning God created the Heaven and the Earth,*
> *and the Earth was without form, and void;*
> *and the darkness was upon the face of the deep.*
> *And the Breath of God*
> *moved upon the face of the water."*
>
> (Genesis, The Holy Bible).

The most impressive aspect of these verses, for a modern cosmologist, is probably the total absence of any reference to the extremely hot, kinetic, explosive state that one could expect at (or immediately after) the "big bang" deflagration. The described state, instead, is somewhat quiet, dark, empty – we can read, indeed, about "void", "darkness", and "the deep" gives us the idea of something enormously desert and empty. In this static configuration there is at most some small fluctuation (the "Breath"), a ripple on the surface of this vacuum.

It is amusing to note that a state of this type (flat, cold and vacuum, only ruffled by quantum fluctuations), can be obtained as the initial state of our Universe, in a string-cosmology context, under the hypothesis that the Universe evolves in a "self-dual" way with respect to the symmetries of the low-energy string effective action [1, 2].

To introduce this result we start considering the gravi-dilaton sector of the low-energy effective action. To lowest order in the α' (higher-derivative) and g_s^2 (higher-loop) expansion the action is the same for all models of superstrings [3, 4], and is given by

$$S = -\frac{1}{2\lambda_{\rm s}^{d-1}} \int_\Omega {\rm d}^{d+1}x \sqrt{-g}\, {\rm e}^{-\phi} \left(R + \partial_\mu\phi\, \partial^\mu\phi + V\right). \qquad (1)$$

dilaton

Einstein-Hilbert

Here ϕ is the dilaton, and $\lambda_{\rm s} = (2\pi\alpha')^{1/2}$ is the fundamental length parameter of string theory. We have written the action using the so-called String frame (S-frame) metric, i.e. the metric to which a "test" string is minimally coupled and in which its evolution is geodesics. We have also included, for completeness and for further applications, a (possibly non-perturbative) dilaton potential, $V = V(\phi)$.

This action should be completed by the source term $S_m(g, \phi)$, describing the matter-fields contributions, and by the Gibbons-Hawking boundary term S_Σ, which is required (as in general relativity) to cancel the variational contributions of the second derivatives of the metric following from the Einstein-Hilbert Lagrangian $\sqrt{-g}\, R$. For the S-frame action (1) the boundary term takes the form [5]

$$S_\Sigma = \frac{1}{2\lambda_{\rm s}^{d-1}} \int_{\partial\Omega} \sqrt{-g}\ {\rm e}^{-\phi} K^\alpha {\rm d}\Sigma_\alpha, \qquad (2)$$

where $K^\alpha = K n^\alpha$. Here K is the trace of the extrinsic curvature of the d dimensional hypersurface $\partial\Omega$ bounding the hypervolume Ω over which we are varying the action, and n^α is the unit vector normal to this hypersurface.

1 The variation of the total action $S + S_m + S_\Sigma$ with respect to $g^{\mu\nu}$ leads then to the equations

$$G_{\mu\nu} + \nabla_\mu\nabla_\nu\phi + \frac{1}{2} g_{\mu\nu} \left(\nabla\phi\right)^2 - g_{\mu\nu}\nabla^2\phi - \frac{1}{2} g_{\mu\nu} V(\phi) = \lambda_{\rm s}^{d-1} {\rm e}^{\phi}\ T_{\mu\nu}, \qquad (3)$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ the gravitational stress tensor of the matter sources, defined as usual by the functional differentiation of S_m as:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}}. \tag{4}$$

2

The variation of the total action with respect to ϕ leads to the dilaton equation of motion,

$$R + 2\nabla^2\phi - (\nabla\phi)^2 + V - \frac{\partial V}{\partial\phi} = \lambda_{\rm s}^{d-1}{\rm e}^{\phi}\sigma, \tag{5}$$

dilaton eom

where σ is the (S-frame) density of dilaton charge of the sources, defined by the functional differentiation of S_m with respect to ϕ:

$$\sigma = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta\phi}. \tag{6}$$

Using the dilaton equation to eliminate the scalar curvature, present in the Einstein tensor, we can eventually rewrite Eq. (3) in the convenient (simplified) form

$$R_{\mu\nu} + \nabla_\mu\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\frac{\partial V}{\partial\phi} = \lambda_{\rm s}^{d-1}{\rm e}^{\phi}\left(T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\,\sigma\right). \tag{7}$$

"dilatonic" Einstein Eq

1.1 Scale factor duality

We will now consider the particular case in which the space-time manifold described by the S-frame metric is spatially flat, homogeneous (but not necessarily isotropic), and in which the matter fields can be phenomenologically described as perfect fluids, at rest in the comoving frame of the given Robertson-Walker geometry. We can thus set, in the synchronous gauge,

$$\begin{aligned} &g_{\mu\nu} = {\rm diag}(1, -a_i^2\delta_{ij}), \qquad a_i = a_i(t), \qquad \phi = \phi(t), \\ &T_\mu{}^\nu = {\rm diag}(\rho, -p_i\delta_i^j), \qquad \rho = \rho(t), \qquad p_i = p_i(t), \qquad \sigma = \sigma(t). \end{aligned} \tag{8}$$

Separating the time and space components of the gravitational equations we then obtain, from the (00) component of Eq. (3),

$$\dot\phi^2 - 2\dot\phi\sum_i H_i + \left(\sum_i H_i\right)^2 - \sum_i H_i^2 - V = 2\lambda_{\rm s}^{d-1}\,{\rm e}^{\phi}\rho \tag{9}$$

(where $H_1 = \dot a_i/a_i$). From the (ii) component of Eq. (7) we have:

$$\dot H_i - H_i\left(\dot\phi - \sum_k H_k\right) + \frac{1}{2}\frac{\partial V}{\partial\phi} = \lambda_{\rm s}^{d-1}\,{\rm e}^{\phi}\left(p_i - \frac{\sigma}{2}\right). \tag{10}$$

From the dilaton equation (5) we are lead, finally, to

$$2\ddot\phi - \dot\phi^2 + 2\dot\phi\sum_i H_i - \sum_i\left(2\dot H_i + H_i^2\right) - \left(\sum_i H_i\right)^2 + V - \frac{\partial V}{\partial\phi} = \lambda_{\rm s}^{d-1}\,{\rm e}^{\phi}\sigma. \tag{11}$$

We have thus obtained a system of $d+2$ equations for the $2d+3$ unknowns $\{a_i, \phi, \rho, p_i, \sigma\}$: its solution requires the input of $d+1$ "equations of state", $p_i = p_i(\rho)$, $\sigma = \sigma(\rho)$, specifying the properties of the considered matter sources.

Let us now consider the symmetries of this system of equations. There are two symmetries, in particular, that are relevant for the discussion of this section. One of them (also present in the cosmological equations of general relativity) is the invariance under the time-reversal transformation $t \to -t$, which implies

$$H_i \to -H_i, \qquad \dot{H}_i \to \dot{H}_i, \qquad \dot{\phi} \to -\dot{\phi}, \qquad \ddot{\phi} \to \ddot{\phi}. \tag{12}$$

Thanks to this invariance property, if the set of variables $S = \{a_i(t), \phi(t), \rho(t)\}$ represents an exact solution of Eqs. (9)-(11), then the time-reversed set $\widetilde{S} = \{a_i(-t), \phi(-t), \rho(-t)\}$ also corresponds to an exact solution of the same equations (with different kinematic properties, in general).

Scale factor duality

The string-cosmology equations, in the particular case $\sigma = 0$ and $V =$ const, are also invariant under other transformations which have no analogue in general relativity, and which include the inversion of an arbitrary number of scale factors of the background geometry (8): the so-called "scale-factor duality" transformations [1, 6]. For a simple illustration of this property we may conveniently rewrite the equations in terms of the "shifted variables" $\overline{\phi}$, $\overline{\rho}$, $\overline{p}_i$, $\overline{\sigma}$, defined by

$$\overline{\phi} = \phi - \ln \prod_i a_i = \phi - \sum_i \ln a_i, \qquad i = 1, \ldots, d,$$
$$\overline{\rho} = \rho \prod_i a_i, \qquad \overline{p}_k = p_k \prod_i a_i, \qquad \overline{\sigma} = \sigma \prod_i a_i. \tag{13}$$

Eqs. (9)-(11) then become:

$$\dot{\overline{\phi}}^2 - \sum_i H_i^2 - V = 2\lambda_{\rm s}^{d-1}\, {\rm e}^{\overline{\phi}}\, \overline{\rho}, \tag{14}$$

$$\dot{H}_i - H_i \dot{\overline{\phi}} + \frac{1}{2}\frac{\partial V}{\partial \phi} = \lambda_{\rm s}^{d-1}\, {\rm e}^{\overline{\phi}} \left(\overline{p}_i - \frac{\overline{\sigma}}{2}\right), \tag{15}$$

$$2\ddot{\overline{\phi}} - \dot{\overline{\phi}}^2 - \sum_i H_i^2 + V - \frac{\partial V}{\partial \phi} = \lambda_{\rm s}^{d-1}\, {\rm e}^{\overline{\phi}}\, \overline{\sigma}. \tag{16}$$

Under the transformation $a \to \widetilde{a} = a^{-1}$, on the other hand, we have:

$$H = a^{-1}\frac{{\rm d}a}{{\rm d}t} \quad \to \quad \widetilde{H} = \widetilde{a}^{-1}\frac{{\rm d}\widetilde{a}}{{\rm d}t} = a\frac{{\rm d}a^{-1}}{{\rm d}t} = -H. \tag{17}$$

We can then easily check that Eqs. (14)-(16), in the particular case $\sigma = 0$ and $\partial V/\partial\phi = 0$, are invariant under the scale-factor duality transformations:

$$a_i \to a_i^{-1}, \qquad \overline{\phi} \to \overline{\phi}, \qquad \overline{\rho} \to \overline{\rho}, \qquad \overline{p}_i \to -\overline{p}_i. \tag{18}$$

This type of transformation is called "dual" as it generalizes to the case of time-dependent backgrounds the T-duality transformation inverting the compactification radius (thus interchanging "winding" and "momentum" modes) in the spectrum of a closed string, quantized in the presence of compact spatial dimensions [7]. For the invariance under the transformations (18), however, there is no need of a compact geometry; what is required, instead, is a non-trivial transformation of the dilaton. Let us suppose, in fact, that we are inverting a number n of scale factors, say $a_1, \ldots, a_n$, with $1 \leq n \leq d$: the condition $\overline{\phi} \to \overline{\phi}$ then implies

$$\phi - \sum_{i=1}^{d} \ln a_i = \widetilde{\phi} - \sum_{i=1}^{d} \ln \widetilde{a}_i = \widetilde{\phi} - \sum_{i=1}^{n} \ln a_i^{-1} - \sum_{i=n+1}^{d} \ln a_i, \tag{19}$$

from which

$$\phi \to \widetilde{\phi} = \phi - 2 \sum_{i=1}^{n} \ln a_i. \tag{20}$$

In the presence of sources, their energy density is also non-trivially transformed: the condition $\overline{\rho} \to \overline{\rho}$ implies, in fact,

$$\rho \prod_{i=1}^{d} a_i = \widetilde{\rho} \prod_{i=1}^{n} a_i^{-1} \prod_{i=1+n}^{d} a_i, \tag{21}$$

from which

$$\rho \to \widetilde{\rho} = \rho \prod_{i=1}^{n} a_i^2. \tag{22}$$

The transformation of the pressure is similar, but with an additional "reflection" of the equation of state along the spatial directions affected by the duality transformation:

$$p_i \to \widetilde{p}_i = -p_i \prod_{k=1}^{n} a_k^2, \qquad\qquad i = 1, \ldots, n. \tag{23}$$

In any case, given a set of variables $S = \{a_i(t), \phi(t), \rho(t), p_i\}$ representing an exact solution of Eqs. (14)-(16), a new solution can be obtained by inverting an arbitrary number n (between 1 and d) of scale factors, and is represented by

$$\widetilde{\mathcal{S}} = \{a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}, a_{n+1}, \ldots, a_d,\ \widetilde{\phi},\ \widetilde{\rho},\ \widetilde{p}_1, \ldots, \widetilde{p}_n,\ p_{n+1}, \ldots, p_d\}, \tag{24}$$

where $\widetilde{\phi}$, $\widetilde{\rho}$, $\widetilde{p}_i$ are given by Eqs. (20), (22), (23), respectively.

The invariance under the transformations (18) is only a particular case of a more general $O(d,d)$ symmetry of the tree-level string cosmology equations [8] (see also the contribution of Meissner [9] to this volume), and can be extended

so as to include the NS-NS two-form $B_{\mu\nu}$ in the effective action. Such an extension is also possible in the presence of fluid sources: a homogeneous gas of strings, in particular, provides a realistic example of source which is automatically compatible with the $O(d,d)$ symmetry of the background equations [10].

In addition, the invariance under the transformations (18) can be extended to the case of non-trivial potentials, $\partial V/\partial\phi \neq 0$, and non-zero dilaton couplings to the matter sources, $\sigma \neq 0$. In both cases, however, we need to generalize those parts of the action describing the self-coupling of the dilaton and the dilaton couplings to the matter fields present in S_m.

In the case of the dilaton potential it is well known [8]-[10] that the invariance under the transformations (18) holds for non trivial V, provided V depends on ϕ through the variable $\overline{\phi}$. Such a variable, unlike ϕ, *is not* a scalar under general coordinate transformations (as evident from the definition (13)): it is thus impossible, in a generic background, to define a potential which is function of $\overline{\phi}$ and which can be directly inserted as a scalar into the covariant action (1). However, as first pointed out in [11], the action and the corresponding equations of motion can be written in a generalized form which is invariant under general coordinate transformations in any metric background, using for the potential a non-local variable which exactly reduces to $\overline{\phi}$ in the limit of a homogeneous geometry.

Here it will be shown that the invariance under the duality transformations (18) can be restored also in the presence of the dilaton charge σ, provided the dilaton coupling to the matter sources is parametrized by a non-local variable, as in the case of the potential. This result is new, and will be explicitly derived in the following subsection.

1.2 Non-local dilaton interactions

The formalism introduced in [11] is based on the non-local variable $\xi(x)$, defined by

$$\xi(x) \equiv \xi[\phi(x)] = -\ln \int \frac{\mathrm{d}^{d+1}y}{\lambda_{\mathrm{s}}^{d}} \left(\sqrt{-g}\,\mathrm{e}^{-\phi}\sqrt{\epsilon(\nabla\phi)^2}\right)_y \delta(\phi_x - \phi_y), \qquad (25)$$

where we have explicitly inserted the parameter

$$\epsilon = \mathrm{sign}\{(\nabla\phi)^2\} = \begin{cases} 1, & (\nabla\phi)^2 > 0, \\ -1, & (\nabla\phi)^2 < 0, \end{cases} \qquad (26)$$

so as to include in the formalism both time-like and space-like dilaton gradients. Note that we are using the convenient notation in which an index appended to round brackets, $(\ldots)_x$, means that all quantities inside the brackets are functions of the appended variable. Similarly, $\phi_x \equiv \phi(x)$. We can immediately check that, for a homogeneous background of the type (8) with spatial sections of finite comoving volume ($\int \mathrm{d}^d y = V_d = \mathrm{const} < \infty$), the variable ξ

exactly reduces to the variable $\overline{\phi}$ of Eq. (13). In that case, in fact, an explicit integration gives

$$\xi = \phi - \ln \prod_i a_i - \ln \left(\frac{V_d}{\lambda_{\rm s}^d} \right), \tag{27}$$

and the constant volume factor can be simply absorbed by rescaling ϕ, so that $\xi \equiv \overline{\phi}$.

Let us now suppose that the matter couplings and the self-coupling of the dilaton are both parametrized by ξ, according to the effective action

$$S = -\frac{1}{2\lambda_{\rm s}^{d-1}} \int {\rm d}^{d+1}x \, \sqrt{-g}\, {\rm e}^{-\phi} \left[R + (\nabla\phi)^2 + V({\rm e}^{-\xi}) \right] + \int {\rm d}^{d+1}x \, \sqrt{-g} \, \mathcal{L}_m({\rm e}^{-\xi}) + S_{\rm GH}, \tag{28}$$

which is a (generally-covariant) scalar functional of the non-local variable ξ. Note that, without loss of generality, we have written both the potential and the matter Lagrangian $\mathcal{L}_m$ as a function of $\exp(-\xi)$. In higher-dimensional manifolds with compact spatial sections, in fact, the exponential of the shifted dilaton plays the role of a "dimensionally-reduced" coupling parameter, and we may thus expect (at least in a perturbative regime) that dilaton interactions appear as a power expansion (or as a a simple function) of such an exponential [11].

The generalized equations of motion can now be obtained by computing the functional derivative of the action (28) with respect to $g^{\mu\nu}$ and ϕ. The derivative with respect to the metric, using the standard definition of gravitational stress tensor, Eq. (4), and the properties of the delta distribution, leads to the (integro-differential) equations of motion

$$G_{\mu\nu} + \nabla_\mu\nabla_\nu\phi + \frac{1}{2} g_{\mu\nu} \left(\nabla\phi^2 - 2\nabla^2\phi - V \right) - \frac{1}{2}\gamma_{\mu\nu} \sqrt{\epsilon(\nabla\phi)^2} \left({\rm e}^{-\phi} I_V - 2\lambda_{\rm s}^{d-1} I_m \right) = \lambda_{\rm s}^{d-1} {\rm e}^{\phi} \, T_{\mu\nu}, \tag{29}$$

which generalize Eq. (3) (see Appendix A for the details of the derivation). Here

$$\gamma_{\mu\nu} = g_{\mu\nu} - \frac{\nabla_\mu\phi\nabla_\nu\phi}{(\nabla\phi)^2}, \tag{30}$$

$$I_V(x) = \lambda_{\rm s}^{-d} \int {\rm d}^{d+1}y \left(\sqrt{-g}\, V' \right)_y \delta(\phi_y - \phi_x), \tag{31}$$

$$I_m(x) = \lambda_{\rm s}^{-d} \int {\rm d}^{d+1}y \left(\sqrt{-g}\, \mathcal{L}'_m \right)_y \delta(\phi_y - \phi_x), \tag{32}$$

where the prime denotes the derivative with respect to the argument $\exp(-\xi)$, namely:

$$V' = \frac{\partial V}{\partial(\mathrm{e}^{-\xi})} = -\mathrm{e}^{\xi}\,\frac{\partial V}{\partial \xi}, \qquad \mathcal{L}'_m = \frac{\partial \mathcal{L}_m}{\partial(\mathrm{e}^{-\xi})} = -\mathrm{e}^{\xi}\,\frac{\partial \mathcal{L}_m}{\partial \xi}. \tag{33}$$

2 The functional derivative with respect to ϕ leads to the dilaton equation of motion,

$$\begin{aligned} &R + 2\nabla^2\phi - (\nabla\phi)^2 + V + \epsilon\frac{\gamma_{\mu\nu}\nabla^\mu\nabla^\nu\phi}{\sqrt{\epsilon(\nabla\phi)^2}}\left(\mathrm{e}^{-\phi}I_V - 2\lambda_{\mathrm{s}}^{d-1}I_m\right) \\ &+ \left(\mathrm{e}^{-\xi} - \mathrm{e}^{-\phi}J\right)\left(V' - 2\lambda_{\mathrm{s}}^{d-1}\mathrm{e}^{\phi}\mathcal{L}'_m\right) = 0, \end{aligned} \tag{34}$$

generalizing Eq. (5). Here

$$J(x) = \lambda_{\mathrm{s}}^{-d}\int \mathrm{d}^{d+1}y\left(\sqrt{-g}\,\sqrt{\epsilon(\nabla\phi)^2}\right)_y \delta'(\phi_x - \phi_y), \tag{35}$$

where δ' denotes the derivative of the delta function with respect to its argument (see Appendix A). The combination of Eqs. (29) and (34) finally leads to the equation

$$\begin{aligned} &R_{\mu\nu} + \nabla_\mu\nabla_\nu\phi \\ &+\frac{1}{2}\left(\mathrm{e}^{-\phi}I_V - 2\lambda_{\mathrm{s}}^{d-1}I_m\right)\left(\epsilon\, g_{\mu\nu}\frac{\gamma_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi}{\sqrt{\epsilon(\nabla\phi)^2}} - \gamma_{\mu\nu}\sqrt{\epsilon(\nabla\phi)^2}\right) \\ &+\frac{1}{2}g_{\mu\nu}\left(\mathrm{e}^{-\xi} - \mathrm{e}^{-\phi}J\right)\left(V' - 2\lambda_{\mathrm{s}}^{d-1}\mathrm{e}^{\phi}\mathcal{L}'_m\right) = \lambda_{\mathrm{s}}^{d-1}\mathrm{e}^{\phi}\,T_{\mu\nu}, \end{aligned} \tag{36}$$

"Einstein" eqs

generalizing Eq. (7).

We can easily check that these new equations, written for a homogeneous background, are invariant under scale-factor duality transformations even in the presence of non-trivial potentials and dilaton couplings, i.e. for $\partial V/\partial\xi \neq 0$, $\partial\mathcal{L}_m/\partial\xi \neq 0$. Consider, for instance, the background configuration of Eq. (8) with time-like dilaton gradients, for which $\epsilon = 1$. From Eq. (30) we obtain:

$$\gamma_0^0 = 0, \qquad \gamma_i^j = \delta_i^j. \tag{37}$$

The $(0,0)$ component of Eq. (29) thus coincides with the $(0,0)$ component of Eq. (3), and is given by Eq. (9), as before.

For the spatial components we first note that, performing the homogeneous limit in which $\xi \to \overline{\phi}$, we are lead to the identities

$$\begin{aligned} \sqrt{\epsilon(\nabla\phi)^2}\left(\mathrm{e}^{-\phi}I_V - 2\lambda_{\mathrm{s}}^{d-1}I_m\right) &\equiv \mathrm{e}^{-\xi}\left(V' - 2\lambda_{\mathrm{s}}^{d-1}\mathrm{e}^{\phi}\mathcal{L}'_m\right) \\ &\longrightarrow -\left(\frac{\partial V}{\partial\overline{\phi}} - 2\lambda_{\mathrm{s}}^{d-1}\mathrm{e}^{\phi}\frac{\partial\mathcal{L}_m}{\partial\overline{\phi}}\right); \end{aligned} \tag{38}$$

$$\begin{aligned} \frac{\gamma_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi}{\sqrt{\epsilon(\nabla\phi)^2}}\left(\mathrm{e}^{-\phi}I_V - 2\lambda_{\mathrm{s}}^{d-1}I_m\right) &\equiv \mathrm{e}^{-\phi}J\left(V' - 2\lambda_{\mathrm{s}}^{d-1}\mathrm{e}^{\phi}\mathcal{L}'_m\right) \\ &\longrightarrow -\frac{\sum_i H_i}{\dot{\phi}}\left(\frac{\partial V}{\partial\overline{\phi}} - 2\lambda_{\mathrm{s}}^{d-1}\mathrm{e}^{\phi}\frac{\partial\mathcal{L}_m}{\partial\overline{\phi}}\right). \end{aligned} \tag{39}$$

Using such identities we find that the dependence on V' and $\mathcal{L}'_m$ completely disappears from the spatial components of Eq. (36) with $\epsilon = 1$, and we obtain the condition

$$R_i{}^j + \nabla_i \nabla^j \phi = \lambda_{\rm s}^{d-1} e^{\phi}\, T_i{}^j. \tag{40}$$

Written explicitly, the new spatial equation is given by

$$\dot{H}_i - H_i \left(\dot{\phi} - \sum_k H_k \right) = \lambda_{\rm s}^{d-1}\, e^{\phi}\, p_i, \tag{41}$$

and is thus crucially simplified with respect to the corresponding (local) spatial equation (10).

The dilaton equation (34) also simplifies in the homogeneous limit, thanks to the identities (38) and (39) from which we obtain

$$R + 2\nabla^2\phi - (\nabla\phi)^2 + V - \frac{\partial V}{\partial \overline{\phi}} = -2\lambda_{\rm s}^{d-1} e^{\phi}\, \frac{\partial \mathcal{L}_m}{\partial \overline{\phi}}. \tag{42}$$

The explicit form is:

$$2\ddot{\phi} - \dot{\phi}^2 + 2\dot{\phi} \sum_i H_i - \sum_i \left(2\dot{H}_i + H_i^2 \right) - \left(\sum_i H_i \right)^2 + \\ + V(\overline{\phi}) - \frac{\partial V}{\partial \overline{\phi}} = \lambda_{\rm s}^{d-1}\, e^{\phi}\, \sigma(\overline{\phi}), \tag{43}$$

where we have defined, by analogy with Eq. (6),

$$\sigma(\overline{\phi}) = -2 \frac{\partial \mathcal{L}_m}{\partial \overline{\phi}}. \tag{44}$$

theorem:

The new set of equations (9), (41), (43) is compatible with scale-factor duality for any V and σ, as can be shown by rewriting the equations in terms of the shifted variables of Eq. (13). With such variables, Eqs. (9), (41), (43) become, respectively,

$$\dot{\overline{\phi}}^2 - \sum_i H_i^2 - V = 2\lambda_{\rm s}^{d-1}\, e^{\overline{\phi}}\, \overline{\rho}, \tag{45}$$

$$\dot{H}_i - H_i \dot{\overline{\phi}} = \lambda_{\rm s}^{d-1}\, e^{\overline{\phi}}\, \overline{p}_i, \tag{46}$$

$$2\ddot{\overline{\phi}} - \dot{\overline{\phi}}^2 - \sum_i H_i^2 + V(\overline{\phi}) - \frac{\partial V}{\partial \overline{\phi}} = \lambda_{\rm s}^{d-1}\, e^{\overline{\phi}}\, \overline{\sigma}(\overline{\phi}). \tag{47}$$

They are manifestly invariant under the generalized transformations

$$a_i \to a_i^{-1}, \qquad \overline{\phi} \to \overline{\phi}, \qquad \overline{\rho} \to \overline{\rho}, \qquad \overline{p}_i \to -\overline{p}_i, \qquad \overline{\sigma} \to \overline{\sigma}, \tag{48}$$

preserving the shifted version of the dilaton-charge density $\overline{\sigma}$.

1.3 The pre-big bang scenario

Let us come back to the cosmological applications of scale-factor duality. Even without using its non-local extensions, the duality symmetry of the equations allows introducing a "dual complement" of the standard cosmological solutions, and suggests new possible scenarios for the primordial evolution of our Universe.

For a simple illustration of this possibility it will be enough to consider a homogeneous, isotropic and spatially flat metric background, sourced by a barotropic perfect fluid with equation of state $p/\rho = \gamma = \text{const}$, with negligible dilaton charge. By imposing $\sigma = 0$, $V = 0$, and assuming $\gamma \neq 0$, one easily finds that Eqs. (45)-(47) are satisfied by the following particular exact solution

$$a = \left(\frac{t}{t_0}\right)^{\frac{2\gamma}{1+d\gamma^2}}, \qquad \overline{\rho} = \rho_0 a^{-d\gamma}, \qquad \overline{\phi} = -\frac{2}{1+d\gamma^2}\ln\left(\frac{t}{t_0}\right) + \text{const}, \tag{49}$$

where $t > 0$, and t_0, ρ_0 are positive integration constants. In terms of the non-shifted variables:

$$\begin{aligned} &a = \left(\frac{t}{t_0}\right)^{\frac{2\gamma}{1+d\gamma^2}}, \qquad \rho = \rho_0 a^{-d(1+\gamma)}, \qquad p = \gamma\rho, \\ &\phi = 2\frac{d\gamma - 1}{1+d\gamma^2}\ln\left(\frac{t}{t_0}\right) + \text{const}, \qquad t > 0. \end{aligned} \tag{50}$$

This solution, defined over the real positive semi-axis $t > 0$, describes a Universe evolving from a past curvature singularity at $t \to 0_-$ to an asymptotically flat configuration at $t \to +\infty$. For $\gamma > 0$ we have a phase of decelerated expansion and decreasing curvature,

$$\dot{a} > 0, \qquad \ddot{a} < 0, \qquad \dot{H} < 0, \tag{51}$$

typical of the standard cosmological scenario. Also, for a "realistic" equation of state with $\gamma d \leq 1$, the dilaton turns out to be non-increasing ($\dot{\phi} \leq 0$); in particular, for a radiation fluid with $\gamma = 1/d$, one recovers the radiation-dominated solution at constant dilaton,

$$p = \frac{\rho}{d}, \qquad a = \left(\frac{t}{t_0}\right)^{\frac{2}{1+d}}, \qquad \rho = \rho_0 a^{-(1+d)}, \qquad \phi = \text{const}, \tag{52}$$

which is also an exact solution of the standard Einstein equations.

Thanks to the symmetries of the string cosmology equations we can now obtain new, different solutions (which have no analogue in the context of the Einstein equations) by performing a time reflection $t \to -t$ and, simultaneously, a dual transformation defined by Eq. (18). Starting in particular from Eq. (50) we are lead to the background

$$a = \left(-\frac{t}{t_0}\right)^{-\frac{2\gamma}{1+d\gamma^2}}, \qquad \rho = \rho_0 a^{-d(1-\gamma)}, \qquad p = -\gamma\rho,$$
$$\phi = -2\frac{1+d\gamma}{1+d\gamma^2}\ln\left(-\frac{t}{t_0}\right) + \text{const}, \qquad t < 0, \tag{53}$$

which is still a particular exact solution of Eqs. (45)-(47). It is defined on the negative real semi-axis $t < 0$, and for $\gamma > 0$ it describes a phase of accelerated (i.e. inflationary) expansion and growing curvature:

$$\dot{a} > 0, \qquad \ddot{a} > 0, \qquad \dot{H} > 0. \tag{54}$$

In this case the Universe evolves from an asimptotically flat initial configuration at $t \to -\infty$ towards a curvature singularity at $t \to 0_-$. The dilaton is always growing ($\dot{\phi} > 0$) for $t \to 0_-$, even if we consider the dual of the radiation-dominated solution (52):

$$p = -\frac{\rho}{d}, \qquad a = \left(-\frac{t}{t_0}\right)^{-\frac{2}{1+d}}, \qquad \rho = \rho_0 a^{-d+1}, \qquad \phi = -\frac{4d}{d+1}\ln\left(-\frac{t}{t_0}\right). \tag{55}$$

This interesting property of the low-energy string-cosmology equations – i.e. the presence of an inflationary "partner" associated to any standard decelerated solution – is also valid in the absence of sources. Consider, for instance, Eqs. (45)-(47) with $p = 0 = \rho$, and $V = 0$. In the isotropic limit we find the particular exact solution

$$a = \left(\frac{t}{t_0}\right)^{1/\sqrt{d}}, \qquad \phi = \left(\sqrt{d} - 1\right)\ln\left(\frac{t}{t_0}\right), \qquad t > 0, \tag{56}$$

describing decelerated expansion and decreasing curvature. By applying the transformations (18) we are lead to the dual solution

$$a = \left(-\frac{t}{t_0}\right)^{-1/\sqrt{d}}, \qquad \phi = -\left(\sqrt{d} + 1\right)\ln\left(-\frac{t}{t_0}\right), \qquad t < 0, \tag{57}$$

describing accelerated expansion, growing curvature and growing dilaton. Actually, both the vacuum and the fluid-dominated solutions can be obtained as asymptotic limits of the general exact solution of the system of equations (45)-(47) (for barotropic sources, with $V = 0$ and $\sigma = 0$), in the large-curvature and small-curvature limits, respectively [12, 13].

It is well known that the decelerated configurations, typical of the standard cosmological scenario, cannot be extended back in time without limits: the range of the time coordinate is bounded from below by the presence of the initial singularity (indeed, going back in time, the growth of H is unbounded, and the curvature blows up to infinity in a finite proper-time interval).

The standard scenario, however, is certainly incomplete because it excludes inflation. The inclusion of inflation, on the other hand, modifies the behavior of the curvature scale: during a phase of "slow-roll" inflation [14], for instance, the background geometry can be approximately described by a de Sitter-like metric where $\dot{H} \simeq 0$, and in which the curvature tends to settle at a constant. One might think, therefore, that a complete (and realistic) cosmological scenario could avoid the initial singularity, replacing it with a primordial inflationary phase at constant curvature.

Unfortunately, however, an epoch of accelerated expansion at constant curvature, described by the Einstein equations, and dominated by the potential energy of some "inflaton" scalar field satisfying causality and weak-energy conditions, cannot be "past eternal", as proved in [15]. Thus, the conventional inflationary scenario mitigates the rapid growth of the curvature typical of the standard cosmological evolution, and shifts back in time the position of the initial singularity, without completely removing it, however (namely, without extending in a geodesically complete way the model, back in time, to infinity).

If a constant-curvature phase is not appropriate to construct a regular model (fully extended over the whole temporal axis), the alternative we are left is a model in which the curvature, as we go back in time, after reaching a maximum, at some point, starts decreasing; in other words, a model in which the standard evolution is completed and complemented by a primordial phase with a specular behavior of H with respect to the standard one. Remarkably, this is exactly what can be obtained assuming that the cosmological evolution satisfies a principle of "self-duality" – i.e. assuming that the past evolution of our Universe is described by the "dual complement" of the present one [2, 22].

self duality

More precisely, if we consider a cosmological model satisfying (at least approximately) the self-dual condition $a(t) = a^{-1}(-t)$, such that the standard decelerated regime at $t > 0$ smoothly evolves, back in time, into the accelerated partner at $t < 0$, we can then obtain a scenario in which the singularity is automatically regularized, and the initial evolution is automatically of the inflationary type. In such a context the big bang singularity is replaced by an epoch of high (but finite) curvature, characterizing the transition between the standard cosmological phase ($\dot{H} < 0$) and its dual ($\dot{H} > 0$): it comes natural, in such a context, to call *"pre-big bang"* the initial phase ($t < 0$) at growing curvature and growing dilaton, in contrast to the subsequent *"post-big bang"* phase ($t > 0$), describing the standard cosmological evolution.

The dilaton, on the other hand, provides an exponential parametrization of the (tree-level) string coupling $g_s = \exp(\phi/2)$, controlling the relative strength of all (gravitational and gauge) interactions [3, 4]. The principle of self-duality thus suggests that the Universe is lead to its present state after a long evolution started from an extremely simple – almost trivial – configuration, characterized by a nearly flat geometry and by a very small coupling parameter,

$$H^2 \to 0, \qquad \phi \to -\infty, \qquad g_s^2 = \mathrm{e}^{\phi} \to 0, \tag{58}$$

the so-called string perturbative vacuum (see Fig. 2). In this case, the initial Universe is characterized by a regime of extremely low energies in which the "curvatures" (i.e. the field gradients) are small ($\lambda_s^2 H^2 \ll 1$, $\lambda_s^2 \dot{\phi}^2 \ll 1$, ...), the couplings are weak ($g_s^2 \ll 1$), and the background dynamics can be appropriately described by the lowest-order string effective action, at tree-level in the α' and quantum loop expansion (also in agreement with the hypothesis of "asymptotic past triviality" [16]). We can talk of "birth of the Universe from the string perturbative vacuum", as also pointed out in a quantum cosmology context (see e.g [17, 18].

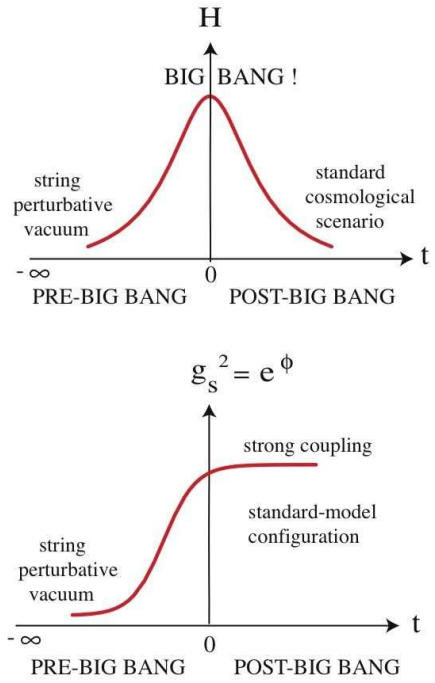

Fig. 2. Qualitative time-evolution of the curvature scale (upper panel) and of the string coupling (lower panel), for a typical self-dual background which smoothly interpolates between the pre-big bang and the post-big bang phase, starting from the string perturbative vacuum.

This picture is in remarkable contrast with the standard (even inflationary) picture in which the Universe starts evolving from a highly-curved geometric state: the more we go back in time, in that context, the more we enter a Planckian and (possibly) trans-Planckian [19] non-perturbative regime of ultra-high energies, requiring the full inclusion of quantum gravity effects, to all orders, for a correct description.

The principle of self-duality, on the contrary, suggests a picture in which the more we go back in time (after crossing the epoch of maximal curvature), the more we approach a *flat, cold* and *vacuum* configuration (strongly reminiscent of the "biblical" scenario quoted at the beginning of Sect. 1), which can be appropriately described by the classical background equations obtained from the action (1). Quantum effects, in the form of higher-curvature and higher-loop contributions, are expected to become important only *towards the end* of the pre-big bang phase, when the background approaches the string scale at $t \to 0_-$. Actually, all studies performed so far have shown that such corrections *must* become dominant, eventually, in order to stop the growth of the curvature [20] and possibly trigger a smooth transition to the post-big bang regime [21].

1.4 A smooth "bounce"

The lowest-order string effective action can appropriately describe the phase of primordial background evolution typical of the pre-big bang scenario, but not the transition to the standard decelerated regime occurring at high curvatures and strong coupling, and requiring the introduction of higher-order corrections. Referring the reader to the existing literature for a detailed review of the transition models studied so far (see for instance [22]), we shall present here only two simple phenomenological examples, by applying, to this purpose, the formalism introduced in subsection 1.2 (and Appendix A). In these examples, in fact, the bouncing transition is induced by the presence of a non-local effective potential $V(\overline{\phi})$, expected to simulate the backreaction of the quantum loop corrections in higher-dimensional manifolds with compact spatial sections [11].

The first example is based on a potential which, in the homogeneous limit, takes the form

$$V(\overline{\phi}) = -V_0 e^{4\overline{\phi}}, \qquad V_0 > 0, \tag{59}$$

four-loop potential

and which may thus perturbatively interpreted as a four-loop potential. With this potential, the duality-invariant equations (45)–(47), in vacuum ($\rho = p = \sigma = 0$), and in the isotropic limit, are solved by the particular exact solution [18]:

$$\begin{aligned} a(t) &= a_0 \left[\frac{t}{t_0} + \left(1 + \frac{t^2}{t_0^2}\right)^{1/2} \right]^{1/\sqrt{d}}, \\ \overline{\phi} &= -\frac{1}{2} \ln \left[t_0 \sqrt{V_0} \left(1 + \frac{t^2}{t_0^2}\right) \right] + \text{const}, \\ \phi &= \ln \frac{\left[t/t_0 + \left(1 + t^2/t_0^2\right)^{1/2} \right]^{\sqrt{d}}}{\left(1 + t^2/t_0^2\right)^{1/2}} + \text{const}, \end{aligned} \tag{60}$$

where t_0 and a_0 are positive integration constants. This regular "bouncing" solution is exactly self-dual – as it satisfies $a(t)/a_0 = a_0/a(-t)$ – and is characterized by a bounded, "bell-like" shape of the curvature and of the dilaton kinetic energy (see Fig. 3). The solution smoothly interpolates between the pre-and post-big bang vacuum solutions (57), (56) (corresponding to the dashed curves of Fig. 3), which are recovered in the asymptotic limits $t \to -\infty$ and $t \to +\infty$, respectively. The bounce of the curvature, and the smooth transition between the two branches of the low-energy solutions, is induced and controlled by the potential (59) which dominates the background evolution in the high-curvature limit $|t| \to 0$, and which becomes rapidly negligible as $t \to \pm\infty$, as illustrated in Fig. 3.

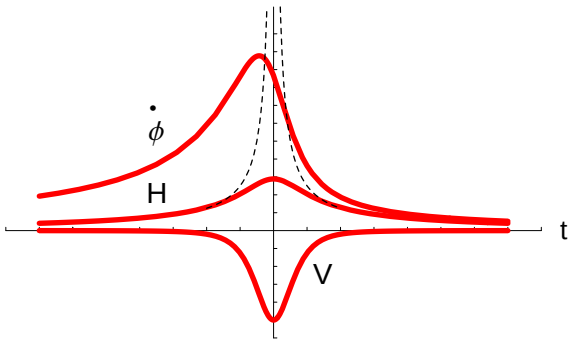

Fig. 3. Plot of the curvature, of the dilaton kinetic energy, and of the potential $V(\overline{\phi})$, for the bouncing solution (60). The dashed curves represent the (singular) vacuum solutions (56), (57), obtained with $V = 0$. All curves are plotted for $t_0 = 1$, $V_0 = 1$, and $d = 3$.

"looking back in time... MG"

It should be noted that in this solution the dilaton keeps growing, monotonically, even in the limit $t \to +\infty$. In more realistic examples, however, such a growth is expected to be damped by the interaction with the matter/radiation post-big bang sources [23], and/or by the action of a suitable non-perturbative potential appearing in the strong coupling regime.

The second example of bounce is based on a general integration of the duality-invariant equations (45)–(47), in the presence of isotropic fluid sources with $\sigma = 0$ and of a two-loop (non-local) potential which in the homogeneous limit takes the form

$$V(\overline{\phi}) = -V_0 e^{2\overline{\phi}}, \qquad V_0 > 0. \tag{61}$$

In this case the equations can be integrated exactly not only for barotropic equations of state ($p/\rho = \gamma = \text{const}$), but also for any ratio p/ρ which is an integrable function of an appropriately defined time-like parameter [2].

An interesting example (motivated by the study of the equation of state of a string gas in rolling backgrounds [24]) is the case in which p/ρ smoothly evolves from the value $\gamma = -1/d$ at $t = -\infty$ to the value $\gamma = 1/d$ at $t = +\infty$, thus connecting the radiation equation of state to its dual partner, according to the law:

$$\frac{p}{\rho} = \frac{1}{d}\frac{x}{\sqrt{x_1^2 + x^2}}. \tag{62}$$

Here x_1 is an arbitrary integration constant, and x is a (dimensionless) time-like coordinate defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{L}{2}\overline{\rho}, \tag{63}$$

where L is a constant with dimensions of length (we are using units in which $2\lambda_{\mathrm{s}}^{d-1} = 1$, so that $[\rho] = L^{-2}$). Using Eqs. (61)–(63), and choosing a simplifying set of integration constants (appropriate to the pedagogical purpose of this paper), we can then obtain the following particular exact solution [2],

$$\begin{aligned}
a &= a_0\left(x + \sqrt{x^2 + x_1^2}\right)^{2/(d-1)},\\
\mathrm{e}^{\phi} &= a_0^d \mathrm{e}^{\phi_0}\left(1 + \frac{x}{\sqrt{x^2 + x_1^2}}\right)^{2d/(d-1)},\\
\rho\mathrm{e}^{\phi} &= \frac{d-1}{dL^2}\mathrm{e}^{2\phi_0}\left(x^2 + x_1^2\right)^{-(d+1)/(d-1)},\\
p\mathrm{e}^{\phi} &= \frac{d-1}{d^2L^2}\mathrm{e}^{2\phi_0} x\left(x^2 + x_1^2\right)^{-(3d+1)/2(d-1)},
\end{aligned} \tag{64}$$

where a_0 and ϕ_0 are integration constants. The smooth and bouncing behavior of this solution is illustrated in Fig. 4.

The above solution is self-dual, in the sense that $\overline{\phi}(x) = \overline{\phi}(-x)$, $\overline{\rho}(x) = \overline{\rho}(-x)$, and

$$\left[\frac{a(x)}{a_0 x_1^{2/(d-1)}}\right] = \left[\frac{a(-x)}{a_0 x_1^{2/(d-1)}}\right]^{-1} \tag{65}$$

(with an appropriate choice of the integration constant a_0 it is always possible to set to 1 the fixed point of scale-factor inversion). The solution satisfies, asymptotically,

$$\begin{aligned}
x \to -\infty \qquad &\Rightarrow \qquad a \sim (-x)^{-2/(d-1)} \sim \overline{\rho} \sim \frac{\mathrm{d}x}{\mathrm{d}t},\\
x \to +\infty \qquad &\Rightarrow \qquad a \sim x^{2/(d-1)} \qquad \sim \frac{1}{\overline{\rho}} \sim \frac{\mathrm{d}t}{\mathrm{d}x}.
\end{aligned} \tag{66}$$

Re-expressing a, ϕ, ρ, p, in the asymptotic limits $x \to \pm\infty$ in terms of the cosmic time t, we can check that this solution smoothly interpolates between the pre-big bang configuration (55) describing accelerated expansion, growing dilaton, negative pressure, and the final post-big bang configuration (52),

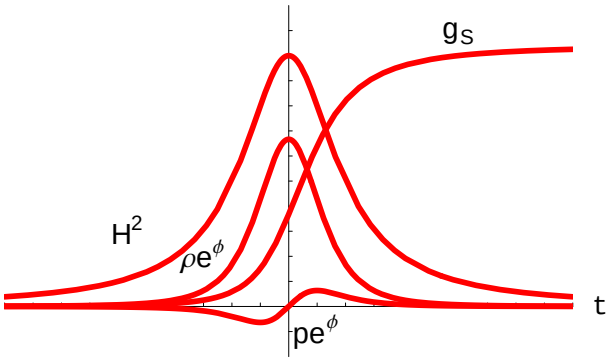

Fig. 4. Plot of the curvature, of the string coupling, of the effective energy density and of the effective pressure for the self-dual solution (64). The curves are plotted for $d = 3$, $L = 1$, $x_1 = 1$, $\phi_0 = 0$, and $a_0 = \exp(-2/3)$.

describing the radiation-dominated state with frozen dilaton and decelerated expansion. As in the previous case the smoothing out of the tree-level singularity, and the appearance of bouncing transition, is a consequence of the effective potential (61).

1.5 Cosmological perturbations

The phase of pre-big bang evolution, being accelerated, can amplify the quantum fluctuations of the metric tensor (and of other background fields) just like any other type of inflationary evolution. However, because of the kinematic properties of pre-big bang inflation (associated to the shrinking of the Hubble horizon H^{-1}), the spectral distribution of the metric fluctuations, after their amplification, tends to grow with frequency [25]. This peculiar aspect of the spectrum may be regarded as representing both an advantage and a difficulty of pre-big bang models with respect to other models of inflation.

????????

The advantage is of phenomenological nature, and refers to the transverse and traceless tensor part of the metric fluctuations. Their amplification leads to the formation of a stochastic background of relic gravitational waves whose spectral energy density, Ω_g, grows wih frequency:

$$\Omega_g(\omega, t) = \left(\frac{H_1}{M_{\rm P}}\right)^2 \Omega_r(t) \left(\frac{\omega}{\omega_1}\right)^{\delta}, \qquad \delta > 0, \qquad \omega \leq \omega_1. \tag{67}$$

Here $M_{\rm P} = 8\pi G = \lambda_{\rm P}^{-1}$ is the Planck mass, $H_1 \simeq M_{\rm s} = \lambda_{\rm s}^{-1}$ the inflation-radiation transition scale (expected to be controlled by the string mass scale $M_{\rm s}$), $\Omega_r = \rho_r/\rho_c$ is the fraction of critical energy density in radiation, ω_1 is the ultraviolet cut-off (i.e. the maximal amplified frequency) of the spectrum, and δ a model-dependent parameter depending on the background kinematics

[25, 26, 27] (see also the contribution of Buonanno and Ungarelli [28] to this volume).

Thanks to the growth of the spectrum, the cosmic graviton background present today as a relic of the inflationary epoch is higher at higher frequencies (in particular, higher than the backgrounds predicted by conventional models of inflation), and thus more easily detectable by current gravitational antennas (see e.g. [22]). Conversely, however, the spectrum is strongly suppressed in the low-frequency regime: we should thus expect, in particular, a negligible contribution of tensor metric perturbations to the observed CMB anisotropy on large scales (as in the case of the ekpyrotic [29] and "new ekpyrotic [30] scenarios where, however, the gravitational background is expected to be low even in the low-frequency regime [31]). It may be stressed, in this connection, that the possible absence of tensor contributions at large scales emerging from (planned) future measurements of the CMB polarization (such as those of WMAP, PLANCK), in combination with a positive signal possibly detected at high frequency by the next generation of gravitational antennas (such as LIGO/VIRGO, LISA, BBO, DECIGO), could represent a strong experimental signal in favor of models of pre-big bang inflation (see e.g. [32]).

DIFFICULTIES!!!

The difficulties associated to a growing spectrum refer to the scalar part of the metric perturbations. In fact, a growing scalar spectrum cannot account for the observed peak structure of the temperature anisotropies of the CMB radiation, which requires, instead, a nearly flat (or "scale-invariant") primordial distribution: $\Omega_s(\omega) \sim \omega^{n_s-1}$, with $n_s \approx 1$. There are two possible ways out of this problem.

1

A first possibility relies on the growth of the dilaton – and thus of the string coupling $g_s^2 = \exp\phi$ – during the phase of pre-big bang inflation. Even starting at weak coupling, a pre-big bang background unavoidably evolves towards the strong coupling regime $g_s \sim 1$. If the bounce is not immediate then the Universe, before the transition to the standard regime, enters a strong-coupling phase where higher-dimensional extended objects like Dirichlet branes and antibranes [4] (whose tension is proportional to the inverse of the string coupling) become light, and can be copiously produced [33]. The cosmic evolution may become dominated by the presence of these higher-dimensional sources [34] and, in that context, a phase of conventional slow-roll inflation can be triggered by the interaction (and eventual collision) of a brane-antibrane pair [35] (see also the contribution of Tye [36] to this volume). This new inflationary regime may efficiently dilute all pre-existing inhomogeneities and generate a new spectrum of scale-invariant, adiabatic scalar perturbations, as required for a successful explanation of the observed anisotropy. This may resolve the incompatibility between a (growing) spectrum of pre-big bang perturbations and present large-scale observations.

2

There is, however, a second possibility which avoids introducing additional inflationary epochs besides the initial dilaton-dominated one, and which is based on the so-called "curvaton mechanism" [37]. According to this mechanism the (flat, adiabatic) spectrum of scalar metric perturbations, responsible

CURVATON
see bibl.
Lyth & Wands

KR axion = curvaton!!!!!!!

for the observed anisotropies, is not produced during the primordial evolution: instead, it is the outcome of the post-inflationary decay of a massive scalar field (the curvaton), whose quantum fluctuations are amplified during inflation with a nearly flat spectrum, and are converted into curvature perturbations after its decay. In the context of the pre-big bang scenario the role of the curvaton is possibly played by the Kalb-Ramond axion σ [38], associated – by space-time duality – to the four components of the NS-NS two-form $B_{\mu\nu}$ present in the massless multiplet of the string spectrum.

For a brief discussion of this possibility we should explain, first if all, why axion fluctuations can be amplified by pre-big bang inflation with a flat spectrum [39], unlike metric fluctuations. The reason is that the slope of the spectrum is directly related to kinematic behavior of the effective "pump field" responsible for the amplification, and that metric and axion fluctuations have different pump fields, even in the same given background.

pump fields

In order to clarify this point let us complete the low-energy action (1) by adding the contribution of the antisymmetric field $B_{\mu\nu}$, considering (for simplicity) a model already dimensionally reduced to four space-time dimensions:

including B mn

$$S = -\frac{1}{2\lambda_s^2}\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\mathrm{e}^{-\phi}\left(R + \partial_\mu\phi\,\partial^\mu\phi + V - \frac{1}{12}H_{\mu\nu\alpha}H^{\mu\nu\alpha}\right),$$
$$H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\alpha\mu} + \partial_\alpha B_{\mu\nu}. \qquad (68)$$

In the absence of sources the equations of motion for $B_{\mu\nu}$ are automatically satisfied by introducing the "dual" axion field σ, such that

$$H^{\mu\nu\alpha} = \frac{\mathrm{e}^{\phi}}{\sqrt{-g}}\,\epsilon^{\mu\nu\alpha\beta}\partial_\beta\sigma, \qquad (69)$$

sigma axion field

and the last term of the action (68) can be replaced by

$$S = \frac{1}{4\lambda_s^2}\int \mathrm{d}^4x\sqrt{-g}\;\mathrm{e}^{\phi}(\nabla\sigma)^2. \qquad (70)$$

Perturbing the metric and the axion field,

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}, \qquad \sigma \to \sigma + \delta\sigma, \qquad (71)$$

around a homogeneous, conformally flat metric background, using the conformal time coordinate η (such that $\mathrm{d}t = a\mathrm{d}\eta$), and applying the standard formalism of linear cosmological perturbations (see e.g. [40]), we obtain for tensor metric and axion fluctuations, respectively, the following quadratic actions:

$$S_h = \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}\eta\,z_h^2(\eta)\left(h'^2 + h\nabla^2 h\right),$$
$$z_h = \frac{a}{\sqrt{2}\,\lambda_s}\,\mathrm{e}^{-\phi/2}, \qquad (72)$$

$$S_\sigma = \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}\eta\, z_\sigma^2(\eta)\left(\delta\sigma'^2 + \delta\sigma\nabla^2\delta\sigma\right),$$
$$z_\sigma = \frac{a}{\sqrt{2}\,\lambda_{\rm s}}\,\mathrm{e}^{\phi/2}. \tag{73}$$

Here h is one of the two physical polarization states of tensor perturbations, the primes denote differentiation with respect to η, and ∇^2 is the flat-space Laplace operator, $\nabla^2 = \delta^{ij}\partial_i\partial_j$. The variation of these actions with respect to h and $\delta\sigma$ leads to the equations of motion, which can be written in terms of the canonical variables $u = (hz_h)$ and $v = (\delta\sigma z_\sigma)$ as follows:

$$(hz_h)'' - \left(\nabla^2 + \frac{z_h''}{z_h}\right)(hz_h) = 0, \tag{74}$$

tensor metric

$$(\delta\sigma z_\sigma)'' - \left(\nabla^2 + \frac{z_\sigma''}{z_\sigma}\right)(\delta\sigma z_\sigma) = 0. \tag{75}$$

axion

The canonical equations are the same for u and v, but the pump fields, z_h and z_σ, are different.

Consider, for instance, the axion equation (75), and recall that during inflation the accelerated evolution of the pump field can be parametrized as a power-law evolution in the negative range of the conformal-time parameter [22, 40], i.e.

$$z_\sigma(\eta) = \frac{M_{\rm P}}{\sqrt{2}}\left(-\frac{\eta}{\eta_1}\right)^{\alpha_\sigma}, \qquad -\infty \le \eta < 0, \tag{76}$$

where $\eta_1 > 0$ is some appropriate reference time-scale. Expanding in Fourier modes, Eq. (75) becomes a Bessel equation for the mode v_k,

$$v_k'' + \left[k^2 - \frac{(\nu_\sigma^2 - 1/4)}{\eta^2}\right]v_k = 0, \qquad \nu_\sigma = \frac{1}{2} - \alpha_\sigma, \tag{77}$$

and its general solution can be conveniently written as a combination of first-kind and second-kind Hankel functions [41], of argument $k\eta$ and index ν_σ, as follows:

$$v_k = (-\eta)^{1/2}\left[A_+(k)H^{(2)}_{\nu_\sigma}(k\eta) + A_-(k)H^{(1)}_{\nu_\sigma}(k\eta)\right]. \tag{78}$$

We shall now canonically normalize this general solution by imposing that the initial state of the fluctuations corresponds to a spectrum of quantum vacuum fluctuations [22, 40]. More explicitly, we shall require that the mode v_k, on the initial spatial hypersurface at $\eta \to -\infty$, may represent freely oscillating, positive frequency modes satisfying the canonical normalization

$$v_k v_k'^* - v_k' v_k^* = i, \tag{79}$$

from which

$$v_k \to \frac{\mathrm{e}^{-ik\eta}}{\sqrt{2k}}, \qquad \eta \to -\infty \tag{80}$$

(modulo an arbitrary phase). Using the large argument limit of the Hankel functions [41],

$$H_\nu^{(2)}(k\eta) = \sqrt{\frac{2}{\pi k\eta}}\, \mathrm{e}^{-ik\eta - i\epsilon_\nu}, \qquad H_\nu^{(1)}(k\eta) = \sqrt{\frac{2}{\pi k\eta}}\, \mathrm{e}^{ik\eta + i\epsilon_\nu} \tag{81}$$

($\epsilon_\nu = -\pi/4 - \nu\pi/2$), we obtain $A_+ = \sqrt{\pi/4}$ and $A_- = 0$. The normalized exact solution for the the axion fluctuations $\delta\sigma_k$ can be finally written as

$$\delta\sigma_k = \frac{v_k}{z_\sigma} = \frac{\mathrm{e}^{i\theta_k}}{M_{\mathrm{P}}} \left(\frac{\pi\eta_1}{2}\right)^{1/2} \left(\frac{\eta}{\eta_1}\right)^{\nu_\sigma} H_{\nu_\sigma}^{(2)}(k\eta), \tag{82}$$

where θ_k is an arbitrary phase determined by the choice of the initial conditions.

In order to determine the spectrum of the fluctuations after their inflationary amplification we must then consider the limit $\eta \to 0_-$, in which $|k\eta| \ll 1$ and the amplitude of the mode k is stretched "outside the horizon". We can use, to this purpose, the small argument limit of the Hankel functions [41], which reads (for $\nu \neq 0$),

$$H_\nu^{(2)}(k\eta) = p_\nu^*(k\eta)^\nu - iq_\nu(k\eta)^{-\nu} + \ldots \tag{83}$$

where q_ν and p_ν are complex (ν-dependent) coefficients (for $\nu = 0$ there are additional logarithmic corrections). We obtain, in this limit,

$$\delta\sigma_k = \frac{v_k}{z_\sigma} \to \frac{\mathrm{e}^{i\theta_k}}{M_{\mathrm{P}}} \left(\frac{\pi\eta_1}{2}\right)^{1/2} \left[-iq_{\nu_\sigma}(k\eta_1)^{-\nu_\sigma} + p_{\nu_\sigma}^*(k\eta_1)^{\nu_\sigma} \left(\frac{\eta}{\eta_1}\right)^{2\nu_\sigma}\right]. \tag{84}$$

The cases we are interested here are limited to "conventional" inflationary backgrounds with $\alpha_\sigma \leq 1/2$, i.e. $\nu_\sigma \geq 0$ (see [32] for a detailed discussion of all possibilities). For such backgrounds the time-dependence of $\delta\sigma_k$ tends to disappear as $\eta \to 0_-$, the fluctuations become frozen, asymptotically, and their (dimensionless) spectral amplitude $k^3|\delta\sigma_k|^2$, controlling the typical amplitude of the perturbations on a comoving length scale $r = k^{-1}$ [40], has the following k-dependence:

$$k^3\,|\delta\sigma_k|^2 \sim k^{3-2\nu_\sigma} = k^{2+2\alpha_\sigma}. \tag{85}$$

This result also holds in the limiting case $\alpha_\sigma = 1/2$ with the only addition of a mild logarithmic correction [26, 27], i.e. $k^3\,|\delta\sigma_k|^2 \sim k^3 \ln^2(k\eta_1)$.

The above calculations can be exactly repeated, in the same form, for the tensor perturbation variable, starting from Eq. (74): the resulting spectrum is formally the same,

$$k^3\,|h_k|^2 \sim k^{3-2\nu_h} = k^{2+2\alpha_h}, \tag{86}$$

with the difference that the spectral slope is now determined by the power α_h, controlling the evolution of the tensor pump field z_h through an equation analogous to Eq. (76).

We are now in the position of discussing the possible pre-big bang production of a flat spectrum of axion fluctuations, even if the associated metric fluctuations are amplified (in the same background) with a growing spectrum. Let us consider, to this purpose, an exact anisotropic solution of the string cosmology equations (9)–(11), in vacuum, and without dilaton potential. The solution describes a phase of pre-big bang inflation characterized by the accelerated (isotropic) expansion of three spatial dimensions, with scale factor $a(\eta)$, and by the accelerated contraction of n "internal" spatial dimensions, with scale factors $b_i(\eta)$, $i = 1, \ldots, n$. In conformal time, such a solution can be parametrized for $\eta \to 0_-$ as [13, 22]

$$a = \left(-\frac{\eta}{\eta_1}\right)^{\beta_0/(1-\beta_0)}, \qquad b_i = \left(-\frac{\eta}{\eta_1}\right)^{\beta_i/(1-\beta_0)},$$
$$\phi_{4+n} = \frac{\sum_i \beta_i + 3\beta_0 - 1}{1-\beta_0} \ln\left(-\frac{\eta}{\eta_1}\right), \tag{87}$$

where the constant coefficients β_0, β_i satisfy the Kasner-like condition

$$\sum_i \beta_i^2 + 3\beta_0^2 = 1, \tag{88}$$

and ϕ_{4+n} is the higher-dimensional dilaton appearing in the full $(4+n)$-dimensional effective action. The four-dimensional dilaton ϕ is related to ϕ_{4+n} by

$$\mathrm{e}^{-\phi} = V_n \, \mathrm{e}^{-\phi_{4+n}} \equiv \mathrm{e}^{-\phi_{4+n}} \prod_i b_i, \tag{89}$$

namely by

$$\phi = \phi_{4+n} - \sum_i \ln b_i = \frac{3\beta_0 - 1}{1-\beta_0} \ln\left(-\frac{\eta}{\eta_1}\right). \tag{90}$$

Let us compute, for this background, the kinematic powers α_h and α_σ controlling the evolution of the pump fields (72), (73):

$$z_h \sim a\mathrm{e}^{-\phi/2} \sim (-\eta)^{\alpha_h}, \qquad \alpha_h = \frac{1}{2}, \tag{91}$$

$$z_\sigma \sim a\mathrm{e}^{\phi/2} \sim (-\eta)^{\alpha_\sigma}, \qquad \alpha_\sigma = \frac{5\beta_0 - 1}{2(1-\beta_0)}. \tag{92}$$

It follows, according to Eq. (86), that the spectrum of tensor (as well as of scalar) metric perturbations is always characterized by a slope which is cubic (modulo log corrections) [13, 26, 27], and which is also "universal", in the sense that it is insensitive to the background parameters (n, β_0, β_i). For the axion fluctuations, on the contrary, we find from Eq. (85) that the spectral slope is strongly dependent on such parameters, and that a scale-invariant spectrum with $2 + 2\alpha_\sigma = 0$ is allowed, in particular, provided $\beta_0 = -1/3$.

We may note, in the special case in which the background is fully isotropic and expanding (i.e., $\beta_0 = \beta_i < 0$), that the Kasner condition (88) implies

$\beta_0 = -1/\sqrt{d}$, so that a scale-invariant spectrum corresponds to $d = 9$, i.e. just to the number of spatial dimensions determined by critical superstring theory [3, 4].

In the less special case in which the spatial geometry can be factorized as the product of a 3-dimensional and a n-dimensional isotropic subspaces we have, instead, $\beta_i = \beta \neq \beta_0$, and $3\beta_0^2 + n\beta^2 = 1$. The spectral slope, in this case, can be expressed in terms of the parameter

$$r = \frac{1}{2}\left(\frac{\dot{V}_n}{V_n}\right)\left(\frac{\dot{V}_3}{V_3}\right)^{-1} = \frac{n\beta}{6\beta_0}, \tag{93}$$

controlling the relative time-evolution of the proper volumes of the internal and external spaces. Eliminating β in terms of β_0 through the Kasner condition, and replacing β_0 with r in Eq. (92), one can then parametrize the deviations from a flat axion spectrum as the relative shrinking or expansion of the two subspaces [42].

Given a sufficiently flat spectrum of axion fluctuations, amplified by the phase of pre-big bang inflation, we are then lead to a post-big bang configuration which is initially characterized (at some given time scale η_i) by a primordial sea of "isocurvature" scalar perturbations, dominated on super-horizon scales by the axion fluctuations $\delta\sigma$ (the metric fluctuations are subdominant on such large scales, being strongly suppressed by the steep slope of their spectrum). The axion can play the role of the curvaton provided that the initial configuration, besides containing the initial fluctuations $\delta\sigma_i$, also contains a non-vanishing axion background, $\sigma_i \neq 0$, whose energy density ρ_σ – even if subdominant – is initially determined by an appropriate potential (possibly approximated by $V_\sigma \sim m^2\sigma^2$). In that case the background evolution, after an initial slow-roll regime, leads to a phase where the axion background starts oscillating with proper frequency m, at a curvature scale $H \sim m$, simulating a dust fluid ($\rho_\sigma \sim a^{-3}$) which may become dominant with respect to the radiation fluid, and eventually decay at the typical scale $H \sim \lambda_{\rm P}^2 m^3$.

In such a type of background the axions fluctuations $\delta\sigma$ become linearly coupled to scalar metric perturbations, and may act as sources for the so-called Bardeen potential Ψ. New metric perturbations can then be generated, starting from $\Psi(\eta_i) = 0$, with the same spectral slope as the axion one, and with a spectral amplitude not smaller, in general, than the axion amplitude. Referring to the literature for a detailed computation [37, 38], we shall recall here that the final spectrum (after the axion decay) of the super-horizon Bardeen potential is related to the initial axion perturbations by

$$\begin{aligned} |\Psi_k| &= \lambda_{\rm P} f(\sigma_i)\,|\delta\sigma_k(\eta_i)|\,, \\ f(\sigma_i) &= c_1\frac{\Omega_\sigma}{\lambda_{\rm P}\sigma_i} + c_2 + c_3\,\lambda_{\rm P}\sigma_i \end{aligned} \tag{94}$$

(the $\lambda_{\rm P}$ factors are due to the canonical normalization of the axion field and of its fluctuations). Here σ_i is the initial amplitude of the axion background,

$\Omega_\sigma \sim 1$ is the axion fraction of critical density at the axion decay epoch, and c_1, c_2, c_3 are dimensionless numbers of order one (Ω_σ cannot be much smaller than one, to avoid a too strong "non-Gaussianity" of the spectrum" [43]). Thanks to its structure, the "form factor" $f(\sigma_i)$ has a minimum of order one around $\lambda_{\rm P}\sigma_i \sim 1$. A (nearly) scale-invariant axion spectrum thus reproduces a (nearly) scale-invariant spectrum of scalar metric perturbations.

As discussed in the literature, a curvaton-induced spectrum of scalar metric perturbations provides the right "adiabatic" initial conditions for reproducing the observed temperature anisotropies of the CMB radiation, exactly as in the case of the slow-roll scenario. The only difference is the "indirect" (i.e., post-inflationary) production of the scalar spectrum, triggered by the presence of a non-vanishing axion background. It must be stressed, however, that the direct connection (94) with the axion spectrum of primordial origin gives us the possibility of extracting, from present CMB observations, important constraints on the parameters of pre-big bang models of inflation [38].

In particular, using the experimental normalization of the anisotropy spectrum, and the direct relation between the pre-big bang inflation scale H_1 and the string scale $M_{\rm s}$, one can speculate about the possibility of "weighing the string mass with the CMB data" [44]. Another application concerns the slope of the scalar perturbation spectrum which, according to most recent WMAP results [45], is given by

$$n_s \equiv 3 + 2\alpha = 0.951^{+0.015}_{-0.019}. \tag{95}$$

Using Eq. (92), and the Kasner condition (88), one obtains

$$\beta_0 \simeq -0.355, \qquad \sum_i \beta_i^2 \simeq 0.62. \tag{96}$$

With $d = 9$ dynamical dimensions this result seems to point out the existence of a small anisotropy between the kinematics of the external and internal spaces during pre-big bang inflation (a fully isotropic expansion would correspond, in fact, to $\beta_0 = -1/\sqrt{9} \simeq -0.33$ and $\sum_i \beta_i^2 = 6/9 \simeq 0.66$). It should be noted, however, that other interpretations of the data are also possible. For instance, the result (95) is also compatible with $\beta_0 = -1/\sqrt{8} \simeq -0.3535$, describing the isotropic expansion of $d = 8$ spatial dimensions! Incidentally, the number (and the kinematics) of the extra spatial dimensions play a crucial role also in the possible production of primordial "seeds" for the large-scale magnetic fields [46].

It should be mentioned, finally, a possible non-Gaussian "contamination" of the statistical properties of the anisotropy spectrum, possibly present in curvaton models with $\Omega_\sigma \ll 1$ [43] (see Eq. (94)). A possible detection of non-Gaussianity, in future CMB measurements, could provide support to the curvaton mechanism, and could be used for a direct discrimination between this scenario and other, more standard scenarios based on slow-roll inflation.

2 The relic dilaton background

The accelerated evolution of the Universe, during the phase of pre-big bang inflation, amplifies the quantum fluctuations of all fields present in the string effective action: thus, in particular, it amplifies the dilaton fluctuations, $\delta\phi \equiv \chi$. The formation of a stochastic background of relic gravitational waves, associated to the amplification of the tensor part of metric fluctuations, is thus accompanied by the simultaneous formation of a cosmic background of relic dilatons [47], whose primordial (high-energy) spectral distribution tends to follow that of tensor metric perturbations [13].

There is, however, a possible important difference in the present intensity of the two cosmic backgrounds, due to the fact that dilatons – unlike gravitons – could become massive in the course of the standard (post-inflationary) evolution. Actually, dilatons *must* become massive if they are non-universally coupled to ordinary matter with gravitational strength (or higher) [48, 49], to avoid the presence of long-range scalar forces which are excluded by the standard gravitational phenomenology (in particular, by the high-precision tests of the equivalence principle). The induced mass may drastically modify the amplitude and the slope of the dilaton spectrum, in the frequency band associated to its non-relativistic sector.

For a simple illustration of the effects of the mass on the spectrum we will consider here the model of vacuum, dilaton-dominated pre-big bang background described by Eq. (57), smoothly joined at $\eta = -\eta_1 < 0$ to the standard radiation-dominated background with frozen dilaton, described by Eq. (52) (we shall work in $d = 3$ spatial dimensions). Perturbing the background equations [13] one finds, in this case, that the dilaton pump field is the same field $z_h \sim a\exp(-\phi/2)$ governing the amplification of metric fluctuations. Taking into account a possible mass contribution, $m^2 = \partial^2 V/\partial\phi^2$, one then obtains for the Fourier modes χ_k the canonical equation:

$$(\chi z_h)_k'' + \left(k^2 + m^2 a^2 - \frac{z_h''}{z_h}\right)(\chi z_h)_k = 0. \tag{97}$$

During the initial pre-big bang regime the potential is negligible ($m^2 = 0$), and the canonically normalized solution for χ_k is that of Eq. (82) (with ν_σ replaced by ν_h). In the subsequent radiation-dominated era ϕ stabilizes to a constant, so that $z_h \sim a \sim \eta$ and the effective potential z_h''/z_h is vanishing. Assuming that the dilaton mass is small enough in string units, and considering the high-frequency sector of the spectrum, associated to the relativistic modes of proper momentum $p = (k/a) \gg m$, we can neglect also the mass term of Eq. (97), to obtain the general solution

$$\chi_k = \frac{1}{a\sqrt{2k}}\left[c_+(k)\mathrm{e}^{-ik\eta} + c_-(k)\mathrm{e}^{ik\eta}\right], \qquad \eta \geq -\eta_1. \tag{98}$$

Matching χ and χ' with the pre-big bang solution (82) at η_1, for super-horizon modes with $(k\eta_1) \ll 1$, we are lead to

$$c_{\pm}(k) = \pm c(k) \mathrm{e}^{\mp ik\eta_1}, \qquad |c(k)| \sim (k\eta_1)^{-\nu_h - 1/2} \tag{99}$$

(modulo numerical factors with modulus of order one). Thus, at large times $\eta \gg \eta_1$,

$$\chi_k \sim \frac{c(k)}{a\sqrt{k}} \sin k\eta. \tag{100}$$

The spectral energy density for the relativistic sector of the dilaton background, in the radiation era, in then determined by

$$\begin{aligned} k\frac{\mathrm{d}\rho}{\mathrm{d}k} &= \frac{k^3}{2a^2}\left(|X_k'|^2 + k^2|X_k|^2\right) \\ &\sim \left(\frac{k}{a}\right)^4 |c(k)|^2 \sim \left(\frac{k}{a}\right)^4 \left(\frac{k}{k_1}\right)^{-2\nu_h - 1} = p^4 \left(\frac{p}{p_1}\right)^{-2\nu_h - 1}, \end{aligned} \tag{101}$$

where $k_1 \sim \eta^{-1}$ is the high-frequency cut-off scale. In units of critical energy density, $\rho_c = 3M_{\mathrm{P}}^2 H^2$,

$$\Omega_\chi(p,t) = \frac{p}{\rho_c}\frac{\mathrm{d}\rho_\chi}{\mathrm{d}p} \sim \left(\frac{H_1}{M_{\mathrm{P}}}\right)^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^4 \left(\frac{p}{p_1}\right)^{\delta}, \qquad m < p < p_1, \tag{102}$$

where we have defined the (model-dependent) slope parameter $\delta = 3 - 2\nu_h > 0$, and we have introduced the (time-dependent) proper momentum associated to the cut-off scale, $p_1 = k_1/a = H_1 a_1/a$, determined by the background curvature scale H_1 at the end of inflation. In general, $(H_1/H)^2(a_1/a)^4 \equiv \rho_r(t)/\rho_c(t) \equiv \Omega_r(t)$, and we may thus conclude that the relativistic sector of the dilaton spectrum, in the radiation era, is exactly the same as the spectrum of tensor metric perturbations (see Eq. (67)), in the same model of background.

However, even if the mass is small, and initially negligible, the proper momentum $p = k/a(t)$ is continuously red-shifted with respect to m during the subsequent cosmological evolution, so that all modes tend to become non-relativistic, $p < m$. For non-relativistic modes the solution (98) is no longer valid, and the correct spectrum must refer to the exact solutions of Eq. (97) with $m \neq 0$. In the radiation era such a solution can be given in terms of the Weber cylinder functions [50], and one finds that the non-relativistic sector of the spectrum splits into two branches, with different slopes: a first branch of modes becoming non-relativistic at a time scale t_{nr} when they are *already inside* the horizon, with proper momentum p such that $p(t_{nr}) \sim m \gg H(t_{nr})$; and a second branch of modes becoming non-relativistic when they are *still outside* the horizon, with $p(t_{nr}) \sim m \ll H(t_{nr})$. The two branches are separated by the momentum scale p_m of the mode becoming non-relativistic just at the time of horizon crossing, i.e. $p(t_{nr}) = m = H(t_{nr})$, and thus related to the cut-of scale p_1 by

$$\frac{p_m}{p_1} = \frac{m\, a_{\mathrm{nr}}}{H_1 a_1} = \frac{m}{H_1}\left(\frac{H_1}{H_{\mathrm{nr}}}\right)^{1/2} = \left(\frac{m}{H_1}\right)^{1/2}. \tag{103}$$

Without applying to the explicit form of the massive solutions of Eq. (97), a quick estimate of the non-relativistic spectrum can be obtained [51] by noting that, if $p_{nr} > H_(t_{nr})$, the number of produced dilatons is the same as in the relativistic case, and the only effect of the non-relativistic transition is a rescaling of the energy density, i.e.

$$\Omega_\chi^{\rm rel} \to \Omega_\chi^{\rm nr} = \left(\frac{m}{p}\right) \Omega_\chi^{\rm rel}. \tag{104}$$

For this branch of the spectrum we then obtain, from Eq. (102),

$$\Omega_\chi(p,t) \sim \left(\frac{m}{H_1}\right) \left(\frac{H_1}{M_{\rm P}}\right)^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3 \left(\frac{p}{p_1}\right)^{\delta-1}, \quad p_m < p < m. \tag{105}$$

In the case $p_{nr} < H_{nr}$, on the contrary, the slope of the spectrum – determined by the background kinematics at the time of horizon exit – has to be the same as that of the relativistic sector, while the time-dependence has to be the non-relativistic one ($\rho_\chi \sim a^{-3}$) of Eq. (105). Continuity with the branch (105) at $p = p_m$ then gives

$$\Omega_\chi(p,t) \sim \left(\frac{m}{H_1}\right)^{1/2} \left(\frac{H_1}{M_{\rm P}}\right)^2 \left(\frac{H_1}{H}\right)^2 \left(\frac{a_1}{a}\right)^3 \left(\frac{p}{p_1}\right)^{\delta}, \quad p_{\rm eq} < p < p_m. \tag{106}$$

The lower limit $p_{\rm eq} < p$ has been inserted here to recall that we are neglecting the effects of the transition to the matter-dominate phase, i.e. we are considering modes re-entering the horizon during the radiation era, with $p > p_{\rm eq} = H_{\rm eq} \sim 10^{-27}$ eV. We should recall, also, that the spectrum has been computed in a radiation-dominated background, and thus is valid, strictly speaking, only for $t > t_{\rm eq}$.

The three branches (102), (105), (106) describe the spectrum (between $p_{\rm eq}$ and p_1) of primordial dilatons produced in the simple example of "minimal" pre-big bang model that we have considered. We refer to the literature for a more detailed computation, for a discussion of its transmission to the present epoch t_0, and for the possible modifications induced by generalized background evolutions (see e.g. [32]). For the pedagogical purpose of this paper this example provides a sufficiently clear illustration of the effects of the mass on the spectrum: in particular, it clearly illustrates the enhancement produced at lower frequencies because of the reduced spectral slope of the branch (105), which may become even decreasing if $\delta < 1$ (see Fig. 5).

In such a context one is naturally lead to investigate whether this enhanced intensity might favor the detection of a non-relativistic dilaton background, with respect to other, relativistic types of cosmic radiation (such as the relic graviton background).

2.1 Light but non-relativistic dilatons

For a phenomenological discussion of this possibility we must start with two important assumptions. The first is that the produced dilaton are light enough

to have survived until the present epoch. Supposing that massive dilatons have dominant decay mode into radiation (e.g., two photons), with gravitational coupling strength, i.e. with a decay rate $\Gamma \sim \lambda_{\rm P}^2 m^3$, it follows that the primordial graviton background is still "alive" in the present Universe (characterized by the time-scale H_0^{-1}) provided $H_0^{-1} < \Gamma^{-1}$, i.e.

$$m \lesssim 10^2\,\mathrm{MeV}. \tag{107}$$

The second assumption we need is that the total energy density of the dilaton background, integrated over all modes, turns out to be dominated by its non-relativistic sector. Only in this case we can evade the stringent bound imposed by the nucleosynthesis, which applies to the relativistic part of any cosmic background of primordial origin.

The energy density of a relativistic background, in fact, evolves in time like the radiation energy density, $\rho^{rel}/\rho_{rad} = \Omega^{rel}/\Omega_{rad} = \text{const}$: the present value of their ratio is thus the same as the value of the ratio at the nucleosynthesis epoch. To avoid disturbing the nuclear processes occurring at that epoch, on the other hand, one must require that $\Omega^{rel}/\Omega_{rad} \lesssim 0.1$ [52]. Using the present value of Ω_{rad} one is lead then to the constraint $\Omega^{rel}(t_0) \lesssim 5 \times 10^{-6}$, which imposes a severe constraint on all relativistic primordial backgrounds. In particular, it imposes an upper limit on the peak value of the graviton background produced in models of pre-big bang inflation, thus determining the minimal level of sensitivity required for its detection [22].

The energy density of a non-relativistic background, on the contrary, evolves like the dark-matter density, and grows in time with respect to the radiation background: $\Omega^{nr}/\Omega_{rad} \sim a$. As a consequence, the value of Ω^{nr} can be very large today, even if negligible at the nucleosynthesis epoch. The only constraint we must apply, in this case, is the critical density bound,

$$h^2 \Omega_\chi(t) = h^2 \int^{p_1} \mathrm{d}(\ln p)\, \Omega_\chi(p,t) < 1, \tag{108}$$

to be imposed at any time t, to avoid a Universe over-dominated by such a cosmic background of dust matter. Here $h \simeq 0.73$ is the present value of the Hubble parameter H_0 in units of 100 km s^{-1} Mpc.

For the dilaton spectrum of Eqs. (102)–(106) there are, in particular, two different cases in which the total energy density is dominated by the non-relativistic modes. A first (obvious) possibility is the case in which all modes of the spectrum are presently non-relativistic, namely $p_1(t_0) < m$ (in this case the branch (105) extends from p_m to p_1). This implies, however, that

$$\begin{aligned} m > \frac{H_1 a_1}{a_0} &= H_1 \frac{a_1}{a_{\rm eq}} \frac{a_{\rm eq}}{a_0} = H_1 \left(\frac{H_{\rm eq}}{H_1}\right)^{1/2} \left(\frac{H_0}{H_{\rm eq}}\right)^{2/3} \\ &\simeq \left(\frac{H_1}{M_{\rm P}}\right)^{1/2} 10^{-4}\mathrm{eV}. \end{aligned} \tag{109}$$

For a typical string-inflation scale, $H_1 \sim M_s$, we obtain a lower limit on m which is well compatible with the upper limit (107), but which requires mass values too high to be compatible with the sensitivity band of present gravitational antennas (see Subsection 2.2).

The second (more interesting) possibility is the case in which $m < p_1(t_0)$, but the parameter δ is smaller than one, and the slope is flat enough, so that the spectrum is peaked not at p_1 but at $p_m = p_1(m/H_1)^{1/2}$ (see Fig. 5). In that case the momentum integral (108) is dominated by the peak value $\Omega_\chi(p_m)$, and the critical density bound can be approximated by the condition $\Omega_\chi(p_m, t_0) \lesssim 1$. Using Eq. (105), and noting that in the matter-dominated era ($t > t_{\rm eq}$) the value of the non-relativistic spectrum keeps frozen at the equality value $\Omega_\chi(t_{\rm eq})$, we are lead to the condition $\Omega_\chi(t_{\rm eq}, p_m) \lesssim 1$, which implies

$$m \lesssim \left(H_{\rm eq}\, M_{\rm P}^4\, H_1^{\delta-4}\right)^{1/(\delta+1)}. \tag{110}$$

For $H_1 \sim M_s$, and $\delta \to 0$, this bound can be saturated by masses as small as

$$m \sim H_{\rm eq} \left(\frac{M_{\rm P}}{M_s}\right)^4 \sim 10^{-23}\,{\rm eV}. \tag{111}$$

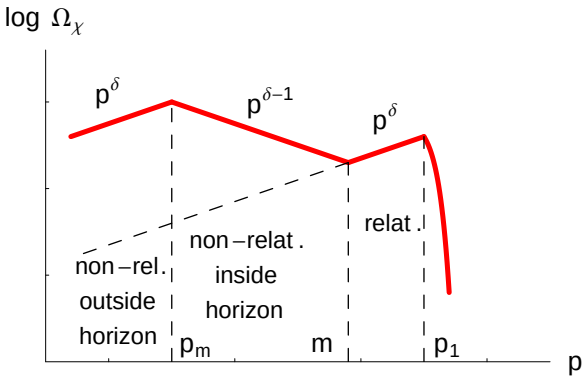

Fig. 5. Example of dilaton spectrum dominated by the non-relativistic sector. The spectrum is peaked at $p = p_m$, and the slope parameter satisfies the condition $\delta < 1$.

It is quite possible, therefore, to have a dilaton mass small enough to fall within the sensitivity range of present gravitational detectors, even if the energy density of the dilaton background is dominated by non-relativistic modes (thus evading the relativistic upper bound $\Omega^{rel} \lesssim 10^{-6}$), and even if the background intensity is large enough to saturate the critical density bound, $\Omega_\chi \sim 1$.

So small mass values, however, are necessarily associated with long-range dilaton forces: in particular, if the mass satisfies the condition $m < p_1(t_0) \sim$

$(M_s/M_P)^{1/2}10^{-4}$ eV (as in the example illustrated in Fig. 5), the corresponding force has a range exceeding the centimeter. This might imply macroscopic violations of the equivalence principle (due to the non-universality of the dilaton coupling [48]), and macroscopic deviations from the standard Newtonian form of the low-energy gravitational interactions (which seem to be excluded, however, by present experimental results [53, 54]).

We should recall, in fact, that in the presence of long-range dilaton fields the motion of a macroscopic test body with nonzero dilaton charge is no longer described by a geodesics. There are forces on the test body due to the gradients of the dilaton field, according to the generalized conservation equation

$$\nabla_\nu T_\mu{}^\nu = \frac{\sigma}{2}\nabla_\mu\phi, \tag{112}$$

following from the application of the contracted Bianchi identity to the gravi-dilaton equations (3) and (5). The integration of this conservation equation over a (space-like) $t = \text{const}$ hypersurface then gives, in the point-particle (or monopole) approximation, the non-geodesic equation of motion [55]

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma_{\alpha\beta}{}^\mu u^\alpha u^\beta = q\nabla^\mu\phi, \tag{113}$$

where q is a dimensionless ratio representing the relative intensity of scalar to tensor forces (i.e., the effective dilaton charge per unit of gravitational mass of the test body).

For the fundamental components of macroscopic matter, such as quark and lepton fields, the value of q (or of the charge density σ) is to be determined from an effective action which includes all relevant dilaton loop corrections [48, 13], and which is of the form

$$S = \frac{1}{2\lambda_s^2}\int \mathrm{d}^4x\sqrt{-g}\Big[- Z_R(\phi)R - Z_\phi(\phi)(\nabla\phi)^2 - V(\phi) + Z_k^i(\phi)(\nabla\psi_i)^2 - M_i^2 Z_m^i(\phi)\psi_i^2\Big]. \tag{114}$$

Here we have used, for simplicity, a scalar model of matter fields ψ_i, and we have called Z the dilaton "form factors" arising from the loop corrections. The effective dilaton charge, therefore, turns out to be frame-dependent (the charge q appearing in Eq. (113), for instance, is referred to the S-frame action and to the S-frame equations (112)). The reason of such a frame dependence is that, in a generic frame, the metric and the dilaton fields are non-trivially mixed through the Z_R and Z_ϕ coupling functions, so that the associated dilaton charge actually controls the matter coupling *not to the pure scalar part*, but *to a mixture of scalar and tensor part* of the gravi-dilaton field.

A frame-independent and unambiguous definition of the dilaton coupling strengths can be given, however, in the canonically rescaled Einstein frame

(E-frame), where the full kinetic part of the action (114) (including the matter and gravi-dilaton sector) is diagonalized in terms of the canonically normalized fields $\widehat{g}_{\mu\nu}$, $\widehat{\phi}$ and $\widehat{\psi}_i$ [13]. Assuming that the dilaton is stabilized by its potential, and expanding the Lagrangian term describing the interaction between $\widehat{\phi}$ and $\widehat{\psi}_i$ around the value ϕ_0 which extremizes the potential, we can define, in this rescaled frame, the effective masses $\widehat{m}_i$ and charges $\widehat{q}_i$ for the canonical fields $\widehat{\psi}_i$. In the weak coupling limit in which $Z_R \simeq Z_\phi \simeq \exp(-\phi)$ one then finds, in particular, that the canonical dilaton charge $\widehat{q}_i$ deviates from the standard "gravitational charge" by the dimensionless factor [13]

$$\overline{q}_i \equiv \frac{\widehat{q}_i}{\sqrt{4\pi G}\,\widehat{m}_i} \simeq 1 + \left[\frac{\partial}{\partial\phi}\ln\left(\frac{Z^i_m}{Z^i_k}\right)\right]_{\phi=\phi_0}. \tag{115}$$

For a pure Brans-Dicke model of scalar-tensor gravity one has, for instance, $\overline{q}_i = 1$ (because there is no dilaton coupling to the matter fields in the Jordan frame, where $\partial Z^i/\partial\phi = 0$). For a string model, on the contrary, the coupling parameters $\overline{q}_i$ deviate from 1 and are non-universal, in general, since the loop form factors Z^i tend to be different for different fields ψ_i. In particular, in the conventional scenario which assumes that the loop corrections determining the coupling are the same determining also the effective mass of the given particle, one obtains large dilaton charges ($\overline{q}_i \sim 50$) for the confinement-generated components of the hadronic masses [48, 49], and smaller charges ($\overline{q}_i \sim 1$) for the leptonic components. In that case, the total dilaton charge of a macroscopic body tends to be large (in gravitational units) and composition-dependent [55], so that a large dilaton mass ($m \gtrsim 10^{-4}$ eV) is required to avoid conflicting with known gravitational phenomenology.

This conclusion can be avoided if the loop corrections combine to produce a cancellation, in such a way that the value of the coupling parameters $\overline{q}_i$ turns out to be highly suppressed with respect to the natural value of order one (a scenario of this type has been proposed, for instance, in [56]). In that case $\overline{q}_i \ll 1$, and light dilaton masses (as required, for instance, for a resonant interaction with gravitational antennas) may be allowed, without clashing with experimental observations.

In the rest of this section we will focus our attention on this possibility, considering the response of the gravitational detectors to a cosmic background of massive, non-relativistic dilatons, assuming that the background energy density corresponds to large fraction of critical density, and that the dilatons are arbitrarily light and very weakly coupled to ordinary matter.

2.2 Dilaton signals in gravitational antennas

The operation mechanism of all gravitational antennnas is based on the so-called equation of "geodesic deviation" (see e.g. [57]), which governs the response of the detector to the incident radiation. Such an equation is obtained by computing the relative acceleration between the world-lines of two nearby

test particles, separated by the infinitesimal space-like vector η^μ, and evolving geodesically in the given gravitational background. The interaction with a dilaton background can be easily included, in this context, by replacing the geodesic paths of the test particles with the world-lines described by Eq. (113): one is lead, in this way, to a generalized equation of deviation [55],

$$\frac{D^2\eta^\mu}{D\tau^2} + R_{\nu\alpha\beta}{}^\mu \eta^\nu u^\alpha u^\beta = q\,\eta^\nu \nabla_\nu \nabla^\mu \phi, \tag{116}$$

which is at the ground of the response of a detector to a background of gravi-dilaton radiation (the symbol D denotes covariant differentiation along a curve parametrized by the affine time-like variable τ).

This equation implies that a gravitational detector can interact with the scalar radiation in two ways: either

i) *directly*, through the *non-geodesic* coupling of its scalar charge to the second derivatives of the scalar background [55, 58]; or

ii) *indirectly*, through the *geodesic* coupling of its gravitational charge to the *scalar part* of the metric fluctuations induced by the dilaton, and contained inside the Riemann tensor [59].

For a precise discussion of the response of the detector we need to compute the "physical strain" $h(t)$ induced by the scalar radiation, which is expressed in terms of the so-called "antenna pattern functions" $F(\theta, \phi)$, describing the detector sensitivity along the different angular directions. To this purpose, we shall rewrite Eq. (116) in the approximation of small displacements ξ^μ around the unperturbed path of the test bodies, by setting $\eta^\mu = L^\mu + \xi^\mu(\tau)$, with $L^\mu = \text{const}$. We then obtain, in the non-relativistic limit,

$$\ddot{\xi}^i = -L^k M_k{}^i, \tag{117}$$

where

$$M_k{}^i = R_{k00}{}^i + q\partial_k \partial^i \phi \tag{118}$$

is the total (scalar-tensor) stress tensor describing the "tidal" forces due to the incident radiation. For the pedagogical purpose of this paper we shall assume that the tensor (i.e., gravity-wave) part of the radiation is absent, and that the scalar radiation can be simply described as a linear fluctuation of the Minkowski metric background $\eta_{\mu\nu}$ and of a constant dilaton background ϕ_0: thus, in the longitudinal gauge,

$$\begin{aligned} ds^2 &= (\eta_{\mu\nu} + \delta g_{\mu\nu})\, dx^\mu dx^\nu = (1+2\psi)dt^2 - (1-2\varphi)\delta_{ij} dx^i dx^j, \\ \phi &= \phi_0 + \chi, \end{aligned} \tag{119}$$

so that

$$M_{ij} = \partial_i \partial_j \varphi - \delta_{ij}\ddot{\psi} - q\partial_i \partial_j \chi. \tag{120}$$

To discuss the detection of a stochastic background of massive scalar radiation it is also convenient to expand the fluctuations in Fourier modes of

proper momentum $\boldsymbol{p} = p\widehat{n}$ and frequency $\nu = E(p) = (p^2 + m^2)^{1/2}$, where the unit vector $\widehat{n}$ specifies the propagation direction of the given mode on the angular two sphere Ω_2. We obtain

$$M_{ij} = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p \int_{\Omega_2} \mathrm{d}^2\widehat{n}\,(2\pi E)^2 \Bigg[\delta_{ij}\psi(p,\widehat{n}) - n_i n_j \varphi(p,\widehat{n}) + \frac{m^2}{E^2} n_i n_j \varphi(p,\widehat{n}) + q\frac{p^2}{E^2} n_i n_j \chi(p,\widehat{n})\Bigg] \mathrm{e}^{2\pi i(p\widehat{n}\cdot\boldsymbol{x} - Et)} + \mathrm{h.c.} \tag{121}$$

(note that we are using "unconventional" units in which $h = 1$, i.e. $\hbar = 1/2\pi$, for an easier comparison with the experimental variables). We will also assume that the dilaton is the only source of scalar metric perturbations, so that $\varphi = \psi$ [40]). Introducing the transverse and longitudinal projectors of the scalar stresses, defined respectively by

$$T_{ij} = \delta_{ij} - n_i n_j, \qquad L_{ij} = n_i n_j, \tag{122}$$

defining $M_{ij} = -\ddot{F}_{ij}$, and projecting the stress tensor onto the detector tensor D^{ij} (specifying the geometric configuration and the orientation of the arms of the detector), we finally obtain the scalar strain as [58, 60, 61]

$$h(t) \equiv D^{ij}F_{ij} = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p \int_{\Omega_2} \mathrm{d}^2\widehat{n} \Bigg[F^{\mathrm{geo}}(\widehat{n})\psi(p,\widehat{n}) + F^{\mathrm{ng}}(\widehat{n})\chi(p,\widehat{n})\Bigg] \mathrm{e}^{2\pi i(p\widehat{n}\cdot\boldsymbol{x} - Et)} + \mathrm{h.c.}. \tag{123}$$

Here

$$F^{\mathrm{geo}} = D^{ij}\left(T_{ij} + \frac{m^2}{E^2}L_{ij}\right), \tag{124}$$

$$F^{\mathrm{ng}} = q\frac{p^2}{E^2}D^{ij}L_{ij}, \tag{125}$$

are the antenna pattern functions corresponding, respectively, to the geodesic (or indirect) and non-geodesic (or direct) interaction of the detector with the scalar radiation background.

It should be noted that the scalar radiation, differently from the case of the tensor component, contributes to the response of the detector also with its longitudinal polarization states. The longitudinal contribution is present also in the ultra-relativistic limit $m \to 0$, $p \to E$, thanks to the non-geodesic coupling (125). In the opposite, non-relativistic limit $p \to 0$, $E \to m$, the geodesic strain tends to become isotropic, $T_{ij} + (m/E)^2 L_{ij} \to \delta_{ij}$, while the non-geodesic one becomes sub-leading.

The results (123) is valid for any type of detector described by the response tensor D^{ij}, and is formally similar to the expression for the strain

obtained in the case of tensor gravitational radiation – modulo the presence of different pattern functions, due to the different polarization properties. The scalar strain (123) can thus be processed, following the standard procedure, to correlate the outputs of two detectors and to extract the so-called signal-to-noise ratio (SNR), representing the experimentally relevant variable for the detection of a stochastic background of cosmic radiation [62].

For our scalar massive background, with spectral energy density $\Omega(p)$, we obtain [58, 60, 61], in particular,

$$SNR = \frac{3NH_0^2}{8\pi^3}\left[2T\int_0^\infty \frac{\mathrm{d}p}{p^3\,(p^2+m^2)^{3/2}}\,\frac{\gamma^2(p)\,\Omega^2(p)}{P_1(\sqrt{p^2+m^2})\,P_2(\sqrt{p^2+m^2})}\right]^{1/2} \tag{126}$$

(see also [32] for a detailed computation). Here T is the total (experimental) correlation time, N an (irrelevant) normalization factor, P_1 and P_2 the noise power spectra of the two detectors, and $\gamma(p)$ the so-called "overlap reduction function", which modulates the correlated signal according to the relative orientation and distance of the detectors, located at the positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$:

$$\gamma(p) = \frac{1}{N}\int_{\Omega_2} \mathrm{d}^2\widehat{n}\; F_1(\widehat{n})\; F_2(\widehat{n})\,\mathrm{e}^{2\pi i p\widehat{n}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}. \tag{127}$$

The overlap is to be calculated with the geodesic pattern function F_i^{geo} of Eq. (124) if we are considering the indirect signal due to a spectrum of scalar metric fluctuations, $\Omega_\psi(p)$; it is to be calculated with the non-geodesic pattern function F_i^{ng} of Eq. (125) if we are considering, instead, the direct signal due to a spectrum of dilaton fluctuations, $\Omega_\chi(p)$.

We are now in the position of stressing another important difference from the case of pure tensor radiation, due to the presence of the mass in the noise power spectra P_i. For a typical power spectrum, in fact, the minimum level of noise is reached around a rather narrow frequency band ν_0: outside that band the noise rapidly diverges, and the signal (126) tends to zero. As $\nu = (p^2+m^2)^{1/2}$ we have, in principle, three possibilities.

1) If $m \gg \nu_0$ then the noise is always outside the sensitivity band $P_i(\nu_0)$, and the signal is always negligible.
2) If $m \ll \nu_0$ then the sensitivity band may only overlap with the relativistic sector of the spectrum, for $p \sim \nu \sim \nu_0$.
3) If $m \sim \nu_0$, finally, the whole non-relativistic part of the spectrum $p \lesssim m$ satisfies the condition $P_i(\nu) \sim P_i(m) \sim P_i(\nu_0)$.

It is thus possible to obtain a resonant response to a massive, non-relativistic background of scalar particles, provided the mass lies in the band of maximal sensitivity of the two detectors [58, 60]. Considering the present, Earth-based gravitational antennas, operating between the Hz and the kHz range, it follows that the maximal sensitivity is presently in the mass range

$$10^{-15}\,\mathrm{eV} \lesssim m \lesssim 10^{-12}\,\mathrm{eV}. \tag{128}$$

Amusingly enough, it turns out that such small values are not so unrealistic if the dilaton mass is perturbatively generated by the mechanism of radiative corrections. For a scalar particle, gravitationally coupled to fermions of mass M_f with dimensionless strength q, there are, in fact, quantum loop corrections to the mass of order $qM_f(\Lambda/M_{\mathrm{P}})$, where Λ is the cut-off, which we shall assume typically localized at the TeV scale (see for instance [63]). Considering the dilaton coupling to ordinary baryonic matter ($M_f \sim 1$ GeV) the induced mass is then:

$$m \sim q\left(\frac{\Lambda}{1\,\mathrm{Tev}}\right)\left(\frac{M_f}{1\,\mathrm{Gev}}\right) \times 10^{-6}\,\mathrm{eV}. \tag{129}$$

Thus, a value of q smaller than (but not very far from) the present upper limits [53] (imposing $q \lesssim 10^{-4}$ in the relevant mass range (128)) is perfectly compatible with the possibility of resonant response of the present detectors.

Quite independently from the possible origin of the dilaton mass, if we assume that the mass is in the resonant range (128), and that the bounds on q are satisfied, we find that a cosmic background of non-relativistic dilatons is possibly detectable by the interferometric antennas of second generation – such as Advanced and Enhanced LIGO – provided the background energy density is sufficiently close to the saturation of the critical density bound [58, 60]. This interesting possibility can be illustrated by considering, for an approximate estimate, the simplified situation of two identical detectors with $P_1 = P_2 = P$, responding non-geodesically with maximal allowed overlap $N\gamma^{ng} \simeq q^2(4\pi/15)$ (the numerical factor is referred to the particular case of interferometric antennas). Let us suppose, also, that the SNR integral (126) is dominated by the peak value Ω_m of the non-relativistic dilaton spectrum, and that such value is reached around $p = m$ (otherwise the response is suppressed by the factor $(p/m)^4$, [60]). Eq. (126) gives, in this case,

$$SNR \sim \frac{\sqrt{2T}}{10\pi^2}\frac{q^2H_0^2\Omega_m}{m^{5/2}P(m)}, \tag{130}$$

and the condition of detactable background (SNR $\gtrsim 1$) implies

$$m^{5/2}P(m) \lesssim q^2h^2\Omega_m\left(\frac{T}{4\times10^7\,\mathrm{s}}\right)^{1/2} \times 10^{-33}\,\mathrm{Hz}^{3/2}. \tag{131}$$

This condition is compared in Fig. 6 with the expected spectral noise of the three LIGO generations (see e.g. [64]), for $T = 4\times10^7$ s. The region of the plane $\{m, P\}$ corresponding to a detectable background is located *above* the bold noise curves (labelled by LIGO I, LIGO II and LIGO III), and *below* the dashed lines, representing the upper limit (131) for different constant values of the parameter $q^2h^2\Omega_m$. This limit may be interpreted either as a constraint on the intensity Ω_m, for backgrounds geodesically coupled ($q^2 = 1$) to the detectors,

or as a limit on the non-geodesic coupling strength q^2, for backgrounds of given energy density Ω_m. As shown in the picture, phenomenologically allowed backgrounds are in principle accessible to the sensitivity of next-generation interferometers (see also [32] for a more detailed discussion).

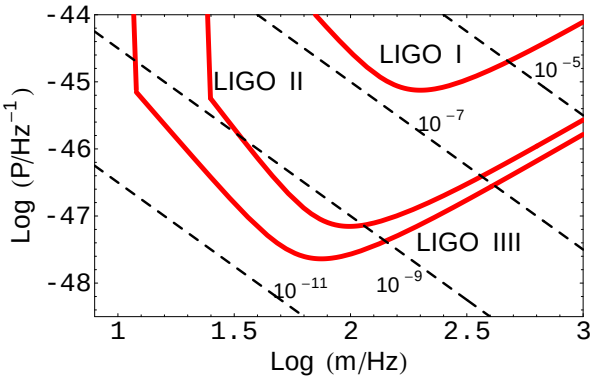

Fig. 6. The noise power spectra of the three LIGO generations (bold curves), and the condition of detectable dilaton background (dashed lines), plotted at different values of the parameter $q^2h^2\Omega_m$ (ranging from 10^{-5} to 10^{-11}).

2.3 Enhanced signals for flat non-relativistic spectra

The result reported in Eq. (130) is generally valid for a growing spectrum with a steep enough slope, as typically obtained in "minimal" models of pre-big bang inflation. However, the cross-correlated signal may result strongly enhanced with respect to Eq. (130) if the dilaton spectrum is sufficiently flat, and if the considered pair of detectors satisfies the condition $\gamma(p) \to$ const $\neq 0$ for $p \to 0$.

Let us consider, in fact, the SNR integral (126), which can be written as

$$(SNR)^2 \sim T \int_0^{p_1} \mathrm{d}p \frac{\gamma^2(p)\Omega^2(p)}{p^3 E^3 P_1(E) P_2(E)}, \tag{132}$$

where $E = (p^2 + m^2)^{1/2}$, and where we can assume that $\Omega(p)$ is a power-law function of p, with an ultraviolet cut-off at $p = p_1$. For a massless spectrum ($p = E$) this integral is always convergent (for any slope), even in the infrared limit $p = E \to 0$: in fact, when $p \to 0$, the physical strains are produced outside the sensitivity band of the detectors, and the noises blow up to infinity, $P_i(E) \to P_i(0) \to \infty$. For $m \neq 0$, on the contrary, in the infrared limit $p \to 0$ the noises keep frozen at the frequency scale determined by the mass of the scalar background, $P_i(E) \to P_i(m) =$ const. In this second case the behavior of the integral dependes on $\gamma(p)$ and $\Omega(p)$.

Suppose now that $\gamma(p) \to \gamma_0 = \text{const}$ for $p \to 0$, and that $\Omega(p) \sim p^\delta$, for $p < m$. For $\delta < 1$ we find that the integral is dominated by the infrared limit, and gives

$$(SNR)^2 \sim \frac{T\gamma_0^2}{m^3 P_1(m) P_2(m)} \int_0^m \frac{\mathrm{d}p}{p^3} \Omega^2(p) = \frac{T\gamma_0^2}{m^3 P_1 P_2} \left[p^{2(\delta-1)}\right]_0^m . \tag{133}$$

Thus, the integral is infrared divergent [65] for all spectra (even if blue, $\delta > 0$) with $\delta < 1$!

This divergence is obviously unphysical, and can be removed by noting that the observation time T is finite, and is thus associated to a minimum resolvable frequency interval $\Delta\nu = \Delta E = \Delta(p^2/2m) \gtrsim T^{-1}$, defining the minimum momentum scale

$$p_{\min} = (2m/T)^{1/2} > 0, \tag{134}$$

acting as effective infrared cut-off for the integral (133). This implies a modified dependence of SNR on the correlation time T in the case of flat enough spectra:

$$SNR \sim T^{1/2} \left[p^{\delta-1}\right]_{p_{\min}}^m \sim \begin{cases} T^{1/2}, & \delta > 1, \\ T^{1-\delta/2}, & \delta < 1. \end{cases} \tag{135}$$

For $\delta < 1$, in particular, there is a faster growth of SNR with T, which may produce an important enhancement of the sensitivity to a cosmic background of non-relativistic scalar particles, as discussed in [61, 65].

It is important to stress that the case $\gamma(p) \to \gamma_0 = \text{const}$ for $p \to 0$ has not been "invented" *ad hoc*: it can be implemented, in practice, with detectors already existing and operative (or with detectors planned to be working in the near future, like resonant spheres). A first simple example, studied in [65], refers in fact to spherical, resonant-mass detectors, whose monopole mode is characterized by the "trivial" response tensor $D^{ij} = \delta^{ij}$. In that case the geodesic pattern function (124) is isotropic,

$$F^{\text{geo}} = \frac{2p^2 + 3m^2}{p^2 + m^2}, \tag{136}$$

and the geodesic overlap function (127), for two identical spheres with spatial separation $|\boldsymbol{x}_1 - \boldsymbol{x}_2| = d$, is given by

$$\gamma(p) = \frac{2}{N} \left(\frac{2p^2 + 3m^2}{p^2 + m^2}\right)^2 \frac{\sin(2\pi p d)}{pd}. \tag{137}$$

This function clearly satisfies the requirement $\gamma(p) \to \gamma_0 = \text{const}$ for $p \to 0$.

A second example, studied in [61], refers to the so-called "common mode" of the interferometric antennas, characterized by the response tensor

$$D_+^{ij} = u^i v^j + v^i u^j, \tag{138}$$

where u^i and v^i are the unit vectors specifying the spatial orientation of the axes of the interferometer. Let us consider, for instance, a geometrical configuration where the vectors $\widehat{u}$ and $\widehat{v}$ are coaligned with the x_1 and x_2 axes of a Cartesian frame, respectively, and the direction $\widehat{n}$ ofthe incident radiation is specified (with respect to the axes x_1, x_2 and x_3) by the polar and azimuthal angles φ and θ. The computation of the geodesic pattern function (124) gives, in that case,

$$F_+^{\rm geo} = 2 - \left(\frac{p}{E}\right)^2 \sin^2\theta. \tag{139}$$

The geodesic overlap function (127), for two coplanar interferometers with spatial separation $|\Delta \boldsymbol{x}| = d$, is [61]

$$\gamma_+^{\rm geo}(p) = \frac{4\pi}{N}\left[\left(4 - 4\frac{p^2}{E^2} + \frac{p^4}{E^4}\right) j_0(\alpha) + \frac{1}{\alpha}\left(4\frac{p^2}{E^2} - 2\frac{p^4}{E^4}\right) j_1(\alpha) + \frac{3}{\alpha^2}\left(\frac{p}{E}\right)^4 j_2(\alpha)\right], \tag{140}$$

where $\alpha = 2\pi p d$, and j_0, j_1, j_2 are spherical Bessel functions. Thus, also in this case, $\gamma \to 16\pi/N = \text{const}$ for $p \to 0$.

3 Late-time cosmology: dilaton dark energy

In this third lecture we will discuss the possibility that a homogeneous, large-scale dilaton field may be the source of the so-called "dark energy" which produces the cosmic acceleration first observed at the end of the last century [66], and confirmed by most recent supernovae data [67, 68].

Let us recall, to this purpose, that the initial phase of pre-big bang inflation is characterized by the monotonic growth of the dilaton and of the string coupling g_s (see Sect. 1.3): the subsequent epoch of standard evolution thus opens up in the strong coupling regime, and should be described by an action which includes all relevant loop corrections. Late enough, i.e. at sufficiently low curvature scales, the higher-derivative corrections can be neglected, and the action can be written in the form of Eq. (114). In that context the loop form factors $Z(\phi)$, and the dilaton potential $V(\phi)$, may play a crucial role in determining the late-time cosmological evolution.

There are, in principle, two possible alternative scenarios.

i) The dilaton is stabilized by the potential at a constant value $\phi = \phi_0$ which extremizes $V(\phi)$. In this case the loop corrections induce a constant renormalization of the effective dilaton couplings (as discussed in Sect. 2.1), and the Universe may approach a late-time configuration dominated by the dilaton potential, with $H^2 \sim V(\phi_0)$.

ii) The dilaton fails to be trapped in a minimum of the potential, and keeps running even during the post-big bang evolution. In this case the late-time cosmological evolution is crucially dependent on the asymptotic behavior of the factors $Z(\phi)$.

These two different possibilities have different impact on the so-called "coincidence problem" (i.e. on the problem of explaining why the dark-matter and dark-energy densities are of the same order just at the present epoch), as we shall discuss in the following subsections.

3.1 Frozen dilaton in the moderate coupling regime

The first type of scenario can be easily implemented [69] using a generic non-perturbative potential which is instantonically suppressed ($V \sim \exp(-1/g_s^2)$) in the weak coupling limit $g_s^2 \to 0$, and which develops a non-trivial structure with a (semi-perturbative) minimum $g_s^2 \sim \alpha_{GUT} \sim (M_s/M_P)^2 \sim 0.1 - 0.01$ in the regime of moderate string coupling. A typical example is the "minimal" potential given, in the E-frame, by [70]

$$\widetilde{V}(\phi) = m_V^2 \left[e^{k_1(\phi-\phi_1)} + \beta e^{-k_2(\phi-\phi_1)} \right] e^{-\epsilon \exp[-\gamma(\phi-\phi_1)]}, \qquad (141)$$

where k_1, k_2, β, ϵ, γ are dimensionless parameters of order one (see Fig. 7). The presence of a local minimum at $\phi_0 \simeq \phi_1$ allows solutions with $\phi = \text{const}$ during the radiation-dominated phase, and (for appropriate values of m_V) may also lead to a late phase of accelerated expansion driven by the potential energy $V(\phi_0)$, *provided* the dilaton is not permanently shifted away from the minimum ϕ_0 by the transition to the matter-dominated epoch [69].

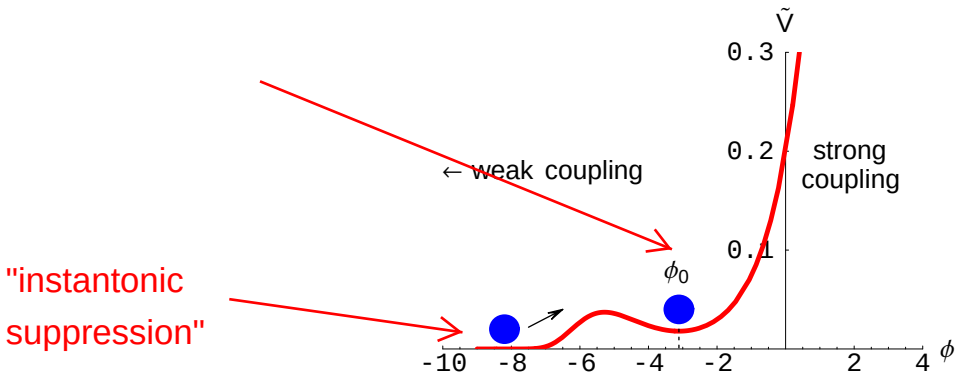

Fig. 7. Plot of the potential (141) for $k_1 = k_2 = \beta = \gamma = 1$, $\epsilon = 0.1$, $\phi_1 = -3$, $m_V = 0.1$, and a local minimum (independent of m_V) at $\phi_0 = -3.112$, corresponding to $g_s^2 = \exp(\phi_0) \simeq 0.045$.

Let us consider, in fact, the equation of motion of a homogeneous dilaton field $\phi(t)$ in the conformally rescaled E-frame (with metric $\widetilde{g}$) where the graviton kinetic energy is canonically normalized, and let us assume that the rescaled matter sources can be described as a perfect fluid of energy density $\widetilde{\rho}$, pressure $\widetilde{p}$, and dilaton charge $\widetilde{\sigma}$. Starting from an action of the type (114) we find that the generalized dilaton equation, for a cosmological background, takes the form

$$A(\phi)\left(\ddot{\phi}+3\widetilde{H}\dot{\phi}\right)+B(\phi)\dot{\phi}^2+\frac{\partial\widetilde{V}}{\partial\phi}+\lambda_{\rm P}^2\left[C(\phi)\left(\widetilde{\rho}-3\widetilde{p}\right)+\widetilde{\sigma}\right]=0, \qquad (142)$$

where A, B and C are functions describing the rescaled (E-frame) loop corrections. For a minimally coupled field, for instance, $A=1$, $B=C=\widetilde{\sigma}=0$; for the dilaton, at tree-level in the string coupling, $A=C=1$, $B=0$. In the most general case we find that a stable dilaton configuration with $\dot{\phi}=0=\ddot{\phi}$ is possible, in the radiation era ($\widetilde{\rho}=3\widetilde{p}$), if the scalar charge of the fluid is negligible, $\widetilde{\sigma}=0$, and the dilaton extremizes the E-frame potential, $\partial\widetilde{V}/\partial\phi=0$.

When the Universe becomes matter-dominated ($\widetilde{p}=0$), however, a new acceleration $\ddot{\phi}=-A^{-1}\lambda_{\rm P}^2C\widetilde{\rho}$ is suddenly generated, which tends to remove the dilaton away from its equilibrium position. Such an acceleration is in competition with the restoring force $\ddot{\phi}=-A^{-1}(\partial\widetilde{V}/\partial\phi)$ (see Eq. (142)). The possibility that the dilaton may bounce back to the stable minimum $\phi=\phi_0$, driving the Universe towards a final phase of accelerated, potential-dominated expansion, thus crucially depends on the values of two parameters: the (loop-corrected) strength $\lambda_{\rm P}^2C(\phi_0)$ of the dilaton coupling to dark matter, and the slope of the dilaton potential (141), determined by the mass scale m_V which also controls the amplitude of the minimum, $V(\phi_0)\sim m_V^2$. Such an amplitude, on the other hand, should correspond to the present Hubble scale ($V(\phi_0)\sim H_0^2$), in a realistic model able to describe the present phase of accelerated expansion.

It can be shown, with a simple numerical analysis, that the values of the coupling strength allowed by present gravitational phenomenology are compatible with a late-time phase dominated by the potential only for a finite range of values of $V(\phi_0)$, depending on the value of the dilaton coupling at the equality epoch [69]. Using the phenomenological upper limit $|C_{\rm eq}|\simeq 0.1$ one finds that the dilaton, after a smal shift at $t=t_{\rm eq}$, bounces back to the minimum provided $10^{-7}H_{\rm eq}\lesssim m_V\lesssim H_{\rm eq}$ (which includes the realistic case $m_V\sim H_0\sim 10^{-6}H_{\rm eq}$) [69]. Smaller values of $|C_{\rm eq}|$ correspond to a larger mass interval. We can say, therefore, that the coincidence problem (i.e., why $V(\phi_0)\sim H_0^2$), in this context remains, but is somewhat alleviated because – thanks to the dynamical correlation between the amplitude $V(\phi_0)$ and the matter-dilaton coupling – only a restricted range of values is allowed for $V(\phi_0)$.

3.2 Running dilaton: saturation of the loop corrections and asymptotic "freezing"

The second possibility, which will be discussed here in more detail, in the case in which the dilaton is not stopped by the structures formed by the potential around $g_s^2 = 1$, and keeps rolling towards $+\infty$ along a smoothly decreasing potential. A possible example of non-perturbative potential of this type is given, in the E-frame, by [71]

$$\widetilde{V} = c_1^4 m_V^2 \left(\frac{\mathrm{e}^{\phi}}{b_1 + c_1^2 \mathrm{e}^{\phi}}\right)^2 \left[\mathrm{e}^{-\beta_1 \exp(-\phi)} - \mathrm{e}^{-\beta_2 \exp(-\phi)}\right], \tag{143}$$

where b_1, c_1, β_1, β_2 are dimensionless parameters, with $0 < \beta_1 < \beta_2$. This potential is instantonically suppressed in the weak coupling limit $\phi \to -\infty$, and is exponentially decaying as

$$\widetilde{V} = m_V^2 \left(\beta_2 - \beta_1\right) \mathrm{e}^{-\phi} + \mathcal{O}\left(\mathrm{e}^{-2\phi}\right) \tag{144}$$

in the limit $\phi \to +\infty$ (see Fig. 8). In this case, as we shall see, we can obtain a scenario of "coupled quintessence" [72] in which the late Universe approaches a (possibly accelerated) state dominated by a mixture of kinetic and potential energy density, and the coincidence problem may find a satisfactory solution thanks to the dilaton-dark matter interactions.

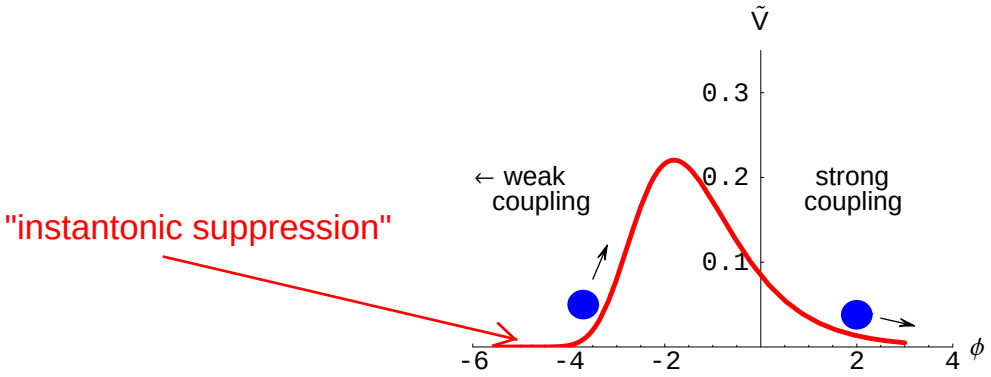

Fig. 8. Plot of the potential (143) for $b_1 = 1$, $c_1 = 10$, $\beta_1 = 0.1$, $\beta_2 = 0.2$ and $m_V = 1$. The dilaton is monotonically growing from the string perturbative vacuum along a "bell-like" non-perturbative potential.

In this case, however, a realistic scenario requires some mechanism of saturation of the loop corrections, so as to keep the present effective values of gravitational and gauge couplings approximately constant and sufficiently "small", even in the large "bare coupling" limit $\phi \to +\infty$. As discussed in

[73], such a saturation can be obtained thanks to the large number of fields (e.g. gauge bosons) entering the loop corrections, assuming (as in models of "induced gravity") that the loop form factors of Eq. (114) have a finite limit for $\phi \to +\infty$, and that can be approximated by a Taylor expansion in powers of the inverse bare coupling $g_s^2 = \exp\phi$. Applying these assumptions to the gravi-dilaton form factors, to the potential, and to the dimensionless parameters $q_i(\phi)$ controlling the dilaton charge-density of the various matter fields, we can set, for $\phi \to +\infty$,

$$\begin{aligned} Z_R(\phi) &= c_1^2 + b_1 \mathrm{e}^{-\phi} + \mathcal{O}(\mathrm{e}^{-2\phi}), \\ Z_\phi(\phi) &= -c_2^2 + b_2 \mathrm{e}^{-\phi} + \mathcal{O}(\mathrm{e}^{-2\phi}), \\ V(\phi) &= V_0 \mathrm{e}^{-\phi} + \mathcal{O}(\mathrm{e}^{-2\phi}), \\ q_i(\phi) &= q_{0i} + \mathcal{O}(\mathrm{e}^{-2\phi}). \end{aligned} \tag{145}$$

The dimensionless coefficients c_1^2 and c_2^2 of this expansion are typically of order $N \sim 10^{-2}$, because of their quantum-loop origin and of the large number N of gauge bosons in GUT groups like E_8. This is in agreement with the fact that c_1^2 controls (according to the action (114)) the asymptotic value of the ratio between the string and the Planck length scale, $c_1^2 = (\lambda_\mathrm{s}/\lambda_\mathrm{P})^2$, which is indeed expected to be a number of the above order. The coefficients $b_1, b_2 \ldots$, on the contrary, are numbers of order one. Note that the expansion of $V(\phi)$ agrees with the asymptotic form of the potential (144).

We should note, finally, that the asymptotic values of the dilaton charges, q_{0i}, have to be strongly suppressed for the ordinary components of matter (such as baryons) and for electromagnetic radiation: if we want a dilaton field active on a cosmological scale of distances, in fact, we need long-range interactions, and we must avoid unacceptable deviations from the standard gravitational phenomenology by suppressing the dilaton couplings, as discussed in Sect. 2.1. For the (possibly exotic) components of dark matter, however, there is no strict phenomenological bound imposing such suppression: in that case, the asymptotic charge q_0 could be nonvanishing, and of order one, leading to interesting late-time deviations from the standard cosmological scenario.

For a simpler illustration of this possibility it is convenient to work in the diagonalized E-frame, obtained from the metric g of Eq. (114) through the rescaling

$$g_{\mu\nu} = c_1^2 \, Z_R^{-1} \, \widetilde{g}_{\mu\nu}. \tag{146}$$

The action (114) becomes, in this new frame

$$S = \frac{1}{2\lambda_\mathrm{P}^2} \int \mathrm{d}^4x \sqrt{-g} \left[-\widetilde{R} + \frac{1}{2} k^2(\phi) \left(\widetilde{\nabla}\phi \right)^2 - \widetilde{V}(\phi) \right] + S_m(\widetilde{g}, \phi, \mathrm{matter}), \tag{147}$$

where

$$k^2(\phi) = 3 \left(\frac{\partial \ln Z_R}{\partial \phi} \right)^2 - 2 \frac{Z_\phi}{Z_R}, \qquad \widetilde{V}(\phi) = c_1^4 Z_R^{-2} \, V. \tag{148}$$

Assuming that the matter action S_m describes a perfect fluid with a dark-matter component $\widetilde{\rho}_m$, a baryon component $\widetilde{\rho}_b$, and a radiation component $\widetilde{\rho}_r = 3\widetilde{p}_r$, the cosmological Einstein equations for the action (147) can then be written (omitting the tilde, and in units $2\lambda_{\rm P}^2 = 1$) as:

$$\begin{aligned} 6H^2 &= \rho_r + \rho_b + \rho_m + \rho_\phi, \\ 4\dot{H} + 6H^2 &= -\frac{\rho_r}{3} - p_\phi, \end{aligned} \tag{149}$$

where

$$\rho_\phi = \frac{k^2(\phi)}{2}\dot{\phi}^2 + V, \qquad p_\phi = \frac{k^2(\phi)}{2}\dot{\phi}^2 - V. \tag{150}$$

The associated dilaton equation, assuming a negligible density of dilaton charge for baryons and radiation ($\sigma_r = 0 = \sigma_b$), can be written as [71]

$$k^2(\ddot{\phi} + 3H\dot{\phi}) + kk'\dot{\phi}^2 + V' + \frac{1}{2}\left[\psi'\left(\rho_b + \rho_m\right) + \sigma_m\right] = 0, \tag{151}$$

where we have defined $\psi = -\ln Z_R$, and the prime denotes differentiation with respect to ϕ. The combination of Eqs. (149)–(151) leads, finally, to the equations of energy-momentum conservation for the various fluid components:

$$\begin{aligned} &\dot{\rho}_r + 4H\rho_r = 0, \\ &\dot{\rho}_b + 3H\rho_b - \frac{\psi'}{2}\dot{\phi}\,\rho_b = 0, \\ &\dot{\rho}_m + 3H\rho_m - \frac{\psi'}{2}\dot{\phi}\,\rho_m - \frac{\sigma_m}{2}\dot{\phi} = 0, \\ &\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) + \frac{1}{2}\dot{\phi}\left[\psi'\left(\rho_b + \rho_m\right) + \sigma_m\right] = 0 \end{aligned} \tag{152}$$

(the last equation is simply the dilaton equation (151), rewritten in fluido-dynamical form).

Let us now concentrate on the coupled dark-matter/dilaton system, and note that there are two types of interactions between these two cosmic sources: a first one, specific to the particular type of dark matter field, generated by the "intrinsic" dilaton charge σ_m; and a second one, more "universal", generated by the standard dilaton coupling to the trace of the stress tensor, and associated to the ψ' terms of the above equations. Both types of coupling are renormalized by the loop corrections, but with opposite effect according to the asymptotic limits of Eq. (145). In fact, the dilaton charge tends to grow, and to reach a constant asymptotic value as $\phi \to +\infty$. The coupling parameter ψ', on the contrary, tends to be exponentially suppressed as

$$\psi' = -\left(\ln Z_R\right)' \to \frac{b_1 \mathrm{e}^{-\phi}}{c_1^2}, \qquad \phi \to +\infty. \tag{153}$$

As a consequence, after the transition to the matter-dominated phase, the Universe may enter two different types of dynamical regimes [71].

1) If the dark-matter charge σ_m is still negligible at the beginning of the matter-dominated phase (as well as the dilaton potential, expected to become important only near the present epoch), then the Universe enters the so-called *"dragging regime"*, in which ρ_m is coupled to ϕ through the ψ' terms of Eqs. (153), and the evolution of the (still subdominant) dilaton kinetic energy ρ_ϕ is "dragged" by ρ_m.

The cosmic evolution, during this regime, can be analytically described (in an approximate way) by noting that the loop factor $k(\phi)$ goes to a constant at late enough time scales,

$$k(\phi) \to k_0 = \sqrt{2}\,\frac{c_2}{c_1}, \qquad\qquad \phi \to +\infty, \tag{154}$$

according to Eqs. (148) and (145). Introducing the canonical variable $\widehat{\phi} = k_0\phi$ (see the action (147)), and neglecting the subdominant contributions of ρ_r and ρ_b, we can then rewrite the coupled equations (151), (152), for the dragging regime, as follows:

$$\ddot{\widehat{\phi}} + 3H\dot{\widehat{\phi}} + \frac{\epsilon}{2}\rho_m = 0, \tag{155}$$

$$\dot{\rho}_m + 3H\rho_m - \frac{\epsilon}{2}\rho_m\dot{\widehat{\phi}} = 0, \tag{156}$$

where $\epsilon = \psi'/k_0 \simeq \mathrm{e}^{-\phi}/(\sqrt{2}c_1c_2) \ll 1$ is the effective coupling parameter. Neglecting the time dependence of ϵ with respect to that of H and $\dot{\widehat{\phi}}$ (for small enough time intervals), we find that the system of equations (149), (155), is satisfied by

$$\dot{\widehat{\phi}} \simeq -2\epsilon H. \tag{157}$$

Thus, from Eq. (156),

$$\begin{aligned} &\rho_m \sim a^{-(3+\epsilon^2)} \sim H^2 \sim \dot{\widehat{\phi}}^{\,2} \sim \rho_\phi, \\ &a \sim t^{2/(3+\epsilon^2)}. \end{aligned} \tag{158}$$

During this phase the dark-matter and the (kinetic) dilaton dark-energy densities are characterized by the same time-evolution, which slightly deviates from the standard behavior of a dust-dominated Universe ($\rho \sim a^{-3}$, $a \sim t^{2/3}$). The kinematics, however, remains decelerated (as $\epsilon \ll 1$).

2) A second, possibly accelerated, *"freezing regime"* is eventually reached in the limit in which the dilaton potential comes into play, and the coupling induced by the intrinsic charge density σ_m becomes dominant with respect to the exponentially suppressed coupling due to ψ'.

Using again the canonical variable $\widehat{\phi}$, assuming that $\sigma_m = q(\phi)\rho_m$ (for a homogeneous fluid), and considering the asymptotic limits $q(\phi) \to q_0$, $V = V_0\exp(-\phi)$ of Eq. (145), we can rewrite the coupled dilaton-dark matter equations (152), for the freezing regime, as follows:

$$\dot\rho_m + 3H\rho_m - \frac{q_0}{2k_0}\rho_m\dot{\widehat\phi} = 0,$$
$$\dot\rho_\phi + 6H\rho_k + \frac{q_0}{2k_0}\rho_m\dot{\widehat\phi} = 0. \tag{159}$$

We have defined the kinetic and potential energy densities, ρ_k and ρ_V, respectively as

$$\rho_k = \frac{\dot{\widehat\phi}^2}{2}, \qquad \rho_V = V(\widehat\phi) = V_0 \mathrm{e}^{-\widehat\phi/k_0}, \qquad \rho_\phi = \rho_k + \rho_V. \tag{160}$$

The system of equations (159), (149) (with $\rho_r = \rho_b = 0$) can be solved by a late-time configuration in which ρ_m, ρ_ϕ, V and H^2 scale in time in the same way, so that the critical fractions of dark-matter density, $\Omega_m = \rho_m/6H^2$, dilaton kinetic energy, $\Omega_k = \rho_k/6H^2$, and potential energy, $\Omega_V = V/6H^2$, are separately frozen at constant values determined by k_0 and q_0 only (i.e. by the parameters c_1, c_2, q_0 of the asymptotic expansion (145)). A simple analysis gives [71]

$$\Omega_k = \frac{3k_0^2}{(q_0+2)^2}, \qquad \Omega_V = \frac{3k_0^2 + q_0(q_0+2)}{(q_0+2)^2},$$
$$\Omega_\phi = \Omega_k + \Omega_V, \qquad \Omega_m = 1 - \Omega_\phi, \tag{161}$$

where k_0 is given by Eq. (154) (see also [32] for a detailed computation).

In this asymptotic state the Universe is thus dominated by a fixed mixture of dark-matter and dilaton (kinetic plus potential) energy density. The dilaton fluid has equation of state

$$w = \frac{p_\phi}{\rho_\phi} = \frac{\Omega_k - \Omega_V}{\Omega_k + \Omega_V} = -\frac{q_0(q_0+2)}{6k_0^2 + q_0(q_0+2)}, \tag{162}$$

and can play the role of the dark energy fluid responsible for the observed cosmic acceleration, provided $q_0 > 1$.

In fact, by rewriting the Einstein equations (149) for $\dot H$ in the form

$$1 + \frac{2\dot H}{3H^2} = \Omega_V - \Omega_k, \tag{163}$$

we obtain

$$\frac{\ddot a}{aH^2} = 1 + \frac{\dot H}{H^2} = \frac{3}{2}\left(\Omega_V - \Omega_k\right) - \frac{1}{2} = \frac{q_0 - 1}{q_0 + 2}. \tag{164}$$

The expansion is accelerated ($\ddot a > 0$) for $q_0 > 1$ or $q_0 < -2$. The second case (corresponding to an acceleration of superinflationary type, with $\dot H > 0$) is to be excluded, however, in our context, as it would imply $\Omega_m < 0$ according to Eqs. (161). Thus, acceleration is only possible for $q_0 > 1$. The explicit form of this asymptotic solution can be finally obtained through the integration of Eq. (164), which gives

$$a \sim t^{(q_0+2)/3}, \qquad H \sim a^{-3/(q_0+2)}, \tag{165}$$

from which

$$\rho_m \sim H^2 \sim \frac{\dot{\widehat{\phi}}^2}{2} \sim V_0\, \mathrm{e}^{-\widehat{\phi}/k_0} \sim a^{-6/(q_0+2)}. \tag{166}$$

To illustrate the smooth background evolution from the initial radiation phase to the intermediate dragging phase, and to the final freezing regime, we shall conclude this subsection by presenting the results of an exact numerical integration of the string cosmology equations (149)–(152). For our illustrative purpose we will assume that Z_R and Z_ϕ are given by the expansion (145) truncated to first order in $\exp(-\phi)$, with $b_1 = b_2 = 1$, $c_1^2 = 100$ and $c_2^2 = 30$. We will adopt the model of dilaton coupling already used in [71], parametrized by the time-dependent charge

$$q(\phi) = q_0\, \frac{\mathrm{e}^{q_0\phi}}{c^2 + \mathrm{e}^{q_0\phi}}, \tag{167}$$

with $c^2 = 150$ and $q_0 = 2.5$. We will also use the E-frame potential (144), with $\beta_1 = 0.1$, $\beta_2 = 0.2$, and $c_1^2 m_V = 10^{-3} H_{\mathrm{eq}}$. The last choice, which implies $m_V \sim H_0$, is crucial to obtain a realistic scenario in which the asymptotic accelerated regime starts at a phenomenologically acceptable epoch (see [71, 32] for a discussion of the mass scale of the non-perturbative dilaton potential, and of the degree of fine-tuning possibly required for realistic cosmological applications). Finally, we will integrate our equations imposing the initial conditions $\rho_\phi(t_i) = \rho_r(t_i)$, $\rho_m(t_i) = 10^{-20}\rho_r(t_i)$, $\rho_b(t_i) = 7 \times 10^{-21}\rho_r(t_i)$, $\phi(t_i) = -2$, at the initial scale $H(t_i) = 10^{40} H_{\mathrm{eq}}$.

The obtained scaling evolution $\rho = \rho(a)$ is illustrated in Fig. 9 for the various cosmic components. We can note that, at large enough times, baryons (full line) and radiation (dotted line) are fully decoupled from the dilaton, and obey the standard scaling behavior ($\rho_r \sim a^{-4}$, $\rho_b \sim a^{-3}$). The late-time dark-matter evolution, on the contrary, is closely tied to the dilaton evolution, and the ratio of their energy densities becomes asymptotically frozen at a constant. With the particular numerical values used in this example we obtain an asymptotic configuration characterized by $\Omega_\phi \simeq 0.73$ and $\Omega_m \simeq 0.27$, with a dark-energy equation of state $w \simeq -0.76$.

3.3 Non-local coupling and pressure backreaction

Another interesting asymptotic configuration can be obtained in the case in which the dilaton is non-locally coupled to the dark matter components, as discussed in subsection 1.2. In that case, the fractions Ω_m and Ω_ϕ may also become frozen at constant asymptotic values, but the background evolution turns out to be decelerated for any values of q_0, as the dark matter develops an effective pressure which tends to compensate the accelerating action of the

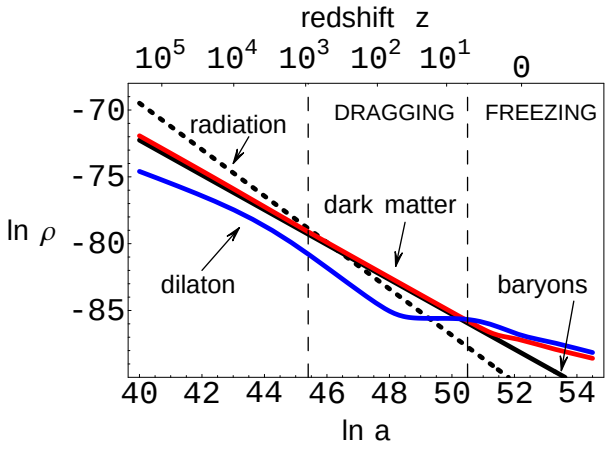

Fig. 9. Late-time evolution (on a logarithmic scale) of the various components of the cosmic energy density. The plots are the result of a numerical integration of Eqs. (149)–(152).

dilaton potential. This effect is new, and will be illustrated in some details in this subsection.

We start assuming that, in the matter part of the action (114), the dilaton is non-locally coupled to the sources through the variable $\xi(\phi)$, as in the action (28). However, differently from the action (28), we will assume (for simplicity) that the dilaton potential is local, $V = V(\phi)$: the gravi-dilaton part of the action is thus identical to that of Eq. (114), and our model is described by

$$S = -\frac{1}{2\lambda_{\rm s}^2}\int {\rm d}^4x\sqrt{-g}\left[Z_R(\phi)R + Z_\phi(\phi)(\nabla\phi)^2 + V(\phi)\right] + \int {\rm d}^4x\sqrt{-g}\,\mathcal{L}_m({\rm e}^{-\xi}). \tag{168}$$

Let us vary this action with respect to g and ϕ, and evaluate the resulting (general covariant) field equations in the limit of a homogeneous, isotropic, spatially flat background, using the results of Subsection 1.2. We obtain a set of equations similar to Eqs. (45)–(47) for what concerns the dilaton charge density $\sigma(\overline{\phi})$, but different for the potential (which now is local), and for the presence of the loop corrections Z_R, Z_ϕ.

Let us finally transform the equations in the E-frame (using the rescaling (146)), and consider the asymptotic limit in which ρ_r, ρ_b are negligible, and the dark matter is coupled to the dilaton only through its intrinsic dilaton charge (namely, the limit in which $\psi' \simeq 0$). The resulting equations (omitting the tilde, in units $2\lambda_{\rm P}^2 = 1$) can be written as

$$6H^2 = \rho_m + \rho_\phi, \tag{169}$$

$$4\dot{H} + 6H^2 = -p_\phi - \frac{\sigma_m}{2}, \tag{170}$$

$$\dot{\rho}_m + 3H\left(\rho_m + \frac{\sigma_m}{2}\right) - \frac{\sigma_m}{2}\dot{\phi} = 0, \tag{171}$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) + \frac{\sigma_m}{2}\dot{\phi} = 0, \tag{172}$$

with ρ_ϕ and p_ϕ defined by Eq. (150), as before. A comparison with the asymptotic limit of Eqs. (149), (152), shows that the genuinely new effect of the non-local interactions is the appearance of an effective pressure term $\sigma_m/2$ for the dark matter component. Indeed, the new terms present in Eqs. (170), (171), can also be obtained from the standard Einstein equations through the shift $p_m \to p_m + \sigma_m/2$.

We are now in the position of asking whether or not this modification (of non-local origin) may change the results of the previous subsection, in particular those concerning the asymptotic freezing configuration. We shall consider, to this purpose, the limit in which $\sigma_m \to q_0\rho_m$ and $V = V_0\exp(-\phi)$, using the canonical variable $\widehat{\phi}$ as in the previous computations.

Let us look for solutions of Eqs. (169)–(172) by requiring for ρ_m, ρ_k and ρ_V the same scaling behavior, and thus imposing, as a first condition, that

$$\frac{\dot{\rho}_m}{\rho_m} = \frac{\dot{\rho}_\phi}{\rho_\phi}. \tag{173}$$

Using Eqs. (171), (172) for ρ_m and ρ_ϕ, and the Einstein equation (169), we obtain

$$\frac{\dot{\widehat{\phi}}}{H} = \frac{6k}{q_0}\left[\overline{\Omega}_V\left(1 + \frac{q_0}{2}\right) - \overline{\Omega}_k\left(1 - \frac{q_0}{2}\right)\right]. \tag{174}$$

We are denoting with a bar the fractions of critical energies for the new configuration associated to the non-local equations, to distinguish it from the "local" freezing solution of Eq. (161). We also impose, as a second condition, that

$$\frac{\dot{\rho}_m}{\rho_m} = \frac{\dot{\rho}_V}{\rho_V}. \tag{175}$$

The definition (160) of ρ_V, together with Eq. (171), gives then

$$\frac{\dot{\widehat{\phi}}}{H} = 3k_0, \tag{176}$$

which, combined with eq. (174), leads to

$$\overline{\Omega}_V = \overline{\Omega}_k\,\frac{2 - q_0}{2 + q_0} + \frac{q_0}{q_0 + 2}. \tag{177}$$

From the definition of $\overline{\Omega}_k$ and Eq. (176), on the other hand, we have

$$\overline{\Omega}_k = \frac{\dot{\widehat{\phi}}}{12\,H^2} = \frac{3}{4}\,k_0^2. \tag{178}$$

The insertion of this result into Eq. (177) finally gives

$$\overline{\Omega}_V = \frac{3k_0^2(2-q_0)+4q_0}{4(2+q_0)}. \tag{179}$$

The combination of $\overline{\Omega}_k$ and $\overline{\Omega}_V$ provides now the values of $\overline{\Omega}_\phi$, $\overline{\Omega}_m$, and the equation of state $\overline{w}$, according to the definitions (161), (162). As we are interested in the kinematical properties of the solution we shall compute, in particular, the acceleration parameter $\ddot{a}/(aH^2)$: dividing by $6H^2$ the modified equation (170) we obtain

$$\frac{\dot{H}}{H^2} = \frac{3}{2}\left(\overline{\Omega}_V - \overline{\Omega}_k\right) - \frac{3}{4}q_0\,\overline{\Omega}_m - \frac{3}{2}, \tag{180}$$

from which

$$\frac{\ddot{a}}{aH^2} \equiv 1 + \frac{\dot{H}}{H^2} = -\frac{1}{2}, \tag{181}$$

quite independently of the values of k_0 and q_0! The integration of $\dot{H}$ finally provides $a \sim t^{2/3}$ and $H^2 \sim \rho \sim a^{-3}$, as in the standard phase of dark-matter dominated evolution.

The considered model of non-local coupling is thus associated to an asymptotic freezing phase which is decelerated, and in which the dilaton energy density has the same dynamical behavior of a dust fluid, $\rho_\phi \sim \rho_m \sim a^{-3}$, in spite of a pressure which is nonvanishing, in general:

$$\overline{w} = \frac{q_q}{2}\,\frac{3k_0^2-2}{3k_0^2+q_0}. \tag{182}$$

This result can be understood by noting Eqs. (180) and (181) together imply

$$\overline{\Omega}_k - \overline{\Omega}_V + \frac{q_0}{2}\,\overline{\Omega}_m \equiv \frac{1}{6H^2}\left(p_\phi + \frac{\sigma_m}{2}\right) = 0, \tag{183}$$

namely a zero total pressure for the coupled dilaton-dark matter fluid (see Eq. (170). The dark matter pressure associated to the non-local effects thus generates a backreaction which exactly compensates – at least in this model – the dilaton pressure, leading the system to restore, asymptotically, the standard dust matter configuration.

3.4 Main differences from uncoupled models

Let us come back to the class of models in which the dilaton is locally coupled to the dark matter components, as discussed in subsection 3.2. If we identify the accelerated freezing phase with our present cosmological phase, and thus the energy density of the dilaton field with the "dark-energy" density responsible for the present cosmic acceleration, we are lead to a dilaton model of

dark energy which is substantially different from the conventional models of quintessence [74] based on a rolling scalar field, uncoupled to dark matter.

A first, important (conceptual) difference concerns the mentioned problem of the cosmic coincidence. In the considered class of dilaton models this problem, if not solved, is at least relaxed: in fact, the dark-energy and dark-matter densities are of the same order not only today but also in the future (forever), and also in the past for a significantly amount of time, depending on the beginning of the freezing epoch (see below).

A second, more phenomenological difference concerns the scaling behavior of the baryonic and dark-matter components of the dust fluid during the freezing epoch. Because of the coupling to the dilaton, the dilution in time of the dark-matter density ρ_m is slower than the standard baryon dilution, $\rho_b \sim a^{-3}$: in particular, the ratio ρ_b/ρ_m decreases in time as

$$\frac{\rho_b}{\rho_m} \sim a^{-3q_0/(2+q_0)} \tag{184}$$

(see Eq. (166)). This could explain why the present fraction of baryons is small ($\sim 10^{-2}$) in critical units – provided the accelerated epoch has an early enough beginning. Direct/indirect measurements of the past value of the ratio ρ_b/ρ_m, compared with its present value, could provide unambiguous tests of this class of models.

Finally, concerning the beginning of the accelerated epoch, it is important to stress that in dilaton models the acceleration can start much earlier than in models of uncoupled quintessence [75, 76].

For a simple illustration of this point we may consider a model in which, during the accelerated regime, there are two types of sources with different dynamical behavior: *i*) an *uncoupled* component ρ_u, with pressure $p_u = 0$ and scaling behavior $\rho_u \sim a^{-3}$ (represented by baryons and, possibly, by a fraction of non-baryonic dark matter uncoupled to the dilaton); *ii*) a *coupled* component ρ_c, with pressure $p_c = w\rho_c$, and a slower scaling behavior $\rho_c \sim a^{-3(1+w)}$ (represented by the dilaton and by the fraction of dark matter coupled to the dilaton). Thus, even if today ρ_c dominates, and drives an accelerated evolution, at ealy enough times the Universe was dominated by ρ_u, and decelerated. From the Einstein equations

$$\begin{aligned} 6H^2 &= \rho_u + \rho_c, \\ 4\dot{H} + 6H^2 &= -p_c = -w\rho_c, \end{aligned} \tag{185}$$

we obtain that the acceleration switches off at the scale a_{acc} such that

$$\left(\frac{\ddot{a}}{aH^2}\right)_{\rm acc} = 1 + \left(\frac{\dot{H}}{H^2}\right)_{\rm acc} = -\frac{1}{2}\left[\Omega_u - (1+3w)(\Omega_u - 1)\right]_{\rm acc} = 0, \tag{186}$$

where $\Omega_u = \rho_u/6H^2$, $\Omega_c = 1 - \Omega_u$. In terms of the present values Ω_u^0, Ω_c^0 of these fractions the above condition becomes

$$\Omega_u^0 \left(\frac{a_{\rm acc}}{a_0}\right)^{-3} = (1+3w)\left(\Omega_u^0 - 1\right)\left(\frac{a_{\rm acc}}{a_0}\right)^{-3(1+w)}, \qquad (187)$$

and fixes the beginning of the acceleration at the redshift scale z_{acc} such that

$$z_{\rm acc} \equiv \frac{a_0}{a_{\rm acc}} - 1 = \left[(1+3w)\left(\frac{\Omega_u^0 - 1}{\Omega_u^0}\right)\right]^{-1/3w} - 1. \qquad (188)$$

Consider now a model of uncoupled quintessence, in which the uncoupled component corresponds to the totality of the dark-matter fluid (plus subdominant contributions), i.e. $\Omega_u^0 = \Omega_m^0$. Using the recent analysis of the SNLS Collaboration [68], based on present observations of supernovae Ia and large-scale structure, one finds $0.2 \leq \Omega_m^0 \leq 0.4$, and $-1.2 \leq w \leq 0.8$. One then obtains, from Eq. (188), $0.4 \lesssim z_{acc} \lesssim 1$ (see Fig. 10, left panel).

If we consider instead a model of dilaton dark energy, then the uncoupled component may range from the baryon component Ω_b to some fraction of the dark matter component Ω_m. In the "maximally coupled" version of the model, in which $\Omega_u^0 = \Omega_b^0$, we can re-apply the supernovae results of SNLS with $\Omega_b^0 \simeq 0.04 - 0.05$, to obtain $w \simeq -0.65 \pm 0.15$. Eq. (188) then implies [76] $z_{acc} \simeq 3 - 4$ (see Fig. 10, right panel).

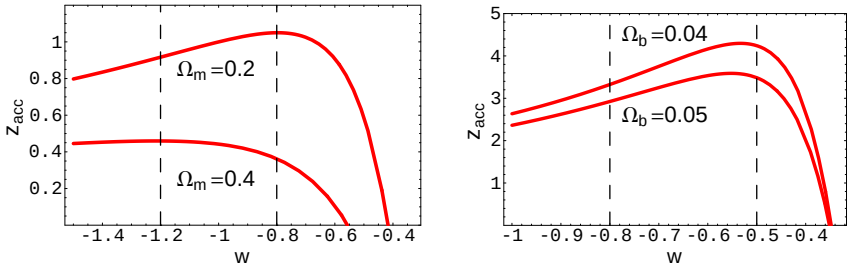

Fig. 10. Beginning of the accelerated epoch for dark-nergy models with uncoupled (left) and fully coupled (right) dark matter, according to the observations of type Ia supernovae in the SNLS dataset. The plotted curves are obtained from Eq. (188), for constant values of the present fraction of the uncoupled dust matter Ω_u^0.

Thus, dilaton models of dark energy are compatible with a beginning of the cosmic accelerations at epochs much earlier than those suggested by other models, according to the most recent supernovae data. The extension back in time of the accelerated regime might have a significant impact on the dilution of baryons, according to Eq. (184). Finally, strongly coupled models tend to be compatible with a "less negative" parameter w (see Fig. 10), thus alleviating the need for "phantom" dark energy [77] with "supernegative" ($w < -1$) equation of state.

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Appendix A

In this Appendix we present a detailed derivation of the equations of motion (29), (34), starting from the action (28) which includes non-local dilaton interactions.

The functional derivation of the action with respect to the metric $g^{\mu\nu}(x)$ contains, besides the standard contributions leading to Eq. (3), the new non-local contributions $V_{\mu\nu}(x)$ and $L_{\mu\nu}(x)$, and can be written as follows,

$$\frac{\delta S}{\delta g^{\mu\nu}(x)} = -\frac{\left(\sqrt{-g}\,\mathrm{e}^{-\phi}\right)_x}{2\lambda_{\mathrm{s}}^{d-1}}\left[G_{\mu\nu}+\nabla_\mu\nabla_\nu\phi+\frac{1}{2}g_{\mu\nu}\left(\nabla\phi^2-2\nabla^2\phi-V\right)\right]_x + \frac{1}{2}\sqrt{-g}\,T_{\mu\nu}(x)+V_{\mu\nu}(x)+L_{\mu\nu}(x), \tag{A.1}$$

where

$$V_{\mu\nu}(x) = -\frac{1}{2\lambda_{\mathrm{s}}^{d-1}}\int \mathrm{d}^{d+1}x'\left(\sqrt{-g}\,\mathrm{e}^{-\phi}V'\right)_{x'}\frac{\delta}{\delta g^{\mu\nu}(x)}\mathrm{e}^{-\xi(x')}, \tag{A.2}$$

$$L_{\mu\nu}(x) = \int \mathrm{d}^{d+1}x'\left(\sqrt{-g}\mathcal{L}'_m\right)_{x'}\frac{\delta}{\delta g^{\mu\nu}(x)}\mathrm{e}^{-\xi(x')} \tag{A.3}$$

(V' and $\mathcal{L}'_m$ are defined by Eq. (33)). We need now to compute the functional derivation of $\exp(-\xi)$. Using the definition (25) we obtain

$$\begin{aligned}\frac{\delta}{\delta g^{\mu\nu}(x)}\mathrm{e}^{-\xi(x')} &= -\frac{1}{\lambda_{\mathrm{s}}^{d}}\int \mathrm{d}^{d+1}y\,\delta^{d+1}(x-y)\delta(\phi_{x'}-\phi_y)\,\mathrm{e}^{-\phi_y} \\ &\quad\left[-\frac{1}{2}\left(\sqrt{-g}\,g_{\mu\nu}\sqrt{\epsilon(\nabla\phi)^2}\right)+\frac{1}{2}\sqrt{-g}\sqrt{\epsilon(\nabla\phi)^2}\,\frac{\partial_\mu\phi\partial_\nu\phi}{(\nabla\phi)^2}\right]_y \\ &= -\frac{1}{2\lambda_{\mathrm{s}}^{d}}\left(\gamma_{\mu\nu}\sqrt{-g}\,\sqrt{\epsilon(\nabla\phi)^2}\,\mathrm{e}^{-\phi}\right)_x\delta(\phi_{x'}-\phi_x),\end{aligned} \tag{A.4}$$

where $\gamma_{\mu\nu}$ is defined in Eq. (30). Thus:

$$V_{\mu\nu} = \frac{1}{2\lambda_{\rm s}^{d-1}}\sqrt{-g}\,{\rm e}^{-2\phi}\frac{1}{2}\gamma_{\mu\nu}\sqrt{\epsilon(\nabla\phi)^2}\,I_V, \tag{A.5}$$

$$L_{\mu\nu} = -\sqrt{-g}\,{\rm e}^{-\phi}\frac{1}{2}\gamma_{\mu\nu}\sqrt{\epsilon(\nabla\phi)^2}\,I_m, \tag{A.6}$$

where I_V and I_m are defined in Eqs. (31), (32). Inserting these results in Eq. (A.1), multiplying by $(-2\lambda_{\rm s}^{d-1})\exp(-\phi)/\sqrt{-g}$, and imposing the condition of zero functional derivative, one is finally lead to Eq. (29).

Let us now consider the functional derivative with respect to $\phi(x)$. Separating the local and non-local terms, as before, we obtain

$$\frac{\delta S}{\delta\phi(x)} = \frac{\left(\sqrt{-g}\,{\rm e}^{-\phi}\right)_x}{2\lambda_{\rm s}^{d-1}}\left[R + 2\nabla^2\phi - (\nabla\phi)^2 + V\right]_x + A(x) + B(x), \tag{A.7}$$

where

$$A(x) = -\frac{1}{2\lambda_{\rm s}^{d-1}}\int {\rm d}^{d+1}x'\left(\sqrt{-g}\,{\rm e}^{-\phi}V'\right)_{x'}\frac{\delta}{\delta\phi(x)}{\rm e}^{-\xi(x')}, \tag{A.8}$$

$$B(x) = \int {\rm d}^{d+1}x'\left(\sqrt{-g}\mathcal{L}'_m\right)_{x'}\frac{\delta}{\delta\phi(x)}{\rm e}^{-\xi(x')}. \tag{A.9}$$

The functional derivative of the variable (25) leads to

$$\begin{aligned}
&\frac{\delta}{\delta\phi(x)}{\rm e}^{-\xi(x')} = \\
&= \frac{1}{\lambda_{\rm s}^d}\int {\rm d}^{d+1}y\Bigg\{ -\left(\sqrt{-g}\,{\rm e}^{-\phi}\sqrt{\epsilon(\nabla\phi)^2}\right)_y\delta(\phi_{x'}-\phi_y)\delta^{d+1}(x-y) \\
&+\left(\sqrt{-g}\,{\rm e}^{-\phi}\sqrt{\epsilon(\nabla\phi)^2}\right)_y\delta'(\phi_{x'}-\phi_y)\left[\delta^{d+1}(x-x') - \delta^{d+1}(x-y)\right] \\
&-\partial_\mu\left[\frac{\sqrt{-g}\,{\rm e}^{-\phi}\epsilon\,\partial^\mu\phi}{\sqrt{\epsilon(\nabla\phi)^2}}\delta(\phi_{x'}-\phi_y)\right]_y\delta^{d+1}(x-y)\Bigg\},
\end{aligned} \tag{A.10}$$

where $\partial_\mu = \partial/\partial y^\mu$, and δ' denotes the derivative of the delta function with respect to its argument.

The first term of this integral exactly cancels the term containing $\partial_\mu{\rm e}^{-\phi}$ in the last part of the integral; also, the third term exactly cancels the term containing $\partial_\mu[\delta(\phi_{x'}-\phi_y)]$ in the last part of the integral. Thus, we are left with:

$$\begin{aligned}
\frac{\delta}{\delta\phi(x)}{\rm e}^{-\xi(x')} = \frac{\delta^{d+1}(x-x')}{\lambda_{\rm s}^d}\int {\rm d}^{d+1}y\left(\sqrt{-g}{\rm e}^{-\phi}\sqrt{\epsilon(\nabla\phi)^2}\right)_y\delta'(\phi_{x'}-\phi_y) \\
-\epsilon\,\frac{{\rm e}^{-\phi}}{\lambda_{\rm s}^d}\delta(\phi_{x'}-\phi_x)\partial_\mu\left(\frac{\sqrt{-g}\,\partial^\mu\phi}{\sqrt{\epsilon(\nabla\phi)^2}}\right)_x.
\end{aligned} \tag{A.11}$$

The second term on the right-hand side of the above equation can be conveniently rewritten as

$$-\epsilon\frac{\mathrm{e}^{-\phi}}{\lambda_{\mathrm{s}}^{d}}\delta(\phi_{x'}-\phi_x)\sqrt{-g}\,\nabla_\mu\left(\frac{\partial^\mu\phi}{\sqrt{\epsilon(\nabla\phi)^2}}\right)_x =$$
$$= -\epsilon\frac{\mathrm{e}^{-\phi}}{\lambda_{\mathrm{s}}^{d}}\delta(\phi_{x'}-\phi_x)\frac{\sqrt{-g}}{\sqrt{\epsilon(\nabla\phi)^2}}\,\gamma_{\mu\nu}\nabla^\mu\nabla^\nu\phi. \tag{A.12}$$

For the first term containing δ' we can exploit the properties of the delta function, and the identities

$$\mathrm{d}y_0=\frac{1}{\dot\phi_y}\mathrm{d}\phi_y, \qquad \frac{\mathrm{d}}{\mathrm{d}\phi_y}=\frac{1}{\dot\phi_y}\frac{\mathrm{d}}{\mathrm{d}y_0}, \tag{A.13}$$

to obtain

$$\lambda_{\mathrm{s}}^{-d}\int\mathrm{d}^dy\,\frac{\mathrm{d}\phi_y}{\dot\phi_y}\left(\sqrt{-g}\,\mathrm{e}^{-\phi}\sqrt{\epsilon(\nabla\phi)^2}\right)_y\delta'(\phi_x-\phi_y)$$
$$=\lambda_{\mathrm{s}}^{-d}\int\mathrm{d}^dy\,\frac{\mathrm{d}\phi_y}{\dot\phi_y}\,\frac{\mathrm{d}}{\mathrm{d}y_0}\left(\frac{\sqrt{-g}\,\mathrm{e}^{-\phi}\sqrt{\epsilon(\nabla\phi)^2}}{\dot\phi}\right)_y\delta(\phi_x-\phi_y)$$
$$=-e^{-\xi(x)}+\lambda_{\mathrm{s}}^{-d}\mathrm{e}^{-\phi(x)}\int\mathrm{d}^dy\,\frac{\mathrm{d}\phi_y}{\dot\phi_y}\,\left(\sqrt{-g}\,\sqrt{\epsilon(\nabla\phi)^2}\right)_y\delta'(\phi_x-\phi_y)$$
$$=-e^{-\xi(x)}+e^{-\phi(x)}J(x), \tag{A.14}$$

where J is the integral defined in Eq. (35). Inserting the results (A.12), (A.14) in Eq. (A.11), using the definitions of A and B, and integrating over x', we finally obtain

$$A(x)+B(x)=\left(\frac{\sqrt{-g}\,\mathrm{e}^{-\phi}}{2\lambda_{\mathrm{s}}^{d-1}}\,V'-\sqrt{-g}\,\mathcal{L}'_m\right)_x\left(e^{-\xi}-e^{-\phi}J\right)_x$$
$$+\,\epsilon\,\frac{\sqrt{-g}\,\mathrm{e}^{-\phi}}{\sqrt{\epsilon(\nabla\phi)^2}}\gamma_{\mu\nu}\nabla^\mu\nabla^\nu\phi\left(\frac{\mathrm{e}^{-\phi}}{2\lambda_{\mathrm{s}}^{d-1}}\,I_V-I_m\right)_x. \tag{A.15}$$

Summing this contribution to the other terms of Eq. (7), multiplying by $(2\lambda_{\mathrm{s}}^{d-1}\exp\phi/\sqrt{-g})$, and imposing the vanishing of the functional derivative, we are lead to the equation of motion (34) for the dilaton.

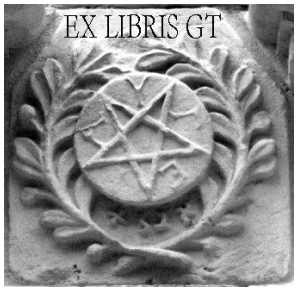
References

1. G. Veneziano: Phys. Lett. B **265**, 287 (1991)
2. M. Gasperini, G. Veneziano: Astropart. Phys. **1**, 317 (1993)
3. M. B. Green, J. Schwartz, E. Witten: *Superstring theory* (Cambridge University Press, Cambridge 1987)

4. J. Polchinski: *String theory* (Cambridge University Press, Cambridge 1998)
5. M. Gasperini: in *Gravitational waves*, ed by I. Ciufolini et al (IOP Publishing, Bristol 2001), p. 280
6. A. A. Tseytlin: Mod. Phys. Lett. A **6** 1721 (1991)
7. K. Kikkawa, M. Y. Yamasaki: Phys. Lett. B **149** 357 (1984)
8. K. A. Meissner, G. Veneziano: Mod. Phys. Lett. A **6**, 3397 (1991); Phys. Lett. B **267**, 33 (1991)
9. K. Meissner: *Dualities in string cosmology*, this volume
10. M. Gasperini, G. Veneziano: Phys. Lett. B **277**, 256 (1992)
11. M. Gasperini, M. Giovannini, G. Veneziano: Phys. Lett. B **569**, 113 (2003); Nucl. Phys. B **694**, 206 (2004)
12. M. Gasperini, G. Veneziano: Mod. Phys. Lett. A **8**, 3701 (1993)
13. M. Gasperini, G. Veneziano: Phys. Rev. D **50**, 2519 (1994)
14. A. D. Linde: Phys. Lett. B **129**, 177 (1983)
15. A. Vilenkin: Phys. Rev. D **46**, 2355 (1992); A. Borde, A. Vilenkin: Phys. Rev. Lett. **72**, 3305 (1994)
16. A. Buonanno, T. Damour, G. Veneziano: Nucl. Phys. B **543**, 275 (1999)
17. M. Gasperini, G. Veneziano: Gen. Rel. Grav. **28**, 1301 (1996)
18. M. Gasperini, J. Maharana, G. Veneziano: Nucl. Phys. B **472**, 349 (1996)
19. R. H. Brandenberger, J. Martin: Phys. Rev. D **71**, 023504 (2005)
20. M. Gasperini, M. Maggiore, G. Veneziano: Nucl. Phys. B **494**, 315 (1997)
21. R. Brustein, R. Madden: Phys. Lett. B **410**, 110 (1997); Phys. Rev. D. **57**, 712 (1998); C. Cartier, E. J. Copeland, R. Madden, JHEP **0001**, 035 (2000)
22. M. Gasperini, G. Veneziano: Phys. Rep. **373** 1 (2003)
23. M. Gasperini: Mod. Phys. Lett. A **14**, 1059 (1999)
24. M. Gasperini, M. Giovannini, K. A. Meissner, G. Veneziano: Nucl. Phys. (Proc. Suppl.) B **49**, 70 (1996)
25. M. Gasperini, M. Giovannini: Phys. Lett. B **282**, 36 (1992); Phys. Rev. D **47**, 1519 (1993)
26. R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov, G. Veneziano: Phys. Rev. D **51**, 6744 (1995)
27. R. Brustein, M. Gasperini, M. Giovannini, G. Veneziano: Phys. Lett. B **361**, 45 (1995)
28. A. Buonanno and C. Ungarelli: *Primordial gravitational radiation in string cosmology*, this volume
29. J. Khoury, B. A. Ovrut, P. J. Steinhardt, N. Turok: Phys. Rev. D **64**, 123533 (2001)
30. E. I. Buchbinder, J. Khoury, B. A. Ovrut: *New ekpyrotic cosmology*, hep-th/0702154
31. L. A. Boyle, P. J. Steinhardt, N. Turok: Phys. Rev. D **69**, 127302 (2002)
32. M. Gasperini: *Elements of string cosmology* (Cambridge University Press, Cambridge 2007), in press
33. M. Maggiore, A. Riotto: Nucl. Phys. B **548**, 427 (1999)
34. S. Alexander, R. Brandenberger, D. Easson: Phys. Rev. D **62**, 103509 (2000)
35. C. Burgess et al: JHEP **0107**, 047 (2001); S. Kachru et al: JCAP **0310**, 013 (2003)
36. H. Tye: *Brane inflation: string theory viewed from the cosmo*, this volume
37. K. Enqvist, M. Sloth: Nucl. Phys. B **626**, 395 (2002); D. H. Lyth, D. Wands: Phys. Lett. B **524**, 5 (2002); T. Moroi, T. Takahashi: Phys. Lett. B **522**, 215 (2001)

38. V. Bozza, M. Gasperini, M. Giovannini, G. Veneziano: Phys. Lett. B **543**, 14 (2002); Phys. Rev. D **67**, 063514 (2003)
39. E. J. Copeland, R. Easther, D. Wands: Phys. Rev. D **56**, 874 (1997); E. J. Copeland, J. E. Lidsey, D. Wands: Nucl. Phys. B **506**, 407 (1997)
40. V. F. Mukhanov, H. A. Feldman, R. H. Brandenberger: Phys. Rep. **215**, 203 (1992)
41. M. Abramowitz, I. A. Stegun: *Handbook of mathematical functions* (Dover, New York 1972)
42. A. Melchiorri, F. Vernizzi, R. Durrer, G. Veneziano: Phys. Rev. Lett. **83**, 4464 (1999)
43. D. Lyth, C. Ungarelli, D. Wands: Phys. Rev. D **67**, 023503 (2003)
44. M. Gasperini: in Proc. of the *Fifth Paris cosmology colloquium*, ed by H. J. De Vega and N. Sanchez (Publication Observatoire de Paris, Paris 1999), p. 317
45. D. N. Spergel et al: astro-ph/0603449
46. M. Gasperini, M. Giovannini, G. Veneziano: Phys. Rev. D **52**, 6651 (1995); M. Gasperini, M. Giovannini, G. Veneziano: Phys. Rev. Lett. **75**, 3796 (1995); D. Lemoine, M. Lemoine: Phys. Rev. D **52**, 1995 (1995); M. Gasperini, S. Nicotri: Phys. Lett. B **633**, 155 (2006)
47. M. Gasperini: Phys. Lett. B **327**, 314 (1994)
48. T. Taylor, G. Veneziano: Phys. Lett. B **213**, 459 (1988)
49. J. Ellis et al: Phys. Lett. B **228**, 264 (1989)
50. R. Durrer, M. Gasperini, M. Sakellariadou, G. Veneziano: Phys. Rev D **59**, 43511 (1999)
51. M. Gasperini, G. Veneziano: Phys. Rev. D **59**, 43503 (1999)
52. R. Brustein, M. Gasperini, G. Veneziano: Phys. Rev. D **55**, 3882 (1997)
53. E. Fischbach, C. Talmadge: Nature **356**, 207 (1992)
54. C. D. Hoyle et al: Phys. Rev. D **70**, 042004 (2004)
55. M. Gasperini: Phys. Lett. B **470**, 67 (1999)
56. T. Damour, A. M. Polyakov: Nucl. Phys. B **423**, 352 (1994); Gen. Rel. Grav. **26**, 1171 (1994)
57. C. Misner, K. Thorne, J. A. Wheeler: *Gravitation* (Freeman, San Francisco 1973)
58. M. Gasperini: Phys. Lett. B **477**, 242 (2000)
59. M. Bianchi et al: Phys. Rev. D **57**, 4525 (1998); M. Maggiore, A. Nicolis: Phys. Rev. D **62**, 024004 (2000)
60. M. Gasperini, C. Ungarelli: Phys. Rev. D **64**, 064009 (2001)
61. N. Bonasia, M. Gasperini: Phys. Rev. D **71**, 104020 (2005)
62. B. Allen, J. D. Romano: Phys. Rev. D **59**, 102001 (1999)
63. M. Doran, J. Jackel: Phys. Rev. D **66**, 043519 (2002)
64. B. J. Owen, B. S. Sathyaprakash: Phys. Rev. D **60**, 022002 (1999)
65. E. Coccia, M. Gasperini, C. Ungarelli: Phys. Rev. D **65**, 067101 (2002)
66. S. Perlmutter et al: Nature **391**, 51 (1998); A. G. Riess et al: Astron. J. **116**, 1009 (1998)
67. A. G. Riess et al: Astrophys. J. **607**, 665 (2004)
68. P. Astier et al: Astron. Astrophys. **447**, 31 (2006)
69. M. Gasperini: Phys. Rev. D**64**, 043510 (2001)
70. N. Kaloper, K. A. Olive: Astropart. Phys. **1**, 185 (1993)
71. M. Gasperini, F. Piazza, G. Veneziano: Phys. Rev. D **65**, 023508 (2002)
72. L. Amendola: Phys. Rev. D **62**, 043511 (2000); L. Amendola, D. Tocchini-Valentini: Phys. Rev. D **64**, 04359 (2001); Phys. Rev. D **66** 043528 (2002)

73. G. Veneziano: JHEP **0206**, 051 (2002)
74. B. Ratra, P. J. E. Peebles: Phys. Rev. D **37** 3406 (1988); C. Wetterich: Nucl. Phys. B **302**, 668 (1988); M. S. Turner, C. White: Phys. Rev. D **56**, 4439 (1997); R. R. Caldwell, R. Dave, P. J. Steinhardt: Phys. Rev. Lett. **80**, 1582 (1998); I. Zlatev, L. Wang, P. J. Steinhardt: Phys. Rev. Lett. **82**, 896 (1999); Phys. Rev. D **59**, 123504 (1999)
75. L. Amendola, M. Gasperini, D. Tocchini-Valentini, C. Ungarelli: Phys. Rev. D **67**, 043512 (2003); L. Amendola, M. Gasperini, F. Piazza: JCAP **09** , 014 (2004)
76. L. Amendola, M. Gasperini, F. Piazza: Phys. Rev. D **74**, 127302 (2006)
77. R. R. Caldwell, M. Kamionkowski, N. N. Weinberg: Phys. Rev. Lett. **91**, 07130 (2003)



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A Model for the Big Bounce

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Abstract

I motivate a proposal for modeling, at weak string coupling, the "Big Bounce" transition from a growing-curvature phase to standard (FRW) cosmology in terms of a pressure-less dense gas of "string-holes" (SH), string states lying on the correspondence curve between strings and black holes. During this phase SH evolve in such a way that temperature and (string-frame) curvature remain $O(M_s)$ and (a cosmological version of) the holographic entropy bound remains saturated. This reasoning also appears to imply a new interpretation of the Hagedorn phase transition in string theory.

SH phase a bridge between pre and post BB

String-black hole transformation

The fate of the big bang singularity of General Relativity (GR) is one of the most important unsolved problems in string theory. Unlike time-like singularities, space-like curvature singularities –such as those related to the big bang or to the interior of black holes– confront string theory with high-curvature, strongly time-dependent backgrounds, a *terra incognita*, to a large extent.

Understanding black hole thermodynamics in string theory from a statistical mechanics viewpoint has been comparatively easier (for a review see e.g. [1]). For extremal and near-extremal black holes analytic treatments are possible and reproduce amazingly well GR expectations. For more usual (e.g. Schwarzschild-like) black holes an analytic treatment is not available but some understanding has been gathered on how string states progressively turn into black holes as the string coupling is increased. In particular, along a correspondence curve (see Fig. 1) given by [2]:

$$M = M_s g_s^{-2} \ , \tag{1}$$

where , $M_s = l_s^{-1}$ and g_s are the string mass and string coupling, respectively, properties of strings and black holes match impressively well [3]. This correspondence also supports an earlier conjecture [4] about the late stage of the evaporation process of black holes in string theory. Such a process inevitably takes a black hole to the correspondence curve (1) where it turns into a string and typically decays into (almost) massless quanta, thus avoiding any singularity at the end of its life. In this note I will argue that the above understanding of black holes can be used to make progress on the issue of the cosmological singularity or, better, on how to describe the transition from a growing curvature phase to a FRW cosmology, transition that I would like to dub the "Big Bounce". The link between these two seemingly unrelated issues is based on the use of of holography and of related cosmological entropy bounds .

Following pioneering work by Bekenstein [5], several cosmological entropy bounds have been proposed and (theoretically) tested during the past few years. At the level at which we shall be working here, they can be simply stated [6], [7] as bounding the entropy of a large region of space by that of a closely packed gas of black holes whose (Schwarzschild) radius coincides with the Hubble radius H^{-1}. Neglecting throughout numerical factors, this leads to an entropy density bound σ_{max} given by [1]:

$$\sigma_{max} = \frac{S_{max}}{V} = H M_P^{D-2} \tag{2}$$

M_P denotes the D-dimensional Planck mass, the inverse of the D-dimensional Planck length, l_P.

In a Universe dominated by a perfect fluid of equation of state $p = w\rho$ this bound can be compared to the actual entropy of the fluid:

$$\sigma_F = \frac{p+\rho}{T} = (1+w)\frac{H^2 \ M_P^{D-2}}{T} = \frac{(1+w)H}{T} \ \sigma_{max} \tag{3}$$

[1] We shall work troughout in a generic number D of space-time dimensions and set $\hbar = c = 1$.

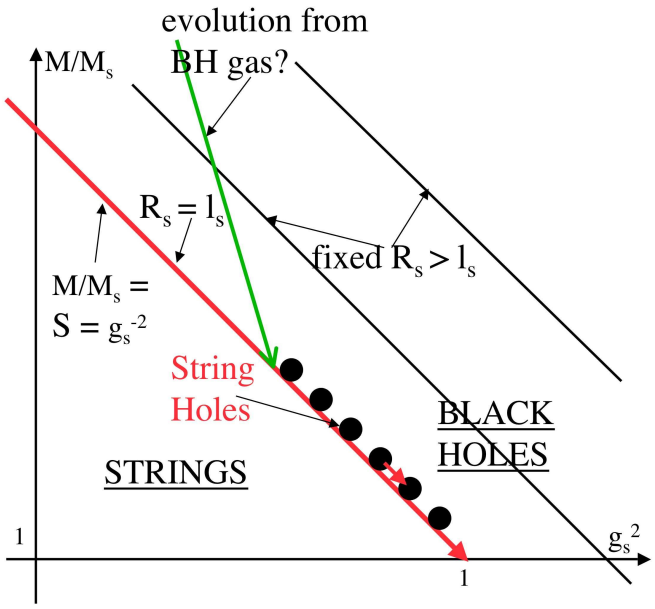

Figure 1: Black-hole/string phase diagram in a log–log g_s vs. M/M_s plot.

"parametric amplification" see 2002

In standard (post-big bang) cosmology one has typically $H/T \ll 1$ and, furthermore, the bound is better and better fulfilled as the Universe expands since H/T decreases with time for $w < 1$ (and remains constant for $w = 1$). In pre-big bang cosmology [8], as well as in other scenarios [9] where a growing-curvature phase precedes the big bang, the situation is completely reversed. A stiff fluid, $w = 1$, is the only safe case, while for $w < 1$ the bound is increasingly threatened as the big bang is approached. Clearly we have to understand what happens as we approach the limit $T \to H$.

In this limit another phenomenon becomes relevant: particle production from the time-dependent geometry or, if we prefer, the amplification of quantum fluctuations in a time-dependent background geometry. This process is far from being adiabatic and produces an entropy density given by [10]:

$$\sigma_Q = g_* H^{D-1} = g_* \left(\frac{H}{M_P}\right)^{D-2} \sigma_{max} \tag{4}$$

where g_* is the effective number of species that are produced. Note that the above entropy density is, parametrically speaking, that of a relativistic gas at $T \sim H$ with g_* species.

Let us now consider the three entropy densities (2), (3), (4) in a string theory context assuming T to be bounded by the Hagedorn temperature, $T \leq T_H \sim M_s$ and curvature to be bounded by l_s^{-2}. Let us also consider (very) weakly coupled string theory, $g_s^2 = \mathrm{e}^{\phi} \ll 1$ and recall the (weak-coupling, closed string) relation:

$$\left(\frac{M_s}{M_P}\right)^{D-2} = g_s^2 = \mathrm{e}^{\phi} \,, \tag{5}$$

where ϕ is the dilaton. In a string cosmology context the dilaton evolves in time and so does, through (5), the ratio of the Planck to the string mass. Furthermore, in the pre-big bang scenario the string coupling starts extremely small and grows towards the bounce while the opposite is true in the ekpyrotic/cyclic Universe. Hereafter we shall concentrate on the former case.

Can we saturate the entropy bound for $T \ll M_s$?. The answer is apparently negative. This looks quite obvious for (4) since $(H/M_P)^{D-2} < (M_s/M_P)^{D-2} = g_s^2 \ll 1$ If we keep g_* fixed as we decrease the coupling there is no way to saturate the bound from σ_Q. We can argue similarly for σ_F since H/T must go to zero as we turn off Newton's constant (i.e. the string coupling). This naive failure to saturate the bound, however, suggests by itself a way to achieve saturation even at very small string coupling: we need the effective number of species g_* to grow as the string coupling is decreased. But how? Actually, we do know already of states that have the desired property: they are precisely the (marginal) black holes that lie on the above-mentioned correspondence curve of Eq. (1) (and Fig. 1) that we now complete as:

$$M = M_s g_s^{-2} = M_s \, \mathrm{e}^{-\phi} = M_P \, \exp\left(-\frac{D-3}{D-2}\phi\right) \tag{6}$$

The Schwarschild radius and Hawking temperature of these black holes are given by the string scale and mass, respectively: we will refer to them in the following as string-holes (SH). The entropy of a SH is given equivalently by the Bekenstein-Hawking or by the weak-coupling string formulae [2]:

"SH"

$$S_{SH} = (l_s/l_P)^{D-2} = g_s^{-2} = S_{string} \equiv M l_s \ , \tag{7}$$

The possibility of forming small primordial black holes in the pre-bang phase of string cosmology, as a result of the blue spectra of tensor and scalar perturbations characteristic of these scenarios [8], was already emphasized in [11], where a possible connection between primordial black hole formation and observation of a stochastic gravitational-wave background at (advanced versions of present) interferometers was also pointed out.

Let us check now that the identification of the final stage of collapse with such a gas of SH satisfies other consistency checks and let us also argue for an evolution of such a gas once it is formed. The following quantities (referring to the string frame) can be readily computed:

$$\begin{aligned} \rho_{SH} &= M_s \ g_s^{-2}/l_s^{D-1} = \mathrm{e}^{-\phi} \ M_s^D \ , \\ \sigma_{SH} &= (l_s/l_P)^{D-2} \ /l_s^{D-1} = \mathrm{e}^{-\phi} \ M_s^{D-1} \ , \end{aligned} \tag{8}$$

and since $H \sim T \sim M_s$, we immediately check that $\sigma = \rho/T$ and that the entropy bound is saturated.

In order to derive an evolution equation for the gas we impose the 2nd law and demand that entropy in a fixed comoving volume (always in the string frame) does not decrease. Since

$$S = \sigma_{SH} V \sim \mathrm{e}^{-\phi} \, V \ \sim \mathrm{e}^{-\bar{\phi}} \ , \tag{9}$$

we see that, necessarily, $\dot{\bar{\phi}} \leq 0$ with the equality sign holding true for an adiabatic evolution. In (9) we have introduced the standard "shifted dilaton" field $\bar{\phi} = \phi - 1/2 \log(g) + \text{const.}$ that plays an important role in string cosmology since it is invariant under scale-factor duality transformations and their continuous generalizations. This result was already obtained some years ago [12] in a less specific framework and is very encouraging since it means that the strong-curvature evolution takes us towards a FRW Universe.

In the present case we expect the evolution of the gas to be almost adiabatic, al least for a while, in which case $\dot{\bar{\phi}} = 0$ and the picture becomes particularly appealing: each SH loses mass while the coupling grows in order to stay on the correspondence curve. The mass that is lost turns out to be *precisely* what is needed to form more SH's so that they keep filling the growing number of Hubble volumes while remaining closely packed. In the string frame the equation of state is that of dust ($p = 0$) since $\rho \sim \exp(-\phi) \sim V^{-1}$. This looks natural since SH are non-relativistic at small g^2. Furthermore, $\dot{\bar{\phi}} = 0$ and $p = 0$ are both

invariant under T-duality transformations, suggesting that, indeed, the SH-phase can serve as bridge between two duality-related pre and post-bounce cosmologies.

We can also ask whether such a phase can be the solution of the equations of motion of a string theoretic effective action. We claim that this is indeed the case. Following [13], consider the tree-level effective action of (say heterotic) string theory including all-order α' corrections (recall that we are interested in a high-curvature small-coupling regime) but add to it a "matter" piece providing the contribution of the dense gas of SH's to the stress-energy tensor. Generalizing slightly the arguments given in [13] we can easily show that a solution with $\dot{\bar{\phi}} = 0$, $p = 0$, and $\rho \sim \exp(-\phi) \sim V^{-1}$ exists whenever a certain β-function, $\beta(H^2)$ with $\beta(0) = 0$, has (at least) one non-trivial zero. This is the case, for instance, in the explicit model discussed in [13]. The location of the smallest non-trivial zero and the derivative of β at that point determine the Hubble parameter and $\rho \exp(\phi)$ during the SH phase, respectively. The latter quantity turns out to be automatically of the right order to correspond to our dense SH gas.

A SH condensate would then be the physical description of the so-called "string phase" of (weak-coupling) PBB cosmology, the horizontal evolution in Fig. 2, during which the string-frame Hubble parameter is fixed (giving de Sitter like evolution) while the dilaton keeps growing at a rate determined by the Hubble parameter via $\dot{\phi} = (D-1)H$. In the absence of matter sources such a solution does not exist, generically, but, as argued above, the opposite is true if we allow for SH production as the Hubble parameter approaches the string scale. In the case of "strongly-coupled" PBB cosmology, instead, the entropy bound is already saturated at weak curvature when the dilaton reaches a critical value (see again Fig. 2) owing to production of light quanta including $D-0$ branes. This complementary situation was described several years ago by Maggiore and Riotto [14]. Let us now compare our model with the "Black Crunch" picture recently considered in [15]. Also there one wishes to keep the bound on entropy saturated and time-independent (for a fixed comoving region of space). This imposes:

$$S_{max} = VHM_P^{D-2} = \text{const.} \tag{10}$$

In [15] it was argued that saturation can be achieved even at sub-string curvature scales. This implies $V\ M_P^{D-2} \sim H^{-1} \sim t$ and selects $p = \rho$ as the relevant equation of state in the Einstein frame. It is straightforward to convert our solution from the string to the Einstein frame using standard relations. One finds that the Einstein-frame metric and Hubble parameter are related to each other and to the E-frame cosmic time by

$$H_E \sim 1/\sqrt{g_E} \sim t_E^{-1}\ , \tag{11}$$

hinting again at a stiff equation of state. Using the appropriate E-frame conservation equation (see, e.g. Eq. (2.117) of [8]), and after inserting our solution for ϕ, one finds indeed the conservation equation for a $p = \rho$ fluid.

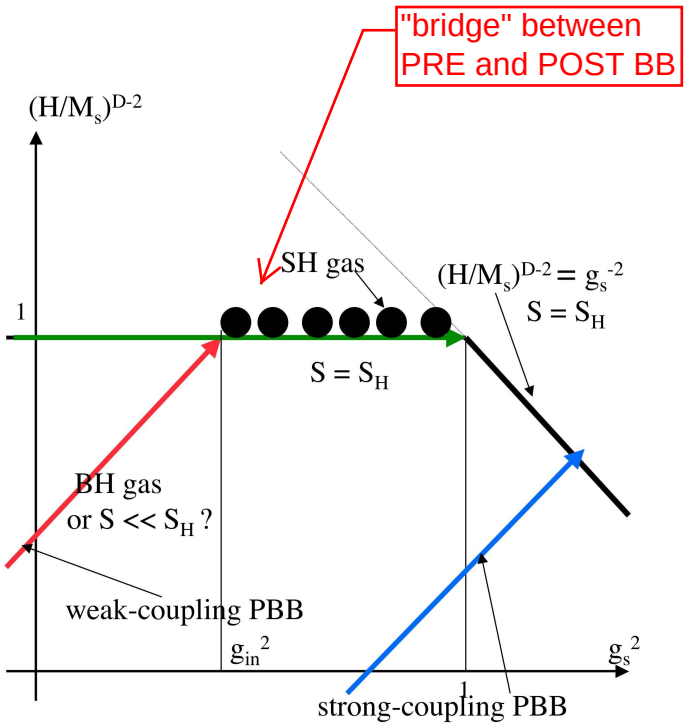

Figure 2: Pre-big bang phase diagram in a log–log g_s vs. H/M_s plot.

Although we have argued that it is unlikely for the entropy bound to be saturated *before* the string scale is reached this is far from being obvious under all circumstances (e.g. for particularly entropic initial states or as a consequence of some highly non-adiabatic classical process). What happens then if the bound becomes saturated at $H \ll M_s$? Since the Hubble radius is shrinking, it is obvious that the evolution of the system will follow trajectories in the $M - g_s^2$ plane that cross the constant $g_s^2 M$ (i.e. constant Schwarzschild radius) lines *downward* (see Figure 1). Thus, eventually, the evolution will take us to the correspondence line and will follow it thereafter (as long as the string coupling is weak). The effective E-frame equation of state will not change. We conclude that the occurrence of the SH phase is a generic outcome of pre-bang evolution in string theory. Indeed, it is likely to be also the final outcome of the chaotic BKL evolution that was argued [16] to occur generically in low-curvature small-coupling string or M-theory cosmology.

What is the ultimate fate of the SH phase we have advocated? The answer is not obvious: as soon as the string coupling reaches values $O(1)$, g_* too becomes $O(1)$ and there is no longer a way to keep the bound saturated while the curvature stays constant. If the dilaton keeps growing one could imagine connecting smoothly to the regime described in [14] staying on the rightmost boundary line in Fig. 2. If instead the dilaton (or rather the ratio M_P/M_s) freezes various possibilities emerge: the SH gas could disintegrate immediately into radiation; it could keep expanding adiabatically, following the $w = 1$ evolution advocated in [17]; or it could expand with considerable entropy release and $w < 1$, since in this case the bound itself grows in time. In any case, ultimately, decay into radiation should take place and the equation of state will become $p = \rho/(D-1)$. This does not mean, however, that there is no memory of what the actual phase preceding the bounce was. Properties of that phase would leave an imprint on large-scale perturbations that should still survive in the SH gas and in the radiation that follows it. Hopefully, these perturbations will have the right properties to produce the observed large-scale structure of our Universe.

Quite apart from cosmology, our considerations also appear to solve a more general puzzle within string theory. Supposedly, both temperature and (string frame) curvature are bounded by one and the same scale, M_s. However, temperature is related to energy density ρ while going from ρ to curvature involves Newton's constant which can be made arbitrarily small by going to the weak-coupling limit. Hence the two bounds appear to be parametrically different. Our point is that they can be simultaneously saturated if the number of effective degrees of freedom near the Hagedorn temperature grows as $G_N^{-1} \sim g_s^{-2}$ i.e. precisely like the entropy of SH's.

This also suggests a novel interpretation of the limiting temperature of string theory. As one increases T at small non zero string coupling, competition between Boltzmann suppression and entropy should lead, already at $T = T_H\,(1 - O(g_s^2))$, to a phase transition whereby a dense gas of SH's of mass M_s/g^2 replaces the more conventional low-temperature gas of light strings/particles. At this point, however, gravitational back-reaction will become important and will cause the proper volume of the gas to become time-dependent. Thus

only a cosmological interpretation of the Hagedorn phase transition will be possible.

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References

[1] J. R. David, G. Mandal and S.R. Wadia, "Microsopic Formulation of Black Holes in String Theory", Phys.Rept. **369** (2002) 549-686 (hep-th/0203048).

[2] G. Veneziano, *Europhys. Lett.* **2** (1986) 133;
L. Susskind, " Some speculations about black hole entropy in string theory" hep-th/9309145;
G. Veneziano *Hot Hadronic Matter: Theory and Experiments*, Divonne, June 1994, eds. J. Letessier, H. Gutbrod and J. Rafelsky, NATO-ASI Series B: Physics, **346** (Plenum Press, New York 1995), p. 63.

[3] E. Halyo, A. Rajaraman and L. Susskind, Phys. Lett. B **392**, 319 (1997);
E.Halyo, B. Kol, A. Rajaraman and L. Susskind, Phys. Lett. B **401**, 15 (1997);
G.T. Horowitz and J. Polchinski, Phys. Rev. D **55**, 6189 (1997);
G.T. Horowitz and J. Polchinski, Phys. Rev. D **57**, 2557 (1998);
T. Damour and G. Veneziano, Nucl. Phys. **B568** (2000) 93.

[4] M. J. Bowick, L. Smolin and L.C.R. Wijewardhana, Gen. Rel. Grav. **19** (1987) 113.

[5] J. D. Bekenstein, Int. J. Theor. Phys. **28** (1989) 967.

[6] W. Fischler and L. Susskind, " Holography and cosmology", hep-th/9806039;
R. Bousso, JHEP **9907** (1999) 004; JHEP **9906** (1999) 028;
E.E. Flanagan, D. Marolf and R.M. Wald, Phys.Rev. **D62** (2000) 084035.

[7] G. Veneziano, Phys. Lett. **B454** (1999) 22;
R. Easther and D.A. Lowe, Phys. Rev. Lett. **82** (1999) 4967;
D. Bak and S. Rey, Class.Quant.Grav. **17** (2000) L83;
N. Kaloper and A. Linde, Phys. Rev. **D60** (1999) 103509;
R. Brustein and G. Veneziano, Phys. Rev. Lett. **84** (2000) 5695.

[8] M. Gasperini and G. Veneziano, Phys. Rept. **373**, (2003) 1;
J. E. Lidsey, D. Wands and E. J. Copeland, Phys. Rept. **337** (2000) 343.

[9] J. Khouri, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. **D66**, 046005 (2002);
J. Khouri, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, Phys. Rev. **D65** (2002) 086007;
P. J. Steinhardt and N. Turok, Phys.Rev. **D65** (2002) 126003.

[10] M. Gasperini and M. Giovannini, Phys. Lett. **B301** (1993) 1519; Class. Q. Grav. **10** (1993) L133;
R. Bandenberger, V. F. Mukhanov and T. Prokopec, Phys. Rev. Lett. **69** (1992) 3606; Phys. Rev. **D 48** (1993) 2443.

[11] E. Copeland, A. R. Liddle, J. E. Lidsey and D. Wands, Phys. Rev. **D58** (1998) 063508.

[12] G. Veneziano, Ref. [7];
R. Brustein, Phys. Rev. Lett. **84** (2000) 2072;
R. Brustein, S. Foffa and R. Sturani, Phys.Lett. **B471** (2000) 352.

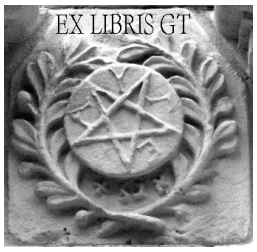
[13] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. **B494** (1997) 315.

[14] M. Maggiore and A. Riotto, Nucl. Phys. **B548** (1999) 427.

[15] T. Banks and W. Fischler , "Black Crunch", hep-th/0212113.

[16] T. Damour and M. Henneaux, Phys. Rev. Lett. **85** (2000) 920; Gen. Rel Grav. **32** (2000) 2339.

[17] T. Banks and W. Fischler, " Holographic cosmology 3.0", hep-th/0310288.



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A SIMPLE/SHORT INTRODUCTION TO PRE-BIG-BANG PHYSICS/COSMOLOGY

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ABSTRACT

A simple, non-technical introduction to the pre-big bang scenario is given, emphasizing physical motivations, considerations, and consequences over formalism.

1. Introduction

It is commonly believed (see e.g.[1]) that the Universe – and time itself – started some 15 billion years ago from some kind of primordial explosion, the famous Big Bang. Indeed, the experimental observations of the red-shift and of the Cosmic Microwave Background (CMB) lead us quite unequivocally to the conclusion that, as we trace back our history, we encounter epochs of increasingly high temperature, energy density, and curvature. However, as we arrive close to the singularity, our classical equations are known to break down. The earliest time we can think about classically is certainly larger than the so-called Planck time, $t_P = \sqrt{G_N \hbar} \sim 10^{-43}$s ($c = 1$ throughout). Hence, the honest answer to the question: Did the Universe and time have a beginning? is: We do not know, since the answer lies in the unexplored domain of quantum gravity.

Besides the initial singularity problem – and in spite of its successes – the hot big bang model also has considerable phenomenological problems. Amusingly, these too can be traced back to the nature of the very early state of the Universe. Let us briefly recall why.

General Relativity, together with the cosmological principle (i.e. the assumption of a homogeneous, isotropic Universe over large scales), allows us to describe the geometry of space-time through the Friedmann-Robertson-Walker (FRW) metric (see e.g.[2]):

FRW

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] , \; d\Omega^2 = d\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 , \qquad (1)$$

where, as usual, the discrete variable $k = 0, 1, -1$ distinguishes the cases of a flat, closed, or open Universe, respectively. In the presence of some matter (fluid), described by an energy density ρ and pressure density p, the evolution of the Universe

is controlled by the Einstein-Friedmann equations

$$\begin{aligned} H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \\ \dot{H} + H^2 = \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) \ , \end{aligned} \tag{2}$$

which, together, imply the energy conservation equation:

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + p) \ . \tag{3}$$

Given a model for the sources, or, more specifically, given a relation between ρ and p (so-called equation of state), one can easily solve these equations and find the usual FRW cosmological solutions. For the matter- and radiation-dominated cases, respectively,

$$\begin{aligned} p = 0 \ &\Rightarrow \ a(t) \sim t^{2/3}, \ \rho \sim a^{-3} \\ p = \rho/3 \ &\Rightarrow \ a(t) \sim t^{1/2}, \ \rho \sim a^{-4} \end{aligned} \tag{4}$$

where, for simplicity, the spatial-curvature term $\frac{k}{a^2}$ was neglected.

The so-called horizon problem arises as follows. The observable part of our Universe, our present horizon, is given by the distance that light has travelled since the big bang, or about 10^{28} cm. At earlier times, the horizon was much smaller. For a hypothetical observer looking at the sky a few Planck times after the Big Bang, the horizon was not much bigger that a few Planck lengths, say about 10^{-32} cm. Instead, as easily seen from (4), the portion of space that corresponds to our present horizon was, for that same hypothetical observer, some 30 orders of magnitude larger than the Planck length, or about 1 mm. In other words, at that time, what has nowadays become our observable Universe consisted of $(10^{30})^3 = 10^{90}$ Planckian-size, causally disconnected regions. There is no reason to expect that conditions in all those regions were initially the same, since there was never any thermal contact between them. Yet, today, all those 10^{90} regions make up our observable Universe and all appear to resemble one another (to within one part in 10^5).

Who prepared such a smooth and very unlikely initial state? Perhaps God, who picked up a very special point in the huge space of all possible initial configurations (see, in this connection, a nice picture in Roger Penrose's book[3]). If, instead, we do not accept God's fine-tuning, or, in more scientific terms, we do not want to attribute homogeneity to some unknown Planckian physics, two logical possibilities are left to our choice:

- Time did have its beginning at the big bang, when initial conditions were rather random, but a period of superluminal expansion (inflation) brought all those 10^{90} patches in causal contact sometimes between the big bang and the present time. This is the standard *post-big bang* inflation paradigm (see e.g.[4]).

- Time did *not* have its beginning with the big bang and some pre-big bang physics cooked up a "good" big bang from a more generic (less fine-tuned) initial state. This is the attitude one takes in the so-called *pre-big bang* scenario.

2 Flat

One can similarly argue that there are two ways of solving the second major problem of standard cosmology, the so-called flatness problem. Today, space is, to a very good approximation, Euclidean. If it does have any spatial curvature (represented by the $\frac{k}{a^2}$ term in the cosmological equations), this is of $O(H^2)$, i.e. extremely small. Given the solutions (4) it is easy to check that, in order to have such conditions today, one has to start, at the Planck time, with an extremely flat Universe, i.e. with a curvature radius a some 30 orders of magnitude larger that the characteristic length scale at the time, $H^{-1} \sim l_P$. Again, two possibilities come to mind: either the Universe was not particularly flat at the beginning – and subsequent inflation stretched out spatial curvature – or some pre-big bang physics prepared a nice spatially-flat big bang.

inflaton

fine tuning

Conventional inflation again chooses the first alternative. To succeed, it needs a weakly-coupled scalar field, the so-called inflaton, which, some time after the big bang, finds itself *homogeneously* displaced from the minimum of its potential, and slowly rolls towards it. While doing so, if certain conditions are met, the effective equation of state is $p \sim -\rho$ and the Universe expands quasi-exponentially. One needs this period of exponential growth to last for a long enough time for all our accessible Universe to come in causal contact. This can be achieved at the price of fine-tuning certain masses or couplings.

My main reservations towards this solution (see[5] for a different criticism) are that no one has a convincing model for what the inflaton ought to be and, even more seriously, that it is not easy to justify the initial conditions that can provide a sufficiently long inflationary phase. One is back somehow to the starting point, since the conditions at the onset of inflation have to come from a previous phase, and this inevitably leads us to giving initial conditions in the mysterious Planckian era. Quantum cosmology has then been invoked; however, I am also sceptical about present quantum cosmology arguments[6] "predicting" inflation since they are based on the so-called minisuperspace truncation of the Wheeler–DeWitt equation. Such an approach is only justified for a fairly homogeneous initial universe, which is just what we do *not* wish to assume.

How come string theory prefers the second way out? In order to explain this I have to open a parenthesis and tell you about some striking properties of quantum (super)strings.

2. Three properties of (super)strings

I will concentrate on just three properties of strings which are crucial to the understanding of their possible cosmological implications:

- 1. There is a fundamental length scale in string theory, providing a characteristic size for strings and thus an ultraviolet cutoff. Thanks to this property, superstring theory can be taken seriously as a candidate finite quantum theory of gravity (and of the other interactions as well). The fundamental length scale λ_s is given in terms of the string tension T by the formula

$$\lambda_s = \sqrt{\hbar/T}\ . \tag{5}$$

Actually λ_s, rather than T, is the fundamental parameter of the theory, providing a meaning for what short and large distances mean. When fields vary little over a string length λ_s one recovers a field-theoretic description given by a local Lagrangian with the smallest number of derivatives.

dilaton

- 2. Couplings are not God-given constants; they are VEVs which, hopefully, become dynamically determined. In particular, a scalar field, the so-called dilaton ϕ, controls all sorts of couplings, gravitational and gauge alike. Since, in our normalizations,

$$T \cdot G_N = l_P^2/\lambda_s^2 \sim \alpha_{gauge} \sim e^{\phi}\ , \tag{6}$$

lambda s

we see that the weak coupling region is $\phi \ll -1$. By contrast, at present, $e^{\phi} \sim 1/25$, implying $\lambda_s \sim 10 l_P \sim 10^{-32}$ cm. In the weak coupling region, perturbative superstring theory is an adequate description of physics, implying that the dilaton itself behaves like a massless particle. As such, the dilaton can easily evolve cosmologically while it is deeply inside the perturbative region. Precision tests of the equivalence principle imply, however, that the dilaton has a mass, i.e. that near its present value $\phi = O(-1)$, its potential has a minimum with finite curvature.

If both the coupling and derivatives are small, physics is adequately described by the tree-level low-energy effective action of string theory, which reads:

$$\begin{aligned}\Gamma_{eff} &= \frac{1}{2}\int d^4x\sqrt{-g}\, e^{-\phi}\Big[\lambda_s^{-2}(\mathcal{R} + \partial_\mu\phi\partial^\mu\phi + H^2_{\mu\nu\rho}) + F^2_{\mu\nu} + \text{higher derivatives}\Big] \\ &+ \Big[\text{higher orders in } e^{\phi}\Big]\ ,\end{aligned} \tag{7}$$

where we have included the contributions of the Kalb-Ramond antisymmetric tensor field through its field strength $H_{\mu\nu\rho}$. Note the two kinds of corrections alluded to in (7). They intervene, respectively, whenever space-time derivatives (i.e. energies) or the string coupling e^{ϕ} become appreciable. Equation (7) will be our starting point to describe pre-big bang cosmology.

- 3. Cosmological field equations exhibit new stringy symmetries whose most interesting representative (scale-factor duality or SFD) acts as follows:

SFD

$$a(t) \to a^{-1}(-t)\ \ ,\ \ \phi(t) \to \phi(-t) - 2d\ \log a(-t)\ \ ,d = 3\ . \tag{8}$$

The interest of this duality transformation lies in the fact that it maps ordinary FRW cosmologies with $H > 0$, $\dot{H} < 0$, and $\dot{\phi} = 0$ at $t > 0$ into inflationary

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cosmologies with $H > 0$, $\dot{H} > 0$, and $\dot{\phi} > 0$ at $t < 0$. Actually, the dual cosmologies are of the so-called super-inflation (or pole-inflation) type, i.e. have a growing – rather than a constant – Hubble parameter. Since many of the distinctive consequences of PBB cosmology originate from this peculiar feature, let me explain in simplified terms where it comes from Eqs. (2) imply that, for an expanding Universe, ρ and H^2, being proportional (take for simplicity the case of $k = 0$), decrease together in time. This is true if G_N is constant. In string theory, where G_N is controlled by the dilaton through Eq. (6), it is perfectly possible to have a growing H while ρ decreases, provided ϕ is also growing.

G/phi

The suggestion from string theory now becomes almost a compelling one: Can one put together a standard cosmology at $t > 0$ and a dual cosmology at $t < 0$ to generate a single scenario containing dilaton-driven pre-big bang inflation and FRW post-big bang behaviour? Since, in such a construct, the Hubble parameter grows for $t < 0$ and decreases for $t > 0$, it should reach its maximum at $t = 0$, instant therefore identified with the occurrence of the Big Bang of standard cosmology. The problem is that this maximum is actually infinite if one works in the context of the low-energy effective theory, i.e. if all corrections in (7) are neglected. The pre- (post-) big bang phases have singularities in the future(past), probably a consequence of the validity, in that approximation, of the assumption leading to the Hawking-Penrose singularity theorems.[7] However, that approximation breaks down as soon as the Hubble parameter reaches values $O(\lambda_s^{-1})$ leading us to expect that the maximal value of H, reached at $t = 0$, should be actually of the order of the fundamental length of string theory.

It is easy, actually, to write down mathematical expressions that interpolate smoothly between the inflationary and the FRW branches. For instance, the ansatz

$$\begin{aligned} \hat{a}(t) &= \left(\frac{t + \sqrt{t^2 + \lambda_s^2}}{\lambda_s} \right)^{1/2} \\ \hat{\phi}(t) &= \phi_0 + \frac{3}{2} \log \left(1 + \frac{t}{\sqrt{t^2 + \lambda_s^2}} \right) \end{aligned} \tag{9}$$

is easily checked to approach a standard radiation-dominated cosmology with constant dilaton at $t \gg \lambda_s$ and to a dual dilaton-driven inflationary cosmology at $t \ll -\lambda_s$. The question is: Does anything like this come from the true (i.e. high-curvature and/or loop-corrected) field equations? Leaving this hard question to the final section, we turn instead to the formulation of the basic pre-big bang postulate.

3. The pre-big bang postulate

DUAL COSMOLOGY

Clearly, if we want to use a dual cosmology for the prehistory of the Universe, given the positive signs of $\dot{H}$ and $\dot{\phi}$, we have to start from (very?) small initial values

for H and e^{ϕ}. Although not strictly necessary, we will also impose, for the sake of simplicity, an almost empty initial Universe. This leads us to the following basic postulate of PBB cosmology:

"phase 1"

> The Universe started its evolution from the most simple initial state conceivable in string theory, its perturbative vacuum. This corresponds to an (almost)
>
> **EMPTY, COLD, FLAT, FREE**

Universe as opposed to the standard

DENSE, HOT, HIGHLY-CURVED

initial state of conventional cosmology.

For this assumption to make sense I will have to argue that the new initial conditions are able to provide, at later times, a hot big bang with the desired characteristics thanks to a long pre-big bang inflationary phase. This looks a priori a very hard task, but I will explain below how it can possibly happen. Before discussing this let me illustrate, in two figures, the qualitative differences between the standard (non-inflationary) model, the standard inflationary scenario, and ours.

In Fig. 1 I am plotting, against cosmic time, the behaviour of the Hubble parameter H measured in Planck units $M_P \sim 10^{19}$ GeV. Note that, while in the first model H is a concave function of time, in the second it has a long flat plateau where $H/M_P \leq 10^{-5}$ (this constraint comes from COBE's data, see e.g.[4]). Finally, in the proposed scenario, H/M_P grows all the way to a maximal value $O(10^{-1})$, but quickly becomes very small at both large positive and large negative t.

In Fig. 2 we can see the consequences of this behaviour of H on the kinematics of horizon crossing. During inflation, increasingly small scales are pushed out of the horizon by the accelerated expansion of the Universe. However, while in standard inflation (Fig. 2a) larger scales cross the horizon at slightly larger values of H, in the pre-big bang scenario (Fig. 2b) it is the other way around: the larger the scale, the smaller the value of H at horizon crossing. As we shall see in Sec. 4, the value of H/M_P at horizon crossing is the determining quantity for evaluating the present magnitude of quantum fluctuations at different length scales. Hence the above kinematics of horizon crossing will have an important bearing on the spectrum of quantum fluctuations.

Before turning to that, we should discuss how dilaton-driven inflation sets in during the pre-big bang phase.

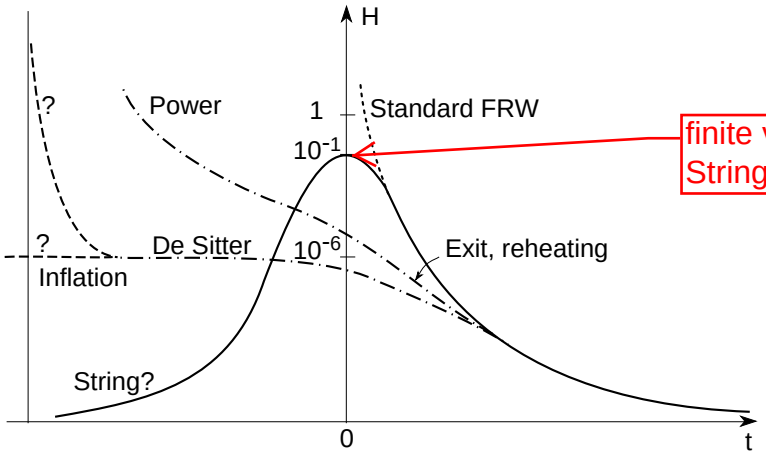
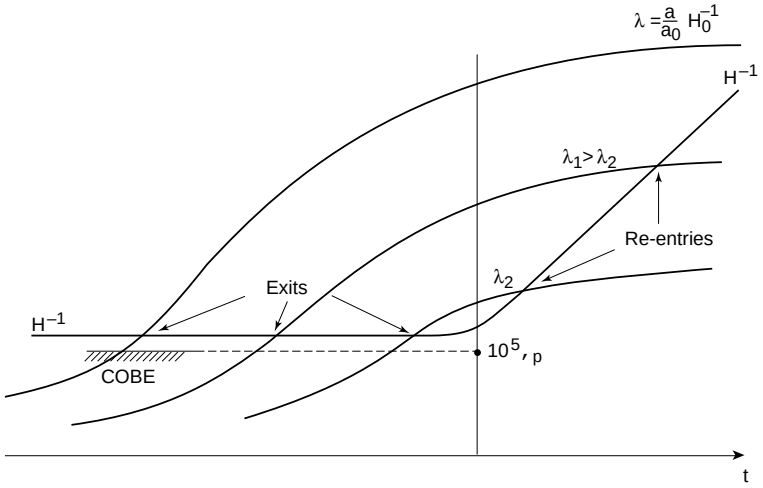
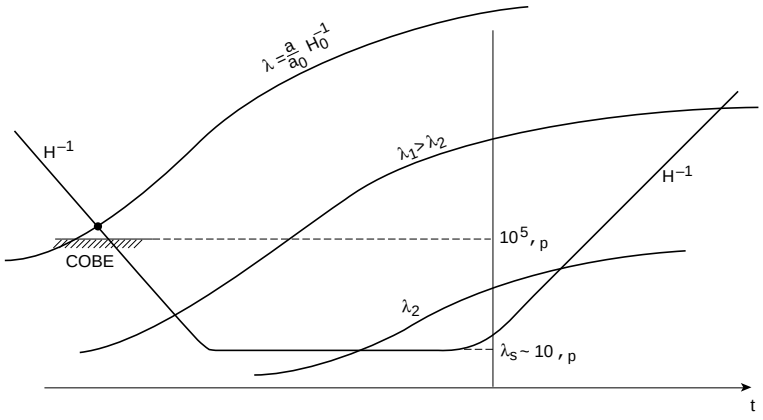


Figure 1



inflation

Figure 2a



string

Figure 2b

4. Pre-big bang inflation as a classical instability

The key to understanding how our apparently innocent initial conditions can give inflation is in the last attribute we assumed for the primordial state: the Universe starts deeply inside the perturbative region, i.e. at very small coupling. In terms of the dilaton this means that ϕ started very large and negative. This entitles us to treat the early history of the Universe classically. Since we have also assumed that it was almost flat, we are also entitled to use the low-energy approximation to string theory. All this can be summarized by saying that we can describe the evolution of the Universe through the classical field equations of the low-energy tree-level effective action (7); in the simplest case used here for illustration, this reduces to:

V- string initial conditions:

$$\Gamma_{eff} = \frac{1}{2\lambda_s^2} \int d^4x \sqrt{-g}\; e^{-\phi} (\mathcal{R} + \partial_\mu \phi \partial^\mu \phi) \; . \qquad (10)$$

It has been known for some time (see[12] for a review) that, if homogeneity and spatial flatness are assumed, then inflationary behaviour automatically follows from the stated initial conditions. Indeed homogeneous, spatially flat solutions fall in *four* categories, of which only *one* satisfies the pre-big bang postulate. The other three exhibit either strong coupling or strong curvature (or both) in the far past.

However, assuming homogeneity from the start is not very satisfactory. If we want to solve in a natural way the homogeneity and flatness problems, we have to start with generic (i.e. not particularly fine-tuned) initial conditions near the perturbative vacuum. During the past year this problem has been tackled by several groups, with somewhat controversial conclusions. Let me try to explain how I see the present situation.

Ansatz: $\frac{1}{2}$

Assume that, at some remote time much before the Big Bang, the Universe was not particularly homogeneous, in the sense that spatial gradients and time derivatives were both of the same order. Assume also, in accordance with the PBB postulate, that both kinds of derivatives were tiny in string units. It can be shown[13] that these initial conditions can lead to a chaotic version of PBB inflation since, as the system evolves, certain patches develop where time derivatives slightly dominate over spatial gradients. Provided this situation is met when the kinetic energy in the dilaton is a non-negligible fraction of the critical density, dilaton-driven inflation sets in,[13,14] blowing up the patch and making it homogeneous, isotropic and spatially flat. The evolution can be studied by analytical methods (gradient expansion[15]) since the approximation of neglecting spatial gradients w.r.t. time derivatives becomes increasingly accurate within the inflating patch.

The controversial issue is that, in order to have sufficient inflation in the patch, dilaton-driven inflation has to last sufficiently long. Its duration is not infinite since it is limited, in the past, by the conditions I just described and, in the future, by the time at which, inevitably, curvatures become of string-size and we can no longer trust the low-energy approximation. Thus, as it was actually noticed from the very beginning,[9] a successful PBB scenario does require very perturbative initial conditions, so that it takes a long time (during which the Universe inflates) to reach the BB singularity.

A particular case of this "fine-tuning" was discussed recently by M. Turner and E. Weinberg.[16] They consider a homogeneous, but not spatially flat Universe and notice that the duration of PBB inflation is limited in the past by the initial value of the spatial curvature. This has to be taken very small in string units if sufficient inflation is to be achieved.

The (almost philosophical) issue is whether this is or is not fine-tuning. String theory has a single length parameter, λ_s, but, fortunately, it has massless states and low-energy vacua (such as Minkowsky space-time) whose characteristic scale is much larger than λ_s. Hence I see nothing wrong in starting the evolution of the Universe in a state of low-energy, small curvatures, and small coupling. Actually, I find it very amusing that a classical instability pushes the Universe from low energy (curvature) and small coupling towards high energy (curvature) and large coupling.

Another result, which has emerged very recently,[13,14] is the behaviour of pre-big bang cosmology in the asymptotic past. If we evolve the system from the initial conditions I described above towards the past, we seem to find two possible behaviours. Either we reach a singularity at some finite cosmic time in the past, or we flow smoothly into a trivial space-time. The first alternative, which looks generic for positive spatial curvature (e.g. a $k = +1$ Universe with $\Omega > 1$) has to be excluded since it contradicts our basic postulate. The second alternative, which looks generic for negative spatial curvature (e.g. a $k = -1$ Universe with $\Omega < 1$), is perfectly consistent with our philosophy and leads to an interesting conjecture[14] for the whole history of time that I will describe below. It would be very interesting if pre-big bang cosmology did predict that the Universe is open, something that appears to be definitely favoured at present (see e.g.[17]) by direct measurements of the red-shift-to-distance relation and by models of large-scale-structure formation.

??

The complete history at which we arrive can be best drawn on a diagram (Fig. 3) reminiscent of (but actually quite different from) a Carter-Penrose diagram,[18] which artificially squashes space-time at infinity. By a simple change of the radial coordinate we can rewrite Eq. (1) as

$$ds^2 = -dt^2 + a(t)^2 \left[dR^2 + r(R)^2 d\Omega^2\right] , \tag{11}$$

and then define the coordinates x, y in Fig. 3 in terms of t and of the proper distance aR by:

$$\tan(y \pm x) = t \pm aR . \tag{12}$$

As $t \to -\infty$, the Universe approaches an exact vacuum of string theory. However, if equal-time hypersurfaces are taken to be those with a roughly constant energy density, we find that triviality is approached "a la Milne" i.e. the metric becomes

$$ds^2 = -dt^2 + t^2 \left[dr^2/(1 + r^2) + r^2 d\Omega^2\right] , \tag{13}$$

while the dilaton approaches an arbitrary constant. For negative time this is just a negative curvature FRW Universe, which contracts linearly in time, $a(t) = -t$. For positive t it is the linearly expanding Universe towards which we are evolving today if $\Omega < 1$ (for a discussion of this late-time behaviour see, for instance, [19]). In the Milne

???????

AdS???

Universe the evolution of the scale factor is driven by the negative spatial curvature ($-k\ a^{-2} \to t^{-2}$). Also shown in Fig. 3 are some time-like geodesics corresponding to a fixed comoving distance R from the origin.

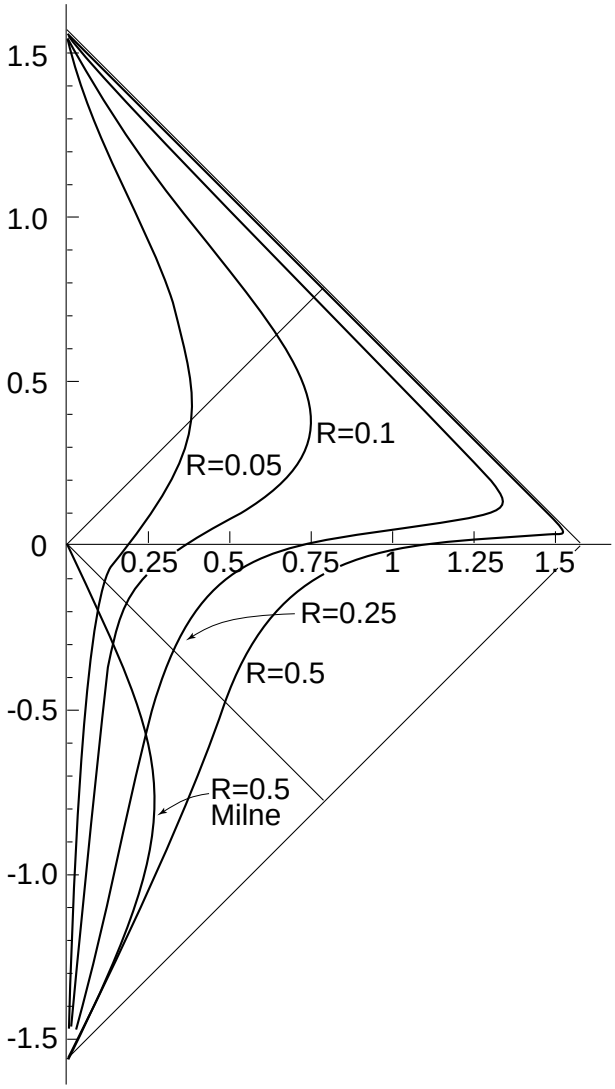


Figure 3

The generic regular solution thus approaches Milne as $t \to -\infty$ but, at any finite large $(-t)$, also contains small dilatonic (and gravitational-wave) perturbations giving $0 < \Omega \ll 1$. As $t \to -\infty$, $\Omega \to 0$. As time goes forward, instead, Ω tends to grow until, at some critical time $-T_0$, Ω becomes $O(1)$, in some region of space. From that moment on, in that "lucky" patch, the metric starts to deviate from Milne (this is shown, in particular, for the $R = 0.5$ geodesic) and dilaton-driven inflation sets in, pushing Ω extremely close to 1 in that patch.

The rest of the story goes as follows: thanks to the string UV cutoff, when the curvature becomes $O(\lambda_s^{-2})$, and/or the coupling becomes $O(1)$, a stringy mechanism

Veneziano "string UV cut-off"

WHICH stringy mech???

prevents reaching the singularity, and a smooth transition to standard hot big bang FRW cosmology follows. An interesting quantum mechanism, described in Sec. 5, is able to provide radiation, temperature and entropy. However, by then, the inflated patch is both homogeneous and spatially flat: we have been able to produce a "good" big bang! Post-big bang evolution is from now on standard, with one qualification. Although we have achieved $\Omega = 1$ through inflation, we had to start from an open Universe and thus $\Omega = 1 - \epsilon$, $\epsilon \ll 1$. Inevitably, FRW evolution will make Ω deviate more and more from 1 until, once more, the Universe will go back to a linearly expanding Milne-like Universe. It would be wrong to think, however, that the Universe will just follow the time-reverse of its original life, firstly because the final coupling is much larger than the initial one, and, secondly, because entropy has kept increasing all the time: presumably, in this scenario, the very final stage of the Universe will consist of an ever increasingly dilute gas of slowly evaporating black holes...

5. Quantum Mechanical Heating of the Universe and Observable PBB relics

Since there are already several review lectures on this subject (e.g.[12,20]), I will limit myself to mention the most recent developments simply after recalling the basic physical mechanism underlying particle production in cosmology.[21] A cosmological (i.e. time-dependent) background coupled to a given type of (small) inhomogeneous perturbation Ψ enters the effective low-energy action in the form:

$$I = \frac{1}{2} \int d\eta \; d^3x \; S(\eta) \left[\Psi'^2 - (\nabla \Psi)^2 \right] . \tag{14}$$

Here η is the conformal time coordinate, and a prime denotes $\partial/\partial\eta$. The function $S(\eta)$ (sometimes called the "pump" field) is, for any given Ψ, a given function of the scale factor, $a(\eta)$, and of other scalar fields (four-dimensional dilaton $\phi(\eta)$, moduli $b_i(\eta)$, etc.) which may appear non-trivially in the background.

Theorem:
q generated by S

KR-axion

While it is clear that a constant pump field S can be reabsorbed in a rescaling of Ψ, and is thus ineffective, a time-dependent S couples non-trivially to the fluctuation and leads to the production of pairs of quanta (with opposite momenta). Looking back at Eq. (7), one can easily determine the pump fields for each one of the most interesting perturbations. The result is:

$$\begin{aligned} \text{Gravity waves, dilaton} &: S = a^2 e^{-\phi} \\ \text{Heterotic gauge bosons} &: S = e^{-\phi} \\ \text{Kalb} - \text{Ramond, axions} &: S = a^{-2} e^{-\phi} \end{aligned} \tag{15}$$

A distinctive property of string cosmology is that the dilaton ϕ appears in some very specific way in the pump fields. The consequences of this are very interesting:

- For gravitational waves and dilatons the effect of ϕ is to slow down the behaviour of a (remember that both a and ϕ grow in the pre-big bang phase). This is the

reason why those spectra are quite steep[22] and give small contributions at large scales.

- For (heterotic) gauge bosons there is no amplification of vacuum fluctuations in standard cosmology, while, in string cosmology, all the "work" is done by the dilaton. In the pre-big bang scenario, the coupling must grow by as large a factor as the one by which the Universe has inflated. This implies a very large amplification of the primordial quantum fluctuation,[23] possibly explaining the long-sought origin of seeds for the galactic magnetic fields.

- Finally, for Kalb-Ramond fields and axions, a and ϕ work in the same direction and spectra can be large even at large scales.[24] Note, incidentally, that the power of a in S is determined by the rank of the corresponding tensor. It is well known, however, that the Kalb-Ramond field can be reduced to a (pseudo)scalar field, the axion, through a duality transformation. This turns out to change S into S^{-1}, i.e. the pump field for the axion is actually $a^2 e^{\phi}$. An interesting duality of cosmological perturbations, reminiscent of electric-magnetic (or strong-weak) duality, can be argued[25] to guarantee the equivalence of the Kalb-Ramond and axion spectra.

- Many other fluctuations, which arise in generic compactifications of superstrings, have also been studied and lead to interesting spectra. For lack of time, I will refer to the existing literature.[26,27]

The possible flatness of axionic spectra in pre-big bang cosmology leads to hopes that, in such a scenario, there is a natural way to generate an interesting spectrum of large-scale fluctuations, one of the much advertised properties of the standard inflationary scenario. Work is still in progress to establish whether this hope is indeed realized.

Before closing this section, I wish to recall how one sees the very origin of the hot big bang in this scenario. One can easily estimate the total energy stored in the quantum fluctuations which were amplified by the pre-big bang backgrounds. The result is, roughly,

$$\rho_{quantum} \sim N_{eff}\ H_{max}^4\ , \tag{16}$$

where N_{eff} is the effective number of species that are amplified and H_{max} is the maximal curvature scale reached around $t = 0$ (this formula has to be modified in case some spectra show negative slopes). We have already argued that $H_{max} \sim \lambda_s^{-1}$ and we know that, in heterotic string theory, N_{eff} is in the hundreds. Yet this rather huge energy density is very far from critical as long as the dilaton is still in the weak-coupling region, justifying our neglect of back-reaction effects. It is very tempting to assume that, precisely when the dilaton reaches a value such that $\rho_{quantum}$ is critical, the Universe will enter the radiation-dominated phase. This too is, at present, the object of active investigation.

6. Conclusion

Pre-big bang cosmology appears to have survived its first 6-7 years of life. Interest in (criticism of) it is clearly growing. It is perhaps time to make a balance sheet.

Conceptual (technical?) and phenomenological problems include:

- Graceful exit from dilaton-driven inflation to FRW cosmology is not fully understood, in spite of recent progress.[28] Possibly, new ideas borrowed from M-theory and D-branes could help in this respect.[29]
- A scale-invariant spectrum of large-scale perturbations does not look automatic, although, for the first time, thanks to the flat axion spectra, it does not look impossible either.

Attractive features include:

- No need to "invent" an inflaton, or to fine-tune potentials.
- Inflation is "natural" thanks to the duality symmetries of string cosmology.
- The initial conditions problem is decoupled from the singularity problem: a solution to the former is already shaping up and looks exciting.
- A classical gravitational instability finds a welcome use in providing inflation; a quantum instability (pair creation) is able to heat up an initially cold Universe and generate a standard hot big bang with the additional features of homogeneity, flatness and isotropy.
- Last but not least: one is dealing with a highly constrained, predictive scheme which can be tested/falsified by low-energy experiments thanks to the fact that a huge red-shift has brought the scale of Planckian physics down to that of human beings:

$$(l_P/H_0)^{1/2} \sim 1 \text{ mm} \tag{17}$$

References

1. S. Hawking in *Proceedings of the Texas/ESO-CERN Symposium on Relativistic Astrophysics, Cosmology, and Fundamental Physics*, Brighton 1990, eds. J.D. Barrow, L. Mestel and P.A. Thomas, *Ann. NY Acad. Sci.* **647**, 315 (1991).
2. S. Weinberg, *Gravitation and Cosmology*, (John Wiley and Sons, Inc., New York, 1972).
3. R. Penrose, *The Emperor's New Mind*, (Oxford University Press, New York, 1989), Fig. (7.19).
4. E. W. Kolb and M. S. Turner, *The Early Universe*, (Addison-Wesley, Redwood City, Ca, 1990).

5. R. Penrose, *Difficulties with inflationary Cosmology* in *Proc. 14th Texas Symposium on Relativistic Astrophysics*, ed. E. J. Fenyves, *Ann. NY Acad. Sci.* **571**, 249 (1989).
6. See e.g. *Euclidean Quantum Gravity*, eds. G. W. Gibbons and S. W. Hawking, (World Scientific,Singapore, 1993), Part III.
7. S. W. Hawking and R. Penrose, *Proc. Roy. Soc.* A **314**, 529 (1970).
8. G. Veneziano, *Quantum Strings and the Constants of Nature*, in *The Challenging Questions*, Erice, 1989, (ed. A. Zichichi, Plenum Press, New York, 1990).
9. G. Veneziano, *Phys. Lett.* B **265**, 287 (1991).
10. M. Gasperini and G. Veneziano, *Astropart. Phys.* **1**, 317 (1993); *Mod. Phys. Lett.* A **8**, 3701 (1993); *Phys. Rev.* D **50**, 2519 (1994).
11. An updated collection of papers on the PBB scenario is available at `http://www.to.infn.it/~gasperin/`.
12. G. Veneziano, *Status of String Cosmology: Basic Concepts and Main Consequences*, in *String Gravity and Physics at the Planck Energy Scale*, Erice, 1995, eds. N. Sanchez and A. Zichichi, (Kluver Academic Publishers, Dordrecht, 1996), p. 285.
13. G. Veneziano, *Phys. Lett.* B **406** , 297 (1997); see also: A. Feinstein, R. Lazkoz and M.A. Vazquez-Mozo, *Closed Inhomogeneous String Cosmologies*, hep-th/9704173;
J. D. Barrow and M. P. Dabrowski, *Is there Chaos in Low-energy String Cosmology?*, hep-th/9711049; K. Saygily, *Hamilton-Jacobi Approach to Pre-Big Bang Cosmology at Long Wavelengths*, hep-th/9710070.
14. A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, *Phys. Rev.* D **57**, 2543 (1998); see also: J. D. Barrow and K. E. Kunze, *Spherical Curvature Inhomogeneities in String Cosmology*, hep-th/9710018.
15. V. A. Belinskii and I.M. Khalatnikov, *Sov. Phys. (JETP)* **36**, 591 (1973);
N. Deruelle and D. Langlois, *Phys. Rev.* D **52**, 2007 (1995);
J. Parry, D. S. Salopek and J. M. Stewart, *Phys. Rev.* D **49**, 2872 (1994).
16. M. Turner and E. Weinberg, *Phys. Rev.* D **56**, 4604 (1997).
17. J. Glanz, *New Light on the Fate of the Universe*, *Science* **278**, 799 (1997) and references therein.
18. See e.g. Ref. 6.
19. Ya. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics*, Vol II (Structure and Evolution of the Universe), Section 2.4.
20. G. Veneziano, Ref. 12; M. Gasperini, *Status of String Cosmology: Phenomenological Aspects*, ibid., p. 305; *Relic gravitons from the pre-big bang: what we know and what we do not know*, hep-th/9607146.
21. See, e.g., V. F. Mukhanov, A. H. Feldman and R. H. Brandenberger, *Phys. Rep.* **215**, 203 (1992).
22. R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Lett.* B **361**, 45 (1995); R. Brustein *et al. Phys. Rev* D **51**, 6744 (1995).
23. M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Rev. Lett.* **75**, 3796 (1995); D. Lemoine and M. Lemoine, *Phys. Rev.* D **52**, 1955 (1995).
24. E.J. Copeland, R. Easther and D. Wands, *Phys. Rev.* D **56**, 874 (1997);
E.J. Copeland, J.E. Lidsey and D. Wands, *S-duality invariant perturbations*

in string cosmology, SUSSEX-TH-97-007 (hep-th/9705050).
25. R. Brustein, M. Gasperini and G. Veneziano, *Duality in Cosmological Perturbation Theory*, CERN-TH/98-53, to appear on hep-th.
26. R. Brustein and M. Hadad, *Phys. Rev.* D **57**, 725 (1998).
27. A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, *Quantum Inhomogeneities in String Cosmology*, *JHEP* **01**, 004 (1998).
28. M. Gasperini, M. Maggiore and G. Veneziano, *Nucl. Phys.* B **494**, 315 (1997); R. Brustein and R. Madden, *Phys. Lett.* B **410**, 110 (1997); *Phys. Rev.* D **57**, 712 (1998).
29. See, e.g. R. Poppe and S. Schwager, *Phys. Lett.* B **393**, 51 (1997);
A. Lukas, B.A. Ovrut and D. Waldram, *Phys. Lett.* B **393**, 65(1997); *String and M-theory cosmological solutions with Ramond forms*, UPR-723T (hep-th/9610238);
N. Kaloper, *Phys. Rev.* D **55**, 3394 (1997);
H. Lu, S. Mukherji and C.N. Pope, *Phys. Rev.* D **55**, 7926 (1997);
N. Kaloper, I. Kogan and K. A. Olive, *Cos(M)ological Solutions in M- and String Theory*, hep-th/9711027.

Note Added

Since I gave this lecture, two relevant papers have appeared:

1) N. Kaloper, A. Linde and R. Bousso (hep-th/9801073) have added further points to the criticism of the PBB scenario expressed in Ref. 16.

2) A numerical study of the spherically symmetric case by J. Maharana, E. Onofri and G. Veneziano (gr-qc/9802001) appears to support the idea discussed in Section 4 that PBB behaviour emerges generically from initial conditions sufficiently close to Milne's trivial vacuum.

These two papers confirm that much more work is still needed to clarify all the relevant issues raised by the new cosmological setup I discussed here.

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hep-th/9902126

Pre-bangian origin of our entropy and time arrow

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Abstract

I argue that, in the chaotic version of string cosmology proposed recently, classical and quantum effects generate, at the time of exit to radiation, the correct amount of entropy to saturate a Hubble (or holography) entropy bound (HEB) and to identify, within our own Universe, the arrow of time. Demanding that the HEB be fulfilled at all times forces a crucial "branch change" to occur, and the so-called string phase to end at a critical value of the effective Planck mass, in agreement with previous conjectures.

CERN-TH/99-32
February 1999

The origin of the present entropy of our Universe, S_0, is one of the deepest cosmological mysteries. The 2.7 K cosmic microwave background (CMB), if it indeed fills our observable Universe uniformly, contributes a gigantic 10^{90} to S_0. However, as repeatedly emphasized by many people, most notably by Roger Penrose [1], such an amount falls very short of what entropy could have been expected to be, even if we go back to the Planckian era, i.e. to $t = t_P \sim 10^{-43}$ s after the big bang. Since entropy can only grow, the entropy of our Universe at $t = t_P$, S_P, must be smaller than S_0; yet, on the basis of the energy content and of the size of the Universe R_P at $t \sim t_P \equiv l_P/c$, we might have expected [1]

$$S_P \sim E_P R_P/c\hbar \sim \rho_P R_P^4 \sim (R_P/l_P)^4 \sim 10^{120} \quad . \tag{1}$$

The fact that the entropy of our Universe must have been at least 30 orders of magnitude smaller than the value in (1) would nicely "explain" our arrow of time, by identifying the beginning of the Universe with this state of incredibly small entropy near the Planck time [1]. In order to solve the problem, Penrose [1] invokes, without much justification, a new "Weyl-curvature hypothesis". The expected value given in Eq. (1) coincides with the so-called Bekenstein entropy bound (BEB) [2], which states that, for any physical system of energy E and physical size R, entropy cannot exceed $S_{BB} = ER/c\hbar$. This bound is saturated by a black hole of mass E and size equal to its Schwarzschild radius $R = GE$. If the newly born Universe were a single black hole its Schwarzschild radius would have been much larger than R_P, and an even higher entropy, $O(10^{180})$, would have resulted. What could have made the initial entropy much smaller than S_{BB} is instead the possibility that the Universe, right after the big bang, was already in a very ordered, homogeneous state. But this is just restating the puzzle in terms of the usual homogeneity problem of standard (non-inflationary) cosmology [3].

The way the two problems are related can be made explicit by introducing a stronger bound on entropy, which, unlike Bekenstein's general bound, should apply to the special case of (fairly) homogeneous cosmological situations. We shall call it the "Hubble entropy bound" and formulate it as follows: Consider a sufficiently homogeneous Universe in which a (local) Hubble expansion (or contraction) rate can be defined, in the sinchronous gauge, as:

$$H \sim 1/6 \; \partial_t(\log g) \quad , \quad g \equiv \det \, (g_{ij}) \quad , \tag{2}$$

with H varying little (percentage-wise) over distances $O(H^{-1})$. In this case H^{-1}, the so-called Hubble radius, is known to correspond to the scale of causal connection, i.e. to the scale within which microphysics can act. In such a context it is hard to imagine that a black hole larger than H^{-1} can form, since, otherwise, different parts of its horizon would be unable to hold together. Thus, the largest entropy we may conceive is the one corresponding to having just one black hole per Hubble volume H^{-3}. Using the Bekenstein–Hawking formula

[1] Throughout this paper we will stress functional dependences while ignoring numerical factors.

for the entropy of a black hole leads to our proposal of a "Hubble entropy bound" (HEB):

$$S < S_{HB} \equiv l_P^{-2} \sum_i H_i^{-2} \sim n_H S_H \quad , \tag{3}$$

where the sum runs over each Hubble-size region. The last estimate in (3) assumes a fairly constant H throughout space, and defines n_H as the number of Hubble-size regions, each one carrying maximal entropy $S_H = l_P^{-2} H^{-2}$. This bound appears to be related, at least in some cases, to the one recently proposed by Fischler and Susskind [4] on the basis of the so-called holography principle. We may thus take S_{HB} to stand for Hubble or Holography entropy bound according to taste. If one applies the HEB to the initial Universe, one finds [4] that, unlike the BEB, it is practically saturated. It is widely fulfilled thereafter [4].

This letter aims at explaining why saturation of the HEB naturally takes place at the big bang in the context of pre-big bang (PBB) cosmology [5]. Before proceeding, we note that, in ordinary inflation, the entropy problem is solved [3] by invoking a non-adiabatic reheating process occurring after inflation. Since inflation has already made the Universe homogeneous, after thermal equilibrium is reached entropy is given by its standard thermodynamic relation to temperature (here the reheating temperature) as

$$S_{RH} \sim T_{RH}^3 R_{RH}^3 \sim S_0 \ . \tag{4}$$

One thus naturally obtains the correct value. However, unlike what will be shown to be the case in the PBB scenario, S_{RH} fails to saturate the HEB since:

$$S_{HB}(t = t_{rh}) = l_P^{-2} H_{RH} R_{RH}^3 = H_{RH}^{-1} T_{RH}^4 R_{RH}^3 = (T_{RH}/H_{RH})\ S_{RH} \gg S_{RH} \ . \tag{5}$$

In order to discuss various forms of entropy in the PBB scenario, let us recall some basic ideas, which have emerged from recent studies of the latter (see [6] for a review). It now looks quite certain that generic –though sufficiently weak– initial conditions lead to a form of stochastic PBB, which, in the Einstein-frame metric, can be seen as a sort of chaotic gravitational collapse [7, 8]. Black holes of different sizes form but, for an observer inside each horizon measuring distances with a stringy meter [2], this is experienced as a pre-big bang inflationary cosmology in which the $t = 0$ (hopefully fake) big bang singularity is identified [8] with the (hopefully equally fake) black hole singularity at $r = 0$.

We are thus led to identifying our observable Universe as what became of a portion of space that was originally inside a large enough black hole. In general, if we want to achieve a very flat and homogeneous Universe, we should better identify our present Universe with just a tiny piece of the collapsing/inflating region. For the purpose of this note, however, this would only complicate the equations without adding new physical information. This is why, hereafter, we shall identify our present Universe with the whole interior of a single initial black hole.

[2]Note that, while we shall work in the string frame throughout, the same results would also follow in the Einstein frame.

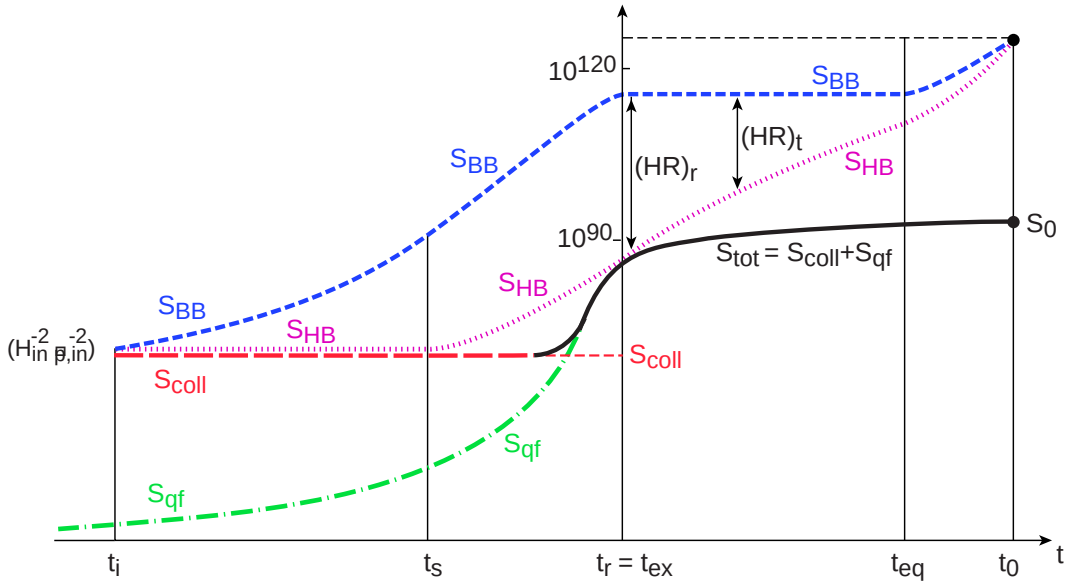

Figure 1: At the beginning of the DDI era ($t = t_i$) the entropy of the just-formed black hole, S_{coll}, coincides with both the BEB, S_{BB}, and the HEB, S_{HB}, while the entropy in quantum fluctuations, S_{qf}, is completely negligible. At the beginning of the string phase, $t = t_s$, both S_{coll} and S_{HB} still have their common initial value. S_{BB} and S_{qf} have grown considerably, but the latter is still negligible if the string coupling is still small at $t = t_s$. During the string phase, S_{qf} catches up with S_{coll} first, and with S_{HB} later, i.e. when the energy in the quantum fluctuations becomes critical and exit to radiation is expected ($t = t_r$). Finally, during the radiation and matter-dominated phases, S_{HB} grows towards S_{BB}, while our own entropy S_{tot} lags far behind and increases only slowly as the result of dissipative phenomena and growth of inhomogeneities.

It is helpful to follow the evolution of various entropies with the help of Fig.1. At time $t = t_i$, corresponding to the first appearance of a horizon, we can use the Bekenstein–Hawking formula to argue that

$$S_{coll} \sim (R_{in}/l_{P,in})^2 \sim (H_{in} l_{P,in})^{-2} = S_{BB} = S_{HB} \ , \tag{6}$$

where we have used the fact [8] that the initial size of the black-hole horizon determines also the initial value of the Hubble parameter. Thus, at the onset of collapse/inflation, there is no hierarchy between the two bounds and entropy is as large as allowed by them. Furthermore, since the collapsing region is large in string (and a fortiori in Planck) units, eq. (6) corresponds to a large number. Incidentally, this number is also close to the number of quanta needed for the collapse to occur [8]. We have also assumed the initial quantum state to be the ground state. Because of the small initial coupling and curvature, quantum fluctuations around it are very small [9], initially, and contribute a negligible amount S_{qf} to the total entropy.

After a short transient phase, dilaton-driven inflation (DDI) should follow and last until t_s, the time at which a string-scale curvature $O(M_s)$ is reached. We expect the process not to generate further entropy (unless more energy flows into the black hole, but this would only increase its total comoving volume), but what happens to the two bounds? This is the crucial observation: while the HEB also stays constant, the BEB grows, causing a large discrepancy between the two at the end of the DDI phase. In order to show this, let us recall one of the equations of string cosmology [5], the conservation law:

$$\partial_t \left(e^{-\phi}\sqrt{g}H\right) = \partial_t \left((\sqrt{g}H^3)\ \ (e^{-\phi}H^{-2})\right) = \partial_t \left(n_H S_H\right) = 0\ . \tag{7}$$

Comparing with (3), we recognize that (7) simply expresses the time independence of the HEB during the DDI phase. While at the beginning of the DDI phase $n_H = 1$, and the whole entropy is in a single Hubble volume, as DDI proceeds the same total amount of entropy becomes equally shared among very many Hubble volumes until, eventually, each one of them contributes a relatively small number. By contrast, it is easy to see that the BEB is increasing fast during the DDI phase since, using (7),

$$S_{BB} \sim MR \sim \rho R^4 \sim H^2 e^{-\phi}\sqrt{g}R = \text{const.} \times (\text{HR})\ , \tag{8}$$

and both R and H grow during DDI. Also, having assumed that the string coupling is still small at the end of DDI, we can easily argue that the entropy in quantum fluctuations remains at a negligible level during that phase.

Something interesting happens if we now consider the string phase [3], characterized by a constant H and $\dot{\phi}$. It is easy to find that, if $\dot{\phi} > 3H$, the HEB starts to decrease while for $\dot{\phi} < 3H$ it increases. Clearly, the first alternative leads to a contradiction with the HEB, since S_{coll} cannot decrease. We are thus led to the amusing result that the HEB demands $\dot{\bar{\phi}} \equiv \dot{\phi} - 3H \leq 0$ during the string phase as opposed to the $\dot{\bar{\phi}} > 0$ condition that characterizes the DDI phase. Thus, the HEB implies a "branch change" occurring between the DDI and the string phase, a well known necessary condition for achieving a graceful exit [11]. The condition $\dot{\bar{\phi}} < 0$ for the string phase also follows from (apparently independent) arguments based on the study of late-time attractors [12, 13].

When will the final exit to the FRW phase occur? It has been assumed [14] that it does when the energy in the quantum fluctuations becomes critical, i.e. when

$$\rho_{qf} \sim N_{eff}\ H_{max}^4 = e^{-\phi_{exit}} M_s^2\ H_{max}^2\ , \tag{9}$$

where N_{eff} is the effective number of particle species produced. Taking $H_{max} \sim M_s$, fixes the value of the dilaton at exit, $e^{\phi_{exit}} \sim 1/N_{eff}$. Using known results on entropy production due to the cosmological squeezing of vacuum fluctuations [15], we find:

$$S_{qf}(\text{ex}) \sim N_{eff}\ H_{max}^3 V \sim e^{-\phi_{exit}} M_s^3\ V \sim \left(l_s^2/l_P^2\right)_{exit}\ V l_s^{-3} \sim S_{HB}(\text{ex})\ , \tag{10}$$

[3] We concentrate here on the standard PBB scenario [5] and not on its variant [10] in which the DDI phase flows directly into a low-energy M-theory phase.

i.e. saturation of the HEB by S_{qf}. Unless exit occurs at this point, the HEB will be violated at later times.

We thus arrive, generically, at the situation shown in Fig.1. At $t = t_{ex} \equiv t_r$, the entropy in the quantum fluctuations has catched up with (and possibly overcome) that of the classical collapse and has become equal to the HEB, $S_{HB} \sim (HR)^3 \sim 10^{90}$. By then, S_{BB} is a factor HR larger, which is precisely the factor 10^{30} that we are running after. From there on, the story is simple: our entropy remains, to date, roughly constant and around 10^{90}, while S_{HB} keeps increasing –with somewhat different rates– during the radiation and the matter-dominated epochs. S_{BB} always remains a factor HR above S_{HB}, but this factor, originally huge, shrinks to unity today, by definition.

In conclusion, the entropy and arrow-of-time problems are neatly solved, in PBB cosmology, by the identification of our observable Universe with (part of) the interior of an original black hole. As such, its initial entropy saturates both the HEB and the BEB and is large because of the assumed large size (in string or Planck units) of the initial black hole. From there on, there is a natural mechanism to provide saturation of the HEB at the beginning of the radiation-dominated phase, i.e. when the BEB lies some thirty orders of magnitude higher. This is precisely what is needed to account for the initial entropy of our Universe, and to unambiguously identify its time arrow.

We do not wish to conceal the fact that our choice of the initial size of the collapsing/inflating region can be objected to, along the lines of Refs. [16], as representing a huge amount of fine-tuning. Our answer to this objection has been expressed elsewhere [7]: the classical collapse/inflation process is a scale-free problem in General Relativity; as such, it should lead to a flattish distribution of horizon sizes, extending from the string length to very large scales, including those appropriate for giving birth to our Universe. No other dimensionless ratio is tuned to a particularly large or small value as evidentiated in Fig.1 by the three upper curves all originating from the same point at $t = t_i$. Finally, we wish to stress again that the entropy considerations discussed in this note appear to provide new general arguments supporting previous conjectures on the way pre-big bang inflation should make a graceful exit into standard, post-big bang FRW cosmology.

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Note added: After completion of this work I became aware of a very recent paper [17] which reaches similar conclusions on the role of H^{-1} in cosmological entropy bounds.

References

[1] See, for instance, R. Penrose, *The Emperor's new mind* (Oxford University Press, New York, 1989), Chapter 7.

[2] J.D. Bekenstein, *Phys. Rev.* **D23** (1981) 287; **D49** (1994) 1912, and references therein.

[3] E.W. Kolb and M.S. Turner, *The early Universe* (Addison-Wesley, Redwood City, CA, 1990); A.D. Linde, *Particle physics and inflationary cosmology* (Harwood, New York, 1990).

[4] W. Fischler and L. Susskind, *Holography and cosmology*, hep-th/9806039; see also D. Bak and S.-J. Rey, *Holographic principle and string cosmology*, hep-th/9811008; A. K. Biswas, J. Maharana and R.K. Pradhan, *The holography principle and pre-big bang cosmology*, hep-th/9811051; S.K. Rama and T. Sarkar, *Holographic principle during inflation and a lower bound on density fluctuations*, hep-th/9812043.

[5] G. Veneziano, *Phys. Lett.* **B265** (1991) 287; K.A. Meissner and G. Veneziano, *Phys. Lett.* . **B267** (1991) 33; *Mod. Phys. Lett.* **A6** (1991) 3397; M. Gasperini and G. Veneziano, *Astropart. Phys.* **1** (1993) 317; *Mod. Phys. Lett.* **A8** (1993) 3701; *Phys. Rev.* **D50** (1994) 2519. An updated collection of papers on the pre-big bang scenario is available at `http://www.to.infn.it/~gasperin/`.

[6] G. Veneziano, *Inflating, warming up, and probing the pre-bangian Universe*, CERN-TH/99-22, hep-th/9902097.

[7] G. Veneziano, *Phys. Lett.* **B406** (1997) 297; A. Buonanno, K.A. Meissner, C. Ungarelli and G. Veneziano, *Phys. Rev.* **D57** (1998) 2543, and references therein.

[8] A. Buonanno, T. Damour and G. Veneziano, *Pre-big bang bubbles from the gravitational instability of generic string vacua*, hep-th/9806230.

[9] A. Ghosh, G. Pollifrone and G. Veneziano, *Phys. Lett.* **B440** (1998) 20.

[10] M. Maggiore and A. Riotto, *D-branes and cosmology*, hep-th/9811089; see also T. Banks, W. Fishler and L. Motl, *Duality versus singularities*, hep-th/9811194.

[11] R. Brustein and G. Veneziano, *Phys. Lett.* **B329** (1994) 429; N. Kaloper, R. Madden and K.A. Olive, *Nucl. Phys.***B452** (1995) 677, *Phys. Lett.* **B371** (1996) 34; R. Easther, K. Maeda and D. Wands, *Phys. Rev.* **D53** (1996) 4247.

[12] M. Gasperini, M. Maggiore and G. Veneziano, *Nucl. Phys.* **B494** (1997) 315.

[13] R. Brustein and R. Madden, *Phys. Lett.* **B410** (1997) 110, *Phys. Rev.* **D57** (1998) 712.

[14] G. Veneziano, in *Effective theories and fundamental interactions*, Erice, 1996, A. Zichichi ed., (World Scientific, Singapore, 1997) p. 300; A. Buonanno, K.A. Meissner, C. Ungarelli and G. Veneziano, *JHEP01*, 004 (1998).

[15] M. Gasperini and M. Giovannini, *Phys. Lett.* **B301** (1993) 334, *Class. Quant. Grav.* **10** (1993) L133; R. Brandenberger, V. Mukhanov and T. Prokopec, *Phys. Rev. Lett.* **69** (1992) 3606, *Phys. Rev.* **D48** (1993) 2443.

[16] M. Turner and E. Weinberg, *Phys. Rev.* **D56** (1997) 4604; N. Kaloper, A. Linde and R. Bousso, *Phys. Rev.* **D59** (1999) 043508.

[17] R. Easther and D.A. Lowe, *Holography, cosmology and the second law of thermodynamics*, hep-th/9902088.

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Quantum hair and the string-black hole correspondence

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Abstract

We consider a thought experiment in which an energetic massless string probes a "stringhole" (a heavy string lying on the correspondence curve between strings and black holes) at large enough impact parameter for the regime to be under theoretical control. The corresponding, explicitly unitary, S-matrix turns out to be perturbatively sensitive to the microstate of the stringhole: in particular, at leading order in l_s/b, it depends on a projection of the stringhole's Lorentz-contracted quadrupole moment. The string-black hole correspondence is therefore violated if one assumes quantum hair to be exponentially suppressed as a function of black-hole entropy. Implications for the information paradox are briefly discussed.

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1 Introduction

The issue of a possible loss of quantum coherence in processes in which a black hole is produced and then evaporates has been the subject of much debate since Hawking's claim [1] that black holes should emit an exactly thermal spectrum of light quanta (see e.g. [2] for a recent review). Progress from string theory on the microscopic understanding of black-hole entropy [3] and on the AdS-CFT correspondence [4], has lent strong support to the belief that no loss of information/quantum-coherence should occur. However, even in the AdS/CFT case, understanding how unitarity on the CFT side teaches about information recovery on the gravity side remains unclear (see [5], [6]).

Another "ab initio" approach to the same problematics consists of the study of trans-planckian-energy collisions of massless strings as a function of center of mass energy (or of the associated gravitational radius R), of impact parameter b, and of the string-length scale l_s, with the relative ratios of these scales defining different regimes for the process [7].

In this framework it has been possible to recover, within a unitarity-preserving S-matrix, both General Relativity expectations and string-size related modifications of it [8], albeit in regimes in which no-black hole formation is expected according to closed-trapped surface criteria[9]. Dealing with the complementary regime (corresponding to $R \gg b, l_s$) has met with more limited success, although some progress has been made in understanding how the threshold of black-hole production can be approached from below [10]. An approximation to deal with the full-collapse regime, proposed a few years ago in [11], appears to predict correctly the existence and rough values of some critical ratios for the onset of collapse, but, unfortunately, has failed so far to provide a unitary description of the process beyond such critical points [12].

Given the above difficulties, the attention has been shifted to a supposedly easier problem [13], that of the scattering of a closed light string off a stack of N $D-p$-branes at small string coupling and large N. Here the equivalent of the black-hole formation regime is the one in which the closed string is absorbed by the brane system and its energy is dissipated in open string excitations of the stack itself. In spite of some progress [13] [14], understanding how information about the initial state gets encoded in the final one is still far from settled.

One problem is that information, if it's to be eventually recovered, has to start coming out, at the latest, by the so-called Page time [15], corresponding roughly to the time by which the evaporating black hole has lost half of its entropy S. In order for this to be possible, the rate of information retrieval cannot be too small, e.g. cannot be of order $\exp(-S)$, at least not after the Page time. Information retrieval should instead be easy if "quantum hair" is inversely proportional to S, as recently proposed in a toy model identifying black holes with a self-sustained critical Bose-Einstein condensate of $N \sim S$ gravitons [16]. Similar claims have been made in [17] on the basis of general uncertainty-principle considerations applied to the geometry itself. Indeed, once an effective classical geometry with an information-free horizon is assumed (even an effective one that corrects the classical horizon), continuous

information loss looks inevitable[2].

In this paper we will address this kind of questions using the correspondence between strings and black holes [18, 19, 20] that occurs when the mass of the former is tuned to the value $M_{SH} = M_s g_s^{-2}$, giving a Schwarzschild radius $R = O(l_s)$. By going to small enough string coupling we can make the entropy of such "string-holes" (SH) arbitrarily large:

$$S_{SH} = \left(\frac{l_s}{l_P}\right)^{D-2} = \left(\frac{M_P}{M_s}\right)^{D-2} = g_s^{-2} \gg 1\,. \tag{1}$$

It is particularly appealing that, for SHs, the question of the size of quantum hair becomes one about whether it is perturbative or not in the string coupling constant. In our case, the role of the parameter N of [16] is played by the string coupling which, for a given string mass, is tuned to a critical value. Unfortunately, and unlike in the simple model of [16], we are presently unable to perform a reliable calculation when g_s and/or M are parametrically larger than their critical values.

Furthermore, in order be able to claim that strings of mass $M_{SH} = M_s g_s^{-2}$ can also be seen as black holes, we have to impose that they are compact enough not to exceed in size their own Schwarzschild radius $R = O(l_s)$, and to check that this restriction does not invalidate the entropy estimate (1). This question was addressed in [20] (see also [19]), where it was argued that the entropy of string states of mass $M \leq M_{SH} = g_s^{-2} M_s$ is shared among states of different size r according to:

$$S(M,r) \sim \frac{M}{M_s}\left(1 - c_1\frac{l_s^2}{r^2}\right)\left(1 - c_2\frac{r^2}{(\alpha' M)^2}\right)\left(1 + c_3\left(\frac{R}{r}\right)^{D-3}\right), \tag{2}$$

where c_i are positive constants of $O(1)$. For $M \ll M_{SH}$ the last terms is negligible and the first two factors give a maximal entropy for $r \sim l_s\sqrt{\frac{M}{M_s}}$, the random-walk value. However, there is still an entropy $O(M/M_s)$ in "compact" strings and, furthermore, as one approaches $M = M_{SH}$, the third term in (2) helps favoring such strings.

Another way of reaching a similar result consists in counting string states at level $N = \alpha' M^2$ produced by oscillators of index larger than K. A simple argument, based on evaluating the corresponding partition function, shows that the entropy of such states is still $O(\sqrt{N})$ if $K \sim \sqrt{N}$. They will generally correspond to occupation numbers $\leq O(1)$ for $O(\sqrt{N})$ oscillators (providing the right value for their mass) and will have a size of order:

$$r^2 \sim l_s^2 \sum_{n>\sqrt{N}} \frac{1}{n}\langle a_n^\dagger a_n\rangle \sim l_s^2\,. \tag{3}$$

This is the kind of states we shall focus our attention on. We recall that, not only entropy, but, qualitatively, many other properties of strings and black holes (decay rates, evaporation time etc.) match on the correspondence line [19, 20].

The idea, therefore, is to consider a thought experiment in which a massless string probes a stringhole target, a process somewhere in between those discussed in [7] (where both projectile and target are massless) and [13] (where the target is infinitely heavy). Studying such a process at sufficiently large impact parameters for the approximations to be under control turns out to be sufficient to reveal whether the quantum hair of such SHs is perturbative or not in $1/S \sim g_s^2$. This appears to be the string-theory counterpart to checking (albeit only at a specific point) whether quantum hair is perturbative in $1/N$ in the approach of [16].

2 A thought experiment revealing quantum hair

We work in flat 10-dimensional spacetime with $(10 - D)$ dimensions compactified at the string-length scale so that the effective large-distance physics lives in D spacetime dimensions. We are also assuming to be working at very small string coupling g_s so that, as already indicated in (1), there is a large hierarchy between the string and Planck mass scales.

Consider now a process in which a massless "probe" string collides with a well-defined heavy (and for the moment generic) "target" string of mass $M \gg M_P \gg M_s$. Let us also take a high-energy limit in which the energy E of the probe string in the rest frame of the heavy one is much larger than M_s and yet much smaller than M,

$$M_s M \ll s - M^2 = -2p \cdot P = 2EM \ll M^2 \,, \tag{4}$$

so that the light string does indeed act (almost) as a probe and yet we can apply a high-energy limit in which graviton exchange dominates.

Following the logic of [7] (see also [21], [13]) we can argue that, at large-enough impact parameter b, the elastic scattering amplitude is given by the semiclassical eikonal formula:

$$S(E, M, b) \sim \exp(i\frac{\mathcal{A}_{cl}}{\hbar}) = \exp\left(i\frac{4GEM}{\hbar}c_D b^{4-D}\right) \equiv e^{2i\delta(E,M,b)} \; ; \; c_D = \Omega_{D-4}^{-1} \equiv \frac{\Gamma(\frac{D-4}{2})}{2\pi^{\frac{D-4}{2}}}. \tag{5}$$

As a consistency check, we note that, when one goes back from b to q-space (or deflection angle θ), one recovers, at the saddle point of the b-integral, the classical Einstein relation (generalized to arbitrary D) between deflection angle, mass, and impact parameter:

$$\theta = \frac{8\pi GM}{\Omega_{D-2} b^{D-3}} \sim \left(\frac{R}{b}\right)^{D-3} \ll 1 \;\; ; \;\; (GM)^{\frac{1}{D-3}} \sim R \ll b \,, \tag{6}$$

where R is the Schwarzschild radius of the heavy string. Obviously, the above formula satisfies the "no-hair" theorem, in the sense that it is sensitive to the mass of the heavy string state but not to its microscopic quantum numbers.

Diagrammatically, the result (5) comes from exponentiating the exchange of a single graviton between the light and the heavy string. Both (5) (and (6)) are indeed only valid at sufficiently large impact parameter (small deflection angle) and suffer from corrections of

higher order in R/b (θ). These will reconstruct, for instance, the deflection formula in the full Schwarzschild (or Kerr if we consider a target with spin) metric. As shown long ago by Duff [22], they correspond, diagrammatically, to exponentiating connected graviton-tree (fan) diagrams in which a single vertex (the trunk of the tree) is attached to the probe string while all the branches terminate on the heavy one, giving the appropriate powers of R and b. These classical correction still satisfy the no-hair condition as well as elastic unitarity. On the other hand if instead"hairy" corrections to $\delta(E, M, b)$ are exponentially suppressed for large black holes, we would expect them to show up in the form:

$$\delta(E, M, b) \to \delta(E, M, b)(1 + \text{classical corrections} + e^{-cS}\hat{Q}) \,, \tag{7}$$

where c is some constant and $\hat{Q}$ represents, schematically, a quantum-hair operator taking different expectation values depending on the black hole microstate. We will check below whether an ansatz like (7) is satisfied for the particular stringhole states introduced in the previous Section.

To address this question recall that, as discussed in [7] and [13] in two different contexts, there are also "string corrections" to the leading eikonal form. These are related to the fact that strings are extended objects and therefore suffer tidal forces when moving in a non trivial geometry[23][1]. Fortunately, at least at small scattering angle, such corrections are fully under control and lead to a unitary S-matrix. Unitarity is now satisfied in a less trivial way: different channels couple, elastic unitarity is violated, but one still obtains a fully unitary S-matrix in the Hilbert space of two arbitrary string states. The question is whether this non-trivial S-matrix contains information about the actual state of the heavy string, and at which level.

Building on the work of [7] and [13] we can be confident that the tidal excitation of both the light and the heavy string are captured, at leading order in θ, by the replacement:

$$\delta(E, M, b) \to \hat{\delta}(E, M, b) = \langle \delta(b + \hat{X}_H - \hat{X}_L) \rangle = 2GEM\hbar^{-1} c_D \langle (b + \hat{X}_H - \hat{X}_L)^{4-D} \rangle \,. \tag{8}$$

Here $\hat{X}_H$ and $\hat{X}_L$ represent the heavy and light string position operators, stripped of their zero modes (which give b), evaluated at $\tau = 0$, and averaged over σ. These operations, together with a normal-ordering prescription, are indicated in (8) by the brackets, i.e.

$$\langle (b + \hat{X}_H - \hat{X}_L)^{4-D} \rangle \equiv \frac{1}{4\pi^2} \int_0^{2\pi} d\sigma_L \int_0^{2\pi} d\sigma_H : \Big(b + \hat{X}_H(\sigma_H, 0) - \hat{X}_L(\sigma_L, 0) \Big)^{4-D} : \,. \tag{9}$$

In words, the classical phase shift is replaced by the average of a *quantum* phase shift in which the impact parameter is affected by a quantum uncertainty encoded in the string position operators.

For what concerns the excitation of the light string, further justification of the above formula comes from the study of string-brane collision discussed in [13], specialized to the

[1] Although all calculations are performed in flat spacetime the effects of an effective non-trivial geometry emerge from the calculation.

case of a stack of 0-branes. For the excitation of the heavy string we can instead appeal to the quantization of the heavy string in the shock-wave metric produced by the light one [24].

Following [7], we now expand (8) to quadratic order in the $\hat{X}$ (the linear order clearly averages out to zero) to get the leading correction in an expansion in $(l_s/b)^2$:

$$2(\hat{\delta}-\delta) = \frac{2\pi GEM(D-2)}{\hbar\Omega_{D-2}b^{D-2}}\langle Q_H^{ij}+Q_L^{ij}\rangle \hat{b}_i\hat{b}_j\,. \tag{10}$$

Here Q_H^{ij} is the $(D-2)$-dimensional (i.e. Lorentz-contracted in the direction of the incoming momentum) quadrupole operator for the heavy string[2].

$$Q_H^{ij} = \hat{X}_H^i\hat{X}_H^j - \frac{\delta_{ij}}{D-2}\sum_{i=1}^{D-2}\hat{X}_H^i\hat{X}_H^i\,, \tag{11}$$

and is projected along the unit vector $\hat{b}$ in the direction of the impact parameter. This projection can also be written in the form:

$$Q_H^{ij}\hat{b}_i\hat{b}_j = \hat{X}_H^i\hat{X}_H^j\left(\hat{b}_i\hat{b}_j - \frac{\delta_{ij}}{D-2}\right) \equiv \Pi_{ij}\hat{X}_H^i\hat{X}_H^j\,. \tag{12}$$

As indicated in (10), we get a similar term for the probe string. At this order in l_s/b the S-matrix thus factorizes in the form:

$$S(E,M,b) = \exp(2i\delta)\ \Sigma_L\ \Sigma_H\ ;\ \Sigma_{L,H} = \exp\left(i(D-2)\Delta\ \tilde{Q}_{L,H}^{ij}\ \hat{b}_i\hat{b}_j\right)\,, \tag{13}$$

where we have defined the dimensionless quantities[3]:

$$\Delta = \frac{2\pi GEMl_s^2}{\hbar\Omega_{D-2}b^{D-2}}\quad ;\quad \tilde{Q}^{ij} = l_s^{-2}Q^{ij}\,, \tag{14}$$

the latter being the quadrupole measured in string-length units. Since the quadrupole operators are hermitian (see also below), each factor appearing in (13) corresponds to a unitary operator. The first two factors are independent of the particular state chosen for the heavy string. Let us therefore concentrate our attention on Σ_H (dropping for simplicity the H suffix). The operator appearing at the exponent in Σ can be easily written down:

$$\tilde{Q}^{ij}\ \hat{b}_i\hat{b}_j = \Pi_{ij}\ \sum_{n=1}^{\infty}\frac{1}{n}\left(a_n^{\dagger i}a_n^j + \tilde{a}_n^{\dagger i}\tilde{a}_n^j + a_n^i\tilde{a}_n^j + a_n^{\dagger i}\tilde{a}_n^{\dagger j}\right)\,. \tag{15}$$

Its diagonal matrix elements are sensitive to the (projected, transverse) quadrupole of the heavy string, while the transitions to other states, induced by terms with two creation or

[2]I am grateful to T. Damour for this interesting remark.

[3]The value of Δ, when compared to unity, determines [7] whether the probe string gets excited or not by tidal forces. However, once more, these effects will *not* depend on the particular state of the target string.

two annihilation operators, correspond to a quadrupole-like excitation of the original string itself. This is hardly surprising in view of the intimate relation between tidal forces and quadrupole moments (see e.g. [25]), and simply appears as a generalization of known facts to an ultra-relativistic situation involving strings (our quadrupole, in particular, is a purely geometrical object).

In order to have an estimate of quantum hair we need to normal order the whole exponential operator occurring in Σ. Following again [7], we find:

$$\begin{aligned}
\Sigma_H &= \Sigma^{(univ)}\ \Sigma^{(hair)}\ ;\ \Sigma^{(univ)} = \Gamma(1+i\Delta)^{D-3}\ \Gamma(1-i(D-3)\Delta) \\
\Sigma^{(hair)} &= :\exp\left(\sum_{n=1}^{\infty}(a_n^{\dagger i}+\tilde{a}_n^i)(a_n^j+\tilde{a}_n^{\dagger j})\left[C_n(\Delta)(\delta_{ij}-\hat{b}_i\hat{b}_j)+\tilde{C}_n(\Delta)\hat{b}_i\hat{b}_j\right]\right): \\
C_n(\Delta) &= -\frac{i\Delta}{n+i\Delta}\ ;\ \tilde{C}_n(\Delta) = C_n(-(D-3)\Delta)\,. \qquad (16)
\end{aligned}$$

At this point the explicit calculation of the S-matrix is simple, in particular between coherent states. $\Sigma^{(univ)}$, being a c-number, does not depend on the internal quantum numbers of the heavy string and, together with similar factors coming from the light string, provides absorption and further contributions to the phase shifts, but no hair. Instead, the operator $\Sigma^{(hair)}$ generates matrix elements that feel the nature of the microstate in which the heavy string actually is. Note that normal ordering has slightly upset the exact quadrupole structure appearing in (11), (12) (which is however recovered for $n \gg \Delta$).

Let us now specify further the process described in the previous section in order to make contact with black-hole physics. To this purpose we shall identify the heavy string with a "stringhole" state described in Sec. 1. The reason for choosing that precise (within factors $O(1)$) value of M is twofold. Choosing M in the range $M_P \ll M \ll M_{SH}$ leads to reliable results, but the string, in this case, is below the correspondence curve, its size is larger than its Schwarzschild radius and therefore is not a collapsed object [18]. On the other hand, various approximations that can be justified for strings of mass up to M_{SH} cease to be valid for strings with $M >> M_{SH}$, i.e. strings that would simulate "large" black holes in string-length units.

Let us first evaluate the quantity Δ in (14) for the SH case. Up to numerical factors:

$$\Delta = \frac{GEMl_s^2}{\hbar\, b^{D-2}} \to \frac{El_s}{\hbar}\left(\frac{l_s}{b}\right)^{D-2} \sim \frac{E}{M_s}\theta^{\frac{D-2}{D-3}}\,. \qquad (17)$$

Given our bounds (4) on E we find:

$$\theta^{\frac{D-2}{D-3}} \ll \Delta \ll g_s^{-2}\theta^{\frac{D-2}{D-3}}\,. \qquad (18)$$

Obviously, even keeping $\theta \ll 1$, but finite and g_s-independent, we can make $\Delta \gg 1$ (yet $\ll g_s^{-2}$) for sufficiently small g_s and with E in a parametrically large region.

In order to estimate the size of quantum hair we note that the coefficients $C_n(\Delta)$ appearing in (16) become $O(1)$ at $n < \Delta$ or of order Δ/n at $n > \Delta$. As already discussed,

typical SHs will have most of the non vanishing occupation numbers of $O(1)$ in oscillators with $n \sim \sqrt{N} \sim g_s^{-2}$. In that case $C_n \sim \Delta/n$ and eq. (16) simplifies further:

$$\Sigma^{(hair)} =: \exp\left(-i(D-2)\Delta \sum_{n=1}^{\infty} \frac{1}{n}(a_n^{\dagger i} + \tilde{a}_n^i)(a_n^j + \tilde{a}_n^{\dagger j})\Pi_{ij}\right) : \ . \tag{19}$$

The basic observation is that the operator appearing in the exponent of (19) is completely unrelated to the one giving the mass of the SH[4] and therefore will distinguish degenerate microstates. It contains positive definite diagonal terms that correspond to the transverse, projected quadrupole of the SH. The non-positive definite terms, corresponding to inelastic transitions, are also microstate-dependent through a similar quadrupole operator. There is also a state-dependent absorption from the real part of C_n. This is suppressed by an extra factor $\Delta/n \sim g_s^2\Delta$ and is not controlled by the quadrupole.

The dominant terms sum up to something $O(1)$ but can still take different values of that same order within the whole SH ensemble. Hence, an experiment measuring the phase of the S-matrix should be able to reduce our ignorance on the state of the SH by a factor $O(2)$[5]. Interpreting this as a reduction on the total number of states e^S, it will correspond to a decrease of $O(1)$ in entropy, meaning that the whole information can be recovered after $O(S)$ experiments. The minimal duration of each experiment being $O(l_s)$, namely the light-crossing time for a SH, the total time needed to recover the information will be of order of the Page/evaporation time $SR \sim g_s^{-2} l_s$.

In our approximation quantum hair also appears to be suppressed with respect to the no-hair terms by a power of the scattering angle. While this is still sufficient for our qualitative discussion, we think that our results should qualitatively extend to scattering angles of O(1). Checking this is not easy since, precisely for a SH target, string-size and classical corrections kick in simultaneously as we increase the scattering angle (although the different θ-dependence should help separate the two kinds of corrections). If so, the quantum hair revealed by our scattering process (with higher multipoles appearing besides Q_{ij}) will indeed approach $O(1)$ for a probe-energy of order of the Hawking/Hagedorn temperature of the SH. Such impact parameters and energies are precisely those typical of Hawking's radiation. Actually, as well known in particle physics (see e.g. [26]), a decay amplitude is usually to be corrected by a "final-state interaction" which basically amounts to multiplying the naive decay amplitude by a factor $S^{1/2}(E,b) \sim \exp(i\delta(E,b))$, where the typical values of E and b will be M_s and l_s respectively. In other words, such a quantum hair may directly leave its imprint in the decay of a SH. Admittedly, all these are hand-waving arguments that should be analyzed more carefully.

In any case, it appears that the quantum-hair amplitude is *not* suppressed, relative to no-hair terms, by $\exp(-S_{SH}) \sim \exp(-g_s^{-2})$ but rather, at most, by a small inverse power of

[4]It is also clearly non-degenerate with the spin of the SH.

[5]If instead we wish to distinguish SH states with differences $O(1)$ in the occupation numbers the sensitivity of (19) will have a suppression factor of $O(\Delta/n) \sim g_s^2\Delta$.

$S \sim g^{-2}$, i.e. is a perturbative effect in the string coupling. Note, however, that a generic individual element of the S-matrix is always suppressed by an $\exp(-\Delta)$ "non-perturbative" factor, which gets compensated by the exponentially large number of final states contributing to inclusive-enough cross sections. Indeed, given that Σ_H is unitary, it is easy to lose its sensitivity to quantum hair if traces over the heavy initial and/or final SH states are taken. Summing individual transition probabilities over final SH states corresponds to considering an inclusive cross section, while tracing/averaging over the initial SH state corresponds to an initial mixed state. In both cases it is quite clear that unitarity of Σ_H washes out all the leading-order SH hair discussed so far[6]. Only subleading terms and/or appropriate interference experiments will be able to leave information about the state of the SH on the probe-string. Whether this is in principle sufficient to retrieve enough information on the SH is not completely obvious.

In conclusion, the results we have presented point in the direction of some perturbative quantum hair being revealed in our thought experiment, very likely something of order $1/S \sim g_s^2$ for a probe of $E \sim M_s$ during a collision (horizon-crossing) time $O(l_s)$. At least naively, this would allow to retrieve the full information about the microstate of the string hole within its evaporation time of order $g_s^{-2} l_s$. This result can be related to the fact that, for a stringhole, the concept of an infomation-free horizon does not make sense (the horizon being as large as the string itself) and, in this sense, it is similar to what is believed to occur for the so-called fuzzballs states of string theory (see [28] and references therein). It would be interesting to see whether thought experiments of the kind discussed here using fuzzballs could reveal a similar amount of quantum hair. Of course the issue of whether or not spacetime around the horizon can be considered to be empty is also very relevant in the recent firewalls debate [29].

Can we reconcile our finding with an exponentially small amount of quantum hair for large black holes (i.e. for black holes much heavier that M_{SH}, for which our simple analysis fails to provide a reliable answer)? Clearly an expression like that of (7) is in contradiction with our findings but one could instead imagine an ansatz like:

$$\delta \to \delta(E, M, b)(1 + \text{classical corrections} + e^{-c\frac{S}{S_{SH}}}\hat{Q}) \,, \tag{20}$$

which would only give an exponential suppression for black holes that are much heavier that those on the correspondence line. Indeed, a single string may fail to represent black holes above the correspondence curve (seen in that case as a critical line separating two phases), in which case $\Sigma^{(hair)}$ could change quite abruptly above the phase transition.

Another possible objection to drawing strong conclusions from our results lies in the possibility[7] that SHs do *not* represent typical black holes but only a tiny fraction of them. In that case, their long hair will make them atypical "hippie-like" black holes within a vast majority of "bald" ones.

[6] At order $l_s^4 b^{-4}$ the eikonal operator will give terms proportional to $X_L^2 X_H^2$ that destroy factorization.

[7] This possible loophole was suggested by M. Porrati.

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References

[1] S. W. Hawking, *Nature* **248** (1974) 30; *Commun. Math. Phys.* **43** (1975) 199.

[2] S. Mathur, *The information paradox : A pedagogical introduction*, arXiv:0909.1038.

[3] A. Strominger and C. Vafa, *Phys.Lett.* **B379** (1996) 99.

[4] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Phys. Lett.* **428** (1998) 105; E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253.

[5] J. M. Maldacena, *J. High Energy Phys.* 0304 (2003) 02; *JHEP* **0304** (2003) 021.

[6] J.L.F. Barbon and E. Rabinovici, *JHEP* **0311** (2003) 047.

[7] D. Amati, M. Ciafaloni and G. Veneziano, *Phys. Lett.* **B197** (1987) 81; *Int. J. Mod. Phys.* **A3** (1988) 1615.

[8] D. Amati, M. Ciafaloni, and G. Veneziano, *Phys. Lett.* **B216** (1989) 41.

[9] D. M. Eardley and S. B. Giddings, *Phys. Rev.* **D66** (2002) 044011; E. Kohlprath and G. Veneziano,*JHEP* **06** (2002) 057; H. Yoshino and Y. Nambu, *Phys. Rev.* **D67** (2003) 024009; S. B. Giddings and V. S. Rychkov, *Phys. Rev.* **D70** (2004) 104026.

[10] G. Veneziano, *JHEP* **11** (2004) 001.

[11] D. Amati, M. Ciafaloni, and G. Veneziano, *JHEP* **02** (2008) 049; G. Marchesini and E. Onofri, *JHEP* **06** (2008) 104; G. Veneziano and J. Wosiek, *JHEP* **09** (2008) 023; G. Veneziano and J. Wosiek, *JHEP* **09** (2008) 024.

[12] M. Ciafaloni and D. Colferai, *JHEP* **11** (2008) 047; *JHEP* **12** (2009) 062.

[13] G. D'Appollonio, P. Di Vecchia, R. Russo and G. Veneziano, *JHEP* **1011** (2010) 100.

[14] W. Black and C. Monni, *Nucl. Phys.* **B859** (2012) 299-320; M. Bianchi and P. Teresi, *Scattering higher spins off D-branes* [arXiv: 1108.1071].

[15] D. N. Page, *Phys. Rev. Lett.* **71** (1993) 3743.

[16] G. Dvali and C. Gomez, *Black Hole's* $1/N$ *Hair*, arXiv:1203.6575; *Black Hole Macro-Quantumness*, to appear, and references therein.

[17] R. Brustein, *Origin of the blackhole information paradox*, arXiv:1209.2686.

[18] G. Veneziano, *Europhys. Lett.* **2** (1986) 133; Proc. Meeting on "Hot hadronic matter: theory and experiments", Divonne, June 1994, eds. J. Letessier, H. Gutbrod and J. Rafelsky, NATO-ASI Series B: Physics (Plenum Press, New York, 1995), **346**, p. 63; L. Susskind, *Some speculations about black hole entropy in string theory*, [hep-th/9309145]; E. Halyo, A. Rajaraman and L. Susskind, *Phys. Lett.* **B392** (1997) 319; E.Halyo, B. Kol, A. Rajaraman and L. Susskind, *Phys. Lett.* **B 401** (1997) 15.

[19] G.T. Horowitz and J. Polchinski, *Phys. Rev.* **D55** (1997) 6189; ibid. **57** (1998) 2557;

[20] T. Damour and G. Veneziano, *Nucl. Phys.* **B568** (2000) 93.

[21] M. Fabbrichesi and R. Iengo, *Phys. Lett.* **B264** (1991) 319.

[22] M. J. Duff , *Phys.Rev.* **D7** (1973) 2317.

[23] S. B. Giddings, *Phys.Rev.* **D74** (2006) 106005.

[24] H. J. de Vega and N. G. Sanchez, *Nucl. Phys.* **B317** (1989) 706–730; G. T. Horowitz and A. R. Steif, *Phys. Rev.* **D42** (1990) 1950–1959; S. B. Giddings, D. J. Gross, and A. Maharana, *Phys. Rev.* **D77** (2008) 046001.

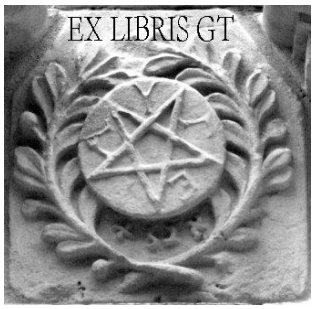
[25] T. Damour and A. Nagar, *Phys. Rev.* **D80** (2009) 084035.

[26] M. Suzuki and L. Wolfenstein, *Phys. Rev.* **D60** (1999) 074019.

[27] P. C. Aichelburg and R. U. Sexl, *Gen. Rel. Grav.* **2** (1971) 303–312; see, e.g. E. Kohlprath and G. Veneziano, *JHEP* **06** (2002) 057 for its generalization.

[28] S. Mathur, *The fuzzball proposal for black holes: an elementary review*, Fortsch. Phys. **53** (2005) 793.

[29] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, *Complementarity or firewalls?*, arXiv: 1207.3123 [hep-th].



CERN–TH-6077/91

SCALE FACTOR DUALITY FOR CLASSICAL AND QUANTUM STRINGS

OK 14.02

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Abstract

Duality under inversion of the cosmological scale factor is discussed both for the classical motion of strings in cosmological backgrounds and for genus zero, low energy effective actions. The string-modified, Einstein-Friedmann equations are then shown to possess physically-inequivalent, duality-related solutions which generically describe the "decay" of an initial, perturbative, flat $D = 10$ superstring vacuum towards a more interesting strong-coupling state through an appealing pre-big-bang cosmological scenario.

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Duality = modular invariance!!!

1 INTRODUCTION

Undoubtedly, one of the deepest quantum symmetries of string theory is (target space) duality. In its simplest form [1], duality says that a (closed) string moving on a circle of radius R is equivalent to one which moves on a circle of radius λ_s^2/R where $\lambda_s^2 = 2\alpha'\hbar$ is the fundamental length parameter (Planck constant) of string theory. Duality has been extended to more complicated situations [2, 3, 4] and is more generally termed modular invariance (in target space). It is believed to be an exact symmetry [5], at least order by order in the string-loop expansion.

Duality appears to have far-reaching consequences, such as introducing a minimal compactification scale [6], restricting the possible form of scalar (or super) potentials [7] and determining some characteristics of non-perturbative supersymmetry breaking [8]. It could also have lots to do with the notion of a minimal observable scale $O(\lambda_s)$ in string collisions [9] and with extended forms [10] of the Uncertainty and Equivalence principles, although these concepts appear to retain their validity irrespectively of compactification.

Under duality the roles of X' and $P \sim \dot{X}$ (winding number and momentum for zero modes) are interchanged. Indeed, a somewhat simplified (see below) derivation of duality [3] consists of performing a canonical transformation on the string's position and momentum variables which are integrated over in the (Hamiltonian) path integral defining the partition function Z. If no anomaly gets in the way, this immediately leads to a symmetry of Z under certain discrete changes of the metric and torsion background fields, which is one of the possible definitions of duality [3].

So far duality has been fully discussed and implemented for a variety of constant background fields $G_{\mu\nu}, B_{\mu\nu}, ...$and there has been some controversy as to the possibility of extending it beyond such static situations[11].

In this paper I wish to present some evidence, both at the classical and at the quantum level, that duality is a very useful concept even for time-dependent backgrounds. Being a symmetry of the effective action (including classical stringy sources), at least to lowest order in derivatives and in the string loop expansion, duality will relate, in general, physically inequivalent cosmological solutions of the string-modified Einstein-Friedmann equations (which include a non-trivial dilaton).

Unlike the usual R-duality, this symmetry does not rest on compactifi-

R- sym/= SFD duality

SFD

cation and connects (physically) expanding to (physically) contracting Universes. To distinguish it R-duality, I shall refer to it as scale-factor-duality (SFD).

In Sect.2 I shall present some heuristic arguments for SFD starting from the classical motion of strings in cosmological backgrounds. In Sect.3 SDF will be substantiated by the analysis of the low-energy, tree-level string effective action. The final result will be a system of modified, SFD-invariant Einstein equations coupled to the SFD-invariant classical sources of Sect.2. In Sect.4 I shall present some explicit solutions and physical considerations, while Sect.5 will contain some conclusions and a rather speculative outlook.

In this paper I shall only present the general ideas and some results. A more detailed account, as well as extensions to more general backgrounds, will be given elsewhere [12, 13].

2 SCALE FACTOR DUALITY AT THE CLASSICAL LEVEL

The possibility of extending R-duality to time-dependent scale factors comes naturally from the study [14, 15, 16] of classical string propagation in homogeneous, isotropic, cosmological backgrounds:

$$g_{\mu\nu} = diag(-1, a^2(t)) \; ; \; \mu, \nu = 0, 1, ... (D-1)$$

In the orthonormal gauge, the corresponding string equations of motion and constraints read (i = 1,2, ..., D-1; $X^0 \equiv t$)

a-factor

$$\ddot{X}^0 - X''^0 = a\frac{da}{dt}\sum_i \left[\left(X'^i\right)^2 - \left(\dot{X}^i\right)^2\right]$$

$$\ddot{X}^i - X''^i = \frac{2}{a}\frac{da}{dt}(X'^0 X'^i - \dot{X}^0 \dot{X}^i),$$

$$\left(\dot{X}^0\right)^2 + \left(X'^0\right) = a^2 \sum_i \left[\left(X'^i\right)^2 + \left(\dot{X}^i\right)^2\right]$$

$$X'^0 \dot{X}^0 = a^2 \sum_i \dot{X}^i X'^i. \qquad (2.1)$$

1
a-->infin

Asymptotic solutions to the system (2.1) were discussed in [16] (see also [14, 15]) for the case $a(t) \to \infty$ in the form of a systematic large-$a(t)$ expansion. Of particular interest was the superinflationary case ($\dot{H} \equiv \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) > 0$) of which a typical representative is

$$\mathrm{a(t)} = (-\mathrm{t})^{-\gamma} \quad (\gamma > 0) \tag{2.2}$$

For such backgrounds, a regime of "high instability" was found to develop inevitably at late times. It is characterized asymptotically by [16]:

1. Proportionality of world-sheet time τ and of conformal time η where, as usual,

$$\mathrm{ad}\eta \equiv \mathrm{dt} \tag{2.3}$$

2. The stretching ("freezing") of spatial string coordinates:

$$\mathrm{X'^i} >> \dot{\mathrm{X}}^{\mathrm{i}} \quad (\mathrm{i} = 1, 2...\mathrm{D}-1) \tag{2.4}$$

3. A negative pressure which, in the ideal gas approximation, takes the asymptotic value:

$$\mathrm{p} = -\frac{\rho}{(\mathrm{D}-1)} \; ; \; \rho = \text{energy density} \tag{2.5}$$

2

In ref. [17] the case of a rapidly contracting Universe, e.g.

$$\tilde{\mathrm{a}}(\mathrm{t}) = (-\mathrm{t})^{\gamma} = \mathrm{a(t)}^{-1} \quad (\gamma > 0) \tag{2.6}$$

was also considered and solved at "late" times ($t \to 0, \tilde{a} \to 0$). In this case, the asymptotic solution was instead characterized by:

$\tilde{1}$. Proportionality between τ and $\tilde{\zeta}$ where:

$$\mathrm{d}\tilde{\zeta} \equiv \tilde{\mathrm{a}}\mathrm{d}\tilde{\mathrm{t}} \tag{2.7}$$

$\tilde{2}$. Very fast shrinkage of the string:

$$\tilde{\mathrm{X}}'^{\mathrm{i}} << \dot{\tilde{\mathrm{X}}}^{\mathrm{i}} \tag{2.8}$$

$\tilde{3}$. An equation of state typical of an ultrarelativistic gas:

$$\tilde{\mathrm{p}} = \frac{\tilde{\rho}}{(\mathrm{D}-1)} \tag{2.9}$$

Comparison of properties 1,2,3 and $\tilde{1}, \tilde{2}, \tilde{3}$ suggests [17] some relation between these two regimes and the corresponding solutions. Since $\tilde{a} = a^{-1}$, the result $\eta = \tilde{\zeta} = \tau$ implies, up to a constant,

$$\mathrm{t} = \tilde{\mathrm{t}} \tag{2.10}$$

A detailed study of the equations of motion and constraints shows that, indeed, one can transform any solution of the system of equations for the background metric (2.1) into a solution for the "dual" metric by the replacements

dual metric

$$\begin{aligned} \mathrm{t} \to \tilde{\mathrm{t}} &= \mathrm{t} \\ \mathrm{X}'^{\mathrm{i}} \to \tilde{\mathrm{X}}'^{\mathrm{i}} &= \mathrm{a}^2 \dot{\mathrm{X}}^{\mathrm{i}} \equiv \mathrm{P}^{\mathrm{i}} \\ \mathrm{P}^{\mathrm{i}} \to \tilde{\mathrm{P}}^{\mathrm{i}} &\equiv \mathrm{a}^{-2} \dot{\tilde{\mathrm{X}}}^{\mathrm{i}} = \mathrm{X}'^{\mathrm{i}} \end{aligned} \tag{2.11}$$

where the latter two equations are consistent with each other thanks to the equations of motion.

¿From (2.11) and from the definition of energy and pressure density for an ideal, isotropic gas of strings:

$$\mathrm{T}^0_0 = \rho \; ; \; \mathrm{T}^{\mathrm{j}}_{\mathrm{i}} = -\delta^{\mathrm{j}}_{\mathrm{i}} \, \mathrm{p}$$

$$\overset{\mu\nu}{\mathrm{T}}(\mathrm{x}) = \frac{1}{\sqrt{-g}\pi\alpha'} \sum_n \int d\sigma d\tau (\dot{X}^\mu_n \dot{X}^\nu_n - X'^\mu_n X'^\nu_n) \delta\left(X - x\right) . \tag{2.12}$$

one finds, after use of (2.11),

$$\sqrt{-g}\rho = \sqrt{-\tilde{g}}\tilde{\rho} \;\; ; \;\; \sqrt{-g}p = -\sqrt{-\tilde{g}}\tilde{p} \tag{2.13}$$

in agreement with (2.5) and (2.9).

A surprise

A "surprise" comes when one writes [17] the usual Einstein-Friedmann equations in the presence of these classical stringy sources: SFD gets badly broken! Indeed, the left- and right-hand sides of the equations transform differently under $a \to \tilde{a} = a^{-1}$. For instance, one combination of Einstein's equations reads:

$$\dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi G_D}{(D-2)} (\rho + p) . \tag{2.14}$$

string modification of Einstein's Eqs

The l.h.s. of (2.14) is clearly odd under $a \to \tilde{a} = a^{-1}$, while the r.h.s., owing to eq. (2.13), has no definite symmetry. As we shall discuss in the next section, the solution of this puzzle lies in the string's modification of Einstein's equations. Since these equations reflect the absence of conformal anomalies in the quantum theory, we can also say that it is the consistent quantization of strings that restores SFD.

3 SCALE FACTOR DUALITY FOR THE EFFECTIVE ACTION

The way duality is implemented at the quantum level is a little subtle even for usual time-independent R-duality. It involves a non-trivial transformation [18] of both the metric $G_{\mu\nu}$ and of the Fradkin-Tseytlin dilaton [19] ϕ which appear in the Euclidean 2D action as:

$$\mathrm{S} = 1/2 \int \mathrm{d}^2\mathrm{z}\sqrt{-\gamma}(\gamma^{\alpha\beta}\partial_\alpha \mathrm{X}^\mu \partial_\beta \mathrm{X}^\nu \mathrm{G}_{\mu\nu} + (\mathrm{R}/4\pi)\phi) \tag{3.1}$$

With this definition of the background fields (and up to a numerical factor), the tree-level string effective action takes, the form [20]:

$$\Gamma_{\mathrm{eff}} = \int \mathrm{d}^{\mathrm{D}}\mathrm{x}\sqrt{-\mathrm{G}}\mathrm{e}^{-\phi}[\mathrm{R} + \partial_\mu\phi\partial^\mu\phi - \mathrm{V} + \text{higher deriv.}]. \tag{3.2}$$

G-->G^-1

phi-->
phi - trlnG

where $V = \frac{(D-10)}{3}$ and R is the scalar curvature constructed out of $G_{\mu\nu}$.

It is clear from (3.2) that even the cosmological term (proportional to $(D-10)$) is not invariant under $G \to G^{-1}$ unless, at the same time, ϕ is also changed:

$$G \to G^{-1};\ \phi \to \phi - trlnG. \tag{3.3}$$

A possible way to understand why ϕ has to transform non-trivially under duality, in spite of the fact it does not couple to $\dot{X}$ or X' in (3.1), is to recall that the fields appearing in Γ_{eff} are "renormalized" fields while those appearing in S are "unrenormalized". The path-integral "proof" of duality

[3] implies that the bare dilaton is left invariant. However, the relation between bare and renormalized ϕ involves, at one loop order, precisely a term proportional to $trlnG$ (see e.g. [21]) This is a generalization of an observation by de Alvis [22] that, at the linearized level (around a flat G), the dilaton mixes with the trace of G.

phi/G mixing

Thus, if the bare dilaton is invariant under duality, the renormalized dilaton will transform as in (3.3). We have actually checked that the coefficient in front of the $trlnG$ term is the correct one if dimensional regularization is used. More generally, since we expect the renormalized and bare dilaton to be related by a local transformation, some generalization of (3.3) should also work at higher orders with just a more complicated, but local, transformation law for ϕ, as recently found to be the case at the next order [23].

Let us now proceed to the case of time-dependent scale factors. Take for instance

$$\lambda_s^2 G_{\mu\nu} = diag(-1, {a_i}^2(t)) \; ; \; B_{\mu\nu} = 0 \; ; \; \phi = \phi(t) \tag{3.4}$$

2.1

.

We may ask if the symmetry under (3.3) survives in this case. Surprisingly perhaps, the answer is in the affirmative at least up to second order in the space-time derivatives (slowly varying fields). Furthermore, for backgrounds of the type (3.4), the symmetry group appears to be extendible from a single Z_2 to ${Z_2}^d$ defined for each $i = 1, 2, ..d$ by:

$$\mathrm{a_i(t)} \rightarrow \mathrm{a_i^{-1}(t)} \; ; \; \phi \rightarrow \phi - 2\mathrm{lna_i} \tag{3.5}$$

This symmetry can be further extended, as shown in [12].

Omega = -1

If one looks at the effective, low energy string action as being a generic Brans-Dicke-type [24] modification of Einstein's gravity, one concludes that the ω parameter of Brans-Dicke is fixed in string theory, by duality, to take the value -1. Such a value is unacceptably small, phenomenologically, unless the dilaton picks up a mass [25], presumably through the same mechanism that breaks supersymmetry.

We may ask at this point whether SFD is really distinguished from R-duality even in the case of compact dimensions. In order to see that this is so, consider the case of a single circle of fixed radius R and a time-dependent scale factor $a(t)$. A naive extension [26] of R-duality would connect this situation to the one in which

$$aR/\lambda_s \rightarrow (aR/\lambda_s)^{-1} \tag{3.6}$$

"We are connecting through SFD a phys expanding to a phys contracting universe"

However, these R-duality-related situations both describe a contracting or expanding circle according to whether aR/λ_s approaches the fixed point value 1 or moves away from it. Instead, by fixing $a(0) = \tilde{a}(0) = 1$ and by letting the two evolve according to (3.5), we are connecting, through SFD, a physically expanding ($aR/\lambda_s \to \infty$) to a physically contracting ($aR/\lambda_s \to 1$) Universe.

It follows from the previous discussion that the correct interpretation of SFD is not that of a true symmetry, but rather of a group acting on the vacuum manifold and transforming solutions of the field equations into other (generally inequivalent) solutions. SFD thus appears to be a generalization of Narain's construction [27] to the case of possibly non-compact, time-dependent backgrounds. This analogy will be made much more complete in ref. [12].

The fact that SFD is a symmetry of the action can be verified immediately at the level of the equations of motion that follow from (3.2). In the general case these read:

$$\begin{aligned} \mathrm{R} - \partial_\mu\phi\partial^\mu\phi + 2\mathrm{D}_\mu\mathrm{D}^\mu\phi - \mathrm{V} = 0 \\ \mathrm{R}_{\mu\nu} + \mathrm{D}_\mu\mathrm{D}_\nu\phi = 0. \end{aligned} \tag{3.7}$$

where D_μ denotes the usual covariant derivative of General Relativity.

For our ansatz (3.4) eqs. (3.7) can be reduced to the following set of independent equations:

$$\begin{aligned} \sum \mathrm{H}_\mathrm{i}^2 - (\dot{\phi} - \sum \mathrm{H}_\mathrm{j})^2 + \mathrm{T}^{-2} = 0 \\ \dot{H}_i - H_i(\dot{\phi} - \sum H_j) = 0 \\ H_i \equiv \dot{a}_i/a_i \ , \ T \equiv \lambda_s V^{-1/2} = \lambda_s \Big(\frac{D-10}{3}\Big)^{-1/2} \end{aligned} \tag{3.8}$$

which are clearly invariant (respectively even and odd) under SFD.

We note, incidentally, that eqs.(3.8) admit, for $D > 10$, the particular solution [28]:

$$\phi = \mathrm{Qt} \ ; \ \mathrm{Q}^2 = \mathrm{T}^{-2} = (\mathrm{D}-10)/3\lambda_\mathrm{s}^{-2} \tag{3.9}$$

D>10???

which is well known to provide an exactly conformal invariant theory to all orders in λ_s. However, as discussed in [15] (see also [29]), these solutions do not seem to describe a physical expansion of the Universe. In this paper

??????????????????????????????????

Gmunu lambda s ^2

we shall stick to the claim [15] that the role of Einstein's metric is played in string theory by $G_{\mu\nu}\lambda_s^2$.

Before giving solutions to the above equations, let us combine the results obtained so far by coupling the classical string sources described in sect.2 to the string-modified Einstein equations of this section. This straightforward excercise yields the following modification of eqs.(3.8)

$$\begin{aligned}
\sum \mathrm{H_i^2} - (\dot{\phi} - \sum \mathrm{H_j})^2 + \mathrm{T}^{-2} &= -\kappa \mathrm{e}^{\phi}\rho \\
\dot{H}_i - H_i(\dot{\phi} - \sum H_j) &= 1/2\kappa e^{\phi} p_i \\
\kappa e^{\phi} = \alpha' e^{\phi} \lambda_s^{D-4} &\sim 8\pi G_N \\
\dot{\rho} + \sum H_i(\rho + p_i) &= 0. \qquad (3.10)
\end{aligned}$$

where we notice that, in this case, one of the field equations, instead of being redundant, yields the energy conservation equation.

It is now clear, after use of (2.13), that, unlike for the case of Einstein's equations, the left- and right-hand sides of (3.10) transform in the same way under SFD.

4 SOLUTIONS AND PHYSICAL CONSIDERATIONS

In this paper I shall consider in some detail the case in which the effect of classical string matter can be neglected, commenting briefly on more general situations. A detailed study of the general case is being made and will appear elsewhere [13]. The general solution of eqs. (3.7) can be written explicitly and reads ($D > 10$)

$$\begin{aligned}
\mathrm{a_i(t)} &= \mathrm{a_i}(-\infty)(\tanh(\pm\tau/2))^{\alpha_i} \\
\bar{\phi}(\mathrm{t}) \equiv \phi(\mathrm{t}) - \sum \ln \mathrm{a_i(t)} &= -\ln\sinh(\pm\tau) + \bar{\phi}_0 \\
\tau = t/T \ , \ \sum \alpha_i^2 &= 1 \qquad (4.1)
\end{aligned}$$

Two remarks are in order:

i) in general the solutions are defined on a half-line in t. Equivalently, there is in general a singularity at some finite value of t here taken conventionally to be $t = 0$. If, for the moment, we consider solutions defined for $t < 0$ the minus sign has to be chosen in eq. (4.1);

ii) the known, all-order solution (3.8) can be recovered formally by taking the limit $t \to -\infty$.

1

-oo< t < 0

The solution (4.1) is general since it depends on $2D - 1$ integration constants (this includes the time at which the singularity occurs) and, indeed, the solution is completely determined once the initial values of a_i, H_i and ϕ are given. It describes a cosmological evolution from $t = -\infty$ to $t = 0$ whereby the Universe is initially very flat (and isotropic):

$$\mathrm{H_i} = -\alpha_\mathrm{i}/\mathrm{T}\sinh^{-1}(-\tau) \to 0 \tag{4.2}$$

and the D-dimensional coupling

$$\lambda_\mathrm{D} = \exp(\phi) \to \mathrm{const.}\exp(\tau) \to 0 \tag{4.3}$$

is very weak. Hence, the initial state is perturbative from the point of view of both the σ-model and the string-loop expansion.

2

tau = O(-1)

As long as $\tau \ll -1$ ($t \ll -T$), things do not change much. However, from $\tau = O(-1)$ onward, scale factors begin to vary faster and faster. Depending upon the signs of the α_i's, some dimensions undergo (super)inflation or very fast contraction, i.e. precisely the kind of behaviours found in [17] from solving Einstein's equations in the presence of highly unstable strings in a self-consistent way. This certainly gives a hint that the addition of classical string sources will not modify qualitatively the solutions of the pure gravity-plus-dilaton system. Notice here that it is SFD that allows, for any solution with an expanding dimension, one with a reciprocally contracting dimension: anisotropic cosmologies are natural alternatives to isotropic ones in SFD-invariant theories.

3

t--> 0-

As $t \to 0^-$, H_i blows up like $\frac{\alpha_i}{t}$ while the behaviour of λ_D depends on the actual value of $\sum \alpha_i$ via

$$\lambda_D \sim (a)^{\sum \alpha_i - 1}(1 - a^2)\ ,\ a \equiv \tanh(-\tau/2) \to 0 \tag{4.4}$$

Thus λ_D goes to zero, to a finite constant or to infinity for $\sum \alpha_i > 1$, $\sum \alpha_i = 1$ and $\sum \alpha_i < 1$, respectively. However, the relevant, effective coupling

in the expanding dimensions after they have become much larger than all the contracting ones is related to λ_D by

$$\lambda_{eff}^{-1} = \lambda_D^{-1} \prod_{contr.\ dim's} a_i \tag{4.5}$$

and is thus easily seen to be always growing as one approaches the singularity. In conclusion, our solutions represent an evolution from flat D-dimensional space-time and weak coupling to a regime of high curvatures and large coupling through a period of super-inflation and dimensional reduction. The duration of inflation and the corresponding "e-folding" factor

$$\mathrm{N} \equiv \ln(\frac{\mathrm{a_{final}}}{\mathrm{a_{initial}}}) \tag{4.6}$$

are determined by requiring that, at the end of inflation, our approximations are still valid, implying, at least:

$$-\mathrm{t_{final}} \geq \lambda_\mathrm{s}\ ;\ |\mathrm{H_{final}}| < \mathrm{O}(\lambda_\mathrm{s}^{-1})\ ;\ \lambda_\mathrm{D} < \mathrm{O}(1). \tag{4.7}$$

Consequently one finds:

$$\mathrm{N} \leq \mathrm{O}(\lambda_\mathrm{s}^{-1}\mathrm{T}) = \mathrm{O}((\mathrm{D}-10)^{-1/2}) \tag{4.8}$$

We thus see that there is no hope of getting a large amount of inflation in the case of a tree-level cosmological constant. Paradoxically perhaps, the amount of inflation increases for decreasing Λ_0, the reason being that the smaller Λ_0, the longer inflation will last.

Case D = 10

The above considerations suggest considering the case of critical dimensions, $D = 10$. In this case, because of non-renormalization theorems, no potential is generated at any finite order in the string loop expansion. Neglecting non-perturbative effects, eqs.(4.1) still give the general solution by letting $\tau \to 0$. One finds:

$$\mathrm{a_i(t)} = \mathrm{a_i}(-\mathrm{t_0})(-\mathrm{t}/\mathrm{t_0})^{\alpha_\mathrm{i}}$$

$$\bar{\phi}(\mathrm{t}) = -\ln(\mathrm{t}/\mathrm{t_0})$$

$$\sum \alpha_i^2 = 1 \quad (D = 10) \tag{4.9}$$

The amount of inflation is only limited now by the initial values of the Hubble constants H_i and of the dilaton. A simple calculation yields:

$$N < min(-\phi_{initial}, -1/2ln(\lambda_s^2 \sum H_i^2)) \tag{4.10}$$

If we imagine to start, at some initial time, with a slight perturbation of the $D = 10$ flat, weakly coupled, supersymmetric vacuum, we can easily achieve large enough values of N for solving the standard cosmological problems.

On the other hand, non-perturbative, supersymmetry breaking effects are expected to produce a non-vanishing V of order [8]

$$\mathrm{V} \sim \exp(-\mathrm{c}\,\exp(-\phi)) \ll 1\ ,\ (\mathrm{c} > 0\ ,\ \phi \ll -1) \tag{4.11}$$

Furthermore, the potential will depend, in general, on the a_i through the combination $r_i(t) \equiv a_i R/\lambda_s$ if the ith. direction is a circle of radius R with a dependence of V on the r_i, itself restricted by modular invariance [7, 8].

In this case eqs. (3.10) can be shown to become:

$$\sum \mathrm{H_i^2} - (\dot{\phi} - \sum \mathrm{H_j})^2 + \mathrm{V}\lambda_\mathrm{s}^{-2} = -\kappa \mathrm{e}^{\phi}\rho$$

$$\dot{H}_i - H_i(\dot{\phi} - \sum H_j) + 1/2(\partial V/\partial\phi) + 1/2(\partial V/\partial ln a_i) = 1/2\kappa e^{\phi} p_i$$

$$\dot{\rho} + \sum H_i(\rho + p_i) = 0. \tag{4.12}$$

The above equations appear to break SFD. Only ordinary R duality is strictly preserved in the compactified dimensions thanks to the symmetry properties of V. Eqs. (4.12) can no longer be solved in closed form for a generic V and their analysis will be investigated elsewhere [13]. Nonetheless, we can already anticipate that the possibility of a long inflation and of a large e-folding factor appears to be preserved even in this case.

G S V

5 CONCLUSIONS AND SOME "PHILOSOPHY"

Obviously, any realistic situation will have to be much more complicated than the one described by the system of eqs. (3.10). Nonetheless, we may

hope some general features of the real world to be shared by the solutions of the simpler system.

Consider, for instance, eqs. (4.12) with some generic V satisfying (4.11) and depending on the "radii" r_i in a modular-invariant way. Let us start evolving from a "very classical" situation, i.e.

$$H_i \ll \lambda_s^{-1} \ , \ \phi \ll -1 \tag{5.1}$$

Under these conditions, eqs. (4.12) imply, at early times,

$$(\dot{\phi} - \sum \mathrm{H_j}) \sim \pm\sqrt{\sum \mathrm{H_i^2}} \ , \ \frac{\mathrm{d}}{\mathrm{dt}}(\sum \mathrm{H_i^2})^{-1/2} \sim \mp 1 \tag{5.2}$$

and thus a two-fold ambiguity.

We thus see that the choice of sign depends, very generally, on whether one wants to describe a late- or an early-time solution. Since, evidently, we do not want to describe today's Universe in terms of a Kaluza-Klein cosmology with a time-dependent gravitational and gauge coupling, we are forced, by physics, to choose the early-time solution (upper signs in eq. (5.2)). We thus obtain an interesting cosmological scenario whereby a classical, weak coupling, small curvature regime evolves naturally into a quantum era, with large curvatures and/or coupling. This is naturally identified with the "big-bang", i.e. with the beginning of our epoch. Of course the (semi)classical description exhibits a singularity and, as such, cannot be continued across $t = 0$. Hopefully, this just reflects the inadequacy of the semiclassical picture, while quantum string theory has a way to go through the singularity on to $t > 0$.

What will happen during the fully quantum era is clearly mere speculation since we do not know, at present, how to tackle such a complicated non-perturbative regime. As already pointed out, SFD is expected to be broken as soon as higher loops and/or compactification effects will be felt. Hopefully, this will select, out of all SFD-related solutions, those evolutions where six dimensions contract while the other three expand. During the quantum era, at least two nice "miracles" should take place: the freezing of the internal dimensions to scales $O(\lambda_s)$ and the freezing of the dilaton at a value $O(1)$ with generation of a dilaton mass. Only under these circumstances the way could be paved for starting a more conventional kind of cosmology at $t > O(\lambda_s)$.

1

2

After completion of this work I became aware of ref. [30] where the solution (4.1) to the β -function equations already appears.

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References

[1] K. Kikkawa and M. Yamasaki, Phys. Lett. B149 (1984)357.
N. Sakai and I. Senda, Progr. Theor. Phys. 75 (1986)692.

[2] V. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. B287 (1987)402;
A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 669.

[3] A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B322 (1989) 167.

[4] M. Dine, P. Huet and N. Seiberg, Nucl. Phys. B322 (1989) 301;
J. Lauer, J. Mas and H. P. Nilles, Phys. Lett. B226 (1989) 251; Nucl. Phys. B351 (1991) 353;
W. Lerche, D. Lüst and N. Warner, Phys. Lett. B231 (1989) 417;
B. R. Greene and M. R. Plesser, Nucl. Phys. B338 (1990) 15;
A. Giveon, N. Malkin and E. Rabinovici, Phys. Lett. B238 (1990) 57.

[5] E. Alvarez and M. Osorio, Phys. Rev. D40 (1989)1150.

[6] G. Veneziano, Europhys. Lett. 2 (1986)133;
T. R. Taylor and G. Veneziano, Phys. Lett. B212 (1988)147;
R. Brandenberger and C. Vafa, Nucl. Phys.B316 (1989) 391.

[7] S. Ferrara, D. Lüst, A. Shapere and S. Theisen, Phys. Lett. B225 (1989)363.
S. Ferrara, D. Lüst, and S. Theisen, Phys. Lett. B233 (1989)147.

[8] A. Font, L. E. Ibañez, D. Lüst and F. Quevedo, Phys. Lett. B245 (1990) 401;
S. Ferrara, N. Magnoli, T. R. Taylor and G. Veneziano, Phys. Lett. B245 (1990) 409;
H. P. Nilles and M. Olechowski, Phys. Lett. B248 (1990) 268;
P. Binetruy and M. K. Gaillard, Phys. Lett. B253 (1991) 119.

[9] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B216 (1989)41;
D. J. Gross and P. F. Mende, Nucl. Phys. B303 (1988) 407.

[10] G. Veneziano, Quantum String Gravity Near the Planck Scale, Proc. 1st Conf. on Particles, Strings and Cosmology, Northeastern University , March 1990 (P. Nath and S. Reucroft eds., World Scientific) p.486, and references therein.

[11] M. Cvetic, J. Molera and B. A. Ovrut, Phys. Lett.B248 (1990) 83;
J. Molera and B. A. Ovrut, Phys. Rev. D40 (1989) 1146; ibid D42 (1990) 2683;
G. Horowitz and A. R. Steif, Phys. Lett. B250 (1990) 49.

[12] K. A. Meissner and G. Veneziano, to appear.

[13] M. Gasperini, N. Sanchez and G. Veneziano, to appear.

[14] N. Turok and P. Bhattacharjee, Phys. Rev. D29 (1984) 1557.

[15] H. J. De Vega and N. Sanchez, Phys. Lett. B197 (1987) 322.
N. Sanchez and G. Veneziano, Nucl. Phys. B333 (1990) 253.

[16] M. Gasperini, N. Sanchez and G. Veneziano, Highly Unstable Fundamental Strings in Inflationary Cosmologies, CERN preprint TH. 5893 (1990).
M. Gasperini, Kinematic Interpretation of String Instability in a Background Gravitational Field, Torino Univ. preprint DFTT-38/90 (1990).
Nguyen, S. H. and G. Veneziano, Inflation-Driven String Instabilities:

Towards a Systematic Large R Expansion, CERN preprint TH. 6009 (1991).

[17] M. Gasperini, N. Sanchez and G. Veneziano, Self Sustained Inflation and Dimensional Reduction from Fundamental Strings, CERN preprint TH. 6010 (1991).

[18] P. Ginsparg and C. Vafa, Nucl. Phys. B289 (1987) 414;
A. Giveon, N. Malkin and E. Rabinovici, Phys. Lett. B220 (1989) 551.

[19] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. 158B (1985) 316.;
E. Witten, Phys. Lett. 149B (1984) 351.

[20] C. Lovelace, Phys. Lett. 135B (1984) 75
E. S. Fradkin and A. A. Tseytlin, ref. [19];
C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, Nucl. Phys. B262 (1985) 593.

[21] A. A. Tseytlin, Int. Journ. Mod. Phys. 4A (1989) 1257, and references therein.

[22] S. P. de Alwis, Phys. Lett. 168B (1986) 59.

[23] A. A. Tseytlin, Duality and the Dilaton, John Hopkins preprint JHU-TIPAC-91008 (1991).

[24] C. Brans and R. H. Dicke, Phys. Rev. 124 (1961) 925;
P. Jordan, Z. Phys. 157 (1959) 112.

[25] T. R. Taylor and G. Veneziano, Phys. Lett. B213 (1988) 450.

[26] R. Brandenberger and C. Vafa, ref.[6];
E. Smith and J. Polchinski, Univ. of Texas preprint UTTG-07-91 (1991).

[27] K. S. Narain, Phys. Lett. B169 (1986) 41;
K. S. Narain, M. H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.

[28] see e.g. I. Antoniadis, C. Bachas, J. Ellis and D. V. Nanopoulos, Nucl. Phys. B238 (1988) 117.

[29] B. Campbell, A. Linde and K. A. Olive, Nucl. Phys. B355 (1991) 146.

[30] M. Mueller, Nucl. Phys. B337 (1990) 37.

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STRING COSMOLOGY: CONCEPTS AND CONSEQUENCES

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ABSTRACT

After recalling a few basic concepts from cosmology and string theory, I will discuss the main ideas/assumptions underlying string cosmology and show how these lead to a two-parameter family of "minimal" models. I will then explain how to compute, in terms of those parameters, the spectrum of scalar, tensor and electromagnetic perturbations, point at their (T and S-type) duality symmetries, and mention their most relevant physical consequences.

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1 Basic Facts about Cosmology and Inflation

It is well known[1] that the Standard Cosmological Model (SCM) works well at "late" times, its most striking successes being perhaps the red shift, the cosmic microwave background (CMB), and primordial nucleosynthesis.

However, the SCM suffers from various problems. At the theoretical level the most serious of these is the initial singularity problem, which basically tells us that we cannot have theoretical control over the initial conditions of the SCM. At a phenomenological level, the SCM cannot explain naturally:

i) the homogeneity and isotropy of our Universe as manifested, in particular, through the small value of $\Delta T/T = O(10^{-5})$ observed with COBE[2];

ii) the flatness problem, i.e. the fact that, within an order of magnitude, $\Omega \equiv \rho \,/\, \rho_{crit} \sim 1$;

iii) the origin of large-scale structure.

Inflation, i.e. a long phase of accelerated expansion of the Universe ($\dot{a}, \ddot{a} > 0$, where a is the scale factor), is the only way known at present of solving the above-mentioned phenomenological problems. Various types of inflationary models have been proposed [for a review, see 3), 4)] each one supposedly mending the problems of the previous version. Particularly severe are the constraints coming from demanding:

a) a graceful exit with the right amount of reheating;

b) the right amount of large-scale inhomogeneities.

In order to satisfy such constraints, fine-tuned initial conditions and/or inflaton potentials are necessary. And this without mentioning the fact that inflation is not addressing at all the initial singularity problem.

Actually, Kolb and Turner, after reviewing the prescriptions for a successful inflation, add[4]:

"Perhaps the most important – and most difficult – task in building a successful inflationary model is to ensure that the inflaton is an integral part of a sensible model of particle physics. The inflaton should spring forth from some grander theory and not vice versa".

I will argue below that superstring theory could be the sought-after grander theory (what could be better than a theory of everything?) naturally providing an inflation-driving scalar field in the general sense defined again in ref. 4):

"It is now apparent that inflation, which was originally so closely related to Spontaneous Symmetry Breaking, is a much more general phenomenon.... Stated in its full generality, inflation involves the dynamical evolution of a very weakly-coupled scalar field that was originally displaced from the minimum of its potential."

I hope to convince you that this will be precisely the picture that we claim takes place in string cosmology. In order to substantiate this claim, I will have to digress and recall a few basic facts in Quantum String Theory.

2 Basic facts in quantum string theory (QST)

I am listing below a few basic properties of strings, emphasizing those that are most relevant for our subsequent discussion. These are:

1. Unlike its classical counterpart, quantum string theory contains a fundamental length scale λ_s representing[5] the ultraviolet, short-distance cut-off (equivalently, a high-momentum cut-off at $E = M_s c^2 \equiv \hbar c/\lambda_s$).

2. Tree-level masses are either zero or $O(M_s)$. Quantum mechanics allows massless strings with non-zero angular momentum[6] while, classically, $M^2 > \text{const.} \times J$. The existence of such states is obviously a crucial property of QST, without which it could not pretend to be a candidate theory of all known interactions.

3. The effective interaction of the massless fields at $E \ll M_s$ takes the form of a classical, gauge-plus-gravity field theory with specified parameters. It is described by an effective action[7),8)] of the (schematic) type:

$$\begin{aligned}\Gamma_{eff} &= \frac{1}{2}\int d^4x\sqrt{-g}\; e^{-\phi}\left[\lambda_s^{-2}(\mathcal{R} + \partial_\mu\phi\partial^\mu\phi) + F_{\mu\nu}^2 + \bar{\psi}\not{D}\psi + R^2 + \ldots\right] \\ &\quad + \Big[\text{higher orders in } e^{\phi}\Big]\ . \end{aligned} \tag{2.1}$$

Equation (2.1) contains two dimensionless expansion parameters. One of them, $g^2 \equiv e^{\phi}$, controls the analogue of QFT's loop corrections, while the other, $\lambda^2 \equiv \lambda_s^2 \cdot \partial^2$, controls string-size effects, which are of course absent in QFT.

4. As indicated in (2.1), QST has (actually needs!) a new particle/field, the so-called dilaton ϕ, a scalar massless particle (at the perturbative level). It appears in Γ_{eff} as a Jordan–Brans–Dicke[9] scalar with a "small" negative ω_{BD} parameter, $\omega_{BD} = -1$.

5. The dilaton's VEV provides[8),10)] a unified value for:

a) The gauge coupling(s) at $E = O(M_s)$.

b) The gravitational coupling in string units.

c) Yukawa couplings, etc., at the string scale.

In formulae:

$$\begin{aligned} \ell_p^2 &\equiv 8\pi G_N \hbar = e^{\phi}\lambda_s^2 \ , \\ \alpha_{GUT}(\lambda_s^{-1}) &\simeq \frac{e^{\phi}}{4\pi} \ , \end{aligned} \tag{2.2}$$

implying (from $\alpha_{GUT} \approx 1/20$) that the string-length parameter λ_s is about 10^{-32} cm. Note, however, that, in a cosmological context in which ϕ evolves in time, the above formulae can only be taken to give the *present* values of α and ℓ_p/λ_s. In the scenario we will advocate, both quantities were much smaller in the very early Universe!

6. Dilaton couplings at large distance are such[11] that a massless dilaton is most likely ruled out[11,12] by precision tests[13] of the equivalence principle [for a possible way out see, however, ref. 14)], i.e.

$$M_{\phi} > 10^{-4} \text{ eV} \ . \tag{2.3}$$

7. Details about the dilaton potential are unknown, yet:

a) On theoretical grounds, in critical superstring theory, the dilaton potential has to go to zero as a double exponential as $\phi \to -\infty$ (weak coupling):

$$V(\phi) \sim \exp\left(-c^2 \exp(-\phi)\right) = \exp\left(-\frac{c^2}{4\pi\alpha_{GUT}}\right) \ , \tag{2.4}$$

with c^2 a positive (but model-dependent) constant.

b) On physical grounds it should have a non-trivial minimum at its present value ($\langle\phi\rangle = \phi_0 \sim 0$) with a vanishing cosmological constant, $V(\phi_0) = 0$.

A typical potential satisfying a) and b) is shown in Fig. 1. The dotted lines at $\phi > 0$ represent our ignorance about strongly coupled string theory. Fortunately, the details of what happens in that region will not be very relevant for our subsequent discussion.

8. There is an exact (all-order) vacuum solution for (critical) superstring theory. Unfortunately, it corresponds to a free theory ($g = 0$ or $\phi = -\infty$) in flat, ten-dimensional, Minkowski space-time, nothing like the world we seem to be living in!

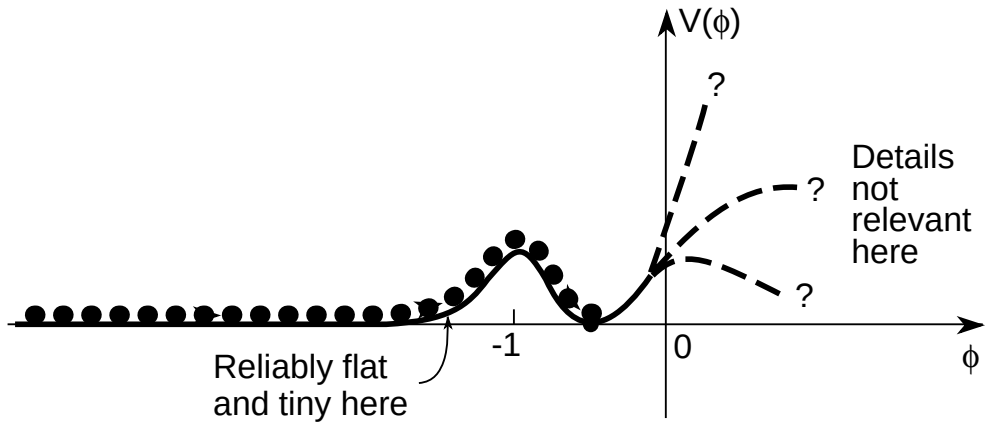

Figure 1: A possible dilaton potential with illustration of an inflation-driving rolling dilaton (large dots).

Before closing this section I would like to comment briefly on a point which appears to be the source of much confusion even among experts: it is the debate between working in the (so-called) String and Einstein "frames" (not to be confused with different coordinate systems). Since the two frames are related by a local field redefinition (a conformal, dilaton-dependent rescaling of the metric to be precise) all physical quantities are independent of the frame one is using. The question is: what should we call the metric? Although, to a large extent, this is a question of taste, one's intuition may work better with one definition than with another. Note also that, since the dilaton is time-independent today, the two frames coincide now.

Let us compare the virtues and problems with each frame.

A) STRING FRAME This is the metric appearing in the fundamental (Polyakov) action for the string. Classical, weakly coupled strings sweep geodesic surfaces with respect to this metric. Also, the dilaton dependence of the low energy effective action takes the simple form indicated in (2.1) only in the string frame. The advantage of this frame is that the string cut-off is fixed and the same is true for the value of the curvature at which higher orders in the σ-model coupling λ become relevant. The main disadvantage is that the gravitational action is not so easy to work with.

B) EINSTEIN FRAME In this frame the pure gravitational action takes the standard Einstein-Hilbert form. Consequently, this is the most convenient frame for studyind the cosmological evolution of metric perturbations. The Planck length is fixed in this frame while the string length is dilaton (hence generally

time) dependent. In the Einstein frame Γ_{eff} takes the form:

$$\Gamma_{eff} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\ \left[R+\partial_\mu\phi\partial^\mu\phi+e^{-\phi}F^2_{\mu\nu}+\partial_\mu A\partial^\mu A+e^{\phi}m^2A^2\right] + \left[G_N e^{-\phi}R^2+\ldots\right], \quad (2.5)$$

showing that, in this frame, masses are dilaton dependent (even at tree level) and so is the value of R at which higher order stringy corrections become important. It is for the above reasons that I will choose to base my discussion (although not always the calculations) in the String frame.

3 Main ideas/assumptions of string cosmology

The very basic postulate of (our own version of) String Cosmology[15),16)] is that the Universe did indeed start near its trivial vacuum mentioned at the end of the previous section.

Fortunately, if one looks at the space of homogeneous (and for simplicity spacially-flat) perturbative vacuum solutions, one finds that the trivial vacuum is a very special, *unstable* solution. This is depicted in Fig. 2a for the simplest case of a ten-dimensional cosmology in which three spatial dimensions evolve isotropically while six "internal" dimensions are static (it is easy to generalize the discussion to the case of dynamical internal dimensions, but then the picture becomes multidimensional).

The straight lines in the $H,\dot{\bar{\phi}}$ plane (where $\dot{\bar{\phi}}\equiv\dot{\phi}-3H$) represent the evolution of the scale factor and of the coupling constant as a function of the cosmic time parameter (arrows along the lines show the direction of the time evolution). As a consequence of a stringy symmetry, known[15),17)] as "Scale Factor Duality (SFD)", there are two branches (two straight lines). Furthermore, each branch is split by the origin in two time-reversal-related parts (time reversal changes the sign of both H and $\dot{\bar{\phi}}$).

The origin (the trivial vacuum) is an "unstable" fixed point: a small perturbation in the direction of positive $\dot{\bar{\phi}}$ makes the system evolve further and further from the origin, meaning larger and larger coupling and absolute value of the Hubble parameter. This means an accelerated expansion or an accelerated contraction, i.e. in the latter case, inflation. It is tempting to assume that those patches of the original Universe that had the right kind of fluctuation have grown up to become (by far) the largest fraction of the Universe today.

In order to arrive at a physically interesting scenario, however, we have to connect somehow the top-right inflationary branch to the bottom-right branch, since the

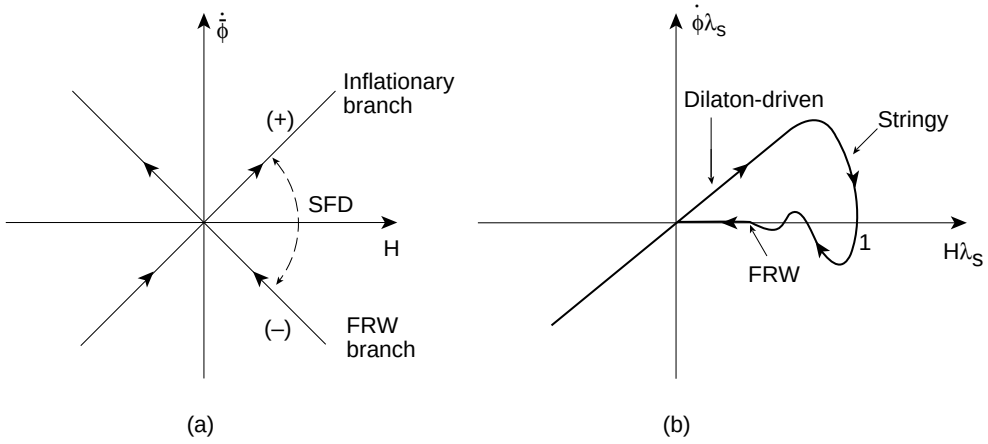

Figure 2: Phase diagrams for the perturbative (a) regime and a conjectured non-perturbative solution (b) to the branch-change problem.

latter is nothing but the standard FRW cosmology, which has presumably prevailed for the last few billion years or so. Here the so-called "exit problem" arises. At lowest order in λ^2 (small curvatures in string units) the two branches do not talk to each other. The inflationary (also called +) branch has a singularity in the future (it takes a finite cosmic time to reach ∞ in our gragh if one starts from anywhere but the origin) while the FRW ($-$) branch has a singularity in the past (the usual big-bang singularity).

It is widely believed that QST has a way to avoid the usual singularities of Classical General Relativity or at least a way to reinterpret them[18),19)]. It thus looks reasonable to assume that the inflationary branch, instead of leading to a non-sensical singularity, will evolve into the FRW branch at values of λ^2 of order unity. This is schematically shown in Fig. 2b, where we have gone back from $\dot{\bar{\phi}}$ to $\dot{\phi}$ and we have implicitly taken into account the effects of a non-vanishing dilaton potential at small ϕ in order to freeze the dilaton at its present value. The need for the branch change to occur at large λ^2, first argued for in[20)], has been recently proved in ref. 21).

There is a rather simple way to parametrize a class of scenarios of the kind defined above. They contain (roughly) three phases and two parameters. Indeed:

In phase I the Universe evolves at $g^2, \lambda^2 \ll 1$ and thus is close to the trivial vacuum. This phase can be studied using the tree-level low-energy effective action (2.1) and is characterized by a long period of dilaton-driven inflation. The accelerated expansion of the Universe, instead of originating from the potential energy of an inflaton field, is driven by the growth of the coupling constant (i.e. by the dilaton's kinetic energy, see ref. 22) for a similar kind of inflationary scenario) with

$\dot{\phi} = 2\dot{g}/g \sim H$ during the whole phase.

Phase I supposedly ends when the coupling λ^2 reaches values of $O(1)$, so that higher-derivative terms in the effective action become relevant. Assuming that this happens while g^2 is still small (and thus the potential is still negligible), the value g_s of g at the end of phase I (the beginning of phase II) is an arbitrary parameter (a modulus of the solution).

During phase II, the stringy version of the big bang, the curvature, as well as $\dot{\phi}$, are assumed to remain fixed at their maximal value given by the string scale (i.e. we expect $\lambda \sim 1$). The coupling g will instead continue to grow from the value g_s until it is its own turn to reach values $O(1)$. At that point, assuming a branch change to have occurred at large curvatures, the dilaton will be attracted to the true non-perturbative minimum of its potential; the standard FRW cosmology can then start, provided the Universe was heated-up and filled with radiation (this is not a problem, see below). The second important parameter of this scenario is the duration of phase II or better the total red-shift, $z_s \equiv a_{end}/a_{beg}$, which has occurred from the beginning to the end of the stringy phase.

Our present ignorance about this most crucial phase (and in particular about the way the exit can be implemented) prevents us from having a better description of this phase which, in principle, should not introduce new arbitrary parameters (z_s should be eventually determined in terms of g_s).

During Phase III, the Universe evolves towards smaller and smaller curvatures but stays at moderate-to-strong coupling. This is the regime in which usual QFT methods are applicable. The details of the particular gauge theory emerging from the string's non-perturbative vacuum will be very important in determining the subsequent evolution and in particular the problem of structure formation, dark matter and the like.

Our scenario contains implicitly an arrow of time, which points in the direction of increasing entropy, inhomogeneity and structure. As a result of the amplification of primordial vacuum fluctuations, the Universe is not coming back to its initial simple (and unique) state (the origin in Fig. 2), but to the much more structured (and interesting) state in which we are living today. Actually, the arrow of time itself should be determined by the direction in which entropy (and complexity) are growing. This will force us to identify (*by definition*) the perturbative vacuum with the initial state of the Universe!

4 Observable consequences

All the observable consequences I will discuss below have something to do with the well-known phenomenon[23] of amplification of vacuum quantum fluctuations in cosmological backgrounds. Any conformally flat cosmological background is known:

a) to amplify tensor perturbations, i.e. to produce a stochastic background of gravitational waves;

b) to induce scalar-metric perturbations from the coupling of the metric either to a fluid or to scalar particles (in our context to the dilaton).

By contrast, because of the scale-invariant coupling of gauge fields in four dimensions, electromagnetic (EM) perturbations are *not* amplified in a conformally flat cosmological background (even if inflationary). In string cosmology, the presence of a time-dependent dilaton in front of the gauge-field kinetic term yields, on top of the two previously mentioned effects,

c) an amplification of EM perturbations corresponding to the creation of macroscopic magnetic (and electric) fields.

Various physically interesting questions arise in connection with the three effects I have just mentioned. These include the following:

1. Does the Universe remain quasi-homogeneous during the whole string-cosmology history?
2. Does one generate a phenomenologically interesting (i.e. measurable) background of GW?
3. Can one produce large enough seeds for generating the observed galactic (and extragalactic) magnetic fields?
4. Can scalar, tensor (and possibly EM) perturbations explain the large-scale anisotropy of the CMB observed by COBE?
5. Do these perturbations have anything to do with the CMB itself?

In the rest of this talk I will first explain, on the toy example of the harmonic oscillator, the common mechanism by which quantum fluctuations are amplified in cosmological backgrounds. I will then give our present answers to the questions listed above. For more details, see Ref. [24].

Consider a one-dimensional (non-relativistic) harmonic oscillator moving in a cosmological background of the simplest kind, characterized by a scale factor $a(t)$. In units in which the mass of the oscillator is 1, the Lagrangian reads:

$$L = \frac{1}{2}a^2(\dot{x}^2 - \omega^2 x^2) \tag{4.1}$$

while the canonical momentum and Hamiltonian are given by

$$p = a^2\dot{x}\ , \qquad H = \frac{1}{2}(a^{-2}p^2 + a^2\omega^2 x^2)\ . \tag{4.2}$$

Let us first discuss the solutions of the classical equations of motion:

$$\ddot{x} + 2\ \dot{a}/a\ \dot{x} + \omega^2 x = 0\ , \qquad \ddot{y} + (\omega^2 - \ddot{a}/a)y = 0\ , \tag{4.3}$$

where $y \equiv ax$ is the proper (physical) amplitude as opposed to the comoving amplitude x.

Solutions to Eqs. (4.3) simplify in two opposite regimes:

a) For $\omega^2 \gg \ddot{a}/a$ there is "adiabatic damping" of the comoving amplitude (the name is clearly unappropriate in the case of contraction):

$$x \sim a^{-1}\ e^{\pm i\omega t}\ , \qquad p \sim a\ \omega\ e^{\pm i\omega t} \tag{4.4}$$

which means that, in this regime, the proper amplitude y and the proper momentum p/a stay constant (and so does the Hamiltonian).

b) For $\omega^2 \ll \ddot{a}/a$ one finds the so-called "freeze-out" regime in which:

$$\begin{aligned} x &\sim B\ +\ C\int_0^t dt' a^{-2}(t') \\ p &\sim C + \ldots \end{aligned} \tag{4.5}$$

where the comoving amplitude and momentum are fixed. In this regime the Hamiltonian (the energy) of the system tends to grow at late times whenever a increases or decreases by a large factor during the freeze-out regime. In the former case the energy is dominated asymptotically by the term proportional to x^2 and is due to the "stretching" of the oscillator caused by the fast expansion, while in the latter case the term proportional to p^2 dominates because of the large blue-shift suffered by the momentum in a contracting background.

Consider now a cosmology such that

$$\begin{aligned} \omega^2 &> \ddot{a}/a\,, \qquad t < t_{ex},\, t > t_{re} \\ \omega^2 &< \ddot{a}/a\,, \qquad t_{ex} < t < t_{re} \end{aligned} \tag{4.6}$$

where, anticipating our subsequent discussion, we have defined the moments of exit and re-entry by the condition $\omega \sim H$. Such an example will be typical of our scenario, since a given scale will be well inside the horizon at the beginning (small Hubble parameter), outside during the high-curvature regime, and then inside again after re-entry. By joining smoothly the two asymptotic solutions, we easily find that the energy of the harmonic oscillator (which is constant during the initial and final phases) has been amplified during the intermediate phase by a factor:

$$|c|^2 = Max\left(\frac{a^2(t_{re})}{a^2(t_{ex})} \quad , \quad \frac{a^2(t_{ex})}{a^2(t_{re})}\right) \tag{4.7}$$

corresponding to the two above-mentioned cases.

The excercise can be repeated at the quantum level starting, for instance, from a harmonic oscillator in its ground state. Quantum mechanics fixes the size of the initial amplitude, momentum and energy:

$$|x| \sim a^{-1}\sqrt{\frac{\hbar}{\omega}}, \quad E \sim \hbar\omega \ . \tag{4.8}$$

The quantum mechanical interpretation of eq. (4.7) is that c is the Bogoliubov coefficient transforming the initial ground state into the final excited quantum state ($|c|^2$ being the average occupation number for the latter). Note that the final state ends up being highly "squeezed", i.e. having a large Δx or Δp depending on the sign of H. If, because of coarse-graining, the squeezed coordinate is not measured, the final state will look like a high-entropy, statistical ensemble of quasi-classical oscillators.

Note, finally, the (Scale-Factor) duality invariance of the resulting amplification. Under $a \to a^{-1}$, position and momentum operators swap their role as the variable in which sqeezing or amplification occurs. Thus the final amplification remains the same.

Up to technical complications, things work out pretty much in the same way for strings[25] and for the three kinds of perturbations mentioned at the beginning of this section. In particular, for each one of the latter, one can define[26] a canonical variable ψ^i (similar to the harmonic oscillator's y) satisfying an equation of the type

$$\psi_k'' + [k^2 - V_i(\eta)]\psi_k = 0 \ , \tag{4.9}$$

where the label i on ψ has been suppressed, k is the comoving wave number, and derivatives with respect to conformal time η are denoted by a prime.

Since, for each i, the "potential" V_i is very small at very early times, grows to a maximum during the stringy era and, finally, drops rapidly to zero at the beginning

of the radiation era, a given scale (k) begins and ends inside the horizon with an intermediate phase outside. Larger scales exit earlier and re-enter later. Also, in our scenario, larger scales exit and re-enter at smaller values of H. Very short scales exit during the stringy era and, for those, our predictions will not be as solid as for the scales that leave the horizon during the perturbative dilatonic phase I. The fact that the amplification of perturbation depends just on some ratios of fields evaluated at exit and re-entry (and not on the details of the evolution in between) makes us believe that our detailed results are trustworthy for those larger scales. This being said, I present below some results on the five issues mentioned above (see, again, ref. 24) for derivations and/or details).

1. **Does the Universe remain quasi-homogeneous during the whole string-cosmology history?**

   The answer to this question turns out to be yes! This is not a priori evident since, in commonly used gauges[26] for scalar perturbations of the metric (e.g. the so-called longitudinal gauge in which the metric remains diagonal), such perturbations appear to grow very large during the inflationary phase and to destroy homogeneity or, at least, to prevent the use of linear perturbation theory. Similar problems had been encountered earlier in the context of Kaluza-Klein cosmology[27].

   In ref. 28) it was shown that, by a suitable choice of gauge (an "off-diagonal" gauge), the growing mode of the perturbation can be tamed. This can be double-checked by using the so-called gauge-invariant variables of Bruni and Ellis[29]. The bottom line is that scalar perturbations in string cosmology behave no worse than tensor perturbations, to which we now turn our attention.

2. **Does one generate a phenomenologically interesting (i.e. measurable) background of GW?**

   The canonical variable ψ for tensor perturbations (i.e. for GW) is defined by:

$$\begin{aligned} g_{\mu\nu} &= a^2[\eta_{\mu\nu} + h_{\mu\nu}] \\ \psi = (a/g)\ h &= a\ e^{-\phi/2} h\ , \end{aligned} \tag{4.10}$$

   where h stands for either of the two transverse-traceless polarizations of the gravitational wave. As long as the perturbation is inside the horizon, ψ remains constant while h is adiabatically damped. By contrast, outside the horizon, ψ is amplified according to

$$\psi_k \sim (a/g)[C_k + D_k \int_{\eta_{ex}}^{\eta} d\eta'\ g^2(\eta')\ a^{-2}(\eta')] \tag{4.11}$$

   where, for each Fourier mode of (comoving) wave number k, $\eta_{ex} = k^{-1}$.

The first term in (4.11) clearly corresponds to the freezing of h itself, while the second term represents the freezing of its associated canonical momentum. In standard (non-dilatonic) inflationary models, the first term dominates since a grows very fast. In our case, the second term dominates since the growth of a is over-compensated by the growth of g (i.e. of ϕ). This is equivalent to saying that, in the Einstein frame, our background describes a contracting Universe.

After matching the result (4.11) with the usual oscillatory, damped behaviour of the radiation-dominated epoch, one arrives at the final result[28),30)] for the magnitude of the stochastic background of GW today:

$$|\delta h_\omega| \equiv k^{3/2}|h_k| \sim \sqrt{\frac{H_0}{M_s}} z_{eq}^{-1/4} z_s \left(\frac{g_s}{g_1}\right) \left(\frac{\omega}{\omega_s}\right)^{1/2} \left[\ln\left(\frac{\omega_s}{\omega}\right) + (z_s)^{-3}\left(\frac{g_s}{g_1}\right)^{-2}\right], \qquad \omega < \omega_s \quad (4.12)$$

where $\omega = k/a$ is the proper frequency, $z_{eq} \sim 10^4, \omega_s \sim z_s^{-1}(g_1)^{1/2} \times 10^{11}$ Hz $\equiv z_s^{-1}\omega_1$.

The above result can be converted into a spectrum of energy density per logarithmic interval of frequency. In critical density units:

$$\frac{d\Omega_{GW}}{d\ln\omega} = z_{eq}^{-1}(g_s)^2 \left(\frac{\omega}{\omega_s}\right)^3 \left[\ln\left(\frac{\omega_s}{\omega}\right) + (z_s)^{-3}\left(\frac{g_s}{g_1}\right)^{-2}\right]^2, \ \omega < \omega_s \ . \quad (4.13)$$

The above spectrum looks quasi-thermal al large scales (i.e. at $\omega < \omega_s$), but is amplified by a large factor relative to a Planckian spectrum of temperature ω_s. In analogy with the harmonic oscillator case, there is a duality symmetry of the spectrum, this time under the transformation $(z_s, g_s) \to (z_s, z_s^{-3} g_s^{-1})$. The transformation corresponds to changing $\bar{\phi}$ into $-\bar{\phi}$ i.e. to what we may call $\bar{S}$-duality. As with the harmonic oscillator, the metric perturbation and its canonically conjugate momentum variable swap their role under such transformation.

In Fig. 3 we show the spectrum of stochastic gravitational waves expected from our two-parameter model. For a given pair g_s, z_s one identifies a point in the $\omega, \delta h_\omega$ plane as illustrated explicitely in the case of $g_s = 10^{-3}, z_s = 10^6$.The resulting point (indicated by a large dot) represents the end-point $\omega_s, \delta h_{\omega_s}$ of the $\omega^{1/2}$ spectrum corresponding to scales crossing the horizon during the dilatonic era.

Although the rest of the spectrum is more uncertain, one can argue that it has to join smoothly the point $\omega_s, \delta h_{\omega_s}$ to the true end-point $\delta h \sim 10^{-30}, \omega \sim 10^{11} Hz$. The latter corresponds to a few gravitons produced at the maximal

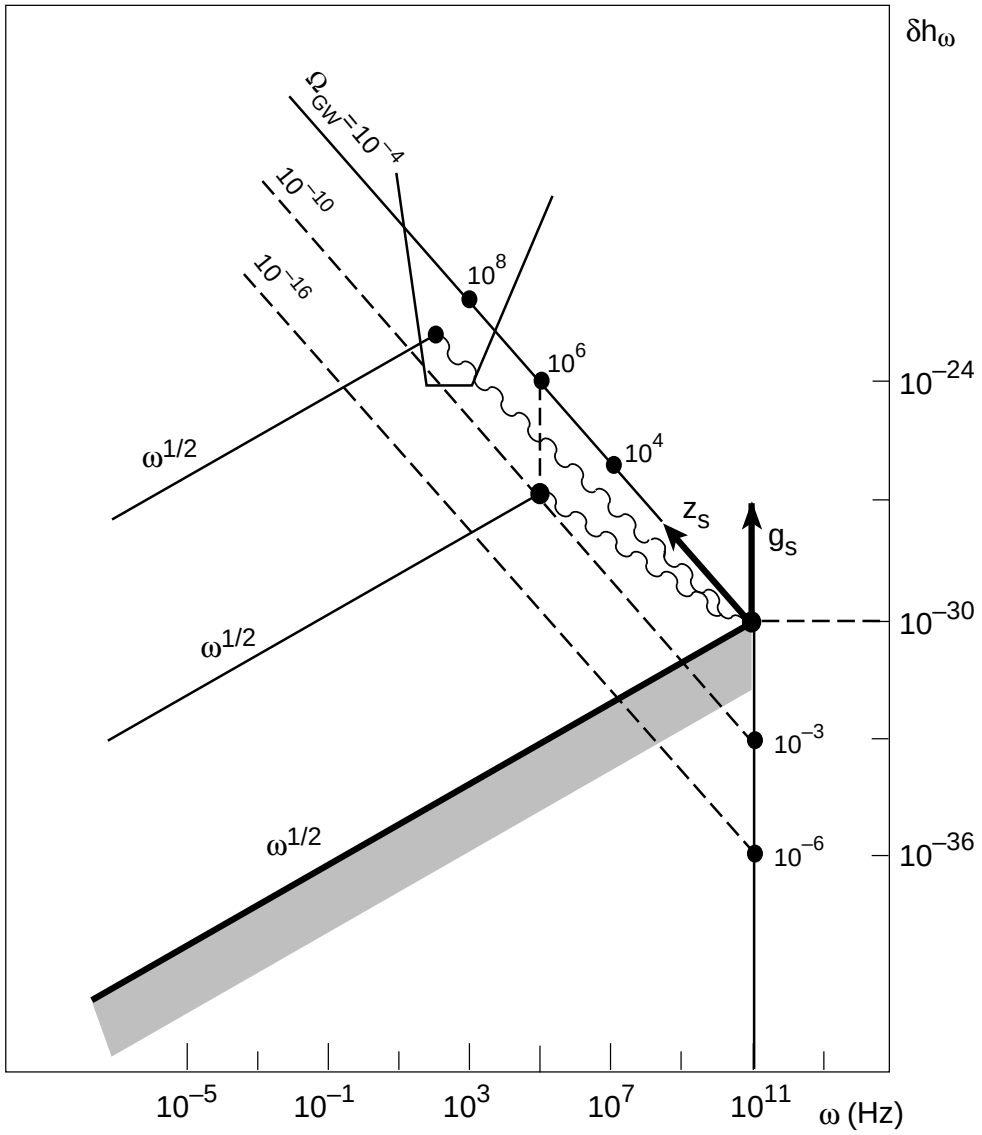

Figure 3: GW spectra from string cosmology against interferometric sensitivity.

amplified frequency ω_1, the last scale to go outside the horizon during the stringy phase. The full spectrum is also shown in the figure for the case $g_s = 10^{-3}, z_s = 10^6$, with the wiggly line representing the less well known high frequency part.

Curves of constant Ω_{GW} are also shown. If $g_s < 1$, as we have assumed, spectra will always lie below the $\Omega_{GW} = 10^{-4}$ line corresponding to as many photons as gravitons been produced. On the other hand, by invoking $\bar{S}$-duality, one can argue that the actual spectrum, by containing two duality-related contributions, will never lie below the self-dual spectrum ending at $\delta h \sim 10^{-30}, \omega \sim 10^{11} Hz$ (the thick line bordering the shaded region). In conclusion all possible spectra sweep the angular wedge inside the two above-mentioned lines.

The odd-shaped region in Fig. 3 shows the expected sensitivity of the so-called "Advanced LIGO" project[31]. While there is no hope to detect our spectrum at LIGO if $g_s = 10^{-3}, z_s = 10^6$, perspectives would be better for, say, $g_s = 10^{-1}, z_s = 10^8$ (the corresponding spectrum is also sketched).

Resonant bars might also be able to reach comparable sensitivity in the kHz region, while microwave cavities, if conveniently developed, could be used in the region 10^6-10^9 Hz [33]. Another interesting possibility consists of coincidence experiments between an interferometer and a bar. The quoted sensitivity[34] to a stochastic background, as a function of the frequency f, of the individual sensitivities of the bar and of the interferometer δh, and of the observation time T_{obs}, is:

$$\delta\Omega_{GW} = 1.5\ 10^{-5}(f/10^3\mathrm{Hz})^3(10^{21}\delta h_{int})(10^{21}\delta h_{bar})(T_{obs}/10^7\ \mathrm{s})^{-1/2}. \quad (4.14)$$

Obviously, detecting a stochastic backgound like ours is a formidable challenge. Also, the physical range of our parameters g_s, z_s could be such that no observable signal will be produced. What is interesting, however, is the mere existence of cosmological models predicting a non-negligible yield of GW in a range of frequencies where other sources predict just a "desert". More complete studies of the sensitivity of various detectors to a stochastic, coloured spectrum of our kind are presently under way.

3. **Can large enough seeds be produced for the generation of observed galactic (and extragalactic) magnetic fields?**

As already mentioned, seeds for generating the galactic magnetic fields through the so-called cosmic dynamo mechanism[35] can be generated in our scenario by the amplification of the quantum fluctuations of the EM field. In this case the canonical variable is just the (Fourier transform of the) usual A_μ potential. In

analogy with (4.11) its amplification, while outside the horizon, is described by the asymptotic solution:

$$A_k \sim g^{-1}\left[C_k + D_k \int_{\eta_{ex}}^{\eta} d\eta' g^2(\eta')\right] , \tag{4.15}$$

which leads[36),37)] to an overall amplification of the electromagnetic field by a factor $|c_k|^2 \sim (g_{re}/g_{ex})^2 + (g_{ex}/g_{re})^2$. This time the spectrum is invariant under $g \to g^{-1}$ i.e. under ordinary S or electric-magnetic duality. In our cosmological scenario we have excluded the possibility of a decreasing coupling constant and, therefore, the main contribution to the amplification comes from the second term on the r.h.s. of eq.(4.15) which gives $|c_k|^2 \sim (g_{re}/g_{ex})^2$.

One can express this result in terms of the fraction of electromagnetic energy stored in a unit of logarithmic interval of ω normalized to the one in the CMB, ρ_γ. One finds:

$$r(\omega) = \frac{\omega}{\rho_\gamma}\frac{d\rho_B}{d\omega} \simeq \frac{\omega^4}{\rho_\gamma}|c_-(\omega)|^2 \equiv \frac{\omega^4}{\rho_\gamma}(g_{re}/g_{ex})^2 . \tag{4.16}$$

The ratio $r(\omega)$ stays constant during the phase of matter-dominated as well as radiation-dominated evolution, in which the Universe behaves like a good electromagnetic conductor[38)]. In terms of $r(\omega)$ the condition for seeding the galactig magnetic field through ordinary mechanisms of plasma physics is[38)]

$$r(\omega_G) \geq 10^{-34} \tag{4.17}$$

where $\omega_G \simeq (1\ \mathrm{Mpc})^{-1} \simeq 10^{-14}$ Hz is the galactic scale. Using the known value of ρ_γ, we thus find, from (4.16, 4.17):

$$g_{ex} < 10^{-33} , \tag{4.18}$$

i.e. a very tiny coupling at the time of exit of the galactic scale.

The conclusion is that string cosmology stands a unique chance in explaining the origin of the galactic magnetic fields. Indeed, if the seeds of the magnetic fields are to be attibuted to the amplification of vacuum fluctuations, their present magnitude can be interpreted as prime evidence that the fine structure constant has evolved to its present value from a tiny one during inflation. The fact that the needed variation of the coupling constant ($\sim 10^{30}$) is of the same order as the variation of the scale factor needed to solve the standard cosmological problems, can be seen as further evidence for scenarios in which coupling and scale factor grow roughly at the same rate during inflation.

4. **Can scalar, tensor (and possibly EM) perturbations explain the large-scale anisotropy of the CMB observed by COBE?**

   The answer here is certainly negative as far as scalar and tensor perturbations are concerned. The reason is simple: for spectra that are normalized to $O(1)$ (at most) at the maximal amplified frequency $\omega_1 \sim 10^{11}$ Hz, and that grow like $\omega^{1/2}$, one cannot have any substantial power at the scales $O(10^{-18}\text{Hz})$ to which COBE is sensitive. The origin of $\Delta T/T$ at large scale would have to be attributed to other effects (e.g. topological defects).

   Fortunately, there is a possibility [39] that the EM perturbations themselves might explain the anisotropies of the CMB since their spectrum turns out to be flatter (and also more model-dependent) than that of metric perturbations. Assuming this to be the case, an interesting relation is obtained[39] between the magnitute of large scale anysotropies and the slope of the power spectrum. Such a relation turns out to be fully consistent, with present bounds on the spectral index.

5. **Do all these perturbations have anything to do with the CMB itself?**

   Stated differently, this is the question of how to arrive at the hot big bang of the SCM starting from our "cold" initial conditions. The reason why a hot universe can emerge at the end of our inflationary epochs (phases I and II) goes back to an idea of L. Parker[40], according to which amplified quantum flluctuations can give origin to the CMB itself if Planckian scales are reached.

   Rephrasing Parker's idea in our context amounts to solving the following bootstrap-like condition: at which moment, if any, will the energy stored in the perturbations reach the critical density?

   The total energy density ρ_{qf} stored in the amplified vacuum quantum fluctuations is given by:

$$\rho_{qf} \sim N_{eff}\, \frac{M_s^4}{4\pi^2}\, (a_1/a)^4 \ , \tag{4.19}$$

   where N_{eff} is the number of effective (relativistic) species, which get produced (whose energy density decreases like a^{-4}) and a_1 is the scale factor at the (supposed) moment of branch-change. The critical density (in the same units) is given by:

$$\rho_{cr} = e^{-\phi} M_s^2 H^2 \ . \tag{4.20}$$

   At the beginning, with $e^{\phi} \ll 1$, $\rho_{qf} \ll \rho_{cr}$ but, in the $(-)$ branch solution, ρ_{cr} decreases faster than ρ_{qf} so that, at some moment, ρ_{qf} will become the dominant sort of energy while the dilaton kinetic term will become negligible. It would be interesting to find out what sort of initial temperatures for the radiation era will come out of this assumption.

5 Conclusions

I want to conclude by listing which are, in my opinion, the pluses and minuses of the scenario I have advocated:

The Goodies

- Inflation comes naturally, without ad-hoc fields and fine-tuning: there is even an underlying symmetry yielding inflationary solutions.
- Initial conditions are natural, yet a simple universe would evolve into a rich and complex one.
- The kinematical problems of the SCM are solved.
- Perturbations do not grow too fast to spoil homogeneity.
- An interesting characteristic spectrum of GW is generated.
- Larger-than-usual electromagnetic perturbations are easily generated and could explain the galactic magnetic fields.
- A hot big bang could be a natural outcome of our inflationary scenario.

The Baddies

- A scale-invariant spectrum is all but automatic (unlike what happens in normal vacuum-energy-driven inflation).
- Our understanding of the high curvature (stringy) phase and of the crucially needed change of branch is still poor in spite of recent progress in Conformal Field Theory.

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References

[1] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, Inc., New York (1972).

[2] G. Smoot et al., *Astrophys. J.* **396** (1992) L1.

[3] L.F. Abbott and So-Young Pi (eds.), *Inflationary Cosmology*, World Scientific, Singapore (1986).

[4] E. Kolb and M. Turner, *The Early Universe*, Addison-Wesley, New York (1990).

[5] G. Veneziano, *Europhys. Lett.* **2** (1986) 133.

[6] G. Veneziano, "Quantum strings and the constants of Nature", in *The Challenging Questions* (Erice, 1989), ed. A. Zichichi, Plenum Press, New York (1990).

[7] C. Lovelace, *Phys. Lett.* **B135** (1984) 75;
C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, *Nucl. Phys.* **B262** (1985) 593.

[8] E.S. Fradkin and A.A. Tseytlin, *Nucl. Phys.* **B261** (1985) 1.

[9] P. Jordan, *Z. Phys.* **157** (1959) 112;
C. Brans and R.H. Dicke, *Phys. Rev.* **124** (1961) 925.

[10] E. Witten, *Phys. Lett.* **B149** (1984) 351.

[11] T.R. Taylor and G. Veneziano, *Phys. Lett.* **B213** (1988) 459.

[12] J. Ellis et al., *Phys. Lett.* **B228** (1989) 264.

[13] See, for instance, E. Fischbach and C. Talmadge, *Nature* **356** (1992) 207.

[14] T. Damour and A. M. Polyakov, *Nucl. Phys.* **B423** (1994) 532.

[15] G. Veneziano, *Phys. Lett.* **B265** (1991) 287; Proceeding 4th PASCOS Conference (Syracuse, May 1994), K.C. Wali ed., World Scientific, Singapore, p. 453.

[16] M. Gasperini and G. Veneziano, *Astropart. Phys.* **1** (1993) 317; *Mod. Phys. Lett.* **A8** (1993) 3701; *Phys. Rev.* **D50** (1994) 2519.

[17] A.A. Tseytlin, *Mod. Phys. Lett.* **A6** (1991) 1721;
A.A. Tseytlin and C. Vafa, *Nucl. Phys.* **B372** (1992) 443.

[18] E. Kiritsis and C. Kounnas, *Phys. Lett.* **B331** (1994) 51;
A.A. Tseytlin, *Phys. Lett.* **B334** (1994) 315.

[19] P. Aspinwall, B. Greene and D. Morrison, *Phys. Lett.* **B303** (1993) 249;
E. Witten, *Nucl. Phys.* **B403** (1993) 159.

[20] R. Brustein and G. Veneziano, *Phys. Lett.* **B329** (1994) 429.

[21] N. Kaloper, R. Madden and K. A. Olive, *Towards a singularity-free inflationary universe?*, Univ. Minnesota preprint UMN-TH-1333/95 (June 1995).

[22] J. Levin and K. Freese, *Nucl. Phys.* **B421** (1994) 635.

[23] L.P. Grishchuk, *Sov. Phys. JEPT* **40** (1975) 409;
A.A. Starobinski, *JEPT Lett.* **30** (1979) 682;
V.A. Rubakov, M. Sazhin and A. Veryaskin, *Phys. Lett.* **B115** (1982) 189;
R. Fabbri and M. Pollock, *Phys. Lett.* **B125** (1983) 445.

[24] M. Gasperini, *Status of string cosmology: phenomenological aspects*, Proceedings of Workshop Erice, September 1995.

[25] M. Gasperini, N. Sanchez and G. Veneziano, *Int. J. Theor. Phys.* **A6** (1991) 3853; *Nucl. Phys.* **B364** (1991) 365;
M. Gasperini, *Phys. Lett.* **B258** (1991) 70;
G. Veneziano, *Helv. Phys. Acta* **64** (1991) 877.

[26] See, e.g. V. Mukhanov, H.A. Feldman and R. Brandenberger, *Phys. Rep.* **215** (1992) 203.

[27] R.B. Abbot, B. Bednarz and S.D. Ellis, *Phys. Rev.* **D33** (1986) 2147.

[28] R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov and G. Veneziano, *Phys. Rev.* **D51** (1995) 6744.

[29] G. F. R. Ellis and M. Bruni, *Phys. Rev.* **D40** (1989) 1804;
M. Bruni, G. F. R. Ellis and P. K. S. Dunsby, *Class. Quant. Grav.* **9** (1992) 921.

[30] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, *Relic gravitational waves from string cosmology*, CERN-TH/95-144 (1995), *Phys. Lett.* in press; see also M. Gasperini and M. Giovannini, *Phys. Rev.* **D47** (1992) 1529.

[31] R.E. Vogt et al., *Laser Interferometer Gravitational-Wave Observatory*, proposal to the National Science Foundation (Caltech, 1989);
C. Bradascia et al., in *Gravitational Astronomy*, eds. D.E. McClelland and H. Bachor, World Scientific, Singapore, (1991).

[32] G. V. Pallottino and V. Pizzella, *Nuovo Cim.* **C4** (1981) 237;
M. Cerdonio et. al, *Phys. Rev. Lett.* **71** (1993) 4107.

[33] F. Pegoraro, E. Picasso and L. Radicati, *J. Phys.* **A11** (1978) 1949;
C. M. Caves, *Phys. Lett.* **B80** (1979) 323;
C. E. Reece et al., *Phys. Lett.* **A104** (1984) 341.

[34] P. Astone, J. A. Lobo and B. F. Schutz, *Class. Quant. Grav.* **11** (1994) 2093.

[35] E. N. Parker, *Cosmical Magnetic Fields*, Clarendon, Oxford (1979);
Y. B. Zeldovich, A. A. Ruzmaikin and D. D. Sokoloff, *Magnetic fields in astrophysics*, Gordon and Breach, New York (1983).

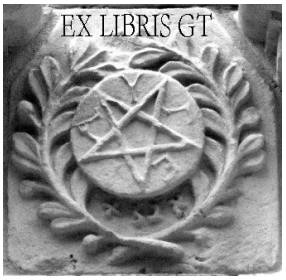
[36] M. Gasperini, M. Giovannini and G. Veneziano, *Primordial magnetic fields from string cosmology*, CERN-TH/95-85 (April 1995) *Phys. Rev. Lett.* in press.

[37] D. Lemoine and M. Lemoine, *Phys. Rev.* **D52** (1995) 1955.

[38] M. S. Turner and L. M. Widrow, *Phys. Rev.* **D37** (1988) 2743.

[39] M. Gasperini, M. Giovannini and G. Veneziano, *Electromagnetic origin of the CMB anisotropy in string cosmology*, CERN-TH/95-102 (April 1995) *Phys. Rev.* in press.

[40] L. Parker, *Nature* **261** (1976) 20.



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STRING COSMOLOGY: CONCEPTS AND CONSEQUENCES

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ABSTRACT

After recalling a few basic concepts from cosmology and string theory, I will discuss the main ideas/assumptions underlying string cosmology and show how these lead to a two-parameter family of "minimal" models. I will then explain how to compute, in terms of those parameters, the spectrum of scalar, tensor and electromagnetic perturbations, point at their (T and S-type) duality symmetries, and mention their most relevant physical consequences.

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String gravity and physics at the Planck scale
(Erice, September 8-19, 1995)
and
Unification: From the weak scale to the Planck scale
(ITP, Santa Barbara, October 23-27, 1995)

CERN-TH/95-301
October 1995

1 Basic Facts about Cosmology and Inflation

It is well known[1] that the Standard Cosmological Model (SCM) works well at "late" times, its most striking successes being perhaps the red shift, the cosmic microwave background (CMB), and primordial nucleosynthesis.

However, the SCM suffers from various problems. At the theoretical level the most serious of these is the initial singularity problem, which basically tells us that we cannot have theoretical control over the initial conditions of the SCM. At a phenomenological level, the SCM cannot explain naturally:

i) the homogeneity and isotropy of our Universe as manifested, in particular, through the small value of $\Delta T/T = O(10^{-5})$ observed with COBE[2];

ii) the flatness problem, i.e. the fact that, within an order of magnitude, $\Omega \equiv \rho \,/\, \rho_{crit} \sim 1$;

iii) the origin of large-scale structure.

Inflation, i.e. a long phase of accelerated expansion of the Universe ($\dot{a}, \ddot{a} > 0$, where a is the scale factor), is the only way known at present of solving the above-mentioned phenomenological problems. Various types of inflationary models have been proposed [for a review, see 3), 4)] each one supposedly mending the problems of the previous version. Particularly severe are the constraints coming from demanding:

a) a graceful exit with the right amount of reheating;

b) the right amount of large-scale inhomogeneities.

In order to satisfy such constraints, fine-tuned initial conditions and/or inflaton potentials are necessary. And this without mentioning the fact that inflation is not addressing at all the initial singularity problem.

Actually, Kolb and Turner, after reviewing the prescriptions for a successful inflation, add[4]:

"Perhaps the most important – and most difficult – task in building a successful inflationary model is to ensure that the inflaton is an integral part of a sensible model of particle physics. The inflaton should spring forth from some grander theory and not vice versa".

I will argue below that superstring theory could be the sought-after grander theory (what could be better than a theory of everything?) naturally providing an inflation-driving scalar field in the general sense defined again in ref. 4):

"It is now apparent that inflation, which was originally so closely related to Spontaneous Symmetry Breaking, is a much more general phenomenon.... Stated in its full generality, inflation involves the dynamical evolution of a very weakly-coupled scalar field that was originally displaced from the minimum of its potential."

I hope to convince you that this will be precisely the picture that we claim takes place in string cosmology. In order to substantiate this claim, I will have to digress and recall a few basic facts in Quantum String Theory.

2 Basic facts in quantum string theory (QST)

I am listing below a few basic properties of strings, emphasizing those that are most relevant for our subsequent discussion. These are:

1. Unlike its classical counterpart, quantum string theory contains a fundamental length scale λ_s representing[5] the ultraviolet, short-distance cut-off (equivalently, a high-momentum cut-off at $E = M_s c^2 \equiv \hbar c/\lambda_s$).

lambda s

2. Tree-level masses are either zero or $O(M_s)$. Quantum mechanics allows massless strings with non-zero angular momentum[6] while, classically, $M^2 > \text{const.} \times J$. The existence of such states is obviously a crucial property of QST, without which it could not pretend to be a candidate theory of all known interactions.

3. The effective interaction of the massless fields at $E \ll M_s$ takes the form of a classical, gauge-plus-gravity field theory with specified parameters. It is described by an effective action[7),8)] of the (schematic) type:

See B B S

$$\Gamma_{eff} = \frac{1}{2}\int d^4x\sqrt{-g}\; e^{-\phi}\left[\lambda_s^{-2}(\mathcal{R} + \partial_\mu\phi\partial^\mu\phi) + F_{\mu\nu}^2 + \bar{\psi}\not{D}\psi + R^2 + \ldots\right] + \left[\text{higher orders in } e^{\phi}\right] . \tag{2.1}$$

Equation (2.1) contains two dimensionless expansion parameters. One of them, $g^2 \equiv e^{\phi}$, controls the analogue of QFT's loop corrections, while the other, $\lambda^2 \equiv \lambda_s^2 \cdot \partial^2$, controls string-size effects, which are of course absent in QFT.

lambda^2

4. As indicated in (2.1), QST has (actually needs!) a new particle/field, the so-called dilaton ϕ, a scalar massless particle (at the perturbative level). It appears in Γ_{eff} as a Jordan–Brans–Dicke[9] scalar with a "small" negative ω_{BD} parameter, $\omega_{BD} = -1$.

5. The dilaton's VEV provides[8),10)] a unified value for:

a) The gauge coupling(s) at $E = O(M_s)$.

Gasperini clear on that. Book.

b) The gravitational coupling in string units.

c) Yukawa couplings, etc., at the string scale.

In formulae:

$$\ell_p^2 \equiv 8\pi G_N \hbar = e^{\phi}\lambda_s^2 \ , \qquad \alpha_{GUT}(\lambda_s^{-1}) \simeq \frac{e^{\phi}}{4\pi} \ , \tag{2.2}$$

"Veneziano assumptions" on string size

implying (from $\alpha_{GUT} \approx 1/20$) that the string-length parameter λ_s is about 10^{-32} cm. Note, however, that, in a cosmological context in which ϕ evolves in time, the above formulae can only be taken to give the *present* values of α and ℓ_p/λ_s. In the scenario we will advocate, both quantities were much smaller in the very early Universe!

6. Dilaton couplings at large distance are such[11] that a massless dilaton is most likely ruled out[11,12] by precision tests[13] of the equivalence principle [for a possible way out see, however, ref. 14)], i.e.

$$M_\phi > 10^{-4} \text{ eV} \ . \tag{2.3}$$

7. Details about the dilaton potential are unknown, yet:

a) On theoretical grounds, in critical superstring theory, the dilaton potential has to go to zero as a double exponential as $\phi \to -\infty$ (weak coupling):

$$V(\phi) \sim \exp\left(-c^2 \exp(-\phi)\right) = \exp\left(-\frac{c^2}{4\pi\alpha_{GUT}}\right) \ , \tag{2.4}$$

with c^2 a positive (but model-dependent) constant.

b) On physical grounds it should have a non-trivial minimum at its present value ($\langle\phi\rangle = \phi_0 \sim 0$) with a vanishing cosmological constant, $V(\phi_0) = 0$.

A typical potential satisfying a) and b) is shown in Fig. 1. The dotted lines at $\phi > 0$ represent our ignorance about strongly coupled string theory. Fortunately, the details of what happens in that region will not be very relevant for our subsequent discussion.

8. There is an exact (all-order) vacuum solution for (critical) superstring theory. Unfortunately, it corresponds to a free theory ($g = 0$ or $\phi = -\infty$) in flat, ten-dimensional, Minkowski space-time, nothing like the world we seem to be living in!

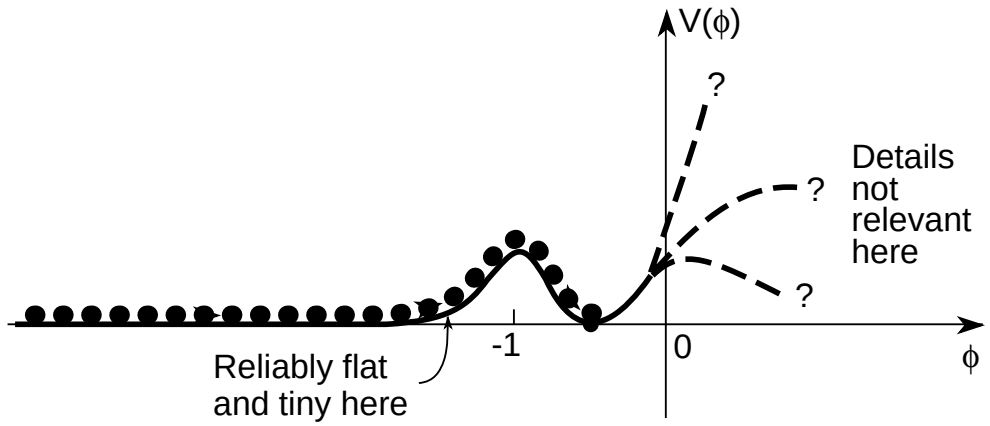

Figure 1: A possible dilaton potential with illustration of an inflation-driving rolling dilaton (large dots).

Before closing this section I would like to comment briefly on a point which appears to be the source of much confusion even among experts: it is the debate between working in the (so-called) String and Einstein "frames" (not to be confused with different coordinate systems). Since the two frames are related by a local field redefinition (a conformal, dilaton-dependent rescaling of the metric to be precise) all physical quantities are independent of the frame one is using. The question is: what should we call the metric? Although, to a large extent, this is a question of taste, one's intuition may work better with one definition than with another. Note also that, since the dilaton is time-independent today, the two frames coincide now.

Let us compare the virtues and problems with each frame.

A) STRING FRAME This is the metric appearing in the fundamental (Polyakov) action for the string. Classical, weakly coupled strings sweep geodesic surfaces with respect to this metric. Also, the dilaton dependence of the low energy effective action takes the simple form indicated in (2.1) only in the string frame. The advantage of this frame is that the string cut-off is fixed and the same is true for the value of the curvature at which higher orders in the σ-model coupling λ become relevant. The main disadvantage is that the gravitational action is not so easy to work with.

B) EINSTEIN FRAME In this frame the pure gravitational action takes the standard Einstein-Hilbert form. Consequently, this is the most convenient frame for studyind the cosmological evolution of metric perturbations. The Planck length is fixed in this frame while the string length is dilaton (hence generally

time) dependent. In the Einstein frame Γ_{eff} takes the form:

$$\Gamma_{eff} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,\left[R+\partial_\mu\phi\partial^\mu\phi+e^{-\phi}F^2_{\mu\nu}+\partial_\mu A\partial^\mu A+e^{\phi}m^2A^2\right] + \left[G_N e^{-\phi}R^2+\ldots\right], \quad (2.5)$$

showing that, in this frame, masses are dilaton dependent (even at tree level) and so is the value of R at which higher order stringy corrections become important. It is for the above reasons that I will choose to base my discussion (although not always the calculations) in the String frame.

3 Main ideas/assumptions of string cosmology

The very basic postulate of (our own version of) String Cosmology[15),16)] is that the Universe did indeed start near its trivial vacuum mentioned at the end of the previous section.

Fortunately, if one looks at the space of homogeneous (and for simplicity spacially-flat) perturbative vacuum solutions, one finds that the trivial vacuum is a very special, *unstable* solution. This is depicted in Fig. 2a for the simplest case of a ten-dimensional cosmology in which three spatial dimensions evolve isotropically while six "internal" dimensions are static (it is easy to generalize the discussion to the case of dynamical internal dimensions, but then the picture becomes multidimensional).

trivial vac

The straight lines in the $H,\dot{\bar{\phi}}$ plane (where $\dot{\bar{\phi}}\equiv\dot{\phi}-3H$) represent the evolution of the scale factor and of the coupling constant as a function of the cosmic time parameter (arrows along the lines show the direction of the time evolution). As a consequence of a stringy symmetry, known[15),17)] as "Scale Factor Duality (SFD)", there are two branches (two straight lines). Furthermore, each branch is split by the origin in two time-reversal-related parts (time reversal changes the sign of both H and $\dot{\bar{\phi}}$).

SFD

1991

The origin (the trivial vacuum) is an "unstable" fixed point: a small perturbation in the direction of positive $\dot{\bar{\phi}}$ makes the system evolve further and further from the origin, meaning larger and larger coupling and absolute value of the Hubble parameter. This means an accelerated expansion or an accelerated contraction, i.e. in the latter case, inflation. It is tempting to assume that those patches of the original Universe that had the right kind of fluctuation have grown up to become (by far) the largest fraction of the Universe today.

In order to arrive at a physically interesting scenario, however, we have to connect somehow the top-right inflationary branch to the bottom-right branch, since the

String cosmo is NOT based on "quantum fluctuations"!!!

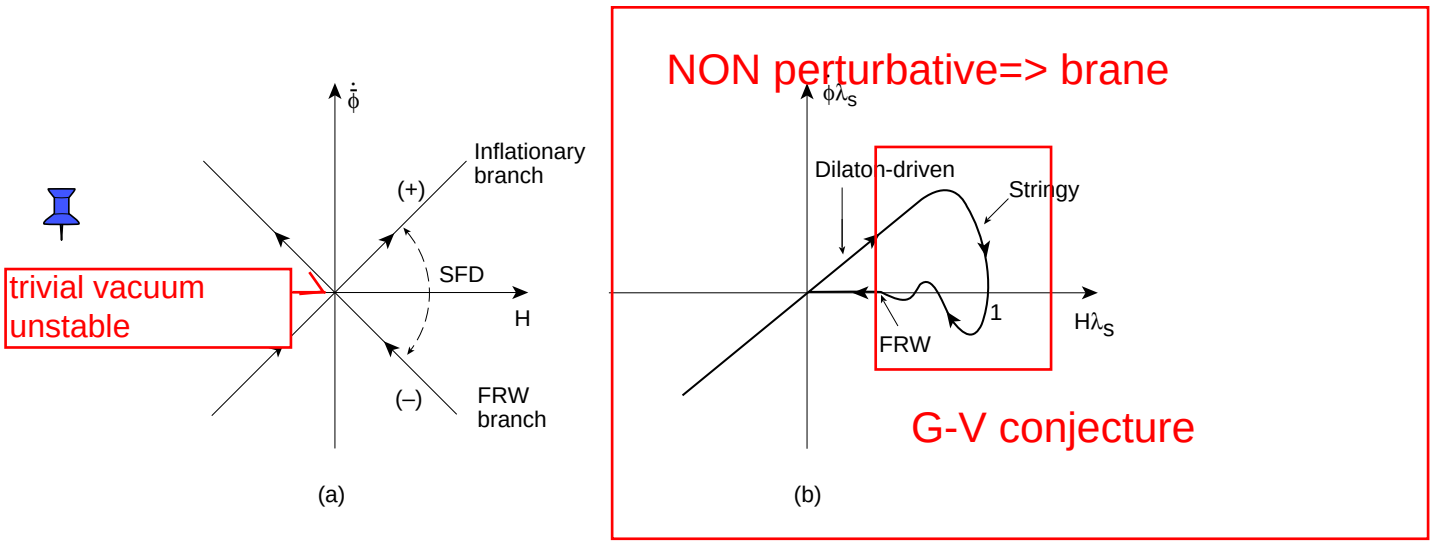

Figure 2: Phase diagrams for the perturbative (a) regime and a conjectured non-perturbative solution (b) to the branch-change problem.

latter is nothing but the standard FRW cosmology, which has presumably prevailed for the last few billion years or so. Here the so-called "exit problem" arises. At lowest order in λ^2 (small curvatures in string units) the two branches do not talk to each other. The inflationary (also called $+$) branch has a singularity in the future (it takes a finite cosmic time to reach ∞ in our gragh if one starts from anywhere but the origin) while the FRW $(-)$ branch has a singularity in the past (the usual big-bang singularity).

exit problem

It is widely believed that QST has a way to avoid the usual singularities of Classical General Relativity or at least a way to reinterpret them[18),19)]. It thus looks reasonable to assume that the inflationary branch, instead of leading to a non-sensical singularity, will evolve into the FRW branch at values of λ^2 of order unity. This is schematically shown in Fig. 2b, where we have gone back from $\dot{\bar{\phi}}$ to $\dot{\phi}$ and we have implicitly taken into account the effects of a non-vanishing dilaton potential at small ϕ in order to freeze the dilaton at its present value. The need for the branch change to occur at large λ^2, first argued for in[20)], has been recently proved in ref. 21).

No sing issue

There is a rather simple way to parametrize a class of scenarios of the kind defined above. They contain (roughly) three phases and two parameters. Indeed:

In phase I the Universe evolves at $g^2, \lambda^2 \ll 1$ and thus is close to the trivial vacuum. This phase can be studied using the tree-level low-energy effective action (2.1) and is characterized by a long period of dilaton-driven inflation. The accelerated expansion of the Universe, instead of originating from the potential energy of an inflaton field, is driven by the growth of the coupling constant (i.e. by the dilaton's kinetic energy, see ref. 22) for a similar kind of inflationary scenario) with

beware: NO inflaton!!

1: A vacuum universe

g^2, lambda^2<<1

$\dot{\phi} = 2\dot{g}/g \sim H$ during the whole phase.

Phase I supposedly ends when the coupling λ^2 reaches values of $O(1)$, so that higher-derivative terms in the effective action become relevant. Assuming that this happens while g^2 is still small (and thus the potential is still negligible), the value g_s of g at the end of phase I (the beginning of phase II) is an arbitrary parameter (a modulus of the solution).

lambda= 1

During phase II, the stringy version of the big bang, the curvature, as well as $\dot{\phi}$, are assumed to remain fixed at their maximal value given by the string scale (i.e. we expect $\lambda \sim 1$). The coupling g will instead continue to grow from the value g_s until it is its own turn to reach values $O(1)$. At that point, assuming a branch change to have occurred at large curvatures, the dilaton will be attracted to the true non-perturbative minimum of its potential; the standard FRW cosmology can then start, provided the Universe was heated-up and filled with radiation (this is not a problem, see below). The second important parameter of this scenario is the duration of phase II or better the total red-shift, $z_s \equiv a_{end}/a_{beg}$, which has occurred from the beginning to the end of the stringy phase.

??????????

see Fig2 b

Our present ignorance about this most crucial phase (and in particular about the way the exit can be implemented) prevents us from having a better description of this phase which, in principle, should not introduce new arbitrary parameters (z_s should be eventually determined in terms of g_s).

During Phase III, the Universe evolves towards smaller and smaller curvatures but stays at moderate-to-strong coupling. This is the regime in which usual QFT methods are applicable. The details of the particular gauge theory emerging from the string's non-perturbative vacuum will be very important in determining the subsequent evolution and in particular the problem of structure formation, dark matter and the like.

"Today"

Our scenario contains implicitly an arrow of time, which points in the direction of increasing entropy, inhomogeneity and structure. As a result of the amplification of primordial vacuum fluctuations, the Universe is not coming back to its initial simple (and unique) state (the origin in Fig. 2), but to the much more structured (and interesting) state in which we are living today. Actually, the arrow of time itself should be determined by the direction in which entropy (and complexity) are growing. This will force us to identify (*by definition*) the perturbative vacuum with the initial state of the Universe!

4 Observable consequences

All the observable consequences I will discuss below have something to do with the well-known phenomenon[23] of amplification of vacuum quantum fluctuations in cosmological backgrounds. Any conformally flat cosmological background is known:

a) to amplify tensor perturbations, i.e. to produce a stochastic background of gravitational waves;

b) to induce scalar-metric perturbations from the coupling of the metric either to a fluid or to scalar particles (in our context to the dilaton).

By contrast, because of the scale-invariant coupling of gauge fields in four dimensions, electromagnetic (EM) perturbations are *not* amplified in a conformally flat cosmological background (even if inflationary). In string cosmology, the presence of a time-dependent dilaton in front of the gauge-field kinetic term yields, on top of the two previously mentioned effects,

c) an amplification of EM perturbations corresponding to the creation of macroscopic magnetic (and electric) fields.

Various physically interesting questions arise in connection with the three effects I have just mentioned. These include the following:

1. Does the Universe remain quasi-homogeneous during the whole string-cosmology history?
2. Does one generate a phenomenologically interesting (i.e. measurable) background of GW?
3. Can one produce large enough seeds for generating the observed galactic (and extragalactic) magnetic fields?
4. Can scalar, tensor (and possibly EM) perturbations explain the large-scale anisotropy of the CMB observed by COBE?
5. Do these perturbations have anything to do with the CMB itself?

In the rest of this talk I will first explain, on the toy example of the harmonic oscillator, the common mechanism by which quantum fluctuations are amplified in cosmological backgrounds. I will then give our present answers to the questions listed above. For more details, see Ref. [24].

Gasperini

Consider a one-dimensional (non-relativistic) harmonic oscillator moving in a cosmological background of the simplest kind, characterized by a scale factor $a(t)$. In units in which the mass of the oscillator is 1, the Lagrangian reads:

$$L = \frac{1}{2}a^2(\dot{x}^2 - \omega^2 x^2) \tag{4.1}$$

while the canonical momentum and Hamiltonian are given by

$$p = a^2\dot{x}\ , \qquad H = \frac{1}{2}(a^{-2}p^2 + a^2\omega^2 x^2)\ . \tag{4.2}$$

Let us first discuss the solutions of the classical equations of motion:

$$\ddot{x} + 2\ \dot{a}/a\ \dot{x} + \omega^2 x = 0\ , \qquad \ddot{y} + (\omega^2 - \ddot{a}/a)y = 0\ , \tag{4.3}$$

where $y \equiv ax$ is the proper (physical) amplitude as opposed to the comoving amplitude x.

Solutions to Eqs. (4.3) simplify in two opposite regimes:

a) For $\omega^2 \gg \ddot{a}/a$ there is "adiabatic damping" of the comoving amplitude (the name is clearly unappropriate in the case of contraction):

$$x \sim a^{-1}\ e^{\pm i\omega t}\ , \qquad p \sim a\ \omega\ e^{\pm i\omega t} \tag{4.4}$$

which means that, in this regime, the proper amplitude y and the proper momentum p/a stay constant (and so does the Hamiltonian).

b) For $\omega^2 \ll \ddot{a}/a$ one finds the so-called "freeze-out" regime in which:

$$\begin{aligned} x &\sim B\ +\ C\int_0^t dt' a^{-2}(t') \\ p &\sim C + \ldots \end{aligned} \tag{4.5}$$

where the comoving amplitude and momentum are fixed. In this regime the Hamiltonian (the energy) of the system tends to grow at late times whenever a increases or decreases by a large factor during the freeze-out regime. In the former case the energy is dominated asymptotically by the term proportional to x^2 and is due to the "stretching" of the oscillator caused by the fast expansion, while in the latter case the term proportional to p^2 dominates because of the large blue-shift suffered by the momentum in a contracting background.

Consider now a cosmology such that

$$\begin{aligned} \omega^2 &> \ddot{a}/a\,, \qquad t < t_{ex},\, t > t_{re} \\ \omega^2 &< \ddot{a}/a\,, \qquad t_{ex} < t < t_{re} \end{aligned} \tag{4.6}$$

where, anticipating our subsequent discussion, we have defined the moments of exit and re-entry by the condition $\omega \sim H$. Such an example will be typical of our scenario, since a given scale will be well inside the horizon at the beginning (small Hubble parameter), outside during the high-curvature regime, and then inside again after re-entry. By joining smoothly the two asymptotic solutions, we easily find that the energy of the harmonic oscillator (which is constant during the initial and final phases) has been amplified during the intermediate phase by a factor:

$$|c|^2 = Max\left(\frac{a^2(t_{re})}{a^2(t_{ex})} \quad , \quad \frac{a^2(t_{ex})}{a^2(t_{re})}\right) \tag{4.7}$$

corresponding to the two above-mentioned cases.

The excercise can be repeated at the quantum level starting, for instance, from a harmonic oscillator in its ground state. Quantum mechanics fixes the size of the initial amplitude, momentum and energy:

$$|x| \sim a^{-1}\sqrt{\frac{\hbar}{\omega}}, \quad E \sim \hbar\omega \ . \tag{4.8}$$

The quantum mechanical interpretation of eq. (4.7) is that c is the Bogoliubov coefficient transforming the initial ground state into the final excited quantum state ($|c|^2$ being the average occupation number for the latter). Note that the final state ends up being highly "squeezed", i.e. having a large Δx or Δp depending on the sign of H. If, because of coarse-graining, the squeezed coordinate is not measured, the final state will look like a high-entropy, statistical ensemble of quasi-classical oscillators.

Note, finally, the (Scale-Factor) duality invariance of the resulting amplification. Under $a \to a^{-1}$, position and momentum operators swap their role as the variable in which sqeezing or amplification occurs. Thus the final amplification remains the same.

Up to technical complications, things work out pretty much in the same way for strings[25] and for the three kinds of perturbations mentioned at the beginning of this section. In particular, for each one of the latter, one can define[26] a canonical variable ψ^i (similar to the harmonic oscillator's y) satisfying an equation of the type

$$\psi_k'' + [k^2 - V_i(\eta)]\psi_k = 0 \ , \tag{4.9}$$

where the label i on ψ has been suppressed, k is the comoving wave number, and derivatives with respect to conformal time η are denoted by a prime.

Since, for each i, the "potential" V_i is very small at very early times, grows to a maximum during the stringy era and, finally, drops rapidly to zero at the beginning

of the radiation era, a given scale (k) begins and ends inside the horizon with an intermediate phase outside. Larger scales exit earlier and re-enter later. Also, in our scenario, larger scales exit and re-enter at smaller values of H. Very short scales exit during the stringy era and, for those, our predictions will not be as solid as for the scales that leave the horizon during the perturbative dilatonic phase I. The fact that the amplification of perturbation depends just on some ratios of fields evaluated at exit and re-entry (and not on the details of the evolution in between) makes us believe that our detailed results are trustworthy for those larger scales. This being said, I present below some results on the five issues mentioned above (see, again, ref. 24) for derivations and/or details).

1. **Does the Universe remain quasi-homogeneous during the whole string-cosmology history?**

   The answer to this question turns out to be yes! This is not a priori evident since, in commonly used gauges[26] for scalar perturbations of the metric (e.g. the so-called longitudinal gauge in which the metric remains diagonal), such perturbations appear to grow very large during the inflationary phase and to destroy homogeneity or, at least, to prevent the use of linear perturbation theory. Similar problems had been encountered earlier in the context of Kaluza-Klein cosmology[27].

   In ref. 28) it was shown that, by a suitable choice of gauge (an "off-diagonal" gauge), the growing mode of the perturbation can be tamed. This can be double-checked by using the so-called gauge-invariant variables of Bruni and Ellis[29]. The bottom line is that scalar perturbations in string cosmology behave no worse than tensor perturbations, to which we now turn our attention.

2. **Does one generate a phenomenologically interesting (i.e. measurable) background of GW?**

   The canonical variable ψ for tensor perturbations (i.e. for GW) is defined by:

$$\begin{aligned} g_{\mu\nu} &= a^2[\eta_{\mu\nu} + h_{\mu\nu}] \\ \psi = (a/g)\ h &= a\ e^{-\phi/2} h\ , \end{aligned} \tag{4.10}$$

   where h stands for either of the two transverse-traceless polarizations of the gravitational wave. As long as the perturbation is inside the horizon, ψ remains constant while h is adiabatically damped. By contrast, outside the horizon, ψ is amplified according to

$$\psi_k \sim (a/g)[C_k + D_k \int_{\eta_{ex}}^{\eta} d\eta'\ g^2(\eta')\ a^{-2}(\eta')] \tag{4.11}$$

   where, for each Fourier mode of (comoving) wave number k, $\eta_{ex} = k^{-1}$.

The first term in (4.11) clearly corresponds to the freezing of h itself, while the second term represents the freezing of its associated canonical momentum. In standard (non-dilatonic) inflationary models, the first term dominates since a grows very fast. In our case, the second term dominates since the growth of a is over-compensated by the growth of g (i.e. of ϕ). This is equivalent to saying that, in the Einstein frame, our background describes a contracting Universe.

After matching the result (4.11) with the usual oscillatory, damped behaviour of the radiation-dominated epoch, one arrives at the final result[28),30)] for the magnitude of the stochastic background of GW today:

$$|\delta h_\omega| \equiv k^{3/2}|h_k| \sim \sqrt{\frac{H_0}{M_s}} z_{eq}^{-1/4} z_s \left(\frac{g_s}{g_1}\right) \left(\frac{\omega}{\omega_s}\right)^{1/2} \left[\ln\left(\frac{\omega_s}{\omega}\right) + (z_s)^{-3}\left(\frac{g_s}{g_1}\right)^{-2}\right], \qquad \omega < \omega_s \quad (4.12)$$

where $\omega = k/a$ is the proper frequency, $z_{eq} \sim 10^4, \omega_s \sim z_s^{-1}(g_1)^{1/2} \times 10^{11}$ Hz $\equiv z_s^{-1}\omega_1$.

The above result can be converted into a spectrum of energy density per logarithmic interval of frequency. In critical density units:

$$\frac{d\Omega_{GW}}{d\ln\omega} = z_{eq}^{-1}(g_s)^2 \left(\frac{\omega}{\omega_s}\right)^3 \left[\ln\left(\frac{\omega_s}{\omega}\right) + (z_s)^{-3}\left(\frac{g_s}{g_1}\right)^{-2}\right]^2, \ \omega < \omega_s \ . \quad (4.13)$$

The above spectrum looks quasi-thermal al large scales (i.e. at $\omega < \omega_s$), but is amplified by a large factor relative to a Planckian spectrum of temperature ω_s. In analogy with the harmonic oscillator case, there is a duality symmetry of the spectrum, this time under the transformation $(z_s, g_s) \to (z_s, z_s^{-3} g_s^{-1})$. The transformation corresponds to changing $\bar{\phi}$ into $-\bar{\phi}$ i.e. to what we may call $\bar{S}$-duality. As with the harmonic oscillator, the metric perturbation and its canonically conjugate momentum variable swap their role under such transformation.

In Fig. 3 we show the spectrum of stochastic gravitational waves expected from our two-parameter model. For a given pair g_s, z_s one identifies a point in the $\omega, \delta h_\omega$ plane as illustrated explicitely in the case of $g_s = 10^{-3}, z_s = 10^6$.The resulting point (indicated by a large dot) represents the end-point $\omega_s, \delta h_{\omega_s}$ of the $\omega^{1/2}$ spectrum corresponding to scales crossing the horizon during the dilatonic era.

Although the rest of the spectrum is more uncertain, one can argue that it has to join smoothly the point $\omega_s, \delta h_{\omega_s}$ to the true end-point $\delta h \sim 10^{-30}, \omega \sim 10^{11} Hz$. The latter corresponds to a few gravitons produced at the maximal

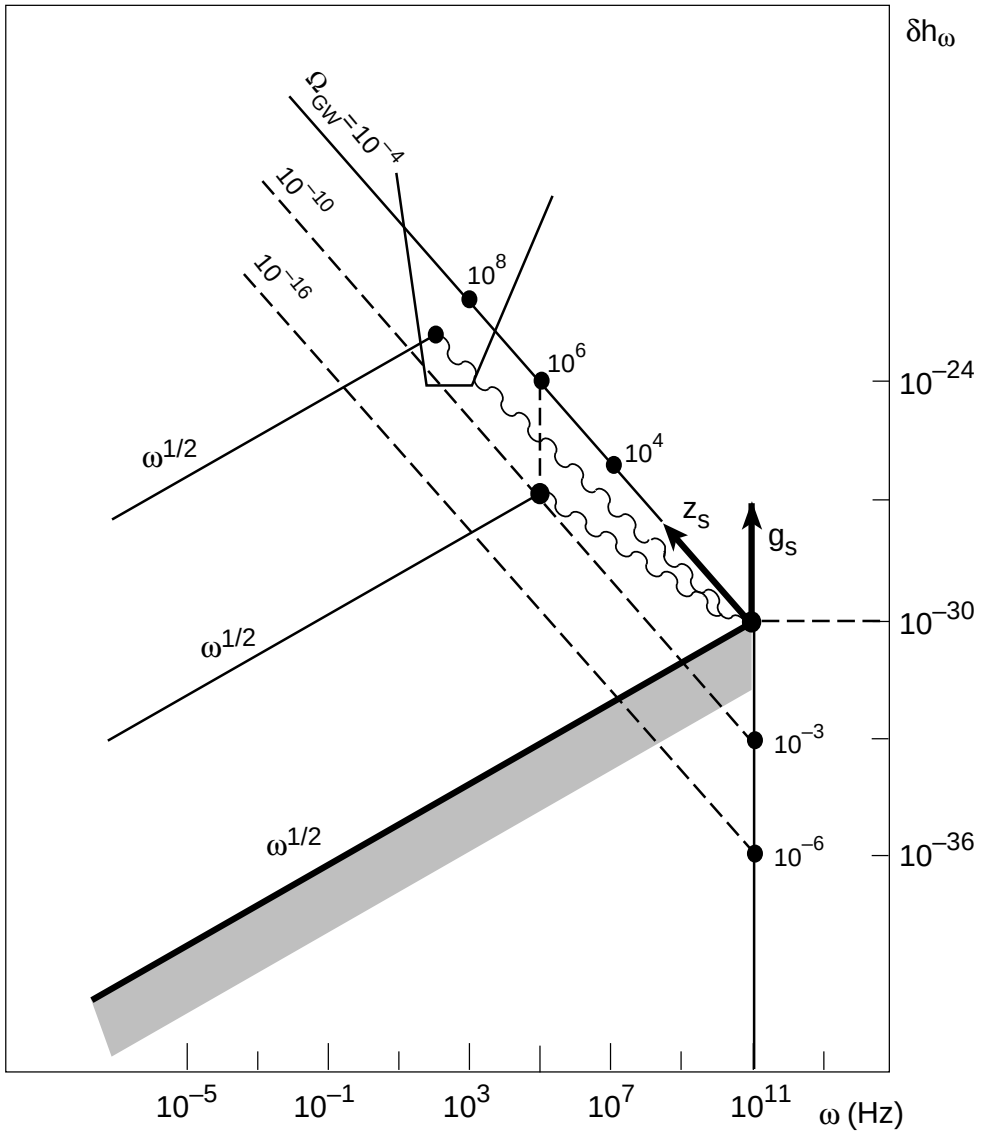

Figure 3: GW spectra from string cosmology against interferometric sensitivity.

amplified frequency ω_1, the last scale to go outside the horizon during the stringy phase. The full spectrum is also shown in the figure for the case $g_s = 10^{-3}, z_s = 10^6$, with the wiggly line representing the less well known high frequency part.

Curves of constant Ω_{GW} are also shown. If $g_s < 1$, as we have assumed, spectra will always lie below the $\Omega_{GW} = 10^{-4}$ line corresponding to as many photons as gravitons been produced. On the other hand, by invoking $\bar{S}$-duality, one can argue that the actual spectrum, by containing two duality-related contributions, will never lie below the self-dual spectrum ending at $\delta h \sim 10^{-30}, \omega \sim 10^{11} Hz$ (the thick line bordering the shaded region). In conclusion all possible spectra sweep the angular wedge inside the two above-mentioned lines.

The odd-shaped region in Fig. 3 shows the expected sensitivity of the so-called "Advanced LIGO" project[31]. While there is no hope to detect our spectrum at LIGO if $g_s = 10^{-3}, z_s = 10^6$, perspectives would be better for, say, $g_s = 10^{-1}, z_s = 10^8$ (the corresponding spectrum is also sketched).

Resonant bars might also be able to reach comparable sensitivity in the kHz region, while microwave cavities, if conveniently developed, could be used in the region 10^6-10^9 Hz [33]. Another interesting possibility consists of coincidence experiments between an interferometer and a bar. The quoted sensitivity[34] to a stochastic background, as a function of the frequency f, of the individual sensitivities of the bar and of the interferometer δh, and of the observation time T_{obs}, is:

$$\delta\Omega_{GW} = 1.5\ 10^{-5}(f/10^3\mathrm{Hz})^3(10^{21}\delta h_{int})(10^{21}\delta h_{bar})(T_{obs}/10^7\ \mathrm{s})^{-1/2}. \quad (4.14)$$

Obviously, detecting a stochastic backgound like ours is a formidable challenge. Also, the physical range of our parameters g_s, z_s could be such that no observable signal will be produced. What is interesting, however, is the mere existence of cosmological models predicting a non-negligible yield of GW in a range of frequencies where other sources predict just a "desert". More complete studies of the sensitivity of various detectors to a stochastic, coloured spectrum of our kind are presently under way.

3. **Can large enough seeds be produced for the generation of observed galactic (and extragalactic) magnetic fields?**

As already mentioned, seeds for generating the galactic magnetic fields through the so-called cosmic dynamo mechanism[35] can be generated in our scenario by the amplification of the quantum fluctuations of the EM field. In this case the canonical variable is just the (Fourier transform of the) usual A_μ potential. In

analogy with (4.11) its amplification, while outside the horizon, is described by the asymptotic solution:

$$A_k \sim g^{-1}\left[C_k + D_k \int_{\eta_{ex}}^{\eta} d\eta' g^2(\eta')\right] , \tag{4.15}$$

which leads[36),37)] to an overall amplification of the electromagnetic field by a factor $|c_k|^2 \sim (g_{re}/g_{ex})^2 + (g_{ex}/g_{re})^2$. This time the spectrum is invariant under $g \to g^{-1}$ i.e. under ordinary S or electric-magnetic duality. In our cosmological scenario we have excluded the possibility of a decreasing coupling constant and, therefore, the main contribution to the amplification comes from the second term on the r.h.s. of eq.(4.15) which gives $|c_k|^2 \sim (g_{re}/g_{ex})^2$.

One can express this result in terms of the fraction of electromagnetic energy stored in a unit of logarithmic interval of ω normalized to the one in the CMB, ρ_γ. One finds:

$$r(\omega) = \frac{\omega}{\rho_\gamma}\frac{d\rho_B}{d\omega} \simeq \frac{\omega^4}{\rho_\gamma}|c_-(\omega)|^2 \equiv \frac{\omega^4}{\rho_\gamma}(g_{re}/g_{ex})^2 . \tag{4.16}$$

The ratio $r(\omega)$ stays constant during the phase of matter-dominated as well as radiation-dominated evolution, in which the Universe behaves like a good electromagnetic conductor[38)]. In terms of $r(\omega)$ the condition for seeding the galactig magnetic field through ordinary mechanisms of plasma physics is[38)]

$$r(\omega_G) \geq 10^{-34} \tag{4.17}$$

where $\omega_G \simeq (1\ \mathrm{Mpc})^{-1} \simeq 10^{-14}$ Hz is the galactic scale. Using the known value of ρ_γ, we thus find, from (4.16, 4.17):

$$g_{ex} < 10^{-33} , \tag{4.18}$$

i.e. a very tiny coupling at the time of exit of the galactic scale.

The conclusion is that string cosmology stands a unique chance in explaining the origin of the galactic magnetic fields. Indeed, if the seeds of the magnetic fields are to be attibuted to the amplification of vacuum fluctuations, their present magnitude can be interpreted as prime evidence that the fine structure constant has evolved to its present value from a tiny one during inflation. The fact that the needed variation of the coupling constant ($\sim 10^{30}$) is of the same order as the variation of the scale factor needed to solve the standard cosmological problems, can be seen as further evidence for scenarios in which coupling and scale factor grow roughly at the same rate during inflation.

4. **Can scalar, tensor (and possibly EM) perturbations explain the large-scale anisotropy of the CMB observed by COBE?**

   The answer here is certainly negative as far as scalar and tensor perturbations are concerned. The reason is simple: for spectra that are normalized to $O(1)$ (at most) at the maximal amplified frequency $\omega_1 \sim 10^{11}$ Hz, and that grow like $\omega^{1/2}$, one cannot have any substantial power at the scales $O(10^{-18}\text{Hz})$ to which COBE is sensitive. The origin of $\Delta T/T$ at large scale would have to be attributed to other effects (e.g. topological defects).

   Fortunately, there is a possibility [39] that the EM perturbations themselves might explain the anisotropies of the CMB since their spectrum turns out to be flatter (and also more model-dependent) than that of metric perturbations. Assuming this to be the case, an interesting relation is obtained[39] between the magnitute of large scale anysotropies and the slope of the power spectrum. Such a relation turns out to be fully consistent, with present bounds on the spectral index.

5. **Do all these perturbations have anything to do with the CMB itself?**

   Stated differently, this is the question of how to arrive at the hot big bang of the SCM starting from our "cold" initial conditions. The reason why a hot universe can emerge at the end of our inflationary epochs (phases I and II) goes back to an idea of L. Parker[40], according to which amplified quantum flluctuations can give origin to the CMB itself if Planckian scales are reached.

   Rephrasing Parker's idea in our context amounts to solving the following bootstrap-like condition: at which moment, if any, will the energy stored in the perturbations reach the critical density?

   The total energy density ρ_{qf} stored in the amplified vacuum quantum fluctuations is given by:

$$\rho_{qf} \sim N_{eff}\, \frac{M_s^4}{4\pi^2}\, (a_1/a)^4 \ , \tag{4.19}$$

   where N_{eff} is the number of effective (relativistic) species, which get produced (whose energy density decreases like a^{-4}) and a_1 is the scale factor at the (supposed) moment of branch-change. The critical density (in the same units) is given by:

$$\rho_{cr} = e^{-\phi} M_s^2 H^2 \ . \tag{4.20}$$

   At the beginning, with $e^{\phi} \ll 1$, $\rho_{qf} \ll \rho_{cr}$ but, in the $(-)$ branch solution, ρ_{cr} decreases faster than ρ_{qf} so that, at some moment, ρ_{qf} will become the dominant sort of energy while the dilaton kinetic term will become negligible. It would be interesting to find out what sort of initial temperatures for the radiation era will come out of this assumption.

5 Conclusions

I want to conclude by listing which are, in my opinion, the pluses and minuses of the scenario I have advocated:

The Goodies

- Inflation comes naturally, without ad-hoc fields and fine-tuning: there is even an underlying symmetry yielding inflationary solutions.
- Initial conditions are natural, yet a simple universe would evolve into a rich and complex one.
- The kinematical problems of the SCM are solved.
- Perturbations do not grow too fast to spoil homogeneity.
- An interesting characteristic spectrum of GW is generated.
- Larger-than-usual electromagnetic perturbations are easily generated and could explain the galactic magnetic fields.
- A hot big bang could be a natural outcome of our inflationary scenario.

The Baddies

- A scale-invariant spectrum is all but automatic (unlike what happens in normal vacuum-energy-driven inflation).
- Our understanding of the high curvature (stringy) phase and of the crucially needed change of branch is still poor in spite of recent progress in Conformal Field Theory.

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References

[1] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, Inc., New York (1972).

[2] G. Smoot et al., *Astrophys. J.* **396** (1992) L1.

[3] L.F. Abbott and So-Young Pi (eds.), *Inflationary Cosmology*, World Scientific, Singapore (1986).

[4] E. Kolb and M. Turner, *The Early Universe*, Addison-Wesley, New York (1990).

[5] G. Veneziano, *Europhys. Lett.* **2** (1986) 133.

[6] G. Veneziano, "Quantum strings and the constants of Nature", in *The Challenging Questions* (Erice, 1989), ed. A. Zichichi, Plenum Press, New York (1990).

[7] C. Lovelace, *Phys. Lett.* **B135** (1984) 75;
C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, *Nucl. Phys.* **B262** (1985) 593.

[8] E.S. Fradkin and A.A. Tseytlin, *Nucl. Phys.* **B261** (1985) 1.

[9] P. Jordan, *Z. Phys.* **157** (1959) 112;
C. Brans and R.H. Dicke, *Phys. Rev.* **124** (1961) 925.

[10] E. Witten, *Phys. Lett.* **B149** (1984) 351.

[11] T.R. Taylor and G. Veneziano, *Phys. Lett.* **B213** (1988) 459.

[12] J. Ellis et al., *Phys. Lett.* **B228** (1989) 264.

[13] See, for instance, E. Fischbach and C. Talmadge, *Nature* **356** (1992) 207.

[14] T. Damour and A. M. Polyakov, *Nucl. Phys.* **B423** (1994) 532.

[15] G. Veneziano, *Phys. Lett.* **B265** (1991) 287; Proceeding 4th PASCOS Conference (Syracuse, May 1994), K.C. Wali ed., World Scientific, Singapore, p. 453.

[16] M. Gasperini and G. Veneziano, *Astropart. Phys.* **1** (1993) 317; *Mod. Phys. Lett.* **A8** (1993) 3701; *Phys. Rev.* **D50** (1994) 2519.

[17] A.A. Tseytlin, *Mod. Phys. Lett.* **A6** (1991) 1721;
A.A. Tseytlin and C. Vafa, *Nucl. Phys.* **B372** (1992) 443.

[18] E. Kiritsis and C. Kounnas, *Phys. Lett.* **B331** (1994) 51;
A.A. Tseytlin, *Phys. Lett.* **B334** (1994) 315.

15: G. V.: Scale factor duality for classical and quantum strings PDF arXiv

16: Gasperini & V: Pre Big Bang in string cosmology arXiv

[19] P. Aspinwall, B. Greene and D. Morrison, *Phys. Lett.* **B303** (1993) 249;
E. Witten, *Nucl. Phys.* **B403** (1993) 159.

[20] R. Brustein and G. Veneziano, *Phys. Lett.* **B329** (1994) 429.

[21] N. Kaloper, R. Madden and K. A. Olive, *Towards a singularity-free inflationary universe?*, Univ. Minnesota preprint UMN-TH-1333/95 (June 1995).

[22] J. Levin and K. Freese, *Nucl. Phys.* **B421** (1994) 635.

[23] L.P. Grishchuk, *Sov. Phys. JEPT* **40** (1975) 409;
A.A. Starobinski, *JEPT Lett.* **30** (1979) 682;
V.A. Rubakov, M. Sazhin and A. Veryaskin, *Phys. Lett.* **B115** (1982) 189;
R. Fabbri and M. Pollock, *Phys. Lett.* **B125** (1983) 445.

[24] M. Gasperini, *Status of string cosmology: phenomenological aspects*, Proceedings of Workshop Erice, September 1995.

[25] M. Gasperini, N. Sanchez and G. Veneziano, *Int. J. Theor. Phys.* **A6** (1991) 3853; *Nucl. Phys.* **B364** (1991) 365;
M. Gasperini, *Phys. Lett.* **B258** (1991) 70;
G. Veneziano, *Helv. Phys. Acta* **64** (1991) 877.

[26] See, e.g. V. Mukhanov, H.A. Feldman and R. Brandenberger, *Phys. Rep.* **215** (1992) 203.

[27] R.B. Abbot, B. Bednarz and S.D. Ellis, *Phys. Rev.* **D33** (1986) 2147.

[28] R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov and G. Veneziano, *Phys. Rev.* **D51** (1995) 6744.

[29] G. F. R. Ellis and M. Bruni, *Phys. Rev.* **D40** (1989) 1804;
M. Bruni, G. F. R. Ellis and P. K. S. Dunsby, *Class. Quant. Grav.* **9** (1992) 921.

[30] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, *Relic gravitational waves from string cosmology*, CERN-TH/95-144 (1995), *Phys. Lett.* in press;
see also M. Gasperini and M. Giovannini, *Phys. Rev.* **D47** (1992) 1529.

[31] R.E. Vogt et al., *Laser Interferometer Gravitational-Wave Observatory*, proposal to the National Science Foundation (Caltech, 1989);
C. Bradascia et al., in *Gravitational Astronomy*, eds. D.E. McClelland and H. Bachor, World Scientific, Singapore, (1991).

[32] G. V. Pallottino and V. Pizzella, *Nuovo Cim.* **C4** (1981) 237;
M. Cerdonio et. al, *Phys. Rev. Lett.* **71** (1993) 4107.

[33] F. Pegoraro, E. Picasso and L. Radicati, *J. Phys.* **A11** (1978) 1949;
C. M. Caves, *Phys. Lett.* **B80** (1979) 323;
C. E. Reece et al., *Phys. Lett.* **A104** (1984) 341.

[34] P. Astone, J. A. Lobo and B. F. Schutz, *Class. Quant. Grav.* **11** (1994) 2093.

[35] E. N. Parker, *Cosmical Magnetic Fields*, Clarendon, Oxford (1979);
Y. B. Zeldovich, A. A. Ruzmaikin and D. D. Sokoloff, *Magnetic fields in astrophysics*, Gordon and Breach, New York (1983).

[36] M. Gasperini, M. Giovannini and G. Veneziano, *Primordial magnetic fields from string cosmology*, CERN-TH/95-85 (April 1995) *Phys. Rev. Lett.* in press.

[37] D. Lemoine and M. Lemoine, *Phys. Rev.* **D52** (1995) 1955.

[38] M. S. Turner and L. M. Widrow, *Phys. Rev.* **D37** (1988) 2743.

[39] M. Gasperini, M. Giovannini and G. Veneziano, *Electromagnetic origin of the CMB anisotropy in string cosmology*, CERN-TH/95-102 (April 1995) *Phys. Rev.* in press.

[40] L. Parker, *Nature* **261** (1976) 20.

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STRING COSMOLOGY: THE PRE-BIG BANG SCENARIO

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ABSTRACT

A review is attempted of physical motivations, theoretical and phenomenological aspects, as well as outstanding problems, of the pre-big bang scenario in string cosmology.

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1 Introduction

These four lectures aim at providing a summary of –and some guidance through– the existing literature dealing with the so-called pre-big bang (PBB) scenario, a new cosmological model largely based on the new symmetries underlying superstring cosmology. The lectures will be pedagogical in nature and will not presuppose an advanced knowledge either of modern inflationary cosmology or of superstring/M-theory. Elements of both will be included in the lectures in order to make them reasonably self-contained. More exhaustive treatments of pre-big bang cosmology are [1] (or will soon be [2]) available elsewhere, while a homepage on the PBB scenario is being kept updated on the Web [3].

The four lectures roughly correspond to the four forthcoming sections and deal, respectively, with:

- BASIC MOTIVATIONS AND IDEAS
- HOW COULD IT HAVE STARTED?
- PHENOMENOLOGICAL CONSEQUENCES
- HOW COULD IT HAVE STOPPED?

In particular, lecture II (Section 3) contains a discussion of the initial conditions, lecture III (Section 4) discusses the phenomenological virtues and shortcomings of the model, while lecture IV (Section 5) deals with the most important open theoretical issues.

2 Basic Motivations and Ideas

2.1 Why string cosmology?

The first question that comes to one's mind when thinking about cosmology and string theory is: Why bother? Indeed, even if string/M-theory is the correct theory of nature, only its effective (low-energy) quantum field theory description appears to be relevant to most of the history of our Universe, i.e. since a very short time after the big bang. This is certainly the case for the standard (hot-big-bang) cosmological model, but it is also true for the standard models of inflation, provided we confine our attention to what happened during the last 70 e-fold of inflation and later (i.e. to what happened *after* our present horizon reached the size of the inflationary Hubble radius). In both instances, one is only confronting situations in which curvatures are very small with respect to the fundamental scale of string theory.

On the other hand, both the hot-big bang model and its inflationary variant suffer from initial condition problems. In the former case, these are just the well-known homogeneity and flatness problems that motivated inflation. In the latter case, although the problems look less

severe, it is still a matter of heated discussion whether or not one should naturally expect a quasi-homogeneous inflaton field highly displaced from the minimum of its potential to emerge from the Planck era. In either case, the question of how to get physically appealing initial conditions lies in the realm of Planck-scale quantum gravity.

At present, the only candidate for a consistent synthesis of general relativity (GR) and quantum mechanics (QM) is superstring theory (see [4] for a recent review, as well as [5] for a non-specialized introduction), or, if we prefer, the mysterious M-theory that reduces to various superstring theories in appropriate limits. It thus seems mandatory to ask whether the above questions on initial conditions do –or do not– find an answer within string theory. Although most string theorists would certainly agree with the above statements –this being after all one of the most selling ads for string theory– many of them would still object to tackling these problems *now*. The "excuse" is that our understanding of string theory, especially at large curvatures, is still largely incomplete. Furthermore, most of the recent progress in non-perturbative string theory has been achieved in the context of "vacua" (i.e. classical solutions to the field equations) that respect a large number of supersymmetries. By definition, a cosmological background (a fortiori one that evolves rapidly in time) breaks (albeit spontaneously) supersymmetry. This is why the Planckian regime of cosmology appears to be intractable for the time being.

There is however a pleasant surprise. About ten years of work on string cosmology have led naturally to considering a scenario –the so-called pre-big bang (PBB) scenario– in which the Universe enjoyed a long perturbative "life" *before* the big bang. Starting from an almost trivial state (asymptotic past triviality, see Section 3), the Universe would have evolved towards stronger and stronger curvature and coupling, thereby inflating, until it entered the non-perturbative phase that replaces the big bang singularity of more standard cosmological models.

The situation is very much reminiscent of QCD and strong interactions. Perturbative QCD has been very successful in predicting a huge number of observables for short-distance-dominated hard processes. Successes in the non-perturbative, large-distance regime have been meagre, by comparison: we still lack a definitive proof of confinement, of spontaneous chiral symmetry breaking, of explicit $U(1)_A$ breaking, etc. Yet, we do believe that QCD is the correct description of hadronic physics down to scales of 10^{-15} cm or so. This is largely based on the belief that large- and short-distance physics "decouple", e.g. on the assumption that the soft hadronization process does not affect certain infrared-safe quantities computed at the quark–gluon level. Fortunately, we did not wait until the confinement problem was solved, to take QCD seriously!

A very similar attitude will be defended here in the case of string cosmology, with one amusing twist: large- and short-distance physics get somehow swapped as we go from QCD to gravity/cosmology. Figure 1 (from Ref. [6]) illustrates this point. The easy regime for gravity is at large distance/small curvatures; the tough one turns out to be the high-curvature regime that replaces here the big bang singularity. Yet, we shall argue that some consequences of string cosmology, those related to scales that were very large with respect to the string scale in the high-curvature regime, should not be affected, other than by a trivial kinematical red-shift, by

the details of the pre- to post-big bang transition ... provided, of course, that such a transition does indeed take place (the counterpart to assuming that confinement does occur in QCD).

The above reasoning does not imply, of course, that one should not address the hard questions *now*. On the contrary, the easy part of the game will give precious information on what the relevant hard questions are (for cosmology) and on how to formulate them. I have already mentioned an example of what I mean: insisting too much on (extended) SUSY vacua appears to be an unacceptable limitation for the problems at hand. Another example is that of demanding stability of an acceptable string vacuum: we shall see (in Section 4) that inflationary string vacua lead to tachyonic, i.e. to growing rather than to oscillating, modes. Such modes appear to horrify most string theorists; however, they are just what inflationary cosmologists happily use all the time in order to generate large-scale structure (LSS), and what PBB cosmology uses to generate heat and entropy from an initially cold Universe (see Section 5).

A completely different criticism of string cosmology comes from the cosmology end: for someone accustomed to a data-driven "bottom-up" approach, string cosmology is too much "top-down". There is certainly a point here. I do not believe that a good model of cosmology is likely to emerge from theoretical considerations alone. Input from the data will be essential in the selection among various theoretical alternatives. We shall see explicit examples of what I mean in Section 5. Yet, it appears that a combination of top-down and bottom-up would be highly desirable. If past history can teach us something in this respect, the construction of the standard model of particle physics (and of QCD in particular) is a perfect example of a fruitful interplay of theoretically sound ideas and beautiful experimental results. Cosmology today resembles the particle physics of the sixties: interesting new data keep coming in at a high pace, while compelling theoretical pillars on which to base our understanding of those data are still missing.

As a final remark, let me turn things around and claim that cosmology could be the only hope that we have for testing string theory in the foreseeable future by using the cosmos itself as the largest conceivable accelerator. The cosmological red-shift since the big bang has kindly brought down Planck-scale physics to a macroscopic scale, thus opening for us a window on the very early Universe. As we shall see in Subsection 2.3, even in this respect, standard and PBB inflation are markedly different.

2.2 Why/Which inflation?

The reasons why the standard hot-big-bang model is unsatisfactory have been repeatedly discussed in the literature. For details, we refer to two excellent reviews [7]. Let me briefly summarize here the basic origin of those difficulties with the simplest Friedmann–Robertson–Walker (FRW) cosmology.

In the FRW framework the size of the (now observable) Universe was about 10^{-2} cm at the start of the classical era, say at $t \sim$ a few times t_P, where $t_P \sim 10^{-43}$ s is the so-called Planck time. This is of course a very tiny Universe w.r.t. its present size ($\sim 10^{28}$ cm), yet it is huge

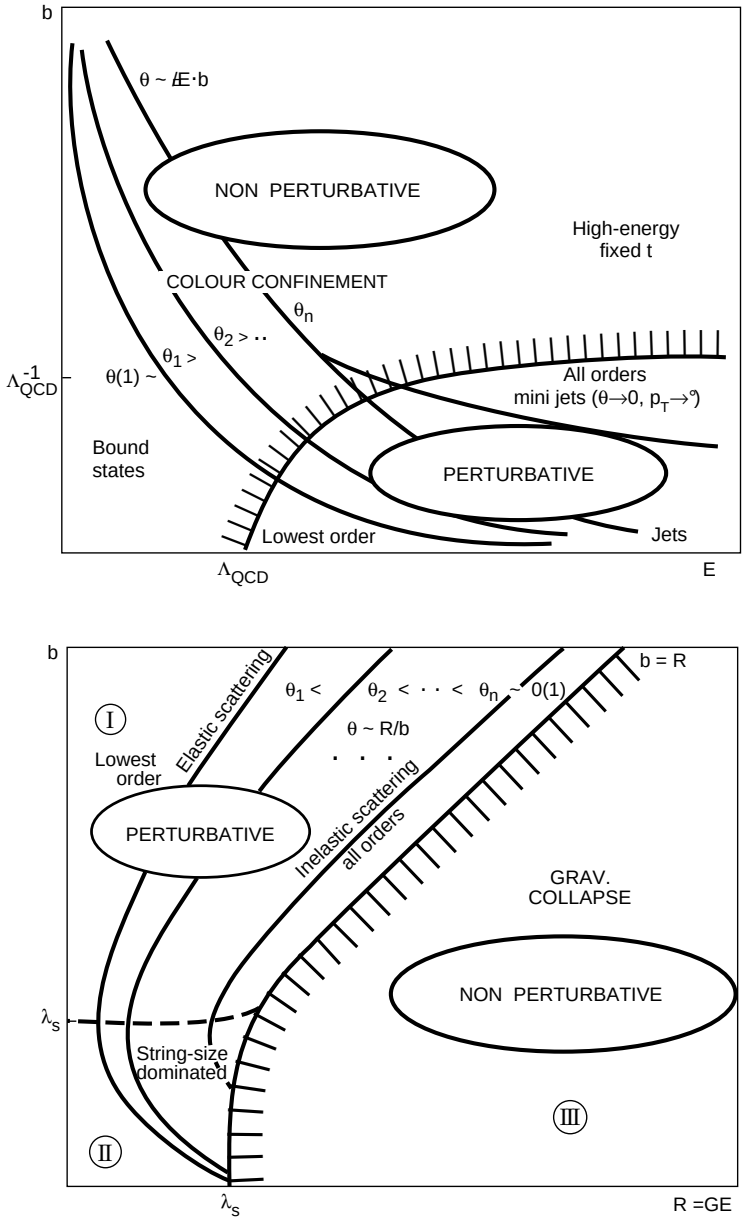
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Figure 1

w.r.t. the horizon (the distance travelled by light) at that time, i.e. to $l_P = ct_P \sim 10^{-33}$ cm. In other words, a few Planck times after the big bang, our observable Universe was much too large! It consisted of $(10^{30})^3 = 10^{90}$ Planckian-size, causally disconnected regions. There had not been, since the beginning, enough time for the Universe to become homogeneous (e.g. to thermalize) over its entire size. Also, soon after $t = t_P$, the Universe was characterized by a huge hierarchy between its Hubble radius on one side and its spatial-curvature radius on the other. The relative factor of (at least) 10^{30} appears as an incredible amount of fine-tuning on the initial state of the Universe, corresponding to a huge asymmetry between time and space derivatives. Was this asymmetry really there? And, if so, can it be explained in any, more natural way?

It should be stressed that, while the above unexplained ratio becomes larger and larger as we approach the Planck time (and would go to infinity at $t = 0$ if we could trust the equations throughout), it represents the ratio of two *classical* length scales. It so happens that one of the two lengths becomes the (quantum) Planck scale *at* $t = t_P$, but the ratio is still huge at much later times when both scales have nothing to do with (and are much larger than) t_P. This comment will be very relevant to the discussion of fine-tuning issues given in Subsection 3.6.

It is well known that a generic way to wash out inhomogeneities and spatial curvature consists in introducing, in the history of the Universe, a long period of accelerated expansion, called inflation [7]. This still leaves two alternatives: either the Universe was generic at the big bang and became flat and smooth because of a long *post*-bangian inflationary phase; or it was already flat and smooth at the big bang as a result of a long *pre*-bangian inflationary phase.

Assuming, dogmatically, that the Universe (and time itself) started at the big bang, leaves only the first alternative. However, that solution has its own problems, in particular those of fine-tuned initial conditions and inflaton potentials. Besides, it is quite difficult [8] to base standard inflation in the only known candidate theory of quantum gravity, superstring theory. Rather, as we shall argue in a moment, superstring theory gives strong hints in favour of the second (pre-big bang) possibility through two of its very basic properties, the first in relation to its short-distance behaviour, the second from its modifications of GR even at large distance.

2.3 Superstring-inspired cosmology

As just mentioned, two classes of properties of string theory are relevant for cosmology. Let us discuss them in turn.

A) Short-distance properties

Since the classical (Nambu–Goto) action of a string is proportional to the area A of the surface it sweeps, its quantization must introduce a quantum of length λ_s through:

$$S/\hbar = A/\lambda_s^2 \, . \tag{1}$$

This fundamental length, replacing Planck's constant in quantum string theory [9], plays the

role of a minimal observable length, of an ultraviolet cut-off. Thus, in string theory, physical quantities are expected to be bound by appropriate powers of λ_s, e.g.

$$\begin{aligned} H^2 \sim R \sim G\rho < \lambda_s^{-2} \\ k_B T/\hbar < c\lambda_s^{-1} \\ R_{comp} > \lambda_s \ . \end{aligned} \tag{2}$$

In other words, in quantum string theory, relativistic quantum mechanics should solve the singularity problems in much the same way as non-relativistic quantum mechanics solved the singularity problem of the hydrogen atom by keeping the electron and the proton a finite distance apart. By the same token, string theory gives us a rationale for asking daring questions such as: What was there before the big bang? Certainly, in no other present theory can such a question be meaningfully asked.

B) Large-distance properties

Even at large distance (low-energy, small curvatures), superstring theory does *not* automatically give Einstein's GR. Rather, it leads to a scalar–tensor theory of the JBD variety. The new scalar particle/field ϕ, the so-called dilaton, is unavoidable in string theory, and gets reinterpreted as the radius of a new dimension of space in so-called M-theory [10]. By supersymmetry, the dilaton is massless to all orders in perturbation theory, i.e. as long as supersymmetry remains unbroken. This raises the question: Is the dilaton a problem or an opportunity? My answer is that it could be both; and while we can try to avoid its potential dangers, we may try to use some of its properties to our advantage ... Let me discuss how.

In string theory, ϕ controls the strength of all forces [11], gravitational and gauge alike. One finds, typically:

$$l_P^2/\lambda_s^2 \sim \alpha_{gauge} \sim e^{\phi} \ , \tag{3}$$

showing the basic unification of all forces in string theory and the fact that, in our conventions, the weak-coupling region coincides with $\phi \ll -1$. In order not to contradict precision tests of the equivalence principle, and of the constancy of the gauge and gravitational couplings in the "recent" past, we require [12] the dilaton to have a mass (see, however, [13] for an amusing alternative) and to be frozen at the bottom of its own potential *today*. This does not exclude, however, the possibility of the dilaton having evolved cosmologically (after all, the metric did!) within the weak coupling region where it was practically massless. The amazing (yet simple) observation [14] is that, by so doing, the dilaton may have inflated the Universe!

A simplified argument, which, although not completely accurate, captures the essential physical point, consists in writing the Friedmann equation (for a spatially flat Universe):

$$3H^2 = 8\pi G\rho \ , \tag{4}$$

and in noticing that a growing dilaton (meaning through (3) a growing G) can drive the growth of H even if the energy density of standard matter decreases in an expanding Universe. This new

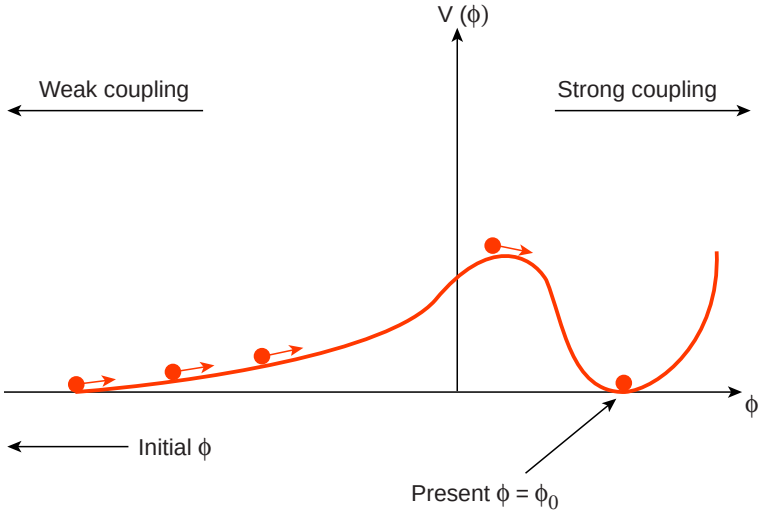

Figure 2

kind of inflation (characterized by growing H and ϕ) has been termed dilaton-driven inflation (DDI). The basic idea of pre-big bang cosmology [14, 15, 16, 17] is thus illustrated in Fig. 2: the dilaton started at very large negative values (where it was practically massless), ran over a potential hill, and finally reached, sometime in our recent past, its final destination at the bottom of its potential ($\phi = \phi_0$). Incidentally, as shown in Fig. 2, the dilaton of string theory can easily roll-up —rather than down— potential hills, as a consequence of its non-standard coupling to gravity.

DDI is not just possible. It exists as a class of (lowest-order) cosmological solutions thanks to the duality symmetries of string cosmology [14], [18], [19]. Under a prototype example of these symmetries, the so-called scale-factor duality (SFD) [14], [18], a FRW cosmology evolving (at lowest order in derivatives) from a singularity in the past is mapped into a DDI cosmology going towards a singularity in the future. Of course, the lowest order approximation breaks down before either singularity is reached. A (stringy) moment away from their respective singularities, these two branches can easily be joined smoothly to give a single non-singular cosmology, at least mathematically. Leaving aside this issue for the moment (see Section 5 for more discussion), let us go back to DDI. Since such a phase is characterized by growing coupling and curvature, it must itself have originated from a regime in which both quantities were very small. We take this as the main lesson/hint to be learned from low-energy string theory by raising it to the level of a new cosmological principle, that of "Asymptotic Past Triviality", to be discussed in the next Lecture.

2.4 Explicit solutions

Many explicit exact PBB-type solutions to the low-energy effective action equations have been constructed and discussed in the literature. For an excellent review, see [1]. Exact solutions can

only be obtained in the presence of symmetries (isometries) and, although they are heuristically very important, they are too special from the point of view of an inflationary cosmology, which, as such, should not accept fine-tuned initial conditions. This is why we shall not go into an exhaustive discussion of explicit solutions here. Instead, in Section 3, we will adress the general problem of the evolution of asymptotically trivial initial data.

Here we shall limit our attention to the simplest Bianchi I-type solutions and to their quasi-homogeneous generalizations, after recalling that many more solutions can be obtained from the former by using the non-compact $O(d,d)$ symmetry of the low-energy string-cosmology equations [19] when the Kalb–Ramond (KR) field $B_{\mu\nu}$ is turned on, or by S-duality transformations (see e.g. [1]) generating a homogeneous axion field (related to $B_{\mu\nu}$ by yet another duality transformation).

The generic homogeneous Bianchi I solution with $B_{\mu\nu}=0$ reads, for $t<0$,

$$\begin{aligned} ds^2 &= -dt^2+\sum_i (-t)^{2\alpha_i} dx^i dx^i \ , \\ \phi &= -(1-\sum_i \alpha_i)\log(-t) \\ 1 &= \sum_i \alpha_i^2 \ . \end{aligned} \tag{5}$$

i.e. represents a generalization of the well-known Kasner solutions (see e.g. [20]) in which one of the two Kasner constraints (the one linear in the α_i) is replaced by the equation giving the time dependence of ϕ (ϕ is absent, or constant, for Kasner, hence the second constraint).

Note that, unlike Kasner's, (5) allows for isotropic solutions ($\alpha_i=\pm 1/\sqrt{d}$ for all i). Also, the quadratic Kasner constraint automatically has 2^d SFD-related branches, obtained by changing the sign of any subset of the $\alpha' s$. Also note that the so-called shifted dilaton defined by:

$$\bar{\phi}=\phi-\frac{1}{2}\log\ (\det\ g_{ij}) \ , \tag{6}$$

which is invariant under the full $O(d,d)$ group, is always given by:

$$\bar{\phi}=-\log(-t) \ . \tag{7}$$

A quasi-homogeneous generalization of (5) was first discussed in [21] (see also [22]) and reads:

$$\begin{aligned} ds^2 &= -dt^2+\sum_a e_i^a(x)\ e_j^a(x)(-t)^{2\alpha_a(x)} dx^i dx^j \ , \\ \phi &= -(1-\sum_i \alpha_i(x))\log(-t) \\ 1 &= \sum_i \alpha_i^2(x) \quad , \quad t<0 \ , \end{aligned} \tag{8}$$

where x stands for the space coordinates. Equation (8) can be shown to be a generic asymptotic solution of the full PDEs near the $t=0$ singularity where spatial gradients become less and

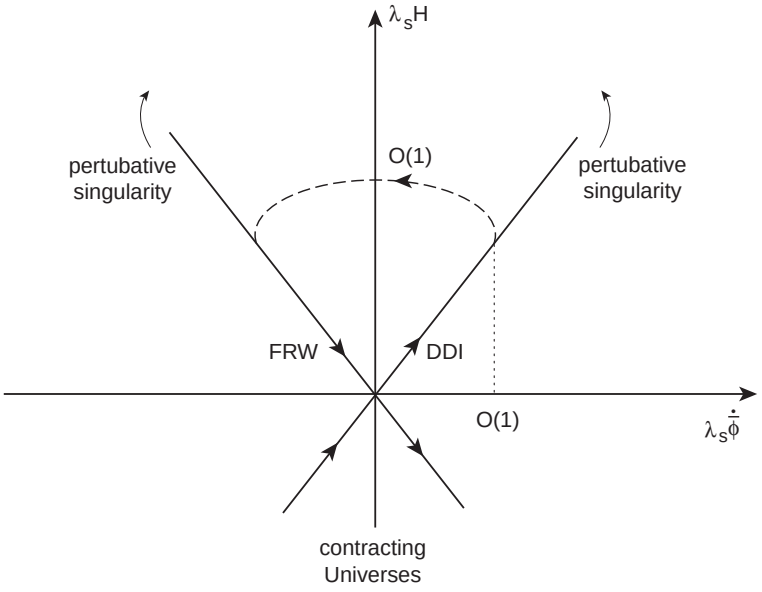

Figure 3

less important w.r.t. time derivatives, justifying the validity of the so-called gradient expansion [23]. Note that Eq. (7) is not modified in the quasi-homogeneous solutions. Besides allowing isotropic cosmologies in the homogeneous case, the presence of the dilaton also removes the necessity of a chaotic (BKL-type [24]) behaviour near the singularity [25].

2.5 Phase diagrams and Penrose-style overview

It is useful to visualize the PBB scenario with the help of some diagrams. Since the actual phase space of the model is multidimensional, each of these diagrams necessarily represents just a cross section of the complete picture.

A very commonly used diagram (Fig. 3) is the flow-diagram in the $\dot{\phi}, H$ plane (time being just a parameter along the flow lines). Since, at lowest order, $\ddot{\phi} \geq 0$, the flow is always from left to right near the origin. The four straight lines represent the four (isotropic for simplicity) solutions connected by SFD and time-reversal. The product of the two transformations represents the physically interesting case, since it maps ordinary decelerating FRW cosmology (top left) to dilaton-driven inflation (top right). Clearly, our scenario needs a high-curvature phase during which the left-to-right flow is inverted (as shown by the dotted line joining the two perturbative branches). This can only happen as the result of higher-order corrections (see Section 5).

A second useful diagram (Fig. 4) is the e^{ϕ}, H plot, i.e. the curvature (energy) coupling plane. The fully perturbative domain (where evolution starts according to the APT postulate) lies, in a log-log plot, to the far left-bottom corner. Sticking again, for simplicity, to the isotropic

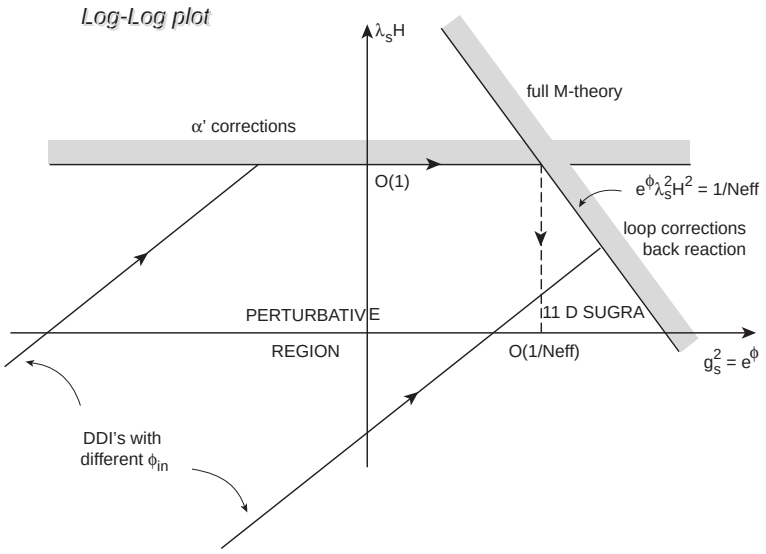

Figure 4

case, DDI evolution is represented by parallel lines distinguished by different initial values of the dilaton (i.e. of the coupling). It is clear that all these solutions run, eventually, into strong curvature or strong coupling (shown as thick solid lines), which one is hit first being determined by the above-mentioned initial coupling. A discussion of what might happen afterwards is given in Section 5.

As a third possibility, let us use a Carter–Penrose style plot [26] (Fig. 5) to represent, on a finite piece of paper, the entire evolution of the Universe. Unlike in ordinary cosmology, where the CP diagram is truncated by the (space-like) hypersurface of the big-bang singularity, here the whole CP diagram, going from past to future time-like and null infinities, is physically meaningful because of our assumption that finite-string-size effects remove the big bang singularity. This diagram will be discussed and used in the following sections.

Finally, let us represent the basic difference between the standard inflation scenario and that of PBB cosmology by plotting, for each cosmological model, the Hubble horizon (H^{-1}) and the physical scale that coincides with it today, as functions of cosmic time. This gives rise to two "wine glasses" (Fig. 6), which are very similar in their upper parts (corresponding to recent epochs) but differ markedly at very early times. The most salient difference appears in the early behaviour of the Hubble horizon, an increasing function of time in the standard inflation, a decreasing one in the PBB case. The figure allows me to stress one phenomenological advantage of PBB inflation: Planck- (or string)-scale physics, being no longer washed out by a long, subsequent inflationary phase, becomes accessible to present (or near-future) experiments at the millimetre (100 GHz) scale. At the same time, larger-scale experiments (such as those on small-angle CMB anisotropies) will test (sub-Planckian-energy) physics during the pre-bangian phase. By contrast, as we have already mentioned in the Introduction, in standard inflation large-scale data probe the Universe as it was seventy e-folds or so before the end of inflation,

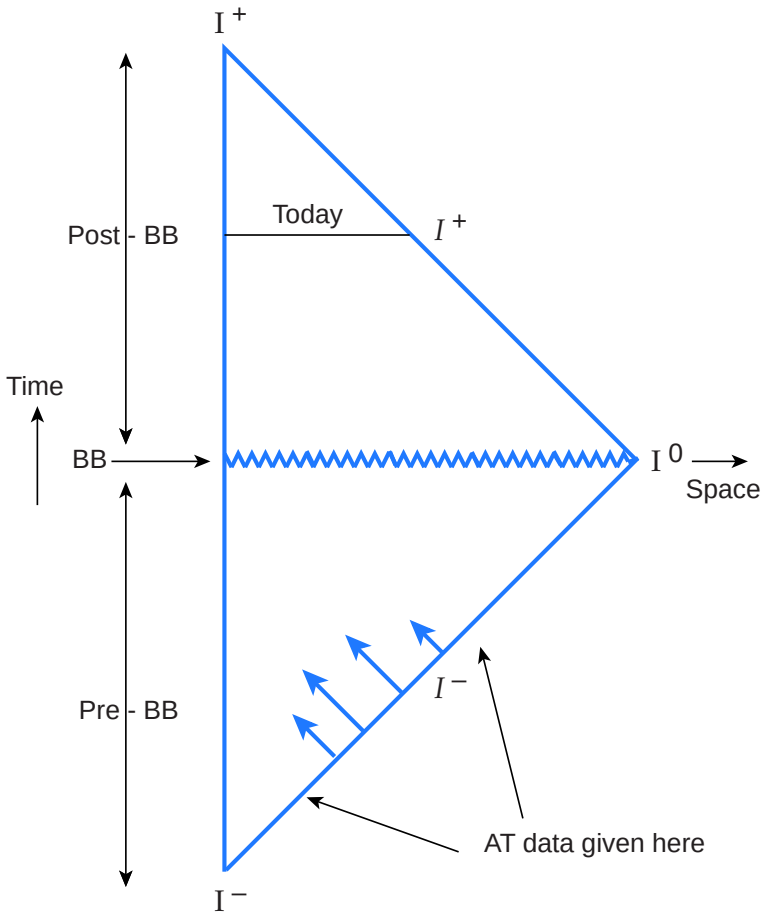

Figure 5

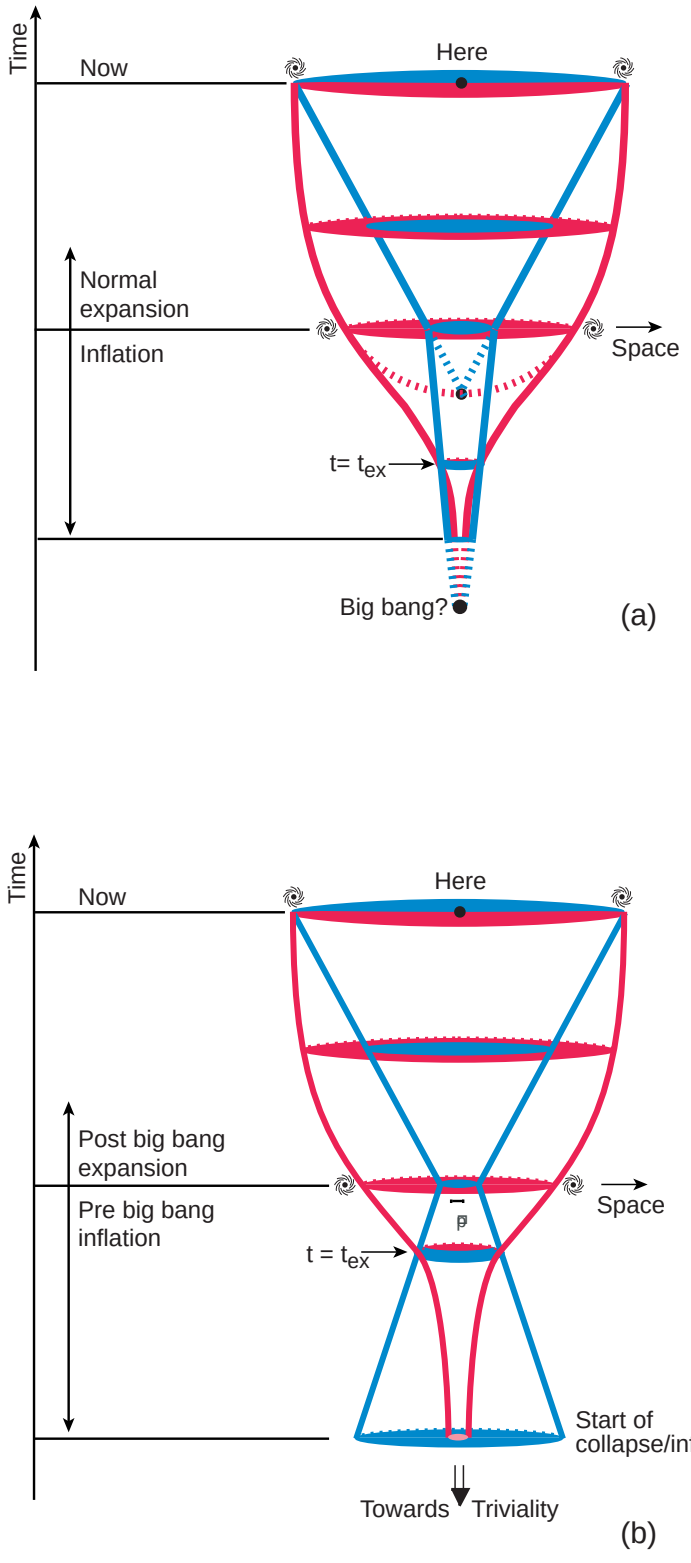

Figure 6
(a) Standard inflation's wineglass; (b) Pre-Big Bang's wineglass

while shorter scales tells us about more recent epochs. Since we know that, seventy e-folds before the end of inflation, H_{infl} was less than $10^{-5}M_P$ (or else excessive large scale anisotropies are created, see Section 4), and that such a scale slowly *decreases* during (slow-roll) inflation, it is clear that, according to standard inflation, physics at energies larger than $10^{-5}M_P$ remains unaccessible.

3 How could it have started?

3.1 Generic asymptotically-trivial past

We have already mentioned that, in standard non-inflationary cosmology, initial conditions have to be fine-tuned to incredible accuracy in the far past (i.e. at $t \sim t_P \sim 10^{-43}$ s). What does this fine-tuning problem look like if we accept hints from scale-factor duality and assume asymptotically trivial, yet generic, initial conditions?

The concept of asymptotic past triviality (APT) is quite similar to that of "asymptotic flatness", familiar from general relativity [27]. The main differences consist in making only assumptions concerning the asymptotic past (rather than future or space-like infinity) and in the additional presence of the dilaton. It seems physically (and philosophically) satisfactory to identify the beginning with simplicity (see e.g. the entropy-related arguments given in Subsection 5.7). What could be simpler than a trivial, empty and flat Universe? Nothing, of course! The problem is that such a Universe, besides being uninteresting, is also non-generic. By contrast, asymptotically flat/trivial Universes are initially simple, yet generic, in a precise mathematical sense that we shall now discuss.

From the point of view of space-time (taken here, for simplicity, to be $(3+1)$-dimensional) the generic solution depends upon four arbitray functions of three coordinates [28] related to the metric, plus two more each for the dilaton and the KR field $B_{\mu\nu}$. Amusingly, there is an exact correspondence between this "target-space" counting and a "world-sheet" counting. In the latter, those eight arbitrary functions correspond to eight arbitrary functions of three-momentum entering the most general physical (i.e. on shell) vertex operator describing gravitons, dilatons, and the KR field (which, in four dimensions, is equivalent to a pseudoscalar, the KR axion). We will see in Subsection 3.4 how these arbitrary functions appear in the asymptotic expansion of our fields.

Can a very rich and complicated Universe, like our own, emerge from such extremely simple initial conditions? This would look much like a miracle. However, as I shall argue below, this is precisely what should be expected, owing to well-known classical and quantum gravitational instabilities.

3.2 The asymptotic past's effective action and different (conformal) frames

The APT postulate implies that the early-time evolution of the Universe can be described in terms of the low-energy tree-level action of string theory. Taking a generic closed superstring theory, this reads:

$$\Gamma_{eff} = \lambda_s^{1-d} \int d^{d+1}x \sqrt{|g|}\, e^{-\phi} \left(R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}(dB)^2 - 2\,\Lambda \right) \quad , \tag{9}$$

where dB is the (three-form) field strength associated with $B_{\mu\nu}$.

A further simplification comes from assuming to be dealing with so-called critical superstring theory, the case in which the tree-level (and actually the all-order perturbative) cosmological constant Λ vanishes. This requires a total of $D = 10$ space-time dimensions. If $D \neq 10$ there will be an effective cosmological constant $O(\lambda_s^{-2})$ preventing any low-curvature solution of the field equations to exist. A similar conclusion is reached if we consider critical, but non-supersymmetric, string theories (see Subsection 3.3).

Equation (9) receives corrections when curvatures become $O(\lambda_s^{-2})$ *or* when the coupling e^ϕ becomes $O(1)$. If such corrections are both negligible, it sometimes becomes useful to perform a change of variable by going to the so-called Einstein frame (not to be confused with different frames in GR). This is done by defining:

$$g_{\mu\nu} = g^{(E)}_{\mu\nu} e^{\frac{2}{d-1}(\phi-\phi_0)} \ . \tag{10}$$

It is relatively easy to rewrite the action (9) using the Einstein metric. The result is simply:

$$\Gamma^E_{eff} = l_P^{1-d} \int d^{d+1}x \sqrt{|g^{(E)}|} \ \left(R - \frac{1}{d-1}\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{-\frac{4}{d-1}\phi}(dB)^2 \right) \ , \tag{11}$$

where $l_P^{d-1} = e^{\phi_0}\lambda_s^{d-1}$ is the present value of the Planck length.

Although the use of the Einstein frame could simplify some calculations, and we shall see examples of this below, it should be kept in mind that the form of the corrections is no longer so simple. For instance, higher-derivative corrections become important when the Einstein-frame curvature is $O(l_P^{-2}e^{-\frac{2}{d-1}\phi} = \lambda_s^{-2})$, i.e. reaches a dilaton-dependent critical value. Similarly, having a constant Newton "constant" in this frame is a mere illusion because (even tree-level) string masses do now depend upon ϕ. For these reasons, although physical results are frame-independent, we shall always describe them with reference to the original string-frame metric in which the stringh length λ_s is constant.

Let us finally remark that the two frames have been made to coincide today, with the dilaton fixed at its present value ϕ_0. Similarly, the assumption of APT would also allow the identification of the two frames in the far past, since the dilaton approaches a constant as $t \to -\infty$. However, the two Einstein frames that coincide with the string frame at $t = \pm\infty$ differ from each other by an enormous conformal factor, i.e. by a huge blowing-up of all physical scales.

3.3 Classical asymptotic symmetries: the importance of SUSY

The classical equations that follow from varying (9) or (11), besides being generally covariant, are also invariant under a two-parameter group of (global) transformations acting as follows:

$$\begin{aligned} \phi &\rightarrow \phi + c \,, \\ g_{\mu\nu} &\rightarrow \lambda^2 g_{\mu\nu} \,. \end{aligned} \tag{12}$$

Indeed (9), (11) are simply rescaled by a constant factor under this group. These two symmetries depend crucially on the validity of the tree-level low-energy approximation and on the absence of a cosmological constant. Loop corrections clearly spoil invariance under dilaton shifts, while lower derivatives (a cosmological constant) or higher derivatives (α') corrections spoil invariance under a rescaling of the metric. Note that, using general covariance, the latter symmetry is equivalent to an overall rescaling of all the coordinates. The relevance of the two classical symmetries on the issue of fine-tuning will become obvious in the next two subsections.

The importance of dealing with critical superstring theory now becomes evident: if one would consider non-supersymmetric string theories, a cosmological constant would almost certainly be generated at some finite order of the loop expansion: this would change completely the large-distance properties and spoil the symmetries of the field equations.

3.4 Dilaton-driven inflation as gravitational collapse

For simplicity, we will only illustrate here the simplest case of gravi-dilaton system already compactified to four space-time dimensions. Through the field redefinition (10), our problem is reduced to the study of a massless scalar field minimally coupled to gravity. It is well known that such a form of matter cannot give inflation (since it has positive pressure). Instead, it can easily lead to gravitational collapse (GC). Thus, in the Einstein frame, the problem becomes that of finding out under which conditions gravitational collapse occurs if asymptotically-trivial initial data are assigned. Gravitational collapse usually means that the (Einstein) metric (hence the volume of 3-space) shrinks to zero at a space-like singularity. However, typically, the dilaton blows up at that same singularity. Given the relation (10) between the Einstein and the (physical) string metric, we can easily imagine that the latter blows up near the singularity, as implied by DDI.

How generically does GC happen? Let us recall the singularity theorems of Hawking and Penrose [29], which state that, under some general assumptions, singularities are inescapable in GR. Looking at the validity of those assumptions in the case at hand, one finds that all but one are automatically satisfied. The only condition to be imposed is the existence of a closed trapped surface (CTS) (a closed surface from which future light cones lie entirely in the region inside the surface). Rigorous results [30] show that this condition cannot be waived: sufficiently weak initial data do not lead to closed trapped surfaces, to collapse, or to singularities. Sufficiently

strong initial data do. But where is the border-line? This is not known in general, but precise criteria do exist for particularly symmetric space-times, e.g. for those endowed with spherical symmetry (see Subsection 3.6).

However, no matter what the general collapse/singularity criterion will eventually turn out to be, we do know, from the classical symmetries described in the previous subsection, that such a criterion cannot depend

- on an over-all additive constant in ϕ, or
- on an over-all multiplicative factor in $g_{\mu\nu}$.

A characterization of APT initial data can be made [31] following the pioneering work [27] of Bondi, Sachs, Penrose, and others. Since our initial quanta are assumed to consist of massless gravitons and dilatons, their past infinity is null: it is the famous $\mathcal{I}^-$ of the Penrose diagram (Fig. 5). APT means that dilaton and metric can be expanded near $\mathcal{I}^-$ in inverse powers of $r \to \infty$, while advanced time v and two angular variables, θ and φ, are kept fixed. We shall thus write:

$$\phi(x^\lambda) = \phi_0 + \frac{f(v,\theta,\varphi)}{r} + o\left(\frac{1}{r}\right) , \tag{13}$$

$$g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + \frac{f_{\mu\nu}(v,\theta,\varphi)}{r} + o\left(\frac{1}{r}\right) . \tag{14}$$

The null wave data on $\mathcal{I}^-$ are: the asymptotic dilatonic wave form $f(v,\theta,\varphi)$, and two polarization components, $f_+(v,\theta,\varphi)$ and $f_\times(v,\theta,\varphi)$, of the asymptotic gravitational wave form $f_{\mu\nu}(v,\theta,\varphi)$, whose other components can be gauged away. The three functions f, f_+, $f_\times$ of v,θ,φ are equivalent to six functions of r,θ,φ with $r \geq 0$, because the advanced time v ranges over the full line $(-\infty,+\infty)$. This is how the six arbitrary functions of the generic solution to the gravi-dilaton system are recovered.

Of particular interest here are the so-called News functions, simply given by

$$N(v,\theta,\varphi) \equiv \partial_v\, f(v,\theta,\varphi) \ , \ N_+ \equiv \partial_v\, f_+ \ , \ N_\times \equiv \partial_v\, f_\times \, , \tag{15}$$

and the "Bondi mass" given by:

$$\begin{aligned} M_-(v) &= \frac{1}{4\pi}\int d^2\Omega M_-(v,\theta,\varphi) \ , \\ g_{vv} &= -\left(1 - \frac{2M_-(v,\theta,\varphi)}{r}\right) + o\left(\frac{1}{r}\right) . \end{aligned} \tag{16}$$

The Bondi mass and the News are connected by the energy–momentum conservation equation, which tells us that the advanced-time derivative of $M_-(v)$ is positive-semidefinite and related to incoming energy fluxes controlled by the News:

$$dM_-(v)/dv = \frac{1}{4}\int d^2\Omega \left(N^2 + N_+^2 + N_\times^2\right) . \tag{17}$$

The physical meaning of $M_-(v)$ is that it represents the energy brought into the system (by massless sources) by advanced time v. In the same spirit one can define the Bondi mass $M_+(u)$ at future null infinity $\mathcal{I}^+$. It represents the energy still present in the system at retarded time u. If only massless sources are present, the so-called ADM mass is given by

$$M_-(+\infty) = M_+(-\infty) = M_{ADM} \ , \tag{18}$$

while $M_-(-\infty) = 0$, and $M_+(+\infty) = M_C$ represents the mass that has not been radiated away even after waiting an infinite time, i.e. the mass that underwent gravitational collapse [32]. Collapse (resp. no-collapse) criteria thus aim at establishing under which initial conditions one expects to find $M_C > 0$ (resp. $M_C = 0$).

Since, as we shall see in the particular case of spherical symmetry, collapse criteria i) do not involve any particularly large number, and ii) do not contain any intrinsic scale but just dimesionless ratios of various classical scales, we expect i) gravitational collapse to be quite a generic phenomenon and ii) that nothing, at the level of our approximations, will be able to fix either the size of the horizon or the value of ϕ at the onset of collapse. Generically, and quite randomly and chaotically, some regions of space will undergo gravitational collapse, will form horizons and singularities therein. When this is translated into the string frame, the region of space-time within the horizon undergoes a period of DDI in which both the initial value of the Hubble parameter and that of ϕ are left arbitrary. In the next subsection we shall see that such arbitrariness provides an answer to the fine-tuning allegations that have been recently moved [33] to the PBB scenario. This section will be concluded with a discussion of how more precisely the case of spherical symmetry can be dealt with.

3.5 Fine-tuning issues

The two arbitrary parameters discussed in the previous subsection are very important, since they determine the range of validity of our description. In fact, since both curvature and coupling increase during DDI, the low-energy and/or tree-level description is bound to break down at some point. The smaller the initial Hubble parameter (i.e. the larger the initial horizon size) and the smaller the initial coupling, the longer we can follow DDI through the effective action equations and the larger the number of reliable e-folds we shall gain.

This does answer, in my opinion, the objections raised recently [33] to the PBB scenario according to which it is fine-tuned. The situation here actually resembles that of chaotic inflation [34]. Given some generic (though APT) initial data, we should ask which is the distribution of sizes of the collapsing regions and of couplings therein. Then, only the "tails" of these distributions, i.e. those corresponding to sufficiently large, and sufficiently weakly coupled, regions will produce Universes like ours, the rest will not. The question of how likely a "good" big bang is to take place is not very well posed and can be greatly affected by anthropic considerations [31].

In conclusion, we may summarize recent progress on the problem of initial conditions by saying that [31]:

Dilaton-Driven Inflation in String Cosmology
is as generic as
Gravitational Collapse in General Relativity.

Furthermore, asking for a sufficiently long period of DDI amounts to setting upper limits on two arbitrary moduli of the classical solutions.

Figure 7 (from Ref. [31]) gives a $(2+1)$-dimensional sketch of a possible PBB Universe: an original "sea" of dilatonic and gravity waves leads to collapsing regions of different initial size, possibly to a scale-invariant distribution of them. Each one of these collapses is reinterpreted, in the string frame, as the process by which a baby Universe is born after a period of PBB inflationary "pregnancy", the size of each baby Universe being determined by the duration of the corresponding pregnancy, i.e. by the initial size of (and coupling in) the corresponding collapsing region. Regions initially larger than 10^{-13} cm can generate Universes like ours, smaller ones cannot.

A basic difference between the large numbers needed in (non- inflationary) FRW cosmology and the large numbers needed in PBB cosmology should be stressed. In the former, the ratio of two classical scales, e.g. of total curvature to its spatial component, which is expected to be $O(1)$, has to be taken as large as 10^{60}. In the latter, the above ratio is initially $O(1)$ in the collapsing/inflating region, and ends up being very large in that same region, thanks to DDI. However, the (common) order of magnitude of these two classical quantities is a free parameter, and it is taken to be much larger than the classically irrelevant quantum scale. Indeed, the smallness of quantum corrections (which would introduce a scale in the problem) was explicitly checked in [35].

We can visualize analogies and differences between standard and pre-big bang inflation by looking again at Figs. 6a and 6b. The common feature in the two pictures is that the fixed comoving scale corresponding to the present horizon was "inside the horizon" for some time during inflation, possibly very deeply inside at its onset. The difference between the two scenarios is just in the behaviour of the Hubble radius during inflation: increasing in standard inflation (a), decreasing in string cosmology (b). Thus, while standard inflation is still facing the initial-singularity question and needs a non-adiabatic phenomenon to reheat the Universe (a kind of small bang), PBB cosmology faces the singularity problem later, combining it with the exit and heating problems (see Section 5).

3.6 The spherically symmetric case

In the spherically symmetric case many authors have studied the problem of gravitational collapse of a minimally coupled scalar field both numerically and analytically. In the former case I will only mention the well-known results of Choptuick [36], pointing at mysterious universalities near critical collapse (i.e. at the border-line situation in which the collapse criteria are just barely met). In this case, a very small black hole forms. This is not the case we are really interested in for the reasons we just explained. We shall thus turn, instead, to what happens

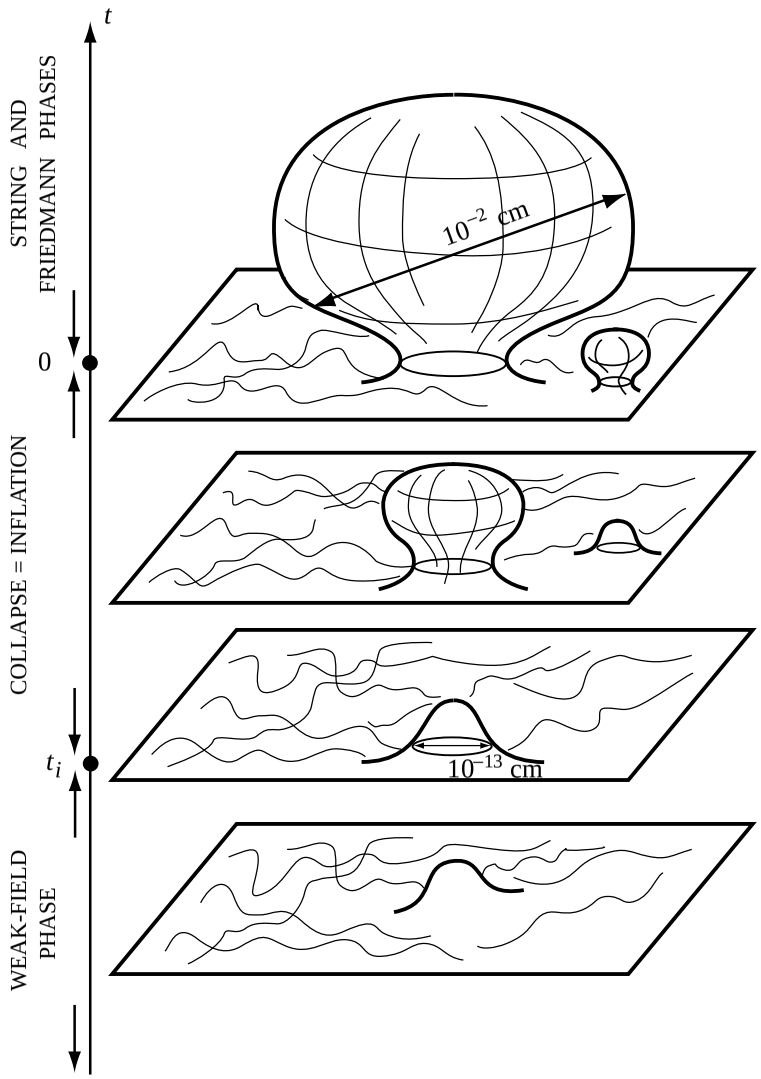
t
STRING AND
FRIEDMANN PHASES
0
COLLAPSE = INFLATION
t_i
WEAK-FIELD
PHASE
10^{-2} cm
10^{-13} cm

Figure 7

when the collapse criteria are largely fulfilled. For this we make use of the rather powerful results due to Christodoulou over a decade of beautiful work [30], [32], [37], [38].

There are no gravitational waves in the spherically symmetric case so that null wave data consist of just an angle-independent asymptotic dilatonic wave form $f(v)$, with the associated scalar News $N(v) = f'(v)$.

A convenient system of coordinates is the double null system, (u, v), such that

$$\phi = \phi(u, v)\,, \tag{19}$$

$$ds^2 = -\Omega^2(u, v)\, du\, dv + r^2(u, v)\, d\omega^2\,, \tag{20}$$

where $d\omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$. The field equations are conveniently re-expressed in terms of the three functions $\phi(u, v)$, $r(u, v)$ and $m(u, v)$, where the local mass function $m(u, v)$ is defined by:

$$1 - \frac{2m}{r} \equiv g^{\mu\nu}\,(\partial_\mu r)\,(\partial_\nu r) = -\frac{4}{\Omega^2}\,\frac{\partial r}{\partial u}\,\frac{\partial r}{\partial v}\,. \tag{21}$$

One gets the following set of evolution equations for m, r and ϕ

$$2\frac{\partial r}{\partial u}\,\frac{\partial m}{\partial u} = \left(1 - \frac{2m}{r}\right)\,\frac{r^2}{4}\,\left(\frac{\partial \phi}{\partial u}\right)^2\,, \tag{22}$$

$$2\frac{\partial r}{\partial v}\,\frac{\partial m}{\partial v} = \left(1 - \frac{2m}{r}\right)\,\frac{r^2}{4}\,\left(\frac{\partial \phi}{\partial v}\right)^2\,, \tag{23}$$

$$r\,\frac{\partial^2 r}{\partial u \partial v} = \frac{2m}{r - 2m}\,\frac{\partial r}{\partial u}\,\frac{\partial r}{\partial v}\,, \tag{24}$$

$$r\,\frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial r}{\partial u}\,\frac{\partial \phi}{\partial v} + \frac{\partial r}{\partial v}\,\frac{\partial \phi}{\partial u} = 0\,. \tag{25}$$

The quantity

$$\mu(u, v) \equiv \frac{2m(u, v)}{r} \tag{26}$$

plays a crucial rôle in the problem. If μ stays everywhere below 1, the field configuration will not collapse but will finally disperse at infinity as outgoing waves. By contrast, if the mass ratio μ can reach anywhere the value 1, this signals the formation of an apparent horizon $\mathcal{A}$. The location of this apparent horizon is indeed defined by the equation

$$\mathcal{A}: \quad \mu(u, v) = 1\,. \tag{27}$$

The above statements are substantiated by some rigorous inequalities [38] stating that:

$$\frac{\partial r}{\partial u} < 0\,, \qquad \frac{\partial m}{\partial v} > 0\,, \tag{28}$$

$$\frac{\partial r}{\partial v}\,(1 - \mu) > 0\,, \qquad \frac{\partial m}{\partial u}\,(1 - \mu) < 0\,. \tag{29}$$

Thus, in weak-field regions ($\mu < 1$), $\partial_v r > 0$, while, as $\mu > 1$, $\partial_v r < 0$, meaning that the outgoing radial null rays ("photons") emitted by the sphere $r = \text{const}$ become convergent, instead of having their usual behaviour. This is nothing else but the signature of a CTS!

In the case of spherical symmetry, it has been possible to prove [37] that the presence of trapped surfaces implies the existence of a future singular boundary $\mathcal{B}$ of space-time where a curvature singularity occurs. Furthermore, the behaviour of various fields near the singularity is just that of a quasi-homogeneous DDI as described by Eqs. (8)! This highly non-trivial result strongly supports the idea that PBB inflation in the string frame is the counterpart of gravitational collapse in the Einstein frame.

Reference [37] gives the following sufficient criterion on the strength of characteristic data, considered at some finite retarded time u

$$\frac{2\Delta m}{\Delta r} \geq \left[\frac{r_1}{r_2}\,\log\left(\frac{r_1}{2\Delta r}\right) + \frac{6r_1}{r_2} - 1\right], \tag{30}$$

where $r_1 \leq r_2$, with $r_2 \leq 3r_1/2$, are two spheres, $\Delta r = r_2 - r_1$ is the width of the "annular" region between the two spheres, and $\Delta m = m_2 - m_1 \equiv m(u, r_2) - m(u, r_1)$ is the mass "contained" between the two spheres, i.e. more precisely the energy flux through the outgoing null cone $u = \text{const}$, between r_1 and r_2. Note the absence of any intrinsic scale (in particular of any short-distance cut-off) in the above criterion. The theorem proved in [37] is not exhausted in the above statement. It contains various bounds as well, e.g.

- an upper bound on the retarded time at which the CTS (i.e. a horizon) is formed,
- a lower bound on the mass, i.e. on the radius of the collapsing region.

The latter quantity is very important for the discussion of the previous subsection since it gives, in the equivalent string-frame problem, an upper limit on the Hubble parameter at the beginning of DDI. Such an upper limit depends only on the size of the advanced-time interval satisfying the CC; since the latter is determined by the scale-invariant condition (30), the initial scale of inflation will be classically undetermined.

The above criterion is rigorous but probably too conservative. It also has the shortcoming that it cannot be used directly on $\mathcal{I}^-$, since $u \to -\infty$ on $\mathcal{I}^-$. In Ref. [31] a less rigorous (or less general) but simpler criterion directly expressible in terms of the News (i.e. on $\mathcal{I}^-$) was proposed on the basis of a perturbative study. It has the following attractive form:

$$\sup_{\substack{v_1, v_2 \\ v_1 \leq v_2}} \mathrm{Var}(N(x))_{x\in[v_1,v_2]} > C = O(1/4)\,, \tag{31}$$

where:

$$\mathrm{Var}\,(N(x))_{x\in[v_1,v_2]} \equiv \langle (N(x) - \langle N\rangle_{[v_1,v_2]})^2\rangle_{x\in[v_1,v_2]}\,. \tag{32}$$

Thus $\mathrm{Var}\,(g)_{[v_1,v_2]}$ denotes the "variance" of the function $g(x)$ over the interval $[v_1, v_2]$, i.e. the average squared deviation from its mean value.

According to this criterion the largest interval satisfying (31) determines the size of the collapsing region and thus, through the collapse inflation connection, the initial value of the Hubble parameter. It would be interesting to confirm the validity of the above criterion and to determine more precisely the value of the constant appearing on its r.h.s. through more analytic or numerical work. Actually, numerical studies of spherically symmetric PBB cosmologies have already appeared [39], while more powerful numerical codes should soon be available [40].

4 Phenomenological Consequences

4.1 Cosmological amplification of vacuum fluctuations: general properties

I will start by recalling the basic physical mechanism underlying particle production in cosmology (for a nice review, see [41]) and by introducing the corresponding (and by now standard) jargon. By the very definition of inflation ($\ddot{a} > 0$) physical wavelengths are stretched past the Hubble scale (H^{-1}) during inflation. After the end of inflation each wavelength grows slower than H^{-1} and thus "re-enters" the horizon. Obviously, the larger the scale the earlier it crosses the horizon outward and the later it crosses it back inward. Hence larger scales "spend" more time "outside the horizon" than smaller ones.

The attentive reader may worry at this point about the way this description applies when distances are measured using the Einsten-frame metric. As we have seen in the previous section, PBB inflation corresponds to accelerated contraction in the Einstein frame. Nonetheless, one can show that physical quantities (that is, typically, dimensionless ratios of physical quantities) do not depend on the choice of the frame: after all, changing frame is nothing more than a local field-redefinition, which is known not to affect the physics. It is amusing to notice, for instance, that physical wavelengths go outside the horizon during the Einstein-frame equivalent of DDI. Indeed, although physical EF scales shrink during the collapse, the horizon H^{-1} shrinks even faster! I refer to the first paper in [16] for further discussion on this point.

Consider now a generic perturbation Ψ on top of a homogeneous background, which includes a cosmological-type metric, a dilaton, and, possibly, other fields, such as another inflaton field, an axion, etc. Since $\Psi = 0$ is, by definition of a perturbation, a classical solution, Ψ intself enters the effective low-energy action quadratically. Soon after the beginning of inflation the background itself becomes homogeneous, isotropic, and spatially flat, so that the perturbed action takes the generic form:

$$I = \frac{1}{2} \int d\eta \; d^3x \; S(\eta) \left[\Psi'^2 - (\nabla \Psi)^2 \right] . \tag{33}$$

Here η is conformal-time ($ad\eta = dt$), and a prime denotes $\partial/\partial\eta$. The function $S(\eta)$ (sometimes called the "pump" field) is, for any given Ψ, a given function of the scale factor $a(\eta)$, and of other scalar fields (four-dimensional dilaton $\phi(\eta)$, moduli $b_i(\eta)$, etc.), which may appear non-trivially in the background.

While it is clear that a constant S may be reabsorbed by rescaling Ψ, and is thus ineffective, a time-dependent S couples non-trivially to Ψ and leads to the production of pairs of quanta (with equal and opposite momenta). In order to see this, it is useful to go over to a Hamiltonian description of the perturbation and of its canonically conjugate momentum Π:

$$\Pi = \frac{\delta I}{\delta \Psi'} = S\ \Psi' \,. \tag{34}$$

The Hamiltonian corresponding to (33) is thus given by

$$H = \frac{1}{2}\int d^3x \Big[S^{-1}\Pi^2 + S(\nabla\Psi)^2 \Big], \tag{35}$$

and the first-order Hamilton equations read

$$\Psi' = \frac{\delta H}{\delta \Pi} = S^{-1}\Pi \,, \qquad\qquad \Pi' = -\frac{\delta H}{\delta \Psi} = S\nabla^2\Psi \,, \tag{36}$$

leading to the decoupled second order equations

$$\Psi'' + \frac{S'}{S}\Psi' - \nabla^2\Psi = 0\,, \qquad\qquad \Pi'' - \frac{S'}{S}\Pi' - \nabla^2\Pi = 0\,. \tag{37}$$

In Fourier space the Hamiltonian (35) is given by

$$H = \frac{1}{2}\sum_{\vec{k}} \Big(S^{-1}\Pi_{\vec{k}}\Pi_{-\vec{k}} + Sk^2\Psi_{\vec{k}}\Psi_{-\vec{k}} \Big), \tag{38}$$

where $\Psi_{-\vec{k}} = \Psi^*_{\vec{k}}$ and $\Pi_{-\vec{k}} = \Pi^*_{\vec{k}}$. The equations of motion become

$$\Psi'_{\vec{k}} = S^{-1}\Pi_{-\vec{k}} \,, \qquad\qquad \Pi'_{\vec{k}} = -Sk^2\Psi_{-\vec{k}} \,, \tag{39}$$

where $k = |\vec{k}|$. The transformation

$$\Pi_{\vec{k}} \to \widetilde{\Pi}_{\vec{k}} = k\Psi_{\vec{k}}\,, \qquad \Psi_{\vec{k}} \to \widetilde{\Psi}_{\vec{k}} = -k^{-1}\Pi_{\vec{k}}\,, \qquad S \to \widetilde{S} = S^{-1} \tag{40}$$

leaves the Hamiltonian, Poisson brackets, and equations of motion unchanged. This symmetry of linear perturbation theory, and its physical consequences, was discussed in [42] under the name of S-duality, since it contains the usual strong–weak coupling (electric–magnetic) duality in the special case of gauge perturbations.

In order to solve the perturbation equations, and to normalize the spectrum, it is convenient to introduce the normalized (but no longer canonically conjugate) variables $\widehat{\Psi}$, $\widehat{\Pi}$, whose Fourier modes are defined by

$$\widehat{\Psi}_k = S^{1/2}\ \Psi_k \,, \qquad\qquad \widehat{\Pi}_k = S^{-1/2}\ \Pi_k \,, \tag{41}$$

so that the Hamiltonian density takes the canonical form:

$$H = \frac{1}{2}\sum_{\vec{k}} \Big(|\widehat{\Pi}_k|^2 + k^2|\widehat{\Psi}_k|^2 \Big)\,. \tag{42}$$

Under S-duality, these new variables transform as the original ones. They satisfy the Schrödinger-like equations

$$\widehat{\Psi}_k{}'' + \left[k^2 - (S^{1/2})'' S^{-1/2}\right] \widehat{\Psi}_k = 0\,, \qquad \widehat{\Pi}_k{}'' + \left[k^2 - (S^{-1/2})'' S^{1/2}\right] \widehat{\Pi}_k = 0. \tag{43}$$

The amplification of perturbations is typically associated with a transition from an inflationary phase in which the pump field is accelerated to a post-inflationary phase in which the pump field is decelerated or constant. In such a class of backgrounds, the "effective potentials", $V_\Psi = (S^{1/2})'' S^{-1/2}$ and $V_\Pi = (S^{-1/2})'' S^{1/2}$, grow during the phase of accelerated evolution, and decrease in the post-inflationary, decelerated epoch, vanishing asymptotically both for very early times, $\eta \to -\infty$, and for very late times, $\eta \to +\infty$.

The initial evolution of perturbations, for all modes with $k^2 > |V_\Psi|$, $|V_\Pi|$, may be described by the WKB-like approximate solutions of Eqs. (43)

$$\begin{aligned} \widehat{\Psi}_k(\eta) &= \left(k^2 - V_\Psi\right)^{-1/4} \; e^{-i\int_{\eta_0}^{\eta} d\eta' \left(k^2 - V_\Psi\right)^{1/2}}\,, \\ \widehat{\Pi}_k(\eta) &= k\left(k^2 - V_\Pi\right)^{-1/4} \; e^{-i\int_{\eta_0}^{\eta} d\eta' \left(k^2 - V_\Pi\right)^{1/2}}\,, \end{aligned} \tag{44}$$

which we have normalized to a vacuum fluctuation, and where the extra factor of k in the solution for $\widehat{\Pi}_k$ comes from consistency with the first order equations (39). We have ignored a possible relative phase in the solutions. Solutions (44) manifestly preserve the S-duality symmetry of the equations, since the potentials V_Ψ, V_Π get interchanged under $S \to S^{-1}$.

Let us now discuss two opposite regimes:

- When the perturbation is deeply inside the horizon ($k/a \gg H$) we find "adiabatic" behaviour, i.e.
$$k\Phi_k \sim S^{-1/2}\,,\; \Pi_k \sim S^{1/2}\,, \tag{45}$$
implying, through (35), that the contribution to the Hamiltonian of modes inside the horizon stays constant.

- When the perturbation is far outside the horizon ($k/a \ll H$), it enters the so-called freeze-out regime in which Ψ and Π stay constant (better have a constant solution, see [42]). Such a behaviour implies, again through (35), that the contribution of super-horizon modes to the Hamiltonian grows in time. If $\dot{S} > 0$, the growth of $\mathcal{H}$ is due to Ψ, while, for $\dot{S} < 0$, it is due to Π. In either case the growth is due to particle production in squeezed states [43], i.e. states in which one canonical variable is very sharply defined and the conjugate one is largely undetermined. Although, strictly speaking, quantum coherence is not lost, in practice the sub-fluctuating variable cannot be measured with unlimited precision (coarse graining) and therefore entropy is produced (see Subsection 4.6).

It is not too hard to join the two extreme regimes mentioned above and to find the qualitative and quantitative features of the solutions. For lack of space we refer the reader to the original literature (see, e.g. [42]).

The above considerations were very general. What is instead typical of the PBB scenario? There are at least two features that are quite unique to string cosmology:

- Pump fields, and in particular their contributions to the evolution equations (43), grow during PBB inflation, while they tend to decay in standard inflation.
- The richer set of backgrounds and fluctuation present in string theory allows for the amplification of new kinds of perturbations.

One can easily determine the pump fields for each one of the most interesting perturbations appearing in the PBB scenario. The result is:

$$\begin{aligned} \text{Gravity waves, dilaton} &: S = a^2 e^{-\phi} \\ \text{Heterotic gauge bosons} &: S = e^{-\phi} \\ \text{Kalb}-\text{Ramond, axions} &: S = a^{-2} e^{-\phi} \, . \end{aligned} \tag{46}$$

In the following subsections we shall briefly describe the characteristics of these four perturbations after their original vacuum fluctuations are amplified by PBB inflation. For further details, see also [44].

4.2 Tensor perturbations: an observable cosmic gravitational radiation background (CGRB)?

It is not surpising to find that, for tensor and dilaton perturbations, the pump field is nothing but the scale factor in the Einstein frame ($a_E = a e^{-\phi/2}$) since, in this frame, the action for gravity and for the dilaton take the canonical form. The Einstein-frame scale factor corresponds to a collapsing Universe (see Section 3), hence to the decreasing pump field $a_E(\eta) \sim \eta^{1/2}$ during DDI. For scales that go outside the horizon during DDI, this implies [45] a Raileigh–Jeans-like spectrum, $d\Omega/d\log k \sim k^3$, up to logarithmic corrections [45].

When the curvature scale reaches the string scale we expect DDI to end, and a high (string scale) curvature phase to follow, before the eventual exit to the FRW phase takes place (see Section 5). Not much is known about the string phase, but, using some physical arguments as well as some quantitative estimates, it can be argued that such a phase will lead to copious GW production at frequencies corresponding to the string scale at the time of exit. After the transition to the FRW phase, all particle production switches off. This is why our GW spectrum has an end point that corresponds to the string/Planck scale at the beginning of the FRW phase. If no inflation takes place after, the end-point frequency corresponds, today, to $\omega = \omega_1 \sim 100$ GHz.

As illustrated in Fig. 8, the GW spectrum can be rather flat below the end point, up to the frequency ω_s, the last scale that went out of the horizon during DDI. Further below ω_s we get

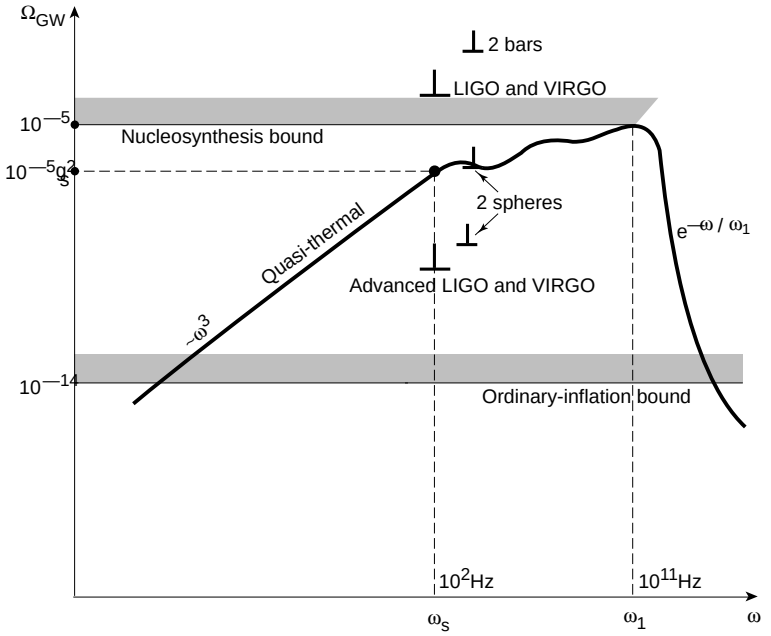

Figure 8

the above-mentioned steep ω^3 spectrum. It thus looks as if the best chances for the detection of our stochastic background lie precisely near ω_s, where a kink (or knee) is expected.

Unfortunately, the position of the knee and the value of Ω_{GW} at that point depend on two background parameters that are, so far, difficult to predict. One corresponds to the duration of (better, the total red-shift during) the string phase, the other to the value of l_P/λ_s at the end of DDI (hence to the value of the dilaton at that time). As shown in Fig. 8, values of Ω_{GW} in the range of 10^{-6}–10^{-7} are possible in some regions of parameter space, which, according to some estimates of sensitivities [46] reported in the same figure for $\omega_s \sim 10^2$Hz, could be inside detection capabilities in the near future.

The signal is predicted to consist of randomly distributed *standing* waves, a feature that has been argued [47] to further help detection. In any case, cross-correlation experiments are mandatory here in order to disentangle this stochastic signal from real noise. Sensitivities to a CGRB of this type have been estimated for a variety of two-detector combinations [46]. A comprehensive review of GW experiments and of their relevance to the early Universe can be found in [48].

4.3 Dilaton perturbations

Since the dilaton is, after all, the inflaton of PBB cosmology, its fluctuations are the most natural source of adiabatic scalar perturbations. We recall that, in standard cosmology, inflaton fluctuations naturally lead to a quasi scale-invariant, Harrison-Zeldovich (HZ) spectrum

of adiabatic perturbations, something highly desirable both to explain CMB anisotropy and for models of LSS formation. Can we get something similar from the dilaton? The answer, unfortunately, is no! Let me spend a moment explaining why.

Unlike tensor perturbations, which do not couple to the scalar field to linear order and are gauge-invariant by themselves, scalar perturbations are contained, a priori, in five functions defined by:

$$\begin{aligned} ds^2 &= a^2(\eta)\left[-(1+2\Phi)d\eta^2 + ((1-2\Psi)\delta_{ij} + \partial_i\partial_j E)\, dx^i dx^j - 2\partial_i B dx^i d\eta\right] \\ \phi &= \phi_0(\eta) + \chi(\eta, \vec{x})\ . \end{aligned} \tag{47}$$

The five functions Φ, Ψ, B, E, χ are not separately gauge-invariant. However, the following "Bardeen" combinations are gauge-invariant:

$$\begin{aligned} \Phi_B &= \Phi + \frac{1}{a}\left[a(B-E')\right]' \quad , \\ \Psi_B &= \Psi - \frac{a'}{a}(B-E') \quad , \\ \chi_{GI} &= \chi + \frac{\phi_0'\, a}{a'}\Psi\ . \end{aligned} \tag{48}$$

Introducing the variable $v \equiv a\chi_{GI}$, the scalar field enters the quadratic action "canonically" i.e.:

$$S_{eff}(v) = \frac{1}{2}\int d\eta d^3x \left[v'^2 - (\vec{\nabla}v)^2 + (z''/z)\ v^2\right] \ , \quad z \equiv \frac{\phi_0' a^2}{a'}\ , \tag{49}$$

giving the evolution equation

$$v_k'' + \left(k^2 - (z''/z)\right)\ v_k = 0\ . \tag{50}$$

In the DDI background, $z \sim a$ and thus the canonical scalar field obeys the same equation as the canonical graviton field, therefore giving identical spectra (as far as the dilaton remains massless, of course). This strongly suggests that adiabatic perturbations in PBB cosmology have a Raleigh–Jeans, rather than HZ, spectrum and that they are unsuitable for generating CMBA or LSS. Before being sure of that, however, we have to analyse the scalar fluctuations of the metric itself in terms of the above-mentioned Bardeen potentials Φ_B, Ψ_B.

A popular gauge (particularly advertised in [41]) is the so-called longitudinal gauge, defined by $B = E = 0$, where $\Phi_B = \Phi$ and $\Psi_B = \Psi$. In this gauge one of the constraints simply reads $\Phi = \Psi$, while a second constraint relates either one of them to v:

$$\Psi_k = -\frac{\phi_0'}{4k^2}(v_k/a)' \sim k^{-3/2}\ \frac{1}{|k\eta|^2}(k/a)_{HC}\ , \tag{51}$$

where we have inserted the small k behaviour of v_k, which is identical to that of tensor perturbations.

Unfortunately, Eq. (51) leads to very large fluctuations of $\Psi = \Phi$ at small $k\eta$, so large that one leaves the linear-perturbation regime for the expansion (47) of the metric much before the

high-curvature scale is reached. Does this mean that the metric becomes very inhomogeneous? It would look to be the case ... unless the growth of Ψ and Φ is in some way a gauge artefact. But how can it be a gauge artefact if Ψ and Φ correspond, in this gauge, to the gauge-invariant Bardeen potentials? The answer to this question was provided in [49]. By going from the longitudinal gauge to an "off-diagonal" gauge with $\Psi = E = 0$, or, even better, to one in which only Φ and E appear, one finds that perturbations of the metric remain small at all η till Planckian/string-scale curvatures are reached.

This is easy to see, for instance, in a gauge with $\Psi = B = 0$, where $\Psi_B \sim (a'/a)E'$. Clearly this gives $E \sim \eta^2 \Psi_B$ and, since E enters the metric with two spatial derivatives, this implies that $h_{ij} \sim (k\eta)^2 \Psi_B$, which is sufficiently small at small $k\eta$ for linear perturbation theory to be valid. One can then look for physical effects of these scalar perturbations (e.g. for contributions to CMBA) and find that they actually remain as small as the tensor contributions. In conclusion, once gauge artefacts are removed, it seems that adiabatic scalar perturbations, as well as their tensor counterparts, remain exceedingly small at large scales.

On the other hand, the rather large yields at short scales also apply to dilatons. This allows for a possible source of scalar waves if the dilaton is very light. However, as recently discussed by Gasperini [50], it is very unlikely that such a signal will be observable, given the constraints on the dilaton mass due to tests of the equivalence principle (see Section 2). Other restrictions on the dilaton mass come from the possibility that their density may become overcritical and close the Universe. This and other possible interesting windows in parameter space are discussed in [17], and will not be reported in any detail here.

4.4 Gauge-field perturbations: seeds for $\vec{B}_{gal}$?

In standard inflationary cosmology there is no amplification of the vacuum fluctuations of gauge fields. This is a straightforward consequence of the fact that inflation makes the metric conformally flat, and of the decoupling of gauge fields from a conformally flat metric precisely in $D = 3 + 1$ dimensions.

As a very general remark, apart from pathological solutions, the only background field that can amplify, through its cosmological variation, e.m. (more generally gauge-field) quantum fluctuations is the effective gauge coupling itself [51]. By its very nature, in the pre-big bang scenario the effective gauge coupling inflates together with space during the PBB phase. It is thus automatic that any efficient PBB inflation brings together a huge variation of the effective gauge coupling, and thus a very large amplification of the primordial e.m. fluctuations [52]. This can possibly provide the long-sought for origin for the primordial seeds of the observed galactic magnetic fields.

To be more quantitative, since the pump field for electromagnetic perturbations is the effective (four-dimensional) gauge coupling itself (see Eq. (46)), the total amplification of e.m. perturbations on any given scale λ is given by α_0/α_{ex}, i.e. by the ratio of the fine structure constant now and the fine structure constant at the time of exit of the scale λ during DDI. It

turns out [52] that, in order to produce sufficiently large seeds for the galactic magnetic fields, such a ratio has to be enormous for the galactic scale, i.e. about 10^{66}. Taken at face value, this would be a very strong indication in favour of the PBB scenario, more particularly of DDI. Indeed, only in such a framework is it natural to expect that the effective gauge coupling grew during inflation by a factor whose logarithm is of the same order as the number of inflationary e-folds.

Notice, however, that, unlike GW, e.m. perturbations interact quite considerably with the hot plasma of the early (post-big bang) Universe. Thus, converting the primordial seeds into those that may have existed at the protogalaxy formation epoch is by no means a trivial exercise (see, e.g. [53]). The question of whether or not the primordial seeds generated in PBB cosmology can evolve into the observed galactic magnetic fields thus remains, to this date, an unsolved, yet very interesting, problem.

4.5 Axion perturbations: seeds for CMBA and LSS?

In four dimensions the curl of $B_{\mu\nu}$, $H_{\mu\nu\rho}$, is equivalent to a pseudoscalar field, the (KR) axion σ, through

$$H_{\mu\nu\rho} = e^{\phi}\ \epsilon_{\mu\nu\rho\tau}\partial^{\tau}\sigma\ . \tag{52}$$

It is easy to see that, while the pump field for $B_{\mu\nu}$ is $a^{-2}\ e^{-\phi}$, that for σ is $a^2\ e^{\phi}$. Indeed their respective perturbations are related by the duality of perturbations discussed in Subsection 4.1. We can use either description with identical physical results. Note that, while a and ϕ worked in opposite directions for tensor and dilaton perturbations, generating strongly tilted (blue) spectra, the two work in the same direction for axions, so that spectra can be flat or even tilted towards large scales (red spectra) [54]. An interesting fact is that, unlike the GW spectrum, that of axions is very sensitive to the cosmological behaviour of internal dimensions during the DDI epoch. On one side, this makes the model less predictive. On the other, it tells us that axions represent a window over the multidimensional cosmology expected generically from string theories, which must live in more that four dimensions. Parametrizing the spectrum by:

$$\Omega_{ax}(k) = \left(\frac{H_{max}}{M_P}\right)^2 (k/k_{max})^{\alpha}\ , \tag{53}$$

and considering the case of three non-compact and six compact dimensions with separate isotropic evolution, one finds:

$$\alpha = \frac{3 + 3r^2 - 2\sqrt{3 + 6r^2}}{1 + 3r^2}\ , \tag{54}$$

where

$$r \equiv \frac{1}{2}\frac{\dot{V}_6\ V_3}{V_6\ \dot{V}_3} \tag{55}$$

is a measure of the relative evolution of the internal and external volumes. Equations (54), (55) show that the axion spectrum becomes exactly HZ (i.e. scale-invariant) when $r = 1$, i.e.

when all nine spatial dimensions of superstring theory evolve in a rather symmetric way [55]. In situations near this particularly symmetric one, axions are able to provide a new mechanism for generating large-scale CMBA and LSS.

Calculation of the effect gives [56], for massless axions:

$$l(l+1)C_l \sim O(1)\left(\frac{H_{max}}{M_P}\right)^4 (\eta_0 k_{max})^{-2\alpha}\frac{\Gamma(l+\alpha)}{\Gamma(l-\alpha)}\,, \tag{56}$$

where C_l are the usual coefficients of the multipole expansion of $\Delta T/T$

$$\langle \Delta T/T(\vec{n})\ \ \Delta T/T(\vec{n}')\rangle \ = \ \sum_l (2l+1)C_l P_l(\cos\theta)\ , \quad \vec{n}\cdot\vec{n}' = \cos\theta\ , \tag{57}$$

and $\eta_0 k_{max} \sim 10^{30}$. In string theory, as repeatedly mentioned, we expect $H_{max}/M_P \sim M_s/M_P \sim 1/10$, while the exponent α depends on the explicit PBB background with the above-mentioned HZ case corresponding to $\alpha = 0$. The standard tilt parameter $n = n_s$ (s for scalar) is given by $n = 1+2\alpha$ and is found, by COBE[57], to lie between 0.9 and 1.5, corresponding to $0 < \alpha < 0.25$ (a negative α leads to some theoretical problems). With these inputs we can see that the correct normalization ($C_2 \sim 10^{-10}$) is reached for $\alpha \sim 0.2$, which is just in the middle of the allowed range. In other words, unlike in standard inflation, we cannot predict the tilt, but when this is given, we can predict (again unlike in standard inflation) the normalization.

With some extra work [58] one can compute the C_l in the acoustic-peak region adding vector and tensor contributions from the seeds. It turns out that the acoustic-peak structure is very sensitive to α, hence to the behaviour of the internal dimensions during the DDI phase. The above-mentioned value, $\alpha = 1$, does not give peaks at all and, as such, looks ruled out by the data. Values of α in the range 0.3–0.4 appear to be preferred (especially in the presence of a cosmological constant with $\Omega_\Lambda \sim 0.7$). We saw, however, that the overall normalization was very sensitive to the value of α. For α in the 0.3–0.4 range, the normalization is off (way too small) by many orders of magnitude. Therefore, if present indications are confirmed, as they seem to be from the recent release of the Boomerang 1997 data analysis [59], one will be forced to a k-dependent α, meaning different phases in the evolution of internal dimensions during DDI.

4.6 Heating up the Universe

Before closing this section, I wish to recall how one sees the very origin of the hot big bang in this scenario. One can easily estimate the total energy stored in the quantum fluctuations, which were amplified by the pre-big bang backgrounds (for a discussion of generic perturbation spectra, see [55, 60]. The result is, roughly,

$$\rho_{quantum} \sim N_{eff}\ H_{max}^4\ , \tag{58}$$

where N_{eff} is the effective number of species that are amplified and H_{max} is the maximal curvature scale reached around $t = 0$. We have already argued that $H_{max} \sim M_s = \lambda_s^{-1}$, and

we know that, in heterotic string theory, N_{eff} is in the hundreds. Yet, this rather huge energy density is very far from critical, as long as the dilaton is still in the weak-coupling region, justifying our neglect of back-reaction effects. It is very tempting to assume [55] that, precisely when the dilaton reaches a value such that $\rho_{quantum}$ is critical, the Universe will enter the radiation-dominated phase. This PBBB (PBB bootstrap) constraint gives, typically:

$$e^{\phi_{exit}} \sim 1/N_{eff} \ , \tag{59}$$

i.e. a value for the dilaton close to its present value.

The entropy in these quantum fluctuations can also be estimated following some general results [61]. The result for the density of entropy S is, as expected,

$$S \sim N_{eff} H_{max}^3 \ . \tag{60}$$

It is easy to check that, at the assumed time of exit given by (59), this entropy saturates recently proposed holography bounds. The discussion of such bounds is postponed to Subsection 5.7 since is has also interesting implications for the exit problem.

5 How could it have stopped?

We have argued that, generically, DDI, when considered at lowest order in derivatives and coupling, evolves *towards* a singularity of the big bang type. Similarly, at the same level of approximation, non-inflationary solutions of the FRW type emerge *from* a singularity. Matching these two branches in a smooth, non-singular way has become known as the (graceful) exit problem in string cosmology [62]. It is, undoubtedly, the most important theoretical problem the PBB scenario is facing today.

Of course, one would not only like to know that a graceful exit does take place: one would also like to describe the transition between the two phases in a quantitative way. Achieving this goal would amount to nothing less than a full description of what replaces the big bang of standard cosmology in the PBB scenario. As mentioned in Section 1, this difficult problem is the analogue, in string cosmology, of the (still not fully solved) confinement problem of QCD. The exit problem is particularly hard because, by its very nature, and by the existing no-go theorems [62], it must occur, if at all, at large curvature and/or coupling and, because of fast time-dependence, must break (spontaneously) supersymmetry. The phenomenological predictions made in the previous section were based on the assumption that i) a graceful exit does take place; ii) sufficiently large scales are only affected by it kinematically, i.e. through an overall red-shift of all scales.

In this section, after recalling some no-go theorems for the exit, we will review various proposals that circumvent those theorems starting from the mathematically simplest, but physically least realistic, proposals and ending with the physically favoured, but harder to analyse, suggestions. The latter proposals suggest possible lines along which a quantitative description of the exit might eventually emerge. Needless to say, in spite of the many encouraging results,

much work remains to be done: perhaps new techniques, and/or a deeper understanding of string theory in its non-perturbative regimes through the construction of the still largely unknown M-theory [10], need to be developed before a full quantitative description can be hoped for.

I should also mention that there have been suggestions [63] that the BB singularity can be avoided if the DDI phase is highly anisotropic. While this is an interesting suggestion, with isotropization taking place later in the non-inflationary regime, we will stick here to the simplest case, in which DDI has already prepared a very homogeneous Universe before exit takes place. This is why our discussion on the exit problem is limited to the case of homogeneous cosmologies. Also, for lack of space, I shall refer the reader to the literature for most of the details.

5.1 No-go theorems

Under some restrictive conditions [62], it was shown that one cannot have a change of branch, i.e. that the Universe cannot make a permanent transition from the inflationary pre-big bang to a FRW post-big bang solution. Perhaps the best way to convey the physical meaning of those theorems is in terms of the necessary conditions for exit recently given by Brustein and Madden [64]. These authors give necessary conditions for two subsequent events to occur: firstly, a branch change in the string frame should take place: this imposes the violation of some energy conditions; secondly, a bounce should occur in the E-frame metric since, as we have seen in Section 3, DDI represents a collapse in the E-frame. This latter transition requires further violation of energy conditions.

Before the reader gets too worried about these violations, I should point out that these refer to the equations of state satisfied by some "effective" sources, which include both higher-derivative and higher-loop corrections. It is well known that such sources generically do lead to violations of the standard energy conditions satisfied by normal matter or radiation-like classical sources.

5.2 Exit via a non-local V

This is perhaps the simplest example of an exit. It was first discussed in [65]. The reason why this is not considered an appealing mechanism for the exit is that the potential it employs depends on $\bar{\phi}$ (instead of ϕ), in order to preserve SFD. By general covariance such a potential, if non-trivial, must be non-local. Unfortunately, there has been no convincing proposal to explain how such non-local potentials might arise within a superstring theory framework.

5.3 Exit via B_{ij}

The antisymmetric KR field may lead to violation of the energy conditions and thus induce an exit. Some amusing examples were given in [66], where a non-trivial B_{ij} is introduced through $O(d,d)$ transformations acting on a pure metric-dilaton cosmology of the type described in Subsection 2.5. It was found that these so-called "boosted" cosmologies were less singular than the original ones. In some cases they were even completely free of singularity and provided examples of exit, albeit in not-so-realistic situations.

It is tempting to speculate that this softening of singularities, due to a non-trivial B_{ij} field, could be related to recent developments in the field of non-commutative geometry [67] induced by a B_{ij} field. Work along these lines is in progress.

5.4 Exit via quantum tunnelling

Several groups [68] have attempted to describe the transition from the pre- to the post-big bang without modifying the low-energy tree-level effective action, by exploiting the quantum cosmology approach based on the Wheeler–De Witt (WDW) equation. In Refs. [68] an $O(d,d)$-invariant WDW equation was derived in the (d^2+1)-dimensional mini-superspace consisting of a homogeneous Bianchi I metric, the antisymmetric tensor, and the dilaton. The $O(d,d)$ symmetry helps avoiding the ordering ambiguities which usually plague the WDW equation. For the time being only the mathematically simpler case of an $O(d,d)$-invariant potential $V(\bar{\phi})$ has been analysed since, in that case, d^2-conserved charges can be defined and the "radial" part of the WDW equation reduces to a one-dimensional Schroedinger equation for a scattering problem.

It is amusing that, from such a point of view, the initial state of the Universe is described by a right-moving plane wave, which later encounters a potential, giving rise to both a transmitted and a reflected (i.e. left-moving) wave. The transmission coefficient gives the probability that the Universe ends up in the pre-big-bang singularity, while the reflection coefficient gives the probability of a successful exit into the post-big-bang decelerating expansion.

For certain forms of $V(\bar{\phi})$ the wave is classically reflected and the WDW approach just confirms this expectation by giving a 100% probability for the exit. However, even when there is no classical exit, the probability of wave-reflection is non-zero because of quantum tunnelling. The quantum probability of a classically forbidden exit turns out to be exponentially suppressed in the coupling constant e^{ϕ}, which is just fine. Unfortunately, it is also exponentially suppressed in the total volume of 3-space (in string units) after the pre-big-bang. Thus, only tiny regions of space have a reasonable chance to tunnel.

5.5 Higher-derivative corrections

While the examples of exit given in the previous subsections are theoretically interesting, they do look somewhat artificial and non-generic. In this and in the following subsection we shall describe two mechanisms for exit that involve very general properties of the lowest order solutions and of string theory. The present feeling is that, if graceful exit occurs, it should be maily induced by some combination of higher-derivative and higher-loop effects. Let us start with the former.

Toy examples have shown [69] that DDI can flow, thanks to higher-curvature corrections, towards a de-Sitter-like phase, i.e. into a phase of constant H (curvature) and constant $\dot{\phi}$. This phase is expected to last until loop corrections become important and give rise to a transition to a radiation-dominated phase (see the next subsection). The idea is to justify the strong curvature transition from the dilatonic to the string phase by proving the existence of an exact de Sitter-like solution to the field equation, which acts as a late time attractor for the perturbative DDI branch. As shown in [69], the existence of such attractors depends on the existence of (non-trivial) solutions for a system of n algebraic equations in n unknowns. In general, we may expect a discrete number of solutions to exist. If at least one of them has some qualitative characteristics, it will act as a late-time attractor for solutions approaching DDI in the far past. An explicit example of this phenomenon was constructed in [69]. In this connection, it is worth mentioning that solutions connecting duality-related low-energy branches through a high-curvature CFT were already proposed in [70].

It was recently pointed out [71] that the reverse order of events is also possible. The coupling may become large *before* the curvature does. In this case, at least for some time, the low-energy limit of M-theory should be adequate: this limit is known [10] to give $D = 11$ supergravity and is therefore amenable to reliable study. It is likely, though not yet clear, that, also in this case, strong curvatures will have to be reached before the exit can be completed.

5.6 Loop corrections and back reaction

The idea here is to invoke the back reaction from particle production as the relevant mechanism. Since the back reaction is an $O(e^{\phi}\alpha' H^2)$ correction, its effect is contained in one-loop $O(R^2)$ contributions to the effective action. A recent calculation [72] shows that, indeed, loop corrections to DDI work in the right direction and become relevant precisely when expected according to the exit criterion (59).

A class of such contributions was analysed some time ago by Antoniadis et al. [73] in the case of a spatially flat ($k = 0$) cosmology and by Easther and Maeda [74] in the case of a closed Universe ($k = 1$). Both groups find non-singular solutions to the loop-corrected field equations. However, neither group is actually able to obtain solutions that start in the dilaton-driven superinflationary regime and later evolve through a branch change.

More recently, several examples of full exit have been constructed [75]. Although they are

based on α' and loop-corrected actions, which have not been derived from reliable calculations, they seem to indicate, at least, that the BM conditions for exit may turn out to be not just necessary but also sufficient. It also appears [76] that exit occurs when the entropy bound becomes threatened by the entropy in the amplified/squeezed quantum fluctuations, as we shall now discuss.

5.7 Entropy considerations

Entropy-related considerations have recently led to model-independent arguments in favour of the occurrence of a graceful exit in string cosmology. As we shall see, those are physically quite close to the arguments based on back-reaction and loop corrections, which we have just discussed in the previous subsection.

Almost twenty years ago Bekenstein [77] suggested that, for a limited gravity system of energy E, and whose size R is larger than its gravitational radius, $R > R_g \equiv 2G_N E$, entropy is bounded by S_{BEB}:

$$S_{BEB} = ER/\hbar = R_g \; R \; l_P^{-2}. \tag{61}$$

Holography [78] suggests that maximal entropy is bounded by S_{HOL},

$$S_{HOL} = A l_P^{-2}, \tag{62}$$

where A is the area of the space-like surface enclosing the region of space whose entropy we wish to bound. For systems of limited gravity, since $R > R_g, A = R^2$, (61) implies the holography bound (62).

Can these entropy bounds be applied to the whole Universe, i.e. to cosmology? A cosmological Universe is *not* a system of limited gravity, since its large-distance behaviour is determined by the gravitational effect of its matter content through Friedmann's equation (4). Furthermore, the holography bound obviously fails for sufficiently large regions of space since, for a given temperature, entropy grows like R^3 while area grows like R^2. The generalization of entropy bounds to cosmology turned out to be subtle.

In 1989, Bekenstein himself [79] gave a prescription for a cosmological extension by choosing R in Eq. (61) to be the particle horizon. Amusingly, he arrived at the conclusion that the bound is violated sufficiently near the big-bang singularity, implying that the latter is fake (if the bound is always valid). About a year ago, Fischler and Susskind (FS) [80] proposed a similar extension of the holographic bound to cosmology, arguing that the area of the particle horizon should bound entropy on the backward-looking light cone according to (62). It was soon realized, however, that the FS proposal requires modifications, since violations of it were found to occur in physically reasonable situations. An improvement of the FS bound applicable to light-like hypersurfaces was later made by Bousso [81].

Of more interest here are the attempts made at deriving cosmological entropy bounds on space-like hypersurfaces [82, 83, 84, 85, 86]. These identify the maximal size of a spatial region

for which holography works: the Hubble radius [82, 83, 85], the apparent horizon [84], or, finally, a so-called causal connection (Jeans) scale [86].

For our purposes here, we do not need to enter into the relative merits of these various proposals. Rather, we will only outline the physical idea behind them. Consider, inside a quasi-homogeneous Universe, a sphere of radius H^{-1}. We may consider "isolated" bodies, in the sense of ref. [77], fully contained in the sphere, i.e. with radius $R < H^{-1}$. For such systems, the usual BB holds and is saturated by a black hole of size R. We may consider next several black holes inside our Hubble volume, each carrying an entropy proportional to the square of its mass. If two, or more, of these black holes merge, their masses will add up, while the total entropy after the merging, being quadratic in the total mass, will exceed the sum of the initial entropies. In other words, in order to maximize entropy, it pays to form black holes as large as possible.

Is there a limit to this process of entropy increase? The suggestion made in [82, 83, 84, 85, 86], which finds support in old results by several groups [87], is that a critical length of order H^{-1} is the upper limit on how large a classically stable black hole can be. If we accept this hypothesis, the upper bound on the entropy contained in a given region $\mathcal{R}$ of space will be given by the number of Hubble volumes in $\mathcal{R}$, $n_H = VH^3$ times the Bekenstein–Hawking entropy of a BH or radius H^{-1}, $H^{-2}l_P^{-2}$. The two factors can be combined in the suggestive formula:

$$S(\mathcal{R}) < l_P^{-2} \int_{\mathcal{R}} d^3x \, \sqrt{h}\tilde{H} \equiv S_{HB} \, , \tag{63}$$

where $\int_{\mathcal{R}} d^3x \, \sqrt{h}$ is the volume of the space-like hypersurface whose entropy we wish to bound, and $\tilde{H}$ differs from one proposal to another [82, 83, 84, 85, 86], but is, roughly, of the order of the Hubble parameter. Actually, since H is proportional to the trace of the second fundamental form on the hypersurface, Eq. (63) reminds us of the boundary term that has to be added to the gravitational action in order to correctly derive Einstein's equations from the usual variational principle. This shows that the bound (63) is generally covariant for $\tilde{H} = H$. It can also be written covariantly for the identification of $\tilde{H}$ made in [86].

For the qualitative discussion that follows, let us simply take $\tilde{H} = H$ and let us convert the bound to string-frame quantities, taking into account the relation between l_P and λ_s, given in Eq. (3). We obtain [83]:

$$S(\mathcal{R}) < (VH^3)(H^{-2}\lambda_s^{-2}e^{-\phi}) = e^{-\bar{\phi}}H\lambda_s^{-2} \, , \tag{64}$$

where we have fixed an arbitrary additive constant in the definition (6) of $\bar{\phi}$. Equation (64) thus connects very simply the entropy bound of a region of fixed comoving volume to the most important variables occurring in string cosmology (see, e.g., the phase diagram of Fig. 3).

An immediate application of the bound (64) was pointed out in [83]. Noting that the bound is initially saturated in the BDV picture [31] of collapse/inflation, the bound itself cannot decrease without a violation of the second law. This gives immediately:

$$\dot{\bar{\phi}} \leq \frac{\dot{H}}{H} \, . \tag{65}$$

It is easy to check that this inequality is obeyed, but just so, during DDI, in the sense that it holds with the equality sign. In other words, the HEB is saturated initially *and throughout* DDI in the BDV picture. The bound also turns out to give a physically acceptable value for the entropy of the Universe just after the big bang: a large entropy on the one hand (about 10^{90}); a small entropy for the total mass and size of the observable Universe on the other, as often pointed out by Penrose [88]. Thus, PBB cosmology neatly explains why the Universe, at the big bang, *looks* so fine-tuned (without being so) and provides a natural arrow of time in the direction of higher entropy [83].

What happens in the mysterious string phase, where we are desperately short of reliable techniques? It is quite clear that Eq. (65) does not allow H to reach saturation ($\dot{H} = 0$) in the first quadrant of Fig. 3 since $\dot{\phi} > 0$ there. Instead, saturation of H in the second quadrant (where $\dot{\phi} \leq 0$) is perfectly all right. But this implies having attained the sought for branch change!

Let us finally look at the loop corrections. Since, physically, these correspond to taking into account the back-reaction from particle production, we may check when the entropy in the cosmologically produced particles starts to threaten the bound. As discussed in Subsection 4.6, the entropy density in quantum fluctuations is given by $\sigma \sim N_{eff} H^3$, which equals the bound $\sigma_{HEB} \sim H l_P^{-2}$ precisely when $l_P^2 H^2 N_{eff} = O(1)$. But, as already pointed out, this is just the line on which the energy density in quantum fluctuations becomes critical (see Eq. (59)) and where, according to [69], the back-reaction becomes $O(1)$. Similar conclusions are reached by applying generalized second law arguments [89].

The picture that finally emerges from all these considerations is best illustrated with reference to the diagram of Fig. 4. Two lines are shown, representing boundaries for the possible evolution. The horizontal boundary is forced upon by the large-curvature corrections, while the tilted line in the first quadrant corresponds to the equation $l_P^2 H^2 N_{eff} = O(1)$ that we have just discussed. Amusingly, this line was also suggested by Maggiore and Riotto [71] as a boundary beyond which copious production of 0-branes would set in. Thus, depending on initial conditions, the PBB bubble corresponding to our Universe would hit first either the high-curvature or the large-entropy boundary and initiate an exit phase. Hopefully, a universal late-time attractor will emerge guiding the evolution into the FRW phase of standard cosmology (shown as a vertical line in Fig. 10).

Needless to say, all this has to be considered, at best, as having heuristic value. If taken seriously, it would suggest that the Universe will never enter the strong-coupling, strong-curvature regime, where the largely unknown M-theory should be used. The low-energy limit of the latter (the much better understood 11-D supergravity) could suffice to deal with the fundamental exit problem of string cosmology. We refer to the literature for several other attempts at M-cosmology [90].

6 Outlook

The outlook for the pre-big bang scenario, as formulated at present, is not necessarily an optimistic one. I am not sure I would bet a lot of money on it being right! But this is not really the issue. We have to remember that the PBB scenario is a top–down approach to cosmology. As stressed in the introduction, it would be quite a miracle if the correct model could be guessed without extensive feed-back from the data. The good news here is that new data are coming in all the time, and will continue to do so with more and more precision in the coming years!

Rather, we should draw some lessons from this new attempt at very early cosmology, whether it succeeds or it fails. As I can see, the main lessons to be drawn are the following:

- Our Universe did not have to emerge, together with space and time, from a singularity; in string theory, the singularity should be fake, either because it is tamed by finite-string-size effects, or because it simply signals the need for new degrees of freedom in the description of physics at very short distances;
- Because string theory is an extension of GR, inflation is possible in that context even in the absence of potential energy (i.e. of an effective cosmological constant); actually, inflation is very natural and easy to achieve, being a consequence of the duality symmetries of the string-cosmology equations;
- Inflation in string cosmology can be related, mathematically, to the problem of gravitational collapse in GR; as such, it is a generic phenomenon, once the assumption of asymptotic past triviality is made; furthermore, the curvature scale and the coupling at the onset of PBB inflation are arbitrary classical moduli;
- The Universe did not have to start hot! A hot Universe can emerge from a cold one thanks to quantum particle production in inflationary backgrounds;
- PBB cosmology predicts a rich spectrum of perturbations with different spectra depending on each perturbation's "pump" field and on its evolution in the PBB era; observable relics of these perturbations may serve as a window on physics in the pre-bangian Universe all the way down to the string/Planck scale;
- The simplest PBB models either predict too small perturbations at large scales, or a spectrum of isocurvature perturbations which may be already "experimentally challenged" (as Rocky Kolb would kindly say);
- The exit problem still remains the hardest theoretical challenge to the whole idea of PBB cosmology;
- Hopefully, the combination of the above-mentioned experimental and theoretical challenges will be able to tell us whether the PBB idea is just doomed, or whether parts of it should be kept while searching for a better scenario; it should also suggest new avenues for physics-driven research in string/M-theory;

- Last but not least, the PBB idea has taught us that we need not lock ourselves into preconceived ideas in cosmology (cf. "the big bang is the beginning of time", "inflation needs a scalar potential"); rather, we should contemplate as wide a range of theoretically sound possibilities as we can in order for Nature to choose, at best, one of them.

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References

[1] Lidsey J. E., Wands David and Copeland E. J., hep-th/9909061, to appear in Phys. Rep.

[2] Gasperini M. and Veneziano G., Phys. Rep., in preparation.

[3] http://www.to.infn.it/~gasperin/.

[4] Polchinski J., "String Theory", Cambridge University Press, 1998.

[5] Greene B., "The Elegant Universe", Norton W. W. and Co., New York & London, 1999.

[6] Veneziano G., "A new approach to semiclassical gravitational scattering", in Proc. 2nd Paris Cosmology Colloquium, Paris, June 1994, eds. de Vega H.J. and Sanchez N., WSPC, Singapore, 1995, p. 322.

[7] Kolb E. W. and Turner M. S., "The Early Universe", Addison-Wesley, Redwood City, CA, 1990;
Linde A. D., "Particle Physics and Inflationary Cosmology", Harwood, New York, 1990.

[8] Brustein R. and Steinhardt P. J., Phys. Lett. B302 (1993) 196.

[9] Veneziano G., Europhys. Lett. 2 (1986) 133; "The Challenging Questions", in Proc. Erice, 1989, ed. Zichichi A., Plenum Press, New York, 1990, p. 199.

[10] See, e.g., Witten E., Nucl. Phys. B443 (1995) 85;
Horava P. and Witten E., Nucl. Phys. B460 (1996) 506.

[11] Witten E., Phys. Lett. B149 (1984) 351.

[12] Taylor T. R. and Veneziano G., Phys. Lett. B213 (1988) 459.

[13] Damour T. and Polyakov A. M., Nucl. Phys. B423 (1994) 532; Gen. Rel. Grav. 26 (1994) 1171.

[14] Veneziano G., Phys. Lett. B265 (1991) 287.

[15] Gasperini M. and Veneziano G., Astropart. Phys. 1 (1993) 317.

[16] Gasperini M. and Veneziano G., Mod. Phys. Lett. A8 (1993) 3701.

[17] Gasperini M. and Veneziano G., Phys. Rev. D50 (1994) 2519.

[18] Tseytlin A. A., Mod. Phys. Lett. A6 (1991) 1721;
Tseytlin A. A. and Vafa C., Nucl. Phys. B372 (1992) 443.

[19] Meissner K.A. and Veneziano G., Phys. Lett. B267 (1991) 33, Mod. Phys. Lett. A6 (1991) 3397;
Sen A., Phys. Lett. B271 (1991) 295;
Hassan S. F. and Sen A., Nucl. Phys. B375 (1992) 103;
Gasperini M. and Veneziano G., Phys. Lett. B277 (1992) 256.

[20] Zeldovich Ya. B. and Novikov I. D., "Structure and Evolution of the Universe", University of Chicago Press, 1982.

[21] Veneziano G., Phys. Lett. B406 (1997) 297.

[22] Buonanno A., Meissner K.A., Ungarelli C. and Veneziano G., Phys. Rev. D57 (1998) 2543;
Feinstein A., Lazkoz R. and Vazquez-Mozo M. A., Phys. Rev. D56 (1997) 5166;
Saygili K., Int. J. Mod. Phys. A14 (1999) 225;
Barrow J. D. and Kunze K. E., Phys. Rev. D56 (1997) 741; ibid. **D57** (1998) 2255;
Clancy D., Feinstein A., Lidsey J.E. and Tavakol R., Phys. Lett. B451 (1999) 303;
Kunze K. E., gr-qc/9906073.

[23] Belinskii V. A. and Khalatnikov I. M., Sov. Phys. (JETP) 36 (1973) 591;
Deruelle N. and Langlois D., Phys. Rev. D52 (1995) 2007;
Parry J., Salopek D. S. and Stewart J. M., Phys. Rev. D49 (1994) 2872.

[24] Belinskii V. A., Lifshitz E. M. and Khalatnikov L. M., Sov. Phys. Usp. 13 (1971) 745;
Misner C. W., Phys. Rev. Lett. 22 (1969) 1071.

[25] Barrow J. D. and Dabrowski M. P., Phys. Rev. D57 (1998) 7204.

[26] Penrose R., "Structure of space-time", in Battelle Rencontres, eds. Dewitt C. and Wheeler J.A., Benjamin, New York, 1968.

[27] Bondi H., van der Burg M. G. J. and Metzner A. W. K., Proc. Roy. Soc. Lond. A269 (1962) 21;
Sachs R. K., Proc. Roy. Soc. A 270 (1962) 103;
Penrose R., Proc. Roy. Soc. Lond. A284 (1965) 159.

[28] See, e.g., Landau L. and Lifshitz E., "The Classical Theory of Fields", Pergamon Press, Oxford, 1962.

[29] Penrose R., Phys. Rev. Lett. 14 (1965) 57;
Hawking S. W. and Penrose R. R., Proc. Roy. Soc A314 (1970) 529.

[30] Christodoulou D., Commun. Math. Phys. 105 (1986) 337.

[31] Buonanno A., Damour T. and Veneziano G., Nucl. Phys. B543 (1999) 275.

[32] Christodoulou D., Comm. Math. Phys. 109 (1987) 613.

[33] Turner M. and Weinberg E., Phys. Rev. D56 (1997) 4604;
Kaloper N., Linde A. and Bousso R., Phys. Rev. D59 (1999) 043508.

[34] Linde A., Phys. Lett. 129B (1983) 177.

[35] Ghosh A., Pollifrone G. and Veneziano G., Phys. Lett. B440 (1998) 20.

[36] Choptuik M. W., Phys. Rev. Lett. 70 (1993) 9.

[37] Christodoulou D., Commun. Pure Appl. Math. 44 (1991) 339.

[38] Christodoulou D., Commun. Pure Appl. Math. 46 (1993) 1131.

[39] Maharana J., Onofri E. and Veneziano G., JHEP 04 (1998) 004;
Chiba T., Phys. Rev. D59 (1999) 083508.

[40] Ungarelli C., private communication.

[41] See, e.g., Mukhanov V. F., Feldman A. H. and Brandenberger R. H., Phys. Rep. 215 (1992) 203.

[42] Brustein R., Gasperini M. and Veneziano G., Phys. Lett. B431 (1998) 277.

[43] Grishchuk L. P. and Sidorov Y. V., Class. Quant. Grav. 6 (1989) L161;
Grishchuk L. P., in Proc. Workshop on Squeezed States and Uncertainty Relations, College Park, Maryland, eds. Han D., Kim Y. K. and Zachary W. W., NASA Conf. Pub. No. 3135, 1992, p. 329.

[44] Veneziano G., in String Gravity and Physics at the Planck Energy Scale, Erice, 1995, eds. Sanchez N. and Zichichi A., Kluwer Academic Publishers, Boston, 1996, p. 285;
Gasperini M. , ibid., p. 305.

[45] Brustein R., Gasperini M., Giovannini M. and Veneziano G., Phys. Lett. B361 (1995) 45;
Brustein R. et al., Phys. Rev. D51 (1995) 6744.

[46] Astone P. et al., Phys. Lett. B385 (1996) 421;
Allen B. and Brustein B., Phys. Rev. D55 (1997) 970;
Allen B. and Romano J.D., Phys. Rev. D59 (1999) 102001;
Ungarelli C. and Vecchio A., gr-qc/9911104.

[47] Grishchuk L. P., gr-qc/9903079.

[48] Maggiore M., gr-qc/9909001, to appear in Phys. Rep.

[49] Brustein R. et al., Phys. Rev. D51 (1995) 6744.

[50] Gasperini M., gr-qc/9910019.

[51] Ratra B., Astrophys. J. Lett. 391 (1992) L1.

[52] Gasperini M., Giovannini M. and Veneziano G., Phys. Rev. Lett. 75 (1995) 3796;
Lemoine D. and Lemoine M., Phys. Rev. D52 (1995) 1955.

[53] Kulsrud R. M., Cen R., Ostriker J. P. and Ryu D., Ap. J. 480 (1997) 481.

[54] Copeland E.J., Easther R. and Wands D., Phys. Rev. D56 (1997) 874;
Copeland E.J., Lidsey J.E. and Wands D., Nucl. Phys. B506 (1997) 407.

[55] Buonanno A., Meissner K. A., Ungarelli C. and Veneziano G., JHEP 1 (1998) 4.

[56] Durrer R., Gasperini M., Sakellariadou M. and Veneziano G., Phys. Lett. B436 (1998) 66, Phys. Rev. D59 (1999) 043511.

[57] Smoot G. F. et al., Ap. J. 396 (1992) L1;
Bennet C. L. et al., Ap. J. 430 (1994) 423.

[58] Melchiorri A., Vernizzi F., Durrer R. and Veneziano G., Phys. Rev. Lett. 83 (1999) 4464.

[59] Mauskopf P. D., et al., astro-ph/9911444.

[60] Brustein R. and Hadad M., Phys. Rev. D57 (1998) 725.

[61] Gasperini M. and Giovannini M., Phys. Lett. B301 (1993) 334, Class. Quant. Grav. 10 (1993) L133;
Brandenberger R., Mukhanov V. and Prokopec T., Phys. Rev. Lett. 69 (1992) 3606, Phys. Rev. D48 (1993) 2443.

[62] Brustein R. and Veneziano G., Phys. Lett. B329 (1994) 429;
Kaloper N., Madden R. and Olive K.A., Nucl. Phys. B452 (1995) 677, Phys. Lett. **B371** (1996) 34;
Easther R., Maeda K. and Wands D., Phys. Rev. D53 (1996) 4247.

[63] Giovannini M., Phys. Rev. D57 (1998) 7223.

[64] Brustein R. and Madden R., Phys. Lett. B410 (1997) 110, Phys. Rev. D57 (1998) 712, JHEP 9907 (1999) 006.

[65] Meissner K.A. and Veneziano G., Mod. Phys. Lett. A6 (1991) 3397.

[66] Gasperini M., Maharana J. and Veneziano G. , Phys. Lett. B296 (1992) 51.

[67] See, e.g., Douglas M. R., hep-th/9901146, and references therein.

[68] Gasperini M., Maharana J. and Veneziano G., Nucl. Phys. **B472** (1996) 349;
Gasperini M. and Veneziano G., Gen. Rel. Grav. 28 (1996) 1301;
Kehagias A.A. and Lukas A., Nucl. Phys. B477 (1996) 549;
Buonanno A. et al., Class. Quant. Grav. 14 (1997) L97.

[69] Gasperini M., Maggiore M. and Veneziano G., Nucl. Phys. B494 (1997) 315.

[70] Kiritsis E. and Kounnas C., gr-qc/9509017, "String Gravity and Cosmology: Some New Ideas", in Proc. of the Four Seas Conference, Trieste, 1995.

[71] Maggiore M. and Riotto A., Nucl. Phys. B548 (1999) 427.

[72] Ghosh A., Madden R. and Veneziano G., hep-th/9908024.

[73] Antoniadis I., Rizos J. and Tamvakis K., Nucl. Phys. B415 (1994) 497.

[74] Easther R. and Maeda K., Phys. Rev. D54 (1996) 7252.

[75] Brustein R. and Madden R., Phys. Rev. D57 (1998) 712;
Brandenberger R., Easther R. and Maia J., JHEP 9808 (1998) 007;
Foffa S., Maggiore M. and Sturani R., Nucl. Phys. B552 (1999) 395;
Cartier C., Copeland E. J. and Madden R., hep-th/9910169.

[76] Madden R., private communication.

[77] Bekenstein J. D., Phys. Rev. D23 (1981) 287 and **D49** (1994) 1912, and references therein.

[78] 't Hooft G. , gr-qc/9321026, in Abdus Salam Festschrift: A Collection of Talks, eds. Ali A., Ellis J. and Randjbar-Daemi S., World Scientific, Singapore, 1993;
Susskind L., J. Math. Phys. 36 (1995) 6377, and references therein.

[79] Bekenstein J. D., Int. J. Theor. Phys. 28 (1989) 967.

[80] Fischler W. and Susskind L., hep-th/9806039.

[81] Bousso R., JHEP 07 (1999) 004 and 06 (1999) 028.
See also Flanagan E.E., Marolf D. and Wald R.M., hep-th/9908070.

[82] Easther R. and Lowe D.A., Phys. Rev. Lett. 82 (1999) 4967.

[83] Veneziano G., Phys. Lett. B454 (1999) 22 and hep-th/9907012.

[84] Bak D. and Rey S., hep-th/9902173.

[85] Kaloper N. and Linde A., Phys. Rev. D60 (1999) 103509.

[86] Brustein R. and Veneziano G., hep-th/9912055.

[87] Carr B. J. and Hawking S. W., Mon. Not. Roy. Astr. Soc. 168 (1974) 399;
Carr B. J., Astrophys. J. 201 (1975) 1;
Novikov I. D. and Polnarev A. G., Astron. Zh. 57 (1980) 250 [Sov. Astron. 24 (1980) 147].

[88] see, e.g., Penrose R., "The Emperor's new mind", Oxford University Press, New York, 1989, Chapter 7.

[89] Brustein R., gr-qc/9904061;
see also Brustein R., Foffa S. and Sturani R. R., hep-th/9907032.

[90] Lukas A., Ovrut B.A. and Waldram D., Phys. Lett. B393 (1997) 65, Nucl. Phys. B495 (1997) 365;
Larsen F. and Wilczek F., Phys. Rev. D55 (1997) 4591;
Kaloper N., Phys. Rev. D55 (1997) 3394;
Lu H., Mukherji S. and Pope C.N., Phys. Rev. D55 (1997) 7926;
Poppe R. and Schwager S., Phys. Lett. B393 (1997) 51;
Lukas A. and Ovrut B. A., Phys. Lett. B437 (1998) 291;
Kaloper N., Kogan I. and Olive K. A., Phys. Rev. D57 (1998) 7340;

Banks T., Fischler W. and Motl L., JHEP 9901 (1999) 019;
Feinstein A. and Vazquez-Mozo M. A., hep-th/9906006.

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STRING COSMOLOGY AND RELIC GRAVITATIONAL RADIATION

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ABSTRACT

String theory counterparts to Einstein's gravity, cosmology and inflation are described. A very tight upper bound on the Cosmic Gravitational Radiation Background (CGRB) of standard inflation is shown to be evaded in string cosmology, while an interesting signal in the phenomenologically interesting frequency range is all but excluded. The generic features of such a stringy CGRB are presented.

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String theory counterparts to Einstein's gravity, cosmology and inflation are described. A very tight upper bound on the Cosmic Gravitational Radiation Background (CGRB) of standard inflation is shown to be evaded in string cosmology, while an interesting signal in the phenomenologically interesting frequency range is all but excluded. The generic features of such a stringy CGRB are presented.

1 Introduction

In this talk I will first explain why string theory offers an interesting alternative to Einstein's gravity and cosmology. The standard post-big-bang picture emerges as just the *late-time* history of a Universe which, in a prehistoric (*pre-big-bang*) era, underwent an inflationary expansion driven by the growth of the universal coupling of the theory.

I will then turn to describing one of the most interesting physical consequences of this new scenario: the production of a Cosmic Gravitational Radiation Background (CGRB), which could by far exceed, in the relevant frequency range, the one predicted by ordinary inflationary models.

I will leave the detailed discussion of the near-future prospects for observability of our CGRB to the following talk by R. Brustein and refer you, for more details on the scenario and the computations, to the collection of papers on string cosmology appearing on WWW under:

http:/www.to.infn.it/teorici/gasperini/

The precious collaboration of Maurizio Gasperini throughout the development of the pre-big-bang scenario, and the additional one of Ramy Brustein, Massimo Giovannini and Slava Mukhanov in working out its consequences for gravitational perturbations, are gratefully acknowledged.

2 Einstein Gravity and Standard Cosmology

In order to introduce string gravity and a cosmological model based on it, I will first recall a few known facts about Einstein gravity and standard cosmology (see for instance [1]).

The well-known Einstein equations:

$$R_{\mu\nu} - 1/2 g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{1}$$

follow from setting to zero the variation of the Einstein–Hilbert action (I will use $c = \hbar = 1$ throughout):

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \ [R - 2\Lambda] + S_{matter} \ . \tag{2}$$

Einstein cosmology follows from Einstein's equations upon insertion of a homogeneous (and, for simplicity, spatially flat) ansatz for the metric:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a(t)^2 dx^i dx^i \tag{3}$$

and after assuming that also matter is homogeneously distributed. The Einstein–Friedman equations (of which only two are independent) then follow:

$$\begin{aligned} H^2 \equiv (\dot{a}/a)^2 &= \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \\ \dot{H} + H^2 \equiv (\ddot{a}/a) &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \\ \dot{\rho} &= -3H(\rho + p) \end{aligned} \tag{4}$$

where the matter energy density ρ and pressure p are defined in terms of $T_{\mu\nu}$ by $T^\nu_\mu = \mathrm{diag}\ (\rho, -p, -p, -p)$. Notice that the effect of a non-vanishing cosmological constant Λ is equivalent to that of a special kind of matter, the "vacuum", with $\rho_{vac} = -p_{vac} = \frac{\Lambda}{8\pi G}$. Normal matter has $(\rho + 3p) > 0$ and therefore leads to a decelerated expansion of the Universe: in particular, a matter-dominated Universe ($p/\rho \sim 0$) expands like $t^{2/3}$, while a radiation-dominated Universe ($p/\rho \sim 1/3$) expands like $t^{1/2}$.

A trivial but important remark for the following discussion: if we regard (as we should) the first of eqs. (4) as expressing the vanishing of the total energy of the matter-plus-gravity system, we see that the expansion of the Universe contributes with a *negative* kinetic energy to such an equation.

Inflation, i.e. a long phase of accelerated expansion of the Universe ($\dot{a}, \ddot{a} > 0$), is badly needed in order to solve the outstanding problems of the standard cosmological model [2]. Unlike ordinary matter, a cosmological constant can easily do the job. The same is true of potential energy originating from a scalar field (the so-called inflaton) which, during some cosmic epoch, was approximately frozen away from the minimum of its potential and thus provided an effective (positive) cosmological constant $\Lambda_{eff} = 8\pi GV$. In this case a (quasi) de Sitter exponential expansion of the Universe takes place:

$$a(t) \sim \exp(Ht) \ , \quad H^2 = \frac{8\pi}{3} GV = \Lambda_{eff}/3 \ . \tag{5}$$

3 The disappointing CGRB of standard inflation

In standard potential-energy-driven inflation, while the inflaton slowly rolls down to the true minimum of the potential (where, by assumption, the potential energy is very small), the Hubble parameter H stays constant or decreases slowly. The Hubble radius H^{-1} thus remains constant (or increases slowly) during the inflationary epoch and then starts to grow like cosmic time t during the radiation- and matter-dominated eras.

In Fig. 1 this behaviour of the Hubble radius is plotted together with the behaviour of different physical scales which, by definition, grow like the scale factor $a(t)$ itself. It is easily found that scales cross the "horizon" outward (exit) during inflation and cross it again inward (re-enter) during the matter- or radiation-dominated epochs. Larger scales exit earlier and re-enter later than shorter scales. In order to solve the homogeneity problem of standard cosmology, it is necessary that the scale corresponding to the present horizon, $O(H_0^{-1})$, once upon a time, was inside the horizon. For this to happen a total red-shift of about

$$z_{infl} \equiv \frac{a_{end}}{a_{beg}} > 10^{30} \tag{6}$$

during inflation is needed [2].

One of the celebrated bonuses of inflation [2] is a natural explanation of the origin of large-scale structure. Let us assume that, initially, there were no inhomogeneities other than the minimal ones due to quantum mechanics. In other words, let us ask ourselves whether the origin of a structure in the Universe can be found in the initial vacuum quantum fluctuations. Vacuum fluctuations of the metric (defined as usual by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) with wavelength

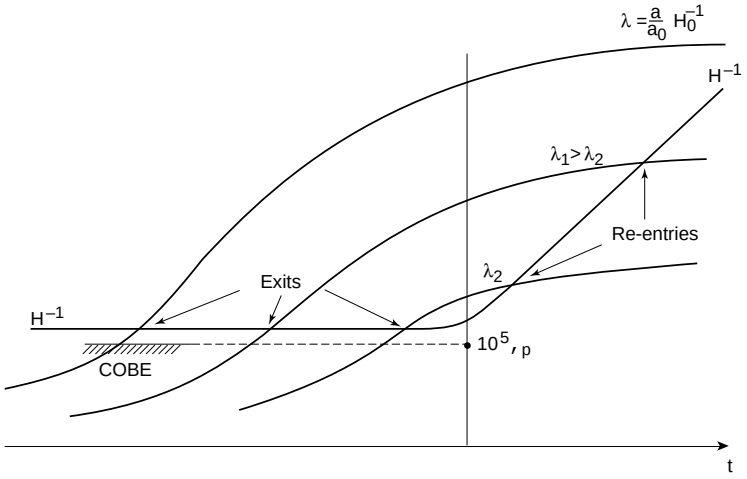

Figure 1

λ have a typical magnitude

$$\delta h(\lambda) \sim \frac{\ell_P}{\lambda} \ , \tag{7}$$

where $\ell_P = \sqrt{G}$ is the Planck length whose magnitude controls the size of quantum gravity effects. We see that the shorter the wavelength the larger the quantum fluctuation.

It is not hard to show that these original perturbations (inhomogeneities) are adiabatically damped (i.e. they follow eq. (7) with $\lambda \sim a$) as long as their physical wavelength stays inside the horizon, while they freeze-out (stay constant) after going outside. Since larger wavelengths spend a longer time outside the horizon (Fig.1) they stay frozen for a longer time. In other words there is a competition between two effects: quantum mechanics favours short scales while classical freeze-out favours large scales.

Combining the two effects leads to a simple and suggestive formula [2] for the present magnitude of tensor metric perturbations i.e of gravitational waves (GW):

$$\rho_\gamma^{-1} \frac{d\rho}{d\log\omega} \equiv \frac{\Omega_{GW}(\omega)}{\Omega_\gamma} \sim (\ell_P H)^2|_{ex} \ , \tag{8}$$

where $\Omega_\gamma = \frac{\rho_\gamma}{\rho_{cr}} \sim 10^{-4}$ and $\Omega_{GW}(\omega) = \rho_{cr}^{-1} \frac{d\rho_{GW}}{d\log\omega}$ and the label ex indicates that $l_P H$ has to be evaluated, for each scale λ, at the time of its exit. This is the

crucial quantity for the GW yield at any given frequency. As we have explained, H is constant or slowly decreasing during inflation, hence the same is true of the GW spectrum as a function of ω. This is the celebrated (quasi) scale-invariant Harrison–Zeldovich spectrum, which appears to be quite efficient for generating the observed large-scale structure (if combined with an appropriate model for dark matter).

Unfortunately, for the purpose of this talk, the above result is bad news, i.e. represents a disappointing spectrum of GW in the relevant frequency region. Indeed, COBE's observation [3] of a $\frac{\Delta T}{T}$ of order 10^{-5} at large angular scales implies $H^{-1} > 10^5 \ell_P$ when scales of the order of the present Hubble radius went out of the horizon, and an even smaller value when shorter scales did (see again Fig.1). Inserting such a limit in eq. (8), we immediately arrive at:

$$\Omega_{GW}(\omega) < 10^{-14} \text{ to } 10^{-15} \tag{9}$$

in the interesting (Hz to MHz) frequency range. This upper limit makes the CGRB produced by ordinary inflation an unobservable signal for some time to come ...

4 String Gravity

Being a theory of extended objects, string theory contains a fundamental length scale λ_s, a built-in ultraviolet (short-distance) cut-off [4]. As a result, string gravity differs from Einstein gravity in a subtle and essential way. Instead of the action (1.1), string theory gives [5]:

$$\begin{aligned} \Gamma_{eff} &= \frac{1}{2}\int d^4x\sqrt{-g}\; e^{-\phi}\left[\lambda_s^{-2}(R+\partial_\mu\phi\partial^\mu\phi)+F_{\mu\nu}^2+\bar{\psi}\not{D}\psi\right] \\ &\quad +\left[\text{higher orders in } \lambda_s^2\cdot\partial^2\right]+\left[\text{higher orders in } e^{\phi}\right]\ . \end{aligned} \tag{10}$$

As indicated in (10), string gravity has (actually needs!) a new particle/field, the so-called dilaton ϕ, a scalar particle. It enters Γ_{eff} as a Jordan–Brans–Dicke [6] scalar with a "small" negative ω_{BD} parameter, $\omega_{BD} = -1$. Bounds on the present rate of variation of α and G imply that, today, $\dot{\phi} < H_0$, while precision tests [7] of the equivalence principle put an upper (lower) limit on the range of the dilaton-exchange force (on the dilaton mass) [8]:

$$m_\phi > 10^{-4} \text{ eV}\ . \tag{11}$$

Both problems are solved by assuming that a non-perturbative dilaton potential has to be added to (10). Such a potential will freeze the dilaton to its present value ϕ_0 and make us recover Einstein's theory (and its experimental successes) at late times.

The value ϕ_0 provides [9] today's unified value of the gauge and gravitational couplings at energy scales of $O(\lambda_s^{-1})$. In formulae:

$$\begin{aligned} \ell_P^2 &\equiv 8\pi G_N = e^{\phi_0}\lambda_s^2 \ , \\ \alpha_{GUT}(\lambda_s^{-1}) &\simeq \frac{e^{\phi_0}}{4\pi} = \frac{\ell_P^2}{\lambda_s^2}, \end{aligned} \tag{12}$$

implying (from $\alpha_{GUT} \approx 1/20$) that the string-length parameter λ_s is about 10^{-32} cm. Note, however, that the above formulae, in a cosmological context in which ϕ evolves in time, can only be taken as giving the *present* values of α and ℓ_P/λ_s. In the scenario we will advocate, both quantities were much smaller in the very early Universe!

Equation (10) contains two dimensionless expansion parameters. One of them, the above-mentioned $g^2 \equiv e^{\phi}$, controls the analogue of loop corrections in quantum field theory (QFT), while the other, $\lambda^2 \equiv \lambda_s^2 \cdot \partial^2$, controls string-size effects, which are of course absent from QFT. Obviously, the expansion in λ^2 is reliable at small curvatures (derivatives), i.e. at energies smaller than the string scale λ_s^{-1}, while higher orders in g^2 will be negligible at weak coupling.

The first and main assumption of our scenario is that the Universe started its evolution in a regime that was perturbative with respect to both expansions, i.e. in a region of weak coupling and small curvatures (derivatives). During that phase the string-gravity equations take the simple form:

$$\begin{aligned} R_{\mu\nu} + \nabla_\mu\nabla_\nu\phi &= -\lambda_s^2\ e^{\phi} T_{\mu\nu} \\ R - \nabla_\mu\phi\nabla^\mu\phi + 2\nabla^2\phi + 2\Lambda &= 0 \ , \end{aligned} \tag{13}$$

which are similar to Einstein's equations, yet substantially different. As already stressed, we wish to recover general relativity at late times; nonetheless, we want to take advantage of the difference for the prehistory of the Universe.

Before closing this section I would like to briefly comment on a point that appears to be the source of much confusion: it is the dilemma between working in the so-called string frame and working in the more conventional Einstein frame. The two frames are not to be confused with different coordinate systems: they are instead related by a local field redefinition, a conformal,

dilaton-dependent rescaling of the metric, to be precise. All physical quantities are independent of the frame one is using. The question is: What should we call the metric? Although, to a large extent, this is a question of taste, one's intuition may work better with one definition than with another. Note also that, since the dilaton is time-independent today, the two frames now coincide.

Let us compare the virtues and problems inherent in each frame.

A) **String Frame.** This is the metric appearing in the original (σ-model) action for the string. Classical, weakly coupled strings sweep geodesic surfaces with respect to this metric [10]. Also, the dilaton dependence of the low-energy effective action takes the simple form indicated in (10) only in the string frame. The advantage of this frame is that the string cut-off is fixed and the same is true of the value of the curvature at which higher orders in the σ-model coupling λ become relevant. The main disadvantage is that the gravitational action is not so easy to work with.

B) **Einstein Frame.** In this frame the pure gravitational action takes the standard Einstein–Hilbert form. Consequently, this is the most convenient frame for studying the cosmological evolution of metric perturbations. The Planck length is fixed in this frame, while the string length is dilaton- (hence generally time-) dependent. In the Einstein frame, Γ_{eff} takes the form:

$$\Gamma_{eff} = \int \frac{d^4x\sqrt{-g}}{16\pi G_N}\left[R + \partial_\mu\phi\partial^\mu\phi + e^{-\phi}F^2_{\mu\nu} + \partial_\mu A\partial^\mu A + e^{\phi}m^2A^2\right] + \left[G_N e^{-\phi}R^2 + \ldots\right], \quad (14)$$

showing that the constancy of G in this frame is only apparent, since masses are dilaton-dependent (even at tree level). The same is true of the value of R at which higher order stringy corrections become important.

For the above reasons I will choose to base the discussion (although not always the calculations) in the string frame.

5 String cosmology

There is an exact (all-order) vacuum solution for (critical) superstring theory. Unfortunately, it corresponds to a free theory ($g = 0$ or $\phi = -\infty$) in flat,

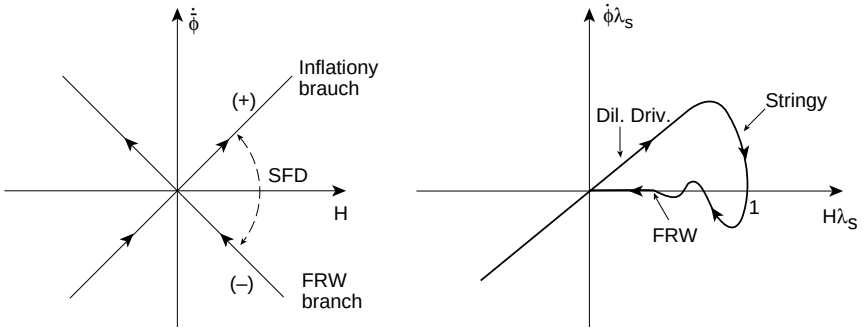

Figure 2

ten-dimensional, Minkowski space-time, nothing like the world we are living in today! Could this instead have been the *original* state of the Universe? The very basic postulate of the pre-big-bang scenario [11], [12] is that this is indeed the case.

Such a postulate is supported by the observation that, in the space of homogeneous (and, for simplicity, spatially-flat) perturbative solutions to the field equations, the trivial vacuum is a very special, *unstable* solution. This is depicted in Fig. 2a for the simplest case of a ten-dimensional cosmology in which three spatial dimensions evolve isotropically while six "internal" dimensions are static (it is easy to generalize the discussion to the case of dynamical internal dimensions, but then the picture becomes multidimensional).

The straight lines in the $H, \dot{\bar{\phi}}$ plane (where $\dot{\bar{\phi}} \equiv \dot{\phi} - 3H$) represent the evolution of the scale factor and of the coupling constant as a function of the cosmic time parameter (arrows along the lines show the direction of the time evolution). As a consequence of a stringy symmetry, known [11], [13] as Scale Factor Duality (SFD), there are two branches (two straight lines). Furthermore, each branch is split by the origin in two time-reversal-related parts (time reversal changes the sign of both H and $\dot{\bar{\phi}}$).

As mentioned, the origin (the trivial vacuum) is an "unstable" fixed point: a small perturbation in the direction of positive $\dot{\bar{\phi}}$ makes the system evolve further and further from the origin, meaning larger and larger coupling and absolute value of the Hubble parameter. This means an accelerated expansion (inflation) or an accelerated contraction. It is tempting to assume that those patches of the original Universe that had the right kind of initial fluctuation

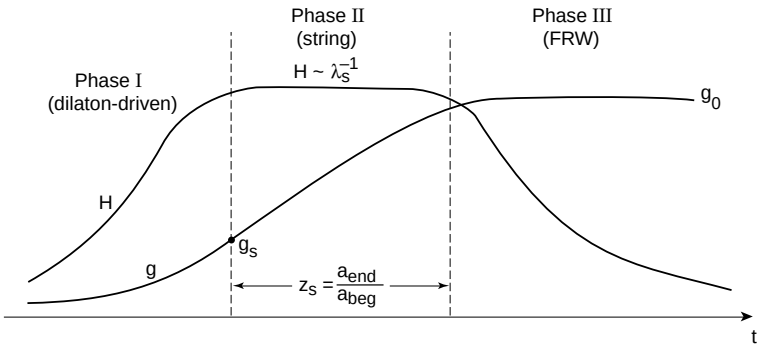

Figure 3

have grown up to become (by far) the largest fraction of the Universe today.

In order to arrive at a physically interesting scenario, however, we have to connect somehow the top-right inflationary branch to the bottom-right branch, since the latter is nothing but the standard FRW cosmology, which has presumably prevailed for the last few billion years or so. Here the so-called *exit problem* of string cosmology arises. At lowest order in λ^2 (small curvatures in string units) the two branches do not talk to each other. The inflationary (also called $+$) branch has a singularity in the future (it takes a finite cosmic time to reach ∞ in our gragh if one starts from anywhere but the origin) while the FRW $(-)$ branch has a singularity in the past (the usual big-bang singularity).

It is widely believed that QST has a way to avoid the usual singularities of classical general relativity or at least a way to reinterpret them [14], [15]. It thus looks reasonable to assume that the inflationary branch, instead of leading to a non-sensical singularity, will evolve into the FRW branch at values of λ^2 of order unity. This is schematically shown in Fig. 2b, where we have gone back from $\dot{\bar{\phi}}$ to $\dot{\phi}$ and we have implicitly taken into account the effects of a non-vanishing dilaton potential at small ϕ in order to freeze the dilaton at its present value. The need for the branch change to occur at large λ^2, first argued for in [16], has recently been proved [17].

There is a rather simple way to parametrize a class of scenarios of the kind defined above. They contain (roughly) three phases and two parameters, which can be easily visualized in Fig. 3.

In phase I the Universe evolves at $g^2, \lambda^2 \ll 1$ and is thus close to the trivial

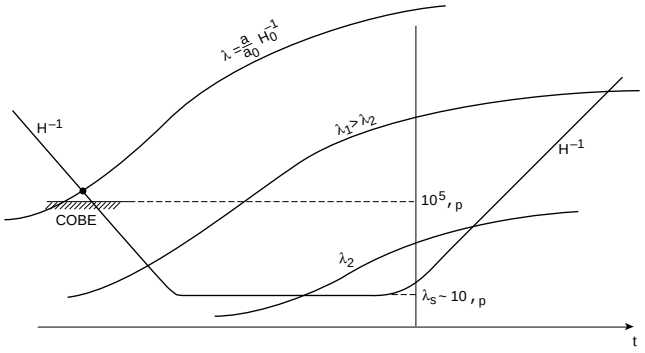

Figure 4

vacuum. This phase can be studied using the tree-level low-energy effective action (10); it is characterized by a long period of dilaton-driven inflation. The accelerated expansion of the Universe, instead of originating from the potential energy of an inflaton field, is driven by the growth of the coupling constant (i.e. by the dilaton's kinetic energy) with $\dot{\phi} = 2\dot{g}/g \sim H$ during the whole phase. Notice that, as for ordinary inflation, the negative value of the kinetic energy associated with an expanding Universe is crucial.

Phase I supposedly ends when the coupling λ^2 reaches values of $O(1)$, so that higher-derivative terms in the effective action become relevant. Assuming that this happens while g^2 is still small (and thus the potential is still negligible), the value g_s of g at the end of phase I (the beginning of phase II) is an arbitrary parameter (a modulus of the solution).

During phase II, the stringy version of the big bang, the curvature as well as $\dot{\phi}$ are assumed to remain fixed at their maximal value, given by the string scale (i.e. we expect $\lambda \sim 1$). The coupling g will instead continue to grow from the value g_s until, in turn, it reaches values $O(1)$. At that point, thanks to a non-perturbative effect in g, the string phase will come to an end and the dilaton will be attracted to the true non-perturbative minimum of its potential; the standard FRW cosmology can then start, provided the Universe was by then heated-up and filled with radiation (see below). The second important parameter of this scenario is the duration of phase II or better the total redshift, $z_s \equiv a_{end}/a_{beg}$, which has occurred from the beginning to the end of the stringy phase.

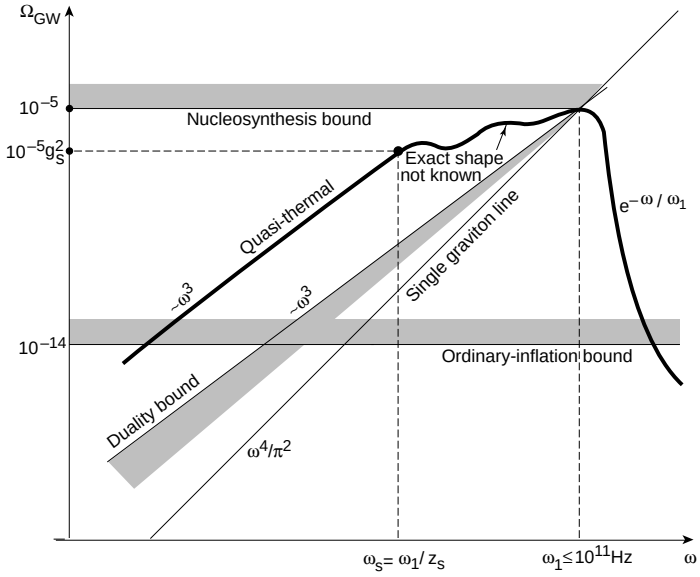

Figure 5

6 Perturbations and CGRB in String Cosmology

Starting from Fig. 3 let us draw the analogue of Fig. 1 for string cosmology, i.e. the behaviour of the horizon and the way different scales cross it. This is depicted in Fig. 4. Recalling eq. (8) we can easily understand why the bound (9) is now easily avoided. All we need in order to satisfy COBE's constraint is that $\ell_P H$ had been small enough at the time when the present horizon's scale crossed the Hubble radius during inflation (see hatched region in Fig. 4). Since the horizon is shrinking during superinflation, this does not prevent $\ell_P H$ from having been much smaller when scales of interest for GW detection (say above 1 Hz) crossed the horizon.

The final outcome for the GW spectrum [18], [19] is shown schematically in Fig. 5 (leaving a more detailed description to Ramy Brustein's talk). For a given pair g_s, z_s one identifies a point in the ω, Ω_ω plane as illustrated explicitly in the case of $g_s = 10^{-3}, z_s = 10^6$. The resulting point (indicated by a large dot) represents the end-point $(\omega_s, \Omega_{\omega_s})$ of the ω^3 spectrum corresponding to scales having crossed the horizon during the dilatonic era.

Although the rest of the spectrum is more uncertain, it can be argued[19] that it must smoothly join the point $(\omega_s, \Omega_{\omega_s})$ to the true end-point $\Omega \leq 10^{-5}, \omega \sim 10^{11}$ Hz. The latter corresponds to a few gravitons produced at the maximal amplified frequency ω_1, the last scale to go outside the horizon during the stringy phase. The full spectrum is also shown in the figure for the case $g_s = 10^{-3}, z_s = 10^6$, with the wiggly line representing the less well-known high-frequency part.

If $g_s < 1$, as we have assumed, spectra will always lie below the $\Omega_{GW} = 10^{-5}$, a line representing also a phenomenological bound for a successful nucleosynthesis to take place [20]. On the other hand, by invoking duality properties of the GW spectrum [21], it can be argued that the actual spectrum will never lie below the self-dual spectrum ending at $\Omega \sim 10^{-5}, \omega \sim 10^{11}$ Hz (the thick line bordering the shaded region). In conclusion all possible spectra sweep the angular wedge inside the two above-mentioned lines and a signal close to the NS bound is all but excluded. The large signal can be attributed to the fact that, in the pre-big-bang scenario, curvatures close to Planck's scale are reached before the end of inflation.

Having left to the next talk the discussion of further details on the GW spectrum and on the future prospects of detecting them, I will use the remaining time to mention a few more encouraging consequences of the pre-big-bang scenario. Like the generation of GW, they have something to do with the well-known phenomenon [22] of amplification of vacuum quantum fluctuations in cosmological backgrounds.

The first concerns scalar perturbations: Do they remain small enough during the pre-big-bang not to destroy the quasi-homogeneity of the Universe? The answer to this question turns out to be yes! This is not a priori evident since, in commonly used gauges (see e.g. [23]) for scalar perturbations of the metric (e.g. the so-called longitudinal gauge in which the metric remains diagonal), such perturbations appear to grow very large during the inflationary phase and to destroy homogeneity or, at least, to prevent the use of linear perturbation theory. In ref. [24] it was shown that, by a suitable choice of gauge (an "off-diagonal" gauge), the growing mode of the perturbation can be tamed. This can be double-checked by using the so-called gauge-invariant variables of Bruni and Ellis [25]. The bottom line is that scalar perturbations in string cosmology behave no worse than tensor perturbations. An interesting question arises here, in connection with the detectability of scalar perturbations of this type by using spherical antennas.

The second point that I wish to mention concerns a rather unique prediction of our scenario: the amplification of EM perturbations. Because of the scale-invariant coupling of gauge fields in four dimensions, electromagnetic (EM) perturbations are *not* amplified in a conformally flat cosmological background (even if inflationary). In string cosmology, the presence of a time-dependent dilaton in front of the gauge-field kinetic term allows the amplification of EM perturbations. Seeds for generating the galactic magnetic fields through the so-called cosmic-dynamo mechanism [26] can thus be obtained.

The final outcome can be expressed [27], [28] in terms of the fraction of electromagnetic energy stored in a unit of logarithmic interval of ω normalized to the one in the CMB, ρ_γ. One finds:

$$r(\omega) = \frac{\omega}{\rho_\gamma}\frac{d\rho_B}{d\omega} \simeq \frac{\omega^4}{\rho_\gamma}|c_-(\omega)|^2 \equiv \frac{\omega^4}{\rho_\gamma}(g_{re}/g_{ex})^2 \ . \tag{15}$$

where g_{ex} (g_{re}) refer to the value of the coupling at exit (re-entry) of the scale ω under consideration.

In terms of $r(\omega)$ the condition for seeding the galactic magnetic field through ordinary mechanisms of plasma physics is [29]

$$r(\omega_G) \geq 10^{-34} \ , \tag{16}$$

where $\omega_G \simeq (1 \text{ Mpc})^{-1} \simeq 10^{-14}$ Hz is the galactic scale. Using the known value of ρ_γ, we thus find, from (15) and (16):

$$g_{ex} < 10^{-33} \ , \tag{17}$$

i.e. a very tiny coupling at the time of exit of the galactic scale.

The conclusion is that string cosmology stands a unique chance to explain the origin of the galactic magnetic fields. Indeed, if the seeds of the magnetic fields are to be attributed to the amplification of vacuum fluctuations, their present magnitude can be interpreted as prime evidence that the fine structure constant has evolved to its present value from a tiny one during inflation. The fact that the needed variation of the coupling constant ($\sim 10^{30}$) is of the same order as the variation of the scale factor needed to solve the standard cosmological problems, can be seen as further evidence for scenarios in which coupling and scale factor grow roughly at the same rate during inflation.

Finally, I would like to mention a more theoretical bonus following from the pre-big-bang picture: a possible explanation of standard cosmology's hot initial state.

The question is: Can one arrive at the hot big bang of the SCM starting from our "cold" initial conditions? The reason why a hot Universe can emerge at the end of our inflationary epochs (phases I and II) goes back to an idea of L. Parker [30], according to which amplified quantum fluctuations can give origin to the CMB itself if Planckian scales are reached.

Rephrasing Parker's idea in our context amounts to solving the following bootstrap-like condition: At which moment, if any, will the energy stored in the perturbations reach the critical density? The total energy density ρ_{qf} stored in the amplified vacuum quantum fluctuations is roughly given by:

$$\rho_{qf} \sim N_{eff} \, \frac{M_s^4}{4\pi^2} \, (a_1/a)^4 \ , \tag{18}$$

where N_{eff} is the number of effective (relativistic) species, which get produced (whose energy density decreases like a^{-4}) and a_1 is the scale factor at the (supposed) moment of branch change. The critical density (in the same units) is given by:

$$\rho_{cr} = e^{-\phi} M_s^2 H^2 \ . \tag{19}$$

At the beginning, with $e^{\phi} \ll 1$, $\rho_{qf} \ll \rho_{cr}$; but, in the $(-)$ branch solution, ρ_{cr} decreases faster than ρ_{qf} so that, at some moment, ρ_{qf} will become the dominant source of energy while the dilaton kinetic term will become negligible. It would be interesting to find out what sort of initial temperatures for the radiation era will come out of this assumption.

7 Conclusions

- All cosmological inflationary models lead to the prediction of a stochastic CGRB that should surround us today very much like its electromagnetic analogue.

- Standard inflationary models must unfortunately satisfy the constraint $\Omega_{GW} < 10^{-10}\Omega_{\gamma}$ in the interesting frequency range.

- Inflationary models, such as those suggested by string theory, in which the Hubble parameter grows during inflation and eventually reaches values $O(\lambda_s^{-1}, \ell_P^{-1})$, evade the above constraint and (may) naturally lead to $\Omega_{GW} \leq 0.1\Omega_{\gamma}$ in the interesting frequency range.

- Observation of such a CGRB would open a unique window on the very early Universe and thus on fundamental physics at the Planck (string) scale.

- Last but not least, as emphasized to me by Emilio Picasso, trying to detect a stochastic CGRB is not just relying on getting a gift from the sky!

References

1. S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, Inc., New York (1972).
2. L.F. Abbott and So-Young Pi (eds.), *Inflationary Cosmology*, World Scientific, Singapore (1986);
   E. Kolb and M. Turner, *The Early Universe*, Addison-Wesley, New York (1990).
3. G. Smoot et al., *Astrophys. J.* **396** (1992) L1.
4. G. Veneziano, "Quantum strings and the constants of Nature", in *The Challenging Questions*, Erice, 1989, ed. A. Zichichi, Plenum Press, New York (1990).
5. C. Lovelace, *Phys. Lett.* **B135** (1984) 75;
   C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, *Nucl. Phys.* **B262** (1985) 593;
   E.S. Fradkin and A.A. Tseytlin, *Nucl. Phys.* **B261** (1985) 1.
6. P. Jordan, *Z. Phys.* **157** (1959) 112;
   C. Brans and R.H. Dicke, *Phys. Rev.* **124** (1961) 925.
7. See, for instance, E. Fischbach and C. Talmadge, *Nature* **356** (1992) 207.
8. T.R. Taylor and G. Veneziano, *Phys. Lett.* **B213** (1988) 459.
9. E. Witten, *Phys. Lett.* **B149** (1984) 351.
10. N. Sanchez and G. Veneziano, *Nucl. Phys.* **B333** (1990) 253;
11. G. Veneziano, *Phys. Lett.* **B265** (1991) 287.
12. M. Gasperini and G. Veneziano, *Astropart. Phys.* **1** (1993) 317; *Mod. Phys. Lett.* **A8** (1993) 3701; *Phys. Rev.* **D50** (1994) 2519.
13. A.A. Tseytlin, *Mod. Phys. Lett.* **A6** (1991) 1721;
    A.A. Tseytlin and C. Vafa, *Nucl. Phys.* **B372** (1992) 443.
14. E. Kiritsis and C. Kounnas, *Phys. Lett.* **B331** (1994) 51;
    A.A. Tseytlin, *Phys. Lett.* **B334** (1994) 315.
15. P. Aspinwall, B. Greene and D. Morrison, *Phys. Lett.* **B303** (1993) 249;
    E. Witten, *Nucl. Phys.* **B403** (1993) 159;
    A. Strominger, *Massless black holes and conifolds in string theory*, hep-

th/9504090.
16. R. Brustein and G. Veneziano, *Phys. Lett.* **B329** (1994) 429.
17. N. Kaloper, R. Madden and K. A. Olive, *Towards a singularity-free inflationary universe?*, Univ. Minnesota preprint UMN-TH-1333/95 (June 1995).
18. R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Lett.* **B361** (1995) 45;
see also M. Gasperini and M. Giovannini, *Phys. Rev.* **D47** (1992) 1529.
19. R. Brustein, M. Gasperini and G. Veneziano, *Peak and end point of the relic graviton background in string cosmology*, hep-th/9604084;
A. Buonanno, M. Maggiore and C. Ungarelli, *Spectrum of relic gravitational waves in string cosmology*, Pisa preprint IFUP-TH 25/96.
20. N. Hata et al., *Phys. Rev. Lett.* **75** (1995) 3977;
C. Copi et al., *Phys. Rev. Lett.* **75** (1995) 3981; and references therein.
21. R. Brustein, M. Gasperini and G. Veneziano, in preparation.
22. L.P. Grishchuk, *Sov. Phys. JEPT* **40** (1975) 409;
A.A. Starobinski, *JEPT Lett.* **30** (1979) 682;
V.A. Rubakov, M. Sazhin and A. Veryaskin, *Phys. Lett.* **B115** (1982) 189;
R. Fabbri and M. Pollock, *Phys. Lett.* **B125** (1983) 445.
23. See, for instance, V. Mukhanov, H.A. Feldman and R. Brandenberger, *Phys. Rep.* **215** (1992) 203.
24. R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov and G. Veneziano, *Phys. Rev.* **D51** (1995) 6744.
25. G. F. R. Ellis and M. Bruni, *Phys. Rev.* **D40** (1989) 1804;
M. Bruni, G. F. R. Ellis and P. K. S. Dunsby, *Class. Quant. Grav.* **9** (1992) 921.
26. E. N. Parker, *Cosmical Magnetic Fields*, Clarendon, Oxford (1979);
Y. B. Zeldovich, A. A. Ruzmaikin and D. D. Sokoloff, *Magnetic fields in astrophysics*, Gordon and Breach, New York (1983).
27. M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Rev. Lett.* **75** (1995) 3796; *Phys. Rev.* **D52** (1995) 6651.
28. D. Lemoine and M. Lemoine, *Primordial magnetic fields in string cosmology*, Inst. d'Astrophysique de Paris preprint (April 1995).
29. M. S. Turner and L. M. Widrow, *Phys. Rev.* **D37** (1988) 2743.
30. L. Parker, *Nature* **261** (1976) 20.

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Quantum Inhomogeneities in String Cosmology

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Abstract

Within two specific string cosmology scenarios –differing in the way the pre- and post-big bang phases are joined– we compute the size and spectral slope of various types of cosmologically amplified quantum fluctuations that arise in generic compactifications of heterotic string theory. By further imposing that these perturbations become the dominant source of energy at the onset of the radiation era, we obtain physical bounds on the background's moduli, and discuss the conditions under which both a (quasi-) scale-invariant spectrum of axionic perturbations and sufficiently large seeds for the galactic magnetic fields are generated. We also point out a potential problem with achieving the exit to the radiation era when the string coupling is near its present value.

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I. INTRODUCTION

Perhaps the most appealing feature of standard inflationary cosmology [1] is its ability to stretch out generic/arbitrary initial classical inhomogeneities and to replace them by a calculable spectrum of cosmologically amplified quantum fluctuations. The latter behave, for all physical purposes, as a set of properly normalized stochastic classical perturbations. A much advertised outcome of slow-roll inflation is a (quasi-) scale-invariant (Harrison-Zeldovich (HZ)) spectrum of density fluctuations, a highly desirable feature for explaining both the CMB temperature fluctuations on large angular scales and the large-scale structure of the visible part of our Universe.

The so-called pre-big bang (PBB) scenario [2,3] offers, within the context of string theory, an alternative to the usual inflationary paradigm. Provided a graceful exit can be achieved (see [4,5] for recent progress on this issue), the PBB scenario exhibits several appealing advantages, e.g.

- it naturally provides inflationary solutions through the duality symmetries [6] of string theory;
- it assumes a natural, simple, initial state for the Universe, which is fully under control: the perturbative vacuum of superstring theory;
- it needs no fine-tuning of couplings and/or potentials: the inflaton is identified with the dilaton, which is ubiquitous in string theory, is effectively massless at weak coupling, and provides inflation through its kinetic energy;
- it can provide a hot big bang initial state as a late-time *outcome* of the pre-big bang phase, through the amplification of vacuum quantum fluctuations generated in this latter phase.

In recent work [7,8] we have discussed the conditions under which classical inhomogeneities get efficiently erased in string cosmology. In general, this does occur provided two moduli of the classical solutions at weak coupling and curvature (basically an initial coupling and an initial curvature scale) are bounded from above. Whether such conditions correspond to an acceptable degree of fine-tuning of the initial conditions or not is still the matter of some controversy [9,10,8].

An interesting outcome of these investigations has been a motivated conjecture [8] that, for negative spatial curvature, the pre-big bang phase itself is generically preceded by a contracting "Milne" phase, corresponding to a particular parametrization of the past light cone of trivial Minkowski space-time with a constant dilaton. Such a background, the trivial all-order classical vacuum of superstring theory, turns out to be an unstable early-time fixed point of the evolution. Thanks to dilaton/metric fluctuations, it appears to lead, inevitably, to pre-big bang-type inflation at later times.

In this paper we shall assume that the above classical picture effectively wipes out, during its long pre-big bang phase, spatial curvature and classical inhomogeneities, and we move on to analyse the second alleged virtue of inflationary cosmology, the generation of an interesting spectrum of amplified quantum fluctuations. As several previous investigations have shown [11–13], achieving this is not at all automatic in string cosmology. It was soon realized that, in the simplest PBB scenario, tensor [11] and scalar-dilaton [12] perturbations tend to have steep spectra (typically a spectral index $n = 4$, as compared to HZ's $n = 1$). Perturbations of gauge fields coming from compactification of the extra 16 bosonic dimensions of heterotic string theory can have somewhat smaller spectral indices [13], but still in the range $3 < n < 4$.

The situation can be improved by assuming [11] that a long string phase (during which the dilaton grew linearly in cosmic time while the Universe expanded exponentially) took place between the dilaton and the usual FRW phase. In such a case, it is possible to get *either* an interesting spectrum of gravitational waves [11] in the range of interest for detection, *or* enough EM perturbations to explain the magnetic fields [13], but not both, apparently. A flat spectrum of EM perturbations, which can possibly provide a new mechanism for generating large scale structure [14] is not excluded either.

Recently, however, Copeland et al. [15] made the interesting observation that axionic perturbations, even in the absence of a string phase, can have a flat spectrum, depending on how the internal dimensions evolve during the dilatonic phase. Unfortunately, Copeland et al. stopped short of computing the axionic spectrum after re-entry. Nonetheless, their result hints at a possible dominance of axionic perturbations over all others and calls for a revision of the whole scenario and of the phenomenological constraints that must be imposed on it.

In this paper we analyse quantum fluctuations of various kinds in two distinct scenarios for the background, with or without an intermediate string phase. We may expect either possibility to occur, depending on the precise mechanism providing the transition (exit) from

the PBB phase into the FRW phase.

An intermediate string phase is natural if we assume [4] that α' corrections provide a non-perturbative fixed point with a high constant curvature and a linearly growing dilaton. In this case we expect the transition to the FRW phase to occur during the string phase as soon as the energy stored in the quantum fluctuations reaches criticality (recall that the condition of criticality depends on the coupling). This is like saying that the final transition to the radiation-dominated era will be induced by string-loop, back-reaction effects (see, e.g. [5]).

We can imagine, instead, that α' corrections are sufficient to provide by themselves a sudden branch-change from the perturbative PBB phase to another duality-related vacuum phase, with the Hubble parameter making a bounce around its maximal value. In the language of [4] this would correspond to a square-root-type vanishing of a β-function. Again, the dual ($-$ branch) phase will gently yield to a FRW Universe as soon as the energy stored in the quantum fluctuations becomes critical.

As already mentioned, an important ingredient of our approach is the (self-consistency) requirement of criticality at the beginning of the radiation era. This provides a new relation between the moduli of the PBB background and the coupling and energy density (or temperature) at the beginning of the radiation era. As we will see, the dilaton at the beginning of this era is generically displaced from its eventual/present value; hence this primordial radiation era is not yet quite the one of standard cosmology. It may take a while before the non-perturbative dilaton potential makes its presence felt and forces the dilaton to its minimum. The detailed study of such post-big bang phase is left to further work.

One of the main conclusions of this paper is that, provided $U(1)_{em}$ has a component in the Kaluza-Klein gauge group produced in the compactification from $D = 10$ to $D = 4$, sufficiently large seeds for galactic magnetic fields can be generated, even in the absence of a string phase. Furthermore, this happens in the same range of moduli for which a nearly scale-invariant spectrum of axionic perturbations is generated. Such a range includes a particularly symmetric point in moduli space, the one corresponding to isotropy (up to T-duality transformations) in all nine spatial dimensions.

The outline of the paper is as follows: In Sec. II we give, for the sake of completeness, the four-dimensional low-energy string-level heterotic effective action that we will work with. In Sec. III we fix our parametrization of the backgrounds for the two previously discussed

scenarios. In Sec. IV we derive general formulae for the spectra of various perturbations, which get amplified by a generic background of the kind discussed in Sec. III. We will verify that our spectra satisfy a "duality" symmetry that can be shown to follow from general arguments [16]. In Sec. V we give the explicit form of the spectra for the two backgrounds discussed in Sec. III, and present them in various tables and plots. We will also impose the criticality condition and discuss its immediate consequences. Finally, Sec. VI contains a discussion of the results and some conclusions.

This paper is somewhat technical in nature and contains explicit general formulae that can be useful to the practitioner but do not carry easy messages: these can be better found in the tables and figures. At any rate, in order to help the reader, we have relegated the most complicated formulae to an appendix.

II. STRING EFFECTIVE ACTION FROM DIMENSIONAL REDUCTION

Following the notations of [17] we consider superstring theory in a space-time $\mathcal{M} \times \mathcal{K}$, where $\mathcal{M}$, with Minkowskian signature, has four non-compact dimensions, and $\mathcal{K}$ consists of six compact dimensions upon which all fields are assumed to be independent. Local coordinates of $\mathcal{M}$ are labelled by $\mu, \nu, \rho = 0, \ldots, 3$, those of $\mathcal{K}$ by $a, b, c = 4, \ldots, 9$. Moreover, all ten-dimensional fields and indices are distinguished by a hat.

We will limit ourselves to the case of a diagonal metric for the internal six-dimensional compact space, of a non-vanishing internal antisymmetric-tensor and of one Abelian heterotic $U(1)$ gauge field A_μ:

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + g_{ab}\, V_\mu{}^a\, V_\nu{}^b & g_{ab}\, V_\mu{}^b \\ g_{ab}\, V_\nu{}^b & g_{ab} \end{pmatrix} , \tag{2.1}$$

$$\hat{B}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} B_{\mu\nu} & W_{\mu a} - B_{ab}\, V_\mu{}^b \\ -W_{\nu a} + B_{ab}\, V_\nu{}^b & B_{ab} \end{pmatrix} . \tag{2.2}$$

In the following we take $g_{ab} = e^{2\sigma_a}\, \delta_{ab}$.

The low-energy four-dimensional effective string action is

$$\mathcal{S}^B_{\rm eff} = \frac{1}{2\lambda_s^2} \int d^4x\, \sqrt{-g}\, e^{-\varphi} \left[\mathcal{R} + g^{\mu\nu}\, \partial_\mu\varphi\, \partial_\nu\varphi - g^{\mu\nu}\, \partial_\mu\sigma_a\, \partial_\nu\sigma_a - \frac{1}{4}\, e^{2\sigma_a}\, V_{\mu\nu}{}^a\, V^{\mu\nu a} \right.$$
$$\left. -\frac{1}{4}\, e^{-2\sigma_a}\, H_{\mu\nu a}\, H^{\mu\nu}{}_a - \frac{1}{12}\, H_{\mu\nu\rho}\, H^{\mu\nu\rho} + \frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} - \frac{1}{4}\, g^{\mu\nu} e^{-2\sigma_b}\, e^{-2\sigma_c}\, \partial_\mu B_{bc}\, \partial_\nu B_{bc} \right] , \tag{2.3}$$

where $\lambda_s = \sqrt{8\pi}/M_s$ is the string-length parameter,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2}\left[V_\mu{}^a\, W_{\nu\rho a} + W_{\mu a}\, V_{\nu\rho}{}^a\right] - \frac{1}{2} A_\mu\, F_{\nu\rho} + \quad \text{cyclic perm.}\,, \tag{2.4}$$

$$H_{\mu\nu a} = W_{\mu\nu a} - B_{ab}\, V_{\mu\nu}{}^b\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{2.5}$$

$$V_{\mu\nu}{}^a = \partial_\mu V_\nu{}^a - \partial_\nu V_\mu{}^a\,, \qquad W_{\mu\nu a} = \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a}\,, \tag{2.6}$$

and φ stands for the effective four-dimensional dilaton field:

$$\varphi = \phi - \sum_a \sigma_a\,.$$

The components of the antisymmetric tensor $H^{\mu\nu\rho}$ with $\mu, \nu, \rho = 0, 1, 2, 3$ can be rewritten in terms of the pseudoscalar axion A as

$$H^{\mu\nu\rho} \equiv E^{\mu\nu\rho\sigma}\, e^{\varphi}\, \partial_\sigma A\,, \tag{2.7}$$

where $E^{\mu\nu\rho\sigma}$ is the covariant full antisymmetric Levi-Civita tensor, which satisfies $\mathcal{D}_\alpha E^{\mu\nu\rho\sigma} = 0$. Using Eq. (2.4), and imposing the Bianchi identity ($d^2 B = 0$), we get the equation of motion for the axion field

$$\partial_\mu(e^{\varphi}\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu A) - \frac{1}{8}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\left[2W_{\mu\nu a}\, V_{\rho\lambda}{}^a + F_{\mu\nu}\, F_{\rho\sigma}\right] = 0\,. \tag{2.8}$$

The reduced action then becomes

$$\begin{aligned}\mathcal{S}^A_{\rm eff} = \frac{1}{2\lambda_s^2}\int d^4x\,\sqrt{-g}\, e^{-\varphi}\,\Big[&\mathcal{R} + g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi - g^{\mu\nu}\,\partial_\mu\sigma_a\,\partial_\nu\sigma_a - \frac{1}{4}\, e^{2\sigma_a}\, V_{\mu\nu}{}^a\, V^{\mu\nu a} \\ &-\frac{1}{4}\, e^{-2\sigma_a}\, H_{\mu\nu a}\, H^{\mu\nu}{}_a - \frac{1}{2}\, e^{2\varphi}\, g^{\mu\nu}\,\partial_\mu A\,\partial_\nu A - \frac{1}{8}\, e^{\varphi}\,\frac{A\,\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\,\left[2W_{\mu\nu a}\, V_{\rho\lambda}{}^a + F_{\mu\nu}\, F_{\rho\sigma}\right] \\ &+\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} - \frac{1}{4}\, g^{\mu\nu} e^{-2\sigma_b}\, e^{-2\sigma_c}\,\partial_\mu B_{bc}\,\partial_\nu B_{bc}\Big]\,.\end{aligned} \tag{2.9}$$

We are interested in fluctuations around a homogeneous background with $\varphi = \varphi(t)$,

$$g_{\mu\nu} = (-1, a^2(t)\,\delta_{ij}, b_c^2(t)\,\delta_{cd})\,, \qquad i,j = 1,2,3\,, \qquad c,d = 4,\ldots,9\,, \qquad \text{all other fields} = 0\,.$$

In the following we will also use the metric $g_{\mu\nu} = (-a^2(\eta), a^2(\eta)\delta_{ij}, b_c^2(\eta)\,\delta_{dc})$, where we have introduced the conformal time η by $d\eta = dt/a$.

III. TWO MODELS FOR THE BACKGROUND

If the initial value of the string coupling is sufficiently small, it is possible for the Universe to reach the high curvature regime, where higher-derivative corrections are important, while the string coupling is still small enough to neglect loop corrections ($g = \exp\varphi/2 \ll 1$). As discussed in the introduction, we will consider two extreme alternatives. In the first, α' corrections "lock" the Universe in a *string phase* with a constant H and a linearly growing dilaton (with respect to cosmic time) [4]; in the second scenario, α' corrections induce a sudden transition from a perturbative (+) branch solution to a perturbative (−) branch phase. We will refer to the latter as the *dual-dilaton phase*.

We will thus consider a PBB cosmological background in which the Universe starts in the perturbative string vacuum, reaches the string curvature scale while in a dilaton-vacuum solution, goes either to the dual-dilaton phase or to the string phase, and finally enters the radiation era as a result of the back-reaction from the amplified quantum fluctuations. We now parametrize these two scenarios for the backgrounds, imposing the continuity of a, a', b_a, b_a', φ.

A. Intermediate dual dilaton phase

1 *Dilaton phase*

For $-\infty < \eta < \eta_s$, with $\eta_s < 0$ we have

$$a(\eta) = -\frac{1}{H_s\,\eta_s}\left|\frac{\eta\,(1-\delta)-\eta_s}{\delta\,\eta_s}\right|^{\delta/(1-\delta)}, \tag{3.1}$$

$$b_a(\eta) = -H_s\,\eta_s\left|\frac{(\eta-\eta_s)\,(1-\delta)-\beta_a\,\eta_s}{\beta_a\,\eta_s}\right|^{\beta_a/(1-\delta)} = e^{\sigma_a}, \tag{3.2}$$

$$\varphi(\eta) = \varphi_s + \frac{3\delta-1}{1-\delta}\log\left|\frac{\eta\,(1-\delta)-\eta_s}{\delta\,\eta_s}\right|, \tag{3.3}$$

$$1 = 3\delta^2 + \sum_a \beta_a^2, \tag{3.4}$$

where $H_s = a'(\eta_s)/a^2(\eta_s)$ is of order M_s. We will consider the case $\delta < 0$ and $\beta_a > 0$, i.e. a superinflationary solution with contracting internal dimensions. Because of the constraint between δ and β_a, if $|\delta| < 1/\sqrt{3}$, some of the β_a must be non-vanishing. In what follows we will pick two extreme cases: i) the most isotropic case, with $\beta_a = \sqrt{(1-3\delta^2)/6}$, or ii) the

most anisotropic one, with $\beta_a = \delta_{a1}\sqrt{(1-3\delta^2)}$. In figures we shall denote these two cases by a subscript 6 and 1, respectively.

2 *Dual-dilaton phase*

For $\eta_s < \eta < \eta_1$, with $\eta_1 > 0$ we take

$$a(\eta) = -\frac{1}{H_s\,\eta_s}\left|\frac{\eta\,(1-\theta)-\eta_s}{\theta\,\eta_s}\right|^{\theta/(1-\theta)}, \tag{3.5}$$

$$b_a(\eta) = -H_s\,\eta_s\left|\frac{(\eta-\eta_s)\,(1-\theta)-\xi_a\,\eta_s}{\xi_a\,\eta_s}\right|^{\xi_a/(1-\theta)}, \tag{3.6}$$

$$\varphi(\eta) = \varphi_s + \frac{3\theta-1}{1-\theta}\log\left|\frac{\eta\,(1-\theta)-\eta_s}{\theta\,\eta_s}\right|, \tag{3.7}$$

where $3\theta^2 + \sum_a \xi_a^2 = 1$ and we will fix $\theta > 0$ and $\xi_a < 0$, i.e. a decelerated expansion for the external scale factor and a decelerated contraction for the internal ones. Again, we distinguish two cases, $\xi_a = -\sqrt{(1-3\theta^2)/6}$ or $\xi_a = -\delta_{a1}\sqrt{(1-3\theta^2)}$.

3 *Radiation phase*

In the region $\eta_1 < \eta < \eta_{\rm eq}$, with $\eta_{\rm eq}$ the time of equivalence between radiation and matter density, we write

$$a(\eta) = \frac{(\eta-\eta_1-\eta_s)}{H_s\,\eta_s^2}\left|\frac{\eta_1\,(1-\theta)-\eta_s}{\theta\,\eta_s}\right|^{\theta/(1-\theta)}, \tag{3.8}$$

$$\varphi(\eta) = \text{const.}\,, \qquad b_a(\eta) = \text{const.} \tag{3.9}$$

B. Intermediate string phase

1 *Dilaton phase*

We parametrize this phase exactly as before. Thus, for $-\infty < \eta < \eta_s < 0$, Eqs. (3.1) to (3.4) hold.

2 *String phase*

For $\eta_s < \eta < \eta_1$, with $\eta_1 < 0$

$$a(\eta) = -\frac{1}{H_s\,\eta}\,, \qquad b_a(\eta) = \text{const.}\,, \qquad \varphi(\eta) = \varphi_s - 2\zeta\,\log\left(\frac{\eta}{\eta_s}\right), \tag{3.10}$$

hence a constant Hubble parameter for the external scale factor.

3 *Radiation phase*

In the range $\eta_1 < \eta < \eta_{\rm eq}$ we have

$$a(\eta) = \frac{1}{H_s\,\eta_1^2}\,(\eta - 2\eta_1)\,, \qquad b_a(\eta) = \text{const.}\,, \qquad \varphi(\eta) = \text{const.} \tag{3.11}$$

An important property of these backgrounds is that the derivative of the field φ is *not* continuous across the two transitions. This reflects the fact that we do not have as yet a satisfactory model for the transitions from one epoch to another. As discussed below, this discontinuity creates a technical problem, which has to be judiciously solved in order to correctly compute the spectrum of perturbations around this kind of backgrounds.

IV. AMPLIFICATION OF VACUUM FLUCTUATIONS

Let us consider a generic massless field, whose quadratic fluctuations are described by the action

$$\delta\mathcal{S} = \int d\eta\,\tilde{a}^2\,\left[(\Psi')^2 - (\nabla\Psi)^2\right]\,, \tag{4.1}$$

where a prime stands for derivative with respect to conformal time η and $\tilde{a}$, the so-called "pump" field, is a homogeneous background field that depends on the particular perturbation under study.

The safest way to analyse the amplification of the vacuum fluctuations of Ψ makes use of a canonical Hamiltonian approach and leads to the derivation [16] of certain duality symmetries of the spectra. We will use instead the simpler Lagrangian method and fix some ambiguity encountered in that approach by demanding agreement with the Hamiltonian treatment. We believe, of course, that our prescription can also be fully justified within the Lagrangian framework.

The equation of motion for the Fourier components of Ψ is

$$\Psi_k'' + 2\,\frac{\tilde{a}'}{\tilde{a}}\,\Psi_k' + k^2\,\Psi_k = 0\,. \tag{4.2}$$

Introducing the canonical variable

$$v_k = \tilde{a}\,\Psi_k\,, \tag{4.3}$$

Eq. (4.2) can be rewritten in the form

$$\frac{d^2 v_k}{d\eta^2} + \left[k^2 - U(\eta)\right] v_k = 0\,, \qquad U(\eta) = \frac{1}{\tilde{a}}\frac{d^2\tilde{a}}{d\eta^2}\,. \tag{4.4}$$

In order to get general formulae for the spectrum we parametrize the pump field in the three epochs as follows[1]

$$\tilde{a} = \left(\frac{\eta}{\eta_s}\right)^{\gamma} \qquad -\infty < \eta < \eta_s < 0\,, \tag{4.5}$$

$$\tilde{a} = \left|\frac{\eta - 2\eta_s}{\eta_s}\right|^{\kappa} \qquad \eta_s < \eta < \eta_1\,,\ \eta_1 > 0\,, \tag{4.6}$$

$$\tilde{a} = \left(\frac{\eta}{\eta_1}\right)^{\alpha} \left|\frac{\eta_1 - 2\eta_s}{\eta_s}\right|^{\kappa} \qquad \eta_1 < \eta < \eta_{\rm eq}\,, \tag{4.7}$$

A. Analytic form for the Bogoliubov coefficients

The solutions of the equation of motion (4.4) for the pump fields (4.5), (4.6) and (4.7) are respectively

$$v_k(\eta) = \sqrt{|\eta|}\, C\, H^{(1)}_{\nu}(k|\eta|)\,, \tag{4.8}$$

$$v_k(\eta) = \sqrt{|(\eta - 2\eta_s)|}\, \left[A_+\, H^{(1)}_{\mu}(k|(\eta - 2\eta_s)|) + A_-\, H^{(2)}_{\mu}(k|(\eta - 2\eta_s)|)\right]\,, \tag{4.9}$$

$$v_k(\eta) = \sqrt{|\eta|}\, \left[B_+\, H^{(1)}_{\rho}(k|\eta|) + B_-\, H^{(2)}_{\rho}(k|\eta|)\right]\,, \tag{4.10}$$

where

$$\nu = |\gamma - 1/2|\,, \quad \mu = |\kappa - 1/2|\,, \quad \rho = |\alpha - 1/2| \tag{4.11}$$

and we have normalized (4.8) allowing only positive frequencies in the flat vacuum state at $\eta \to -\infty$, so that

$$v_k(\eta) \to \frac{C}{\sqrt{k}} e^{-ik\eta} \tag{4.12}$$

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[1] In order to simplify the final expression of the Bogoliubov coefficients, we have slightly changed the constant parameters appearing in the pump field in the three phases. With the original parameters of Sect. 3, the Bogoliubov coefficients would just change by numerical factors $O(1)$, but the spectral slopes and the "duality" symmetry (see sect. IV B) would still be the same.

and $|C| = 1$. For reasons explained below we impose the continuity of Ψ_k and of its first derivative at η_s, η_1, *not* the continuity of the canonical field v_k. Using the relation

$$H_\mu^{(2)\prime}(z)\, H_\mu^{(1)}(z) - H_\mu^{(1)\prime}(z)\, H_\mu^{(2)}(z) = -4i/(\pi z)\,,$$

we obtain

$$\begin{aligned} A_+ = C\,\frac{i\pi}{4}\,\Big\{&(k|\eta_s|)\,\Big[H_\nu^{(1)}(k|\eta_s|)\,H_\mu^{(2)\prime}(k|\eta_s|) - H_\nu^{(1)\prime}(k|\eta_s|)\,H_\mu^{(2)}(k|\eta_s|)\Big] + \\ &(\gamma-\kappa)\,H_\nu^{(1)}(k|\eta_s|)\,H_\mu^{(2)}(k|\eta_s|)\Big\}\,, \end{aligned} \tag{4.13}$$

$$\begin{aligned} A_- = C\,\frac{i\pi}{4}\,\Big[&(k|\eta_s|)\,\Big[H_\mu^{(1)}(k|\eta_s|)\,H_\nu^{(1)\prime}(k|\eta_s|) - H_\mu^{(1)\prime}(k|\eta_s|)\,H_\nu^{(1)}(k|\eta_s|)\Big] + \\ &+(\kappa-\gamma)\,H_\nu^{(1)}(k|\eta_s|)\,H_\mu^{(1)}(k|\eta_s|)\Big\}\,, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} B_+ = \frac{i\pi}{4}\,\Big\{&\Big[A_+\,H_\mu^{(1)}(k\,|\eta_1|) + A_-\,H_\mu^{(2)}(k\,|\eta_1|)\Big]\,\Big[(k\,\eta_1)\,H_\rho^{(2)\prime}(k\,|\eta_1|) + (\kappa-\alpha) \\ &H_\rho^{(2)}(k\,|\eta_1|)\Big] - (k\,\eta_1)\,\Big[A_+\,H_\mu^{(1)\prime}(k\,|\eta_1|) + A_-\,H_\mu^{(2)\prime}(k\,|\eta_1|)\Big]\,H_\rho^{(2)}(k\,|\eta_1|)\Big\}\,, \end{aligned} \tag{4.15}$$

$$\begin{aligned} B_- = -\frac{i\pi}{4}\,\Big\{&\Big[A_+\,H_\mu^{(1)}(k\,|\eta_1|) + A_-\,H_\mu^{(2)}(k\,|\eta_1|)\Big]\,\Big[(k\,\eta_1)\,H_\rho^{(1)\prime}(k\,|\eta_1|) + (\kappa-\alpha) \\ &H_\rho^{(1)}(k\,|\eta_1|)\Big] - (k\,\eta_1)\,\Big[A_+\,H_\mu^{(1)\prime}(k\,|\eta_1|) + A_-\,H_\mu^{(2)\prime}(k\,|\eta_1|)\Big]\,H_\rho^{(1)}(k\,|\eta_1|)\Big\}\,, \end{aligned} \tag{4.16}$$

where the prime stands for derivative with respect to the argument of the Hankel function. From the condition $|C| = 1$ we get $|A_+|^2 - |A_-|^2 = 1$ and $|B_+|^2 - |B_-|^2 = 1$, as needed for generic Bogoliubov coefficients.

B. "Duality" of the Bogoliubov coefficients

In this section we analyse the behaviour of the Bogoliubov coefficients B_- and A_- under the "duality" transformation $\gamma \to -\gamma$, $\kappa \to -\kappa$ and $\alpha \to -\alpha$, under which the pump fields are reversed. We will thus check that, with a careful choice of the matching conditions, the symmetry that can be shown to be exact in the Hamiltonian approach [16] is preserved.

In our context we need the following relations among Hankel functions (see e.g. [18])

$$H_{-\xi}^{(1,2)}(z) = e^{\pm i\pi\xi}\,H_\xi^{(1,2)}(z)\,, \tag{4.17}$$

$$z\,H_{\xi-1}^{(1,2)}(z) + z\,H_{\xi+1}^{(1,2)}(z) = 2\xi\,H_\xi^{(1,2)}(z)\,, \tag{4.18}$$

$$z\,\frac{d}{dz}\,H_\xi^{(1,2)}(z) + z\,H_\xi^{(1,2)}(z) = \xi\,H_{\xi-1}^{(1,2)}(z)\,. \tag{4.19}$$

Independently of the range of frequencies we get

$$\frac{A_-(\gamma,\kappa)}{A_-(-\gamma,-\kappa)} = \exp\left[i\pi + \frac{i\pi}{2}\left(\left|\gamma+\frac{1}{2}\right| + \left|\kappa+\frac{1}{2}\right| - \left|\gamma-\frac{1}{2}\right| - \left|\kappa-\frac{1}{2}\right|\right)\right], \tag{4.20}$$

$$\frac{B_-(\gamma,\kappa,\alpha)}{B_-(-\gamma,-\kappa,-\alpha)} = \exp\left[i\pi + \frac{i\pi}{2}\left(\left|\gamma+\frac{1}{2}\right| + \left|\alpha+\frac{1}{2}\right| - \left|\gamma-\frac{1}{2}\right| - \left|\alpha-\frac{1}{2}\right|\right)\right]. \tag{4.21}$$

There are two important comments to be made on the above formulae. The first is that $B_-(-\gamma,-\kappa,-\alpha)$ differs from $B_-(\gamma,\kappa,\alpha)$ by just a phase. Hence the spectrum (being proportional to $|B_-|^2$) is identical for a given pump field or for its inverse. We stress that this duality property holds independently of the number and characteristics of the intermediate phases and thus, as argued in [16], is generally valid. The second observation is that duality depends crucially on having imposed the continuity of the field and of its derivative on Ψ_k and *not* on the canonical field v_k. The difference in imposing continuity of Ψ_k or of v_k arises from the discontinuous nature of the background itself (actually of $\dot{\varphi}$) and from the fact that Ψ_k and v_k obey equations containing first and second time-derivatives of the pump field, respectively. This gives rise to δ-function contributions in the case of v_k, making the requirement of continuity suspicious for that variable.

One welcome consequence of "duality" is the fact that the antisymmetric tensor field and the axion have identical spectra since their pump fields are the inverse of each other (see below). This must be so since they are just different descriptions of the same physical degree of freedom.

C. General form of the spectral slopes

The parameters η_s and η_1 in the formulae for B_- define two characteristic comoving frequencies, $k_s = |\eta_s|^{-1}$, $k_1 = |\eta_1|^{-1}$, which can be traded for two proper frequencies f_s and f_1 by the standard relations

$$2\pi f_s = \left(\frac{k_s}{a}\right), \qquad 2\pi f_1 = \left(\frac{k_1}{a}\right). \tag{4.22}$$

It is easy to see that the two scenarios for the background, intermediate dual dilaton phase and intermediate string phase, lead to $f_s > f_1$ and $f_s < f_1$, respectively. In the case $f \ll f_s, f_1$ (fluctuations that exit in the dilaton phase and re-enter in the radiation

Exponents	*Bogoliubov coefficient*	*Leading contribution*	*Power of f (ϵ_Υ)*
$\gamma > 1/2, \alpha > 1/2, \forall\kappa$	$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \neq 0$	$\mathcal{C}_1$	$1 - \lvert\gamma\rvert - \lvert\alpha\rvert$
$\gamma > 1/2$, $\alpha < 1/2$, $\forall\kappa$	$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_4 \neq 0$, $\mathcal{C}_3 = 0$	$\mathcal{C}_1$	$-\lvert\gamma - \alpha\rvert$
$\gamma < 1/2$, $\alpha > 1/2$, $\forall\kappa$	$\mathcal{C}_1, \mathcal{C}_3, \mathcal{C}_4 \neq 0$, $\mathcal{C}_2 = 0$	$\mathcal{C}_1$	$-\lvert\gamma - \alpha\rvert$
$\gamma < \alpha$, $\gamma < 1/2$, $\lvert\alpha\rvert < 1/2$, $\forall\kappa$	$\mathcal{C}_1 = 0$, $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \neq 0$	$\mathcal{C}_2$	$-\lvert\gamma - \alpha\rvert$
$\gamma > \alpha$, $\alpha < 1/2$, $\lvert\gamma\rvert < 1/2$, $\forall\kappa$	$\mathcal{C}_1 = 0$, $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \neq 0$	$\mathcal{C}_3$	$-\lvert\gamma - \alpha\rvert$
$\gamma < -1/2$, $\alpha < -1/2$, $\forall\kappa$	$\mathcal{C}_1 = 0$, $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \neq 0$	$\mathcal{C}_4$	$1 - \lvert\gamma\rvert - \lvert\alpha\rvert$

TABLE I. *Power of f in the Bogoliubov coefficient B_- for $f \ll f_1, f_s$*

phase), which is common to both scenarios, we can approximate the exact result (4.16) for the Bogoliubov coefficient B_- in the following way[2]

$$B_- = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 + \cdots$$

where

$$\mathcal{C}_1 = \mathcal{N}_1 \left(\frac{f}{2f_s}\right)^{-\nu} \left(\frac{f}{2f_1}\right)^{-\rho} \left[\mathcal{C}_1^1 \left(\frac{f_1}{f_s}\right)^{\mu} + \mathcal{C}_1^2 \left(\frac{f_s}{f_1}\right)^{\mu}\right], \tag{4.23}$$

$$\mathcal{C}_2 = \mathcal{N}_2 \left(\frac{f}{2f_s}\right)^{-\nu} \left(\frac{f}{2f_1}\right)^{\rho} \left[\mathcal{C}_2^1 \left(\frac{f_1}{f_s}\right)^{\mu} + \mathcal{C}_2^2 \left(\frac{f_s}{f_1}\right)^{\mu}\right], \tag{4.24}$$

$$\mathcal{C}_3 = \mathcal{N}_2 \left(\frac{f}{2f_s}\right)^{\nu} \left(\frac{f}{2f_1}\right)^{-\rho} \left[\mathcal{C}_3^1 \left(\frac{f_1}{f_s}\right)^{\mu} + \mathcal{C}_3^2 \left(\frac{f_s}{f_1}\right)^{\mu}\right], \tag{4.25}$$

$$\mathcal{C}_4 = \mathcal{N}_4 \left(\frac{f}{2f_s}\right)^{1-\nu} \left(\frac{f}{2f_1}\right)^{1-\rho} \left[\mathcal{C}_4^1 \left(\frac{f_s}{f_1}\right)^{\mu-1} + \mathcal{C}_4^2 \left(\frac{f_1}{f_s}\right)^{\mu+1} + \mathcal{C}_4^3 \left(\frac{f_1}{f_s}\right)^{\mu-1} + \mathcal{C}_4^4 \left(\frac{f_s}{f_1}\right)^{\mu+1}\right]. \tag{4.26}$$

Since, by their definition (4.11), $\nu, \rho > 0$, $\mathcal{C}_1$ gives the leading contribution unless the coefficients appearing in front of it vanish. Table I shows which one of the $\mathcal{C}_i$ is dominant for different choices of the background parameters.

[2]We have used the following relations for ν not integer: $H_\nu^{(1,2)} = (1 \pm i \cot\nu\pi)\, J_\nu \mp \frac{i}{\sin\nu\pi}\, J_{-\nu}$, with $J_\nu = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k}$, assuming $\nu \neq 0$.

Exponents	*Bogoliubov coefficient*	*Leading contribution*	*Power of f (ϵ_Υ)*
$\gamma > 1/2,\ \kappa > 1/2$	$\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4 \neq 0$	$\mathcal{D}_1$	$1 - \lvert\gamma\rvert - \lvert\kappa\rvert$
$\gamma > 1/2,\ \kappa < 1/2$	$\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_4 \neq 0,\ \mathcal{D}_3 = 0$	$\mathcal{D}_1$	$-\lvert\gamma - \kappa\rvert$
$\gamma < 1/2,\ \kappa > 1/2$	$\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_4 \neq 0, \mathcal{D}_2 = 0$	$\mathcal{D}_1$	$-\lvert\gamma - \kappa\rvert$
$\gamma < \kappa,\ \gamma < 1/2,\ \lvert\kappa\rvert < 1/2$	$\mathcal{D}_1 = 0,\ \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4 \neq 0$	$\mathcal{D}_2$	$-\lvert\gamma - \kappa\rvert$
$\gamma > \kappa,\ \kappa < 1/2,\ \lvert\gamma\rvert < 1/2$	$\mathcal{D}_1 = 0,\ \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4 \neq 0$	$\mathcal{D}_3$	$-\lvert\gamma - \kappa\rvert$
$\gamma < -1/2,\ \kappa < -1/2$	$\mathcal{D}_1 = 0,\ \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4 \neq 0$	$\mathcal{D}_4$	$1 - \lvert\gamma\rvert - \lvert\kappa\rvert$

TABLE II. *Power of f in the Bogoliubov coefficient B_- for an intermediate dual-dilaton phase and $f_1 \ll f \ll f_s$*

In the case $f_1 \ll f \ll f_s$ (fluctuations that exit in the dilaton phase and re-enter in the dual-dilaton phase) we get instead:

$$B_- = \mathcal{D}_1 \left(\frac{f}{2f_s}\right)^{-\mu-\nu} + \mathcal{D}_2 \left(\frac{f}{2f_s}\right)^{\mu-\nu} + \mathcal{D}_3 \left(\frac{f}{2f_s}\right)^{-\mu+\nu} + \mathcal{D}_4 \left(\frac{f}{2f_s}\right)^{2-\mu-\nu} . \tag{4.27}$$

Table II shows the leading contribution to B_- in this case. The explicit form of the coefficients $\mathcal{N}_i$, $\mathcal{C}_i^j$ and $\mathcal{D}_i$ for both cases is given in the appendix.

Tables (I and II) also show the leading power of f appearing in the Bogoliubov coefficient B_-, in the two above-mentioned cases, i.e. re-entry in the dual-dilaton or in the radiation phase. We can summarize this behaviour as follows

$$|B_-|^2 \sim f^{2-2|\gamma|-2|\alpha|} \qquad \gamma > 1/2\,, \alpha > 1/2 \ \text{ or } \ \gamma < -1/2\,, \alpha < -1/2\,, \tag{4.28}$$

$$|B_-|^2 \sim f^{-2|\gamma-\alpha|} \qquad \text{all other cases}\,. \tag{4.29}$$

In the case of an intermediate string phase the Bogoliubov coefficient for $f_s \ll f \ll f_1$ (fluctuations that exit in the string phase and re-enter in the radiation phase) is given by Eq. (4.27) after the substitution $\gamma \to \kappa, \kappa \to \alpha$ (hence $\nu \to \mu, \mu \to \rho$).

The spectrum of fluctuations for a generic field Υ is

$$\Omega_\Upsilon = \frac{1}{\rho_c} \frac{d\rho_\Upsilon}{d\log f} = \mathcal{N}_\Upsilon \frac{8\pi^2}{\rho_c} f^4 \, |B_-^\Upsilon|^2 \,, \tag{4.30}$$

Particles (Υ)	*Pump field* ($\tilde{a}$)	*Spectral slope* n_Υ	
		re-entry dual phase	*re-entry radiation phase*
Gravitons	$a\, e^{-\varphi/2}$	4	3
Axions	$a\, e^{\varphi/2}$	$4-4\left\vert\frac{\delta}{1-\delta}-\frac{\theta}{1-\theta}\right\vert$	$4-2\left\vert\frac{3}{2}-\frac{2\delta}{1-\delta}\right\vert$
Heterotic photons	$e^{-\varphi/2}$	$4-2\left\vert\frac{\delta}{1-\delta}-\frac{\theta}{1-\theta}\right\vert$	$4-2\left\vert\frac{1}{2}-\frac{\delta}{1-\delta}\right\vert$
$V_{\mu\nu}$	$e^{-\varphi/2+\sigma}$	$4-2\left\vert\frac{(\delta-\beta_a)}{1-\delta}-\frac{(\theta-\xi_a)}{1-\theta}\right\vert$	$4-2\left\vert\frac{1}{2}-\frac{(\delta-\beta_a)}{1-\delta}\right\vert$
$W_{\mu\nu}$	$e^{-\varphi/2-\sigma}$	$4-2\left\vert\frac{(\delta+\beta_a)}{1-\delta}-\frac{(\theta+\xi_a)}{1-\theta}\right\vert$	$4-2\left\vert\frac{1}{2}-\frac{(\delta+\beta_a)}{1-\delta}\right\vert$
B_{ab}	$a\, e^{-\varphi/2-2\sigma}$	$4-4\left\vert\frac{\beta_a}{1-\delta}-\frac{\xi_a}{1-\theta}\right\vert$	$3-\frac{4\beta_a}{1-\delta}$

TABLE III. *Spectral slopes for an intermediate dual-dilaton phase in the range $f \ll f_1$ (re-entry in radiation phase) and in the range $f_1 \ll f \ll f_s$ (re-entry in dual-dilaton phase)*

where $\mathcal{N}_\Upsilon$ is the number of polarization states. We have found it convenient to use a "spectral slope" parameter n_Υ defined by the relation

$$n_\Upsilon = \frac{d\log\Omega_\Upsilon}{d\log f} = 4 + 2\epsilon_\Upsilon\,, \tag{4.31}$$

where ϵ_Υ is the exponent appearing in the f-dependence of $|B_-^\Upsilon|$ (see Tables I and II). The spectral slope, which is simply related to the usual spectral index by $slope = (index - 1)$, is more convenient to describe the main property of the spectrum, since its sign tells us whether the spectrum is increasing or decreasing with f. We will now apply the above general results to various possible backgrounds and perturbations occurring in string theory.

V. APPLICATION TO OUR SPECIFIC SITUATIONS

We now discuss the explicit form of the Schrödinger-like equation (4.4) for the fields occurring in the action (2.9). This amounts to finding, for each perturbation, the relevant pump field and canonical variable. For gravitons and dilatons we refer to [11,12]. For V_μ and W_μ the equations of motion are

$$\partial_\mu\left(\sqrt{-g}\, e^{-\varphi}\, e^{2\sigma_a}\, V^{\mu\nu a}\right) = 0\,, \tag{5.1}$$

$$\partial_\mu\left(\sqrt{-g}\, e^{-\varphi}\, e^{-2\sigma_a}\, W^{\mu\nu a}\right) = 0\,. \tag{5.2}$$

Particles (Υ)	*Pump field* ($\tilde{a}$)	*Spectral slope* n_Υ	
		Exit in dilaton phase	*Exit in the string phase*
Gravitons	$a\, e^{-\varphi/2}$	3	$\begin{cases} 6-2\zeta & \zeta > \frac{3}{2} \\ 2\zeta & \zeta < \frac{3}{2} \end{cases}$
Axions	$a\, e^{\varphi/2}$	$4 - 2\left\vert\frac{3}{2} - \frac{2\delta}{1-\delta}\right\vert$	-2ζ
Heterotic photons	$e^{-\varphi/2}$	$4 - 2\left\vert\frac{1}{2} - \frac{\delta}{1-\delta}\right\vert$	$4-2\zeta$
$V_{\mu\nu}$	$e^{-\varphi/2+\sigma}$	$4 - 2\left\vert\frac{1}{2} - \frac{(\delta-\beta_a)}{1-\delta}\right\vert$	$4-2\zeta$
$W_{\mu\nu}$	$e^{-\varphi/2-\sigma}$	$4 - 2\left\vert\frac{1}{2} - \frac{(\delta+\beta_a)}{1-\delta}\right\vert$	$4-2\zeta$
B_{ab}	$a\, e^{-\varphi/2-2\sigma}$	$3 - \frac{4\beta_a}{1-\delta}$	$\begin{cases} 6-2\zeta & \zeta > \frac{3}{2} \\ 2\zeta & \zeta < \frac{3}{2} \end{cases}$

TABLE IV. *Spectral slopes for an intermediate string phase in the frequency ranges* $f \ll f_s$ *(exit in dilaton phase) and* $f_s \ll f \ll f_1$ *(exit during the string phase)*

Using the gauge $W_{0a} = 0 = V_{0a}$ eliminates one unphysical degree of freedom. However, since the dilaton depends only on time, we can use the equations of motion to further require $\nabla \cdot \vec{V}_a = 0, \nabla \cdot \vec{W}_a = 0$. The equations for the vector fields then take the form (4.1) and their canonical variables are simply:

$$\psi^j_{Va} = V^j_a\, e^{-\varphi/2+\sigma_a}\,, \tag{5.3}$$

$$\psi^j_{Wa} = W^j_a\, e^{-\varphi/2-\sigma_a}\,. \tag{5.4}$$

The same procedure has been applied for heterotic photons in [13]. For the axion field the equation of perturbations around the zero field solution is [15]

$$A_k'' + 2\frac{a'}{a}\, A_k' + \varphi'\, A_k' + k^2\, A_k = 0\,, \tag{5.5}$$

and the canonical variable therefore is $v_k = e^{\varphi/2}\, a\, A_k$. It is straightforward to obtain the equation of perturbations for the $B_{\mu\nu}$-field and its canonical variable, i.e. $v_k = e^{-\varphi/2}\, a^{-1}\, B_k$.

Note that, since the pump fields of the axion and the antisymmetric tensor are duality related, the spectrum of their fluctuations will be the same. For the internal B-field we get instead $v_k = a\, e^{-\varphi/2-2\sigma}\, B_k$. A list of all relevant "pump" fields can be found in Tables III

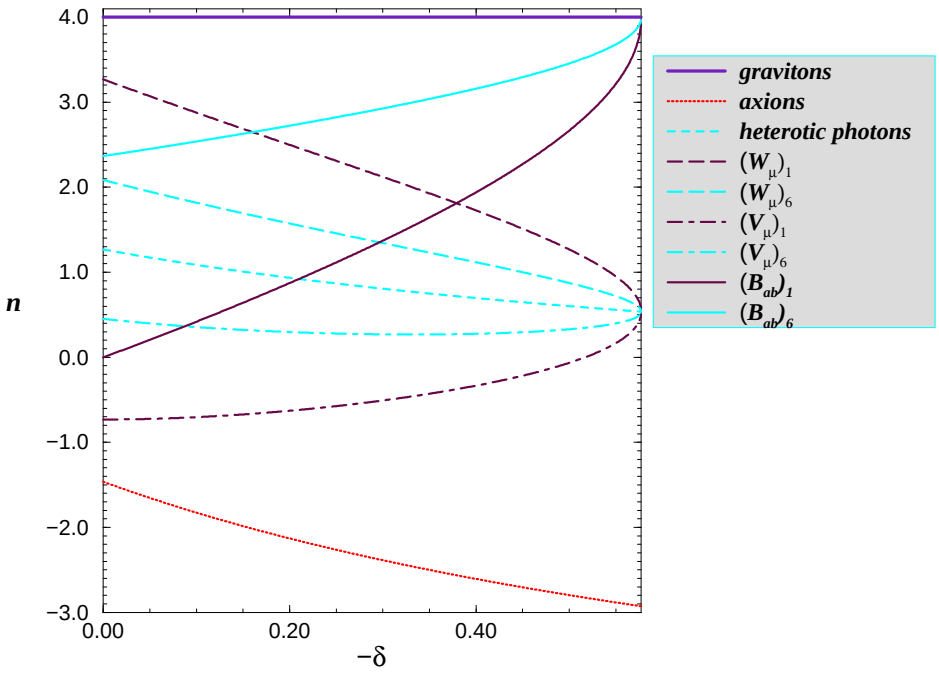

FIG. 1. *Spectral slopes for fluctuations that re-enter in the dual-dilaton phase when the latter is characterized by* $\theta = 1/\sqrt{3}$ *(i.e. by constant moduli).*

and IV. In the first we give the spectral slope for various fluctuations in the case of an intermediate dual-dilaton phase. The same is done in Table IV for an intermediate string phase. We now turn to discussing perturbations in the two scenarios.

A. Intermediate dual-dilaton phase

In this scenario, the super-inflationary phase ends at time $\eta = \eta_s$. Since we assume such a phase to have washed out any initial spatial curvature, the energy density must always be critical. At $\eta = \eta_s$ the dominant source of energy is the kinetic energy of the dilaton, $\rho_\varphi(\eta_s) \sim M_s^4\, e^{-\varphi_s}$.

Consider now the energy stored in the amplified perturbations during the dual-dilaton phase. Figs. 1, 2 and 3 give the spectral slopes in various cases for the two relevant frequency ranges. Fig. 4 gives the normalization of the spectra in the whole frequency range for the particular case $\theta = 1/\sqrt{3}, \delta = -0.3$. Since all perturbations are of the same order at the

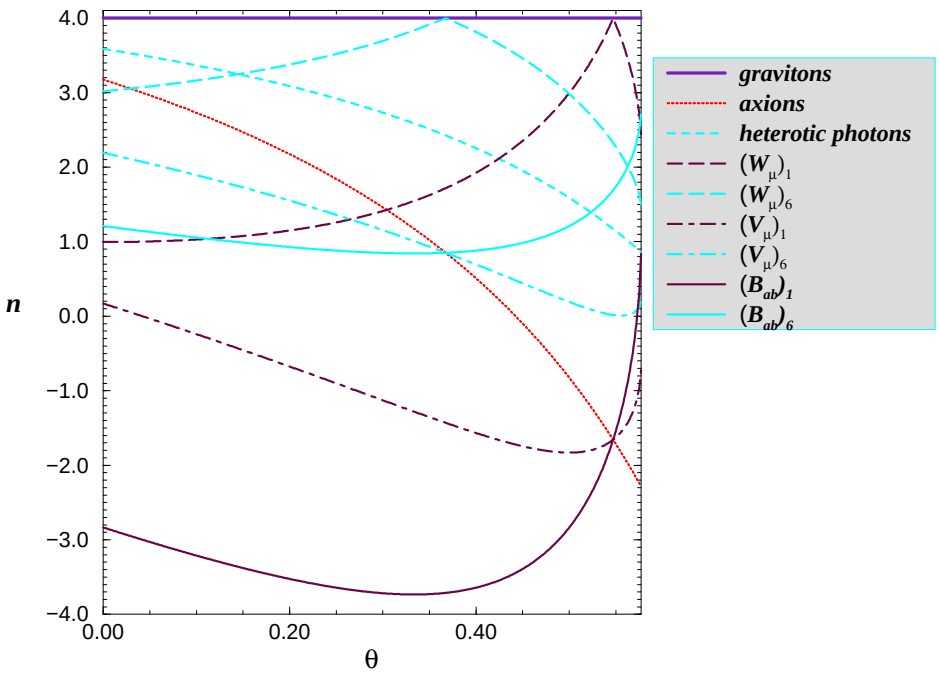

FIG. 2. *Same as Fig. 1 where, instead of fixing θ, we fix $\delta = -0.26$ in the intermediate phase.*

maximal amplified frequency (here f_s), perturbations with (the most) negative spectral slope dominate over all others. From the above mentioned figures we see that the spectral slope of axionic fluctuations re-entering during the dual dilaton phase is generally the most negative one (at least if we consider isotropic compactifications): we thus ignore contributions to the energy density from perturbations other than the axion's. The basic idea is to assume that the transition from the dual-dilaton phase to the radiation phase occurs precisely when the energy density in the perturbations becomes critical and starts to dominate over the kinetic energy of the coherent dilaton field.

Let us fix for simplicity $\theta = 1/\sqrt{3}$ (i.e. frozen internal dimensions in the dual-dilaton phase) and then impose criticality at the end of the dual-dilaton phase in the form:

$$M_s^2\, H_1^2 \simeq e^{\varphi_1}\, \rho_A(\eta_1)\,. \tag{5.6}$$

Using the equations of motion and assuming $|\eta_1| \gg |\eta_s|$, we have

$$H_1 \simeq M_s \left|\frac{\eta_s}{\eta_1}\right|^{(3+\sqrt{3})/2}, \qquad e^{\varphi_1} \simeq e^{\varphi_s} \left|\frac{\eta_s}{\eta_1}\right|^{-\sqrt{3}}. \tag{5.7}$$

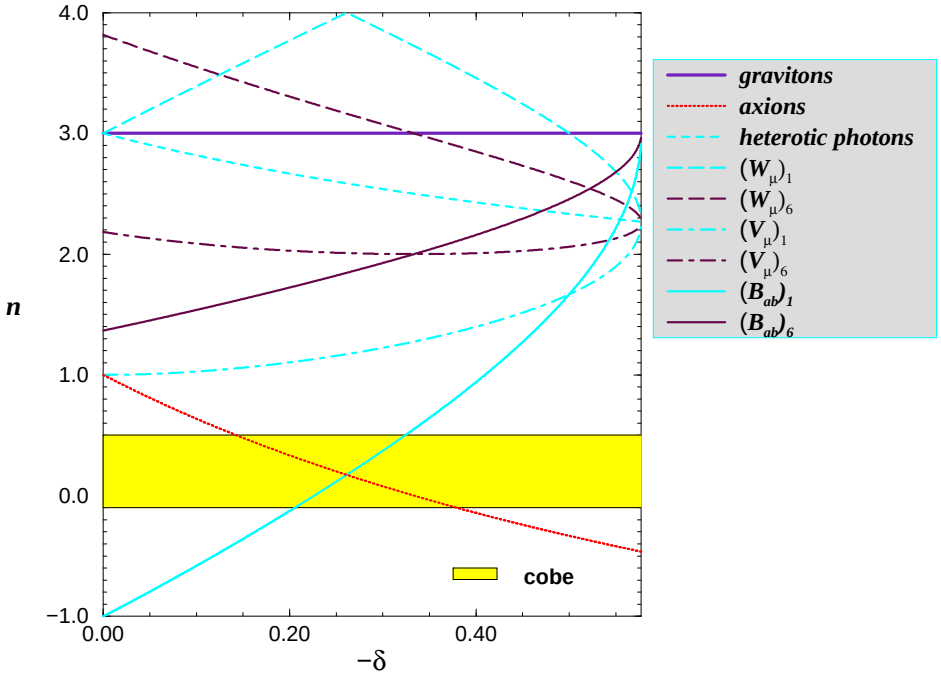

FIG. 3. *Spectral slopes for fluctuations that re-enter in the radiation phase. For comparison we also show the experimental constraint from COBE* [20]*:* $n + 1 = 1.2 \pm 0.3$.

Taking into account the results of Sec. IV we get, apart from factors $O(1)$,

$$\rho_A(\eta_1) \simeq f_s^4(\eta_s) \left(\frac{a_s}{a_1}\right)^4 \left(\frac{f_1}{f_s}\right)^{n_A} \simeq f_s^4(\eta_s) \left|\frac{\eta_s}{\eta_1}\right|^{4/(1-\delta)} \quad , \; n_A < 0 \, , \tag{5.8}$$

where we have restricted ourselves to the case $n_A < 0$ for the reasons explained above.

The dependence of the value of the dilaton background at $\eta = \eta_1$ on the parameter δ is completely fixed by the criticality condition Eq. (5.6). Indeed, inserting Eqs. (5.7) and (5.8) in Eq. (5.6), we obtain

$$e^{\varphi_1} \simeq \left|\frac{\eta_s}{\eta_1}\right|^{3+\sqrt{3}-4/(1-\delta)} \, . \tag{5.9}$$

If we define an effective temperature $T_{\rm eff}$ at the beginning of the radiation era by

$$\rho_A(\eta_1) \simeq T_{\rm eff}^4 \, , \tag{5.10}$$

we get

$$\left|\frac{\eta_s}{\eta_1}\right| \simeq \left(\frac{T_{\rm eff}}{M_s}\right)^{1-\delta} , \tag{5.11}$$

where we have assumed $f_s(\eta_s) \sim M_s$, and thus

$$e^{\varphi_1} \simeq \left(\frac{T_{\rm eff}}{M_s}\right)^{(1-\delta)(3+\sqrt{3})-4} . \tag{5.12}$$

It is important to stress that this effective temperature may have nothing to do with the actual temperature of a relativistic gas in thermal equilibrium at $\eta = \eta_1$. In particular, if the coupling is still very small, axions may not thermalize at all, in spite of dominating the energy and of driving a radiation-dominated era. For the same reason, the fact that $T_{\rm eff}$ can be large in string units should not be a matter of concern.

Let us now estimate the value of the frequencies f_s and f_1 at present time. If we assume that the CMB photons we observe today carry the (red-shifted) energy of the primordial perturbations (in particular from axion decay), we have

$$2\pi f_1 = \frac{k_1}{a_0} \simeq \frac{a_1}{a_0} H_1 , \tag{5.13}$$

$$\Omega_\gamma(t_0) = \frac{1}{G(\eta_1) M_{\rm Pl}^2} \frac{H_1^2}{H_0^2} \left(\frac{a_1}{a_0}\right)^4 , \tag{5.14}$$

$$f_1^4 \simeq G(\eta_1)\, M_{\rm Pl}^2\, H_0^2\, H_1^2\, \Omega_\gamma(t_0) \simeq e^{\varphi_1}\, H_0^2\, H_1^2\, \Omega_\gamma(t_0) M_{\rm Pl}^2/M_{\rm s}^2 , \tag{5.15}$$

where H_0 and $\Omega_\gamma(t_0)$ ($\sim 10^{-4}$) are respectively the Hubble parameter and the fraction of the critical energy stored in radiation at the present time t_0. Using Eq. (5.7) and Eq. (5.11) we finally get

$$f_1 \simeq \sqrt{H_0 M_{\rm Pl}}\, e^{\varphi_1/4}\, (\Omega_\gamma(t_0))^{1/4} \left(\frac{T_{\rm eff}}{M_s}\right)^{(1-\delta)(3+\sqrt{3})/4} , \tag{5.16}$$

$$f_s \simeq f_1 \left(\frac{T_{\rm eff}}{M_s}\right)^{\delta-1} . \tag{5.17}$$

Note that, if we choose $T_{\rm eff} = 10^{15}\,{\rm GeV}$, corresponding to a relatively short dual-dilaton phase, and we fix $\delta = -0.3$ in order to have an almost flat axion spectrum in the low-frequency region (see Fig. 3), we get $f_1 \sim 10^5\,{\rm Hz}, f_s \sim 10^9\,{\rm Hz}$ and $\varphi_s \simeq -23, \varphi_1 \simeq -11$. Therefore, the value of the dilaton at the beginning of the radiation era is still far from the present value ($\varphi_0 \sim -1$).

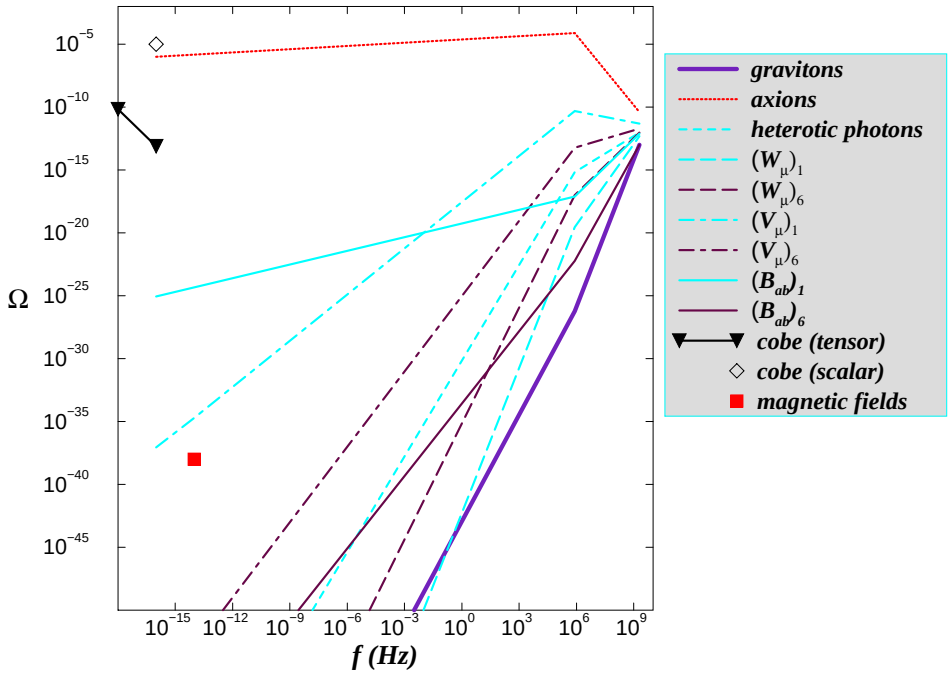

FIG. 4. *Spectrum for all the fluctuations considered in the case of a dual-dilaton intermediate phase, with the following choice:* $\delta = -0.3$, $\theta = 1/\sqrt{3}$, $f_1 \sim 10^5\,\mathrm{Hz}, f_s \sim 10^9\,\mathrm{Hz}$, $\varphi_s \sim -23$, $\varphi_1 \sim -11$, $T_{\rm eff} = 10^{15}\,\mathrm{GeV}$.

Typical spectra for all the fields we have considered are shown in Fig 4. In particular, for axionic fluctuations that re-enter in the dual-dilaton phase ($f_1 \ll f \ll f_s$), we get a decreasing spectrum

$$\Omega_A \simeq G(\eta_1)\, H_1^2\, \Omega_\gamma(t_0) \left(\frac{f}{f_s}\right)^{n_A} \left(\frac{f_s}{f_1}\right)^4 . \tag{5.18}$$

On the contrary, for fluctuations that re-enter in the radiation phase ($f \ll f_1 \ll f_s$), we get

$$\Omega_A \simeq G(\eta_1)\, H_1^2\, \Omega_\gamma(t_0) \left(\frac{f}{f_1}\right)^{n_A} \left(\frac{f_s}{f_1}\right)^{2(\nu+\mu)} , \tag{5.19}$$

$$\simeq \Omega_\gamma(t_0) \left(\frac{f}{f_1}\right)^{n_A} , \tag{5.20}$$

which includes the possibility of a scale-invariant flat spectrum.

As can be seen from Fig. 4, the Kaluza-Klein "photons" $V_\mu{}^a$, can give sufficiently large seeds for galactic magnetic fields ($\Omega > 10^{-38}$ for $f_M \approx 10^{-14}$ Hz [19]) in this case, provided of course that the true electromagnetic field has a non-vanishing component along this direction in group space. Amusingly enough, this can be achieved in a range of moduli where axionic perturbations have a nearly flat spectrum.

B. Intermediate string phase

In this scenario the Bogoliubov coefficients are still expressed by Eqs. (4.13), (4.14), (4.15) and (4.16), η_s is the time at which the string phase starts, and we again assume that the radiation phase, dominated by the energy stored in the amplified vacuum fluctuations, begins at $\eta = \eta_1$. We recall that, in this case, $f_1 > f_s$ and that we expect $f_1(\eta_1) \sim M_s$. Since axions have the most negative spectral slope, we impose again that their energy density becomes critical at the beginning of the radiation phase:

$$H_1^2 = G(\eta_1)\, \rho_A(\eta_1)\,, \tag{5.21}$$

Using then

$$\rho_A(\eta_1) \simeq f_1^4(\eta_1)\, \left(\frac{f_s}{f_1}\right)^{n_A} \tag{5.22}$$

and assuming again that the photons we observe today originate from the amplified vacuum fluctuations, we can fix the present value of f_1 to be

$$f_1(t_0) \simeq \sqrt{H_0 M_{\rm Pl}}\, e^{\varphi_1/4}\, (\Omega_\gamma(t_0))^{1/4}\,, \tag{5.23}$$

and relate φ_1 to the duration of the string phase $z_s = a_1/a_s$ (a free parameter)

$$e^{\varphi_1} \simeq z_s^{-2\zeta}\,. \tag{5.24}$$

If we define again

$$\rho_A(\eta_1) \simeq T_{\rm eff}^4\,, \tag{5.25}$$

we obtain

$$z_s \simeq \left(\frac{T_{\rm eff}}{M_s}\right)^{2/\zeta}\,. \tag{5.26}$$

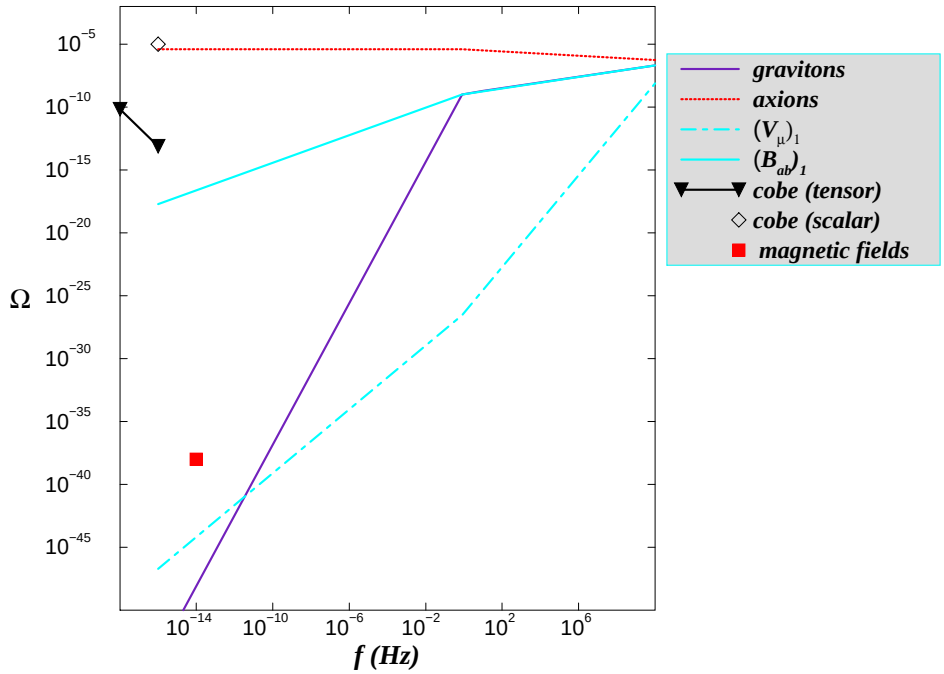

FIG. 5. *Some fluctuation spectra in the case of a string intermediate phase. The following choice of parameters was made:* $\zeta = 0.08$, $\delta = -1/3$, $f_1 \sim 10^{10}\,\mathrm{Hz}$, $f_s \sim 8 \cdot 10^{-1}\,\mathrm{Hz}$, $\varphi_s \sim -8, \varphi_1 \sim -5$, $T_{\rm eff} = 2 \times 10^{18}\,\mathrm{GeV}$.

With the choice $T_{\rm eff} = 2 \times 10^{18}\,\mathrm{GeV}, \zeta = 0.08$, corresponding to a very long string phase, and fixing $\delta = -1/3$ in order to have a flat axion spectrum in the low-frequency region (see Fig. 3), we obtain

$$f_1 \sim 10^{10}\,\mathrm{Hz},\ \ f_s \sim 8 \cdot 10^{-1}\,\mathrm{Hz}\ , \varphi_s \simeq -8,\ \varphi_1 \simeq -5\ .$$

As in the scenario with a dual-dilaton intermediate phase, we find the unpleasant result that the dilaton is still far from its present value at the beginning of the radiation era. In Fig. 5 we summarize the results of the spectra for some perturbations. The spectrum of the fluctuations that exit in the dilaton phase is given in the limit $f \ll f_s \ll f_1$, using the coefficients $\mathcal{C}_i$, shown in Table I. For fluctuations that exit in the string phase we consider instead the limit $f_s \ll f \ll f_1$, and the Bogoliubov coefficients are expressed by the quantities $\mathcal{D}_i$ (see Table II after substituting $\gamma \to \kappa$ and $\kappa \to \alpha$).

Note that, for fluctuations of the axion field that exit in the string phase, we have

$$\Omega_A \simeq G(\eta_1)\, H_1^2\, \Omega_\gamma(t_0)\, \left(\frac{f}{f_s}\right)^{n_A}\, \left(\frac{f_1}{f_s}\right)^{2(\mu+\rho)-4} ,$$
$$\simeq \Omega_\gamma(t_0)\, \left(\frac{f}{f_s}\right)^{n_A} . \tag{5.27}$$

Substituting the parameters of Table I we get a decreasing spectrum

$$\Omega_A \simeq G(\eta_1)\, H_1^2\, \Omega_\gamma(t_0)\, \left(\frac{f}{f_1}\right)^{-2\zeta} . \tag{5.28}$$

In this example, a long string phase produces a gravitational spectrum of order $\sim 10^{-9}$ in the range of detection of LIGO/VIRGO, but a very steep spectrum of Kaluza-Klein photons $V_\mu{}^a$ at high frequencies and consequently a value of perturbations at $f_M \sim 10^{-14}$ Hz far below the lower limit necessary to seed the dynamo mechanism for galactic magnetic fields [19].

VI. DISCUSSION

Our main results can be summarised as follows: Eqs. (4.28), (4.29), Tables III and IV, Figs. 1, 2 and 3 give our main conclusions concerning the spectral slopes of the various spectra in the two scenarios, while Figs. 4 and 5 illustrate the spectra of all perturbations for certain typical choices of the background's moduli. Rather than discussing the fine details, we would like to draw some conclusions, which appear to be relatively robust with respect to (slight?) variations of the moduli.

♢ Our calculations are based on the use of the low-energy effective action both for the backgrounds and for the perturbations. Since in the pre-big bang scenario a high-curvature phase is necessary before any exit to standard cosmology can be achieved, such a procedure is often criticized (see e.g. [21]) and requires some justification. We have seen in our explicit computations that the spectrum of long-wavelength perturbations, which exit and re-enter at small curvatures (in string units), does not depend on the details of the high curvature phase. Also, the use of higher-derivative-corrected perturbation equations has recently been shown [22] not to change the low-frequency spectra by more than a number $O(1)$. Thus predictions for the low-energy end of the spectra appear to be robust. Why? The physical explanation almost certainly lies in the freezing-out of super-horizon-scale perturbations. The occurrence of a constant mode at sufficiently large wavelengths can be shown without

reference to the low-energy approximation [16] and, by a canonical transformation argument, should also apply to the constant-momentum mode. By contrast, the high frequency spectra are expected to depend quite crucially on the details of the strong curvature transition. We expect our naive formulae to give "lower bounds" for those parts of the spectra.

♢ The main result of our investigation is the confirmation of the suggestion found in ref. [15] that positive spectral slopes are by no means a must in pre-big bang cosmology. By computing the spectra after re-entry, we have confirmed that axions do have, more often than not, negative slopes (decreasing spectra). However, other fields, such as KK gauge fields and scalars, can also exhibit negative slopes. A particularly promising case is the one shown in Fig. 3, since, in a region around the one with nine-dimensional symmetry ($\delta = \pm\beta_a = -1/3$), the axion spectrum and that of a KK scalar field are nearly flat, while the slope of the spectrum of some KK gauge field is positive but sufficiently small to produce large enough seeds for the galactic magnetic fields.

♢ Unfortunately, the promising results of Fig. 3 are somewhat spoiled when a long dual-dilaton or string phase is inserted in the background between the dilaton and radiation phases. In this case, the spectral slopes grow somewhat wild in the negative direction (see e.g. Fig. 2 for the dual-dilaton case), making some of the spectra peak at very low frequency. The generic consequence of this phenomenon is a huge increase in the total, integrated energy density in the perturbations. If the string coupling (the dilaton) is not very small throughout the intermediate phase, the energy in these perturbations soon becomes critical and the intermediate phase stops. The only way to have a long intermediate phase is therefore to force the dilaton to be very perturbative until the end of the intermediate phase, be it the string or the dual-dilaton phase. In this case, however, at the beginning of the radiation phase the dilaton is still very much displaced from its present value (where supposedly the minimum of its non-perturbative potential is) and may have a hard time reaching it later. In other words, the most appealing possible scenarios appear to be those with a sudden transition between the dilaton and radiation phases occurring at "realistic" values for the string coupling (roughly $1/N$, if N is the number of effectively amplified distinct species). Although this appears at present as some kind of fine tuning of the ratio of two moduli, it is not excluded that a better understanding of the initial conditions leading to PBB behaviour along the lines of Ref. [8] may tie together the initial values of the coupling and the curvature

so that such conditions are naturally realised.

♢ If the latter picture is adopted, it is possible to have a nearly scale-invariant dilaton/moduli spectrum. This could lead to an interesting mechanism to generate large-scale anisotropy along the lines given in Ref. [14]. In the same region of moduli space one obtains reasonably large fluctuations of the KK gauge fields to provide sizeable seeds for the galactic magnetic fields. On the negative side, in this region of parameter space, the situation would be quite discouraging for generating a large enough gravitational-wave signal in the interesting frequency range.

NOTE ADDED

While completing this work we became aware of a paper by Brustein and Hadad [23] which is also dealing with generic perturbations in string cosmology. Their method is somewhat different from ours: instead of working within a specific parametrization of the high-curvature phase, they have assumed the freezing of the fluctuation and of its conjugate momentum for super-horizon scales. Also, they have not imposed our criticality condition and thus have not obtained predictions on the value of the dilaton at the beginning of the radiation phase. We have checked that our results agree with theirs wherever a comparison is possible.

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APPENDIX A:

Here we give the explicit form of the coefficients entering the Bogoliubov formulae of Sec. IV C. In the case of the limit $f \ll f_1 \ll f_s$ we get:

$$
\begin{aligned}
\mathcal{N}_1 &= \frac{i}{8\pi}\,\frac{\Gamma(\mu)\,\Gamma(\nu)\,\Gamma(\rho)}{\Gamma(1+\mu)}\,,\\
\mathcal{C}_1^1 &= (\gamma-\kappa+\mu+\nu)\;(-\alpha+\kappa+\mu-\rho)\;,\\
\mathcal{C}_1^2 &= (\gamma-\kappa-\mu+\nu)\;(\alpha-\kappa+\mu+\rho)\;;
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}_2 &= -\frac{i}{8\pi}\,\frac{\Gamma(\mu)\,\Gamma(\nu)\,\Gamma(-\rho)}{\Gamma(1+\mu)}\;[\cos(\pi\,\rho)-i\,\sin(\pi\,\rho)]\;,\\
\mathcal{C}_2^1 &= (\gamma-\kappa+\mu+\nu)\;(\alpha-\kappa-\mu-\rho)\;,\\
\mathcal{C}_2^2 &= (\gamma-\kappa-\mu+\nu)\;(-\alpha+\kappa-\mu+\rho)\;;
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}_3 &= -\frac{i}{8\pi}\,\frac{\Gamma(\mu)\,\Gamma(-\nu)\,\Gamma(\rho)}{\Gamma(1+\mu)}\;[\cos(\pi\,\nu)-i\,\sin(\pi\,\nu)]\;,\\
\mathcal{C}_3^1 &= (\gamma-\kappa+\mu-\nu)\;(\alpha-\kappa-\mu+\rho)\;,\\
\mathcal{C}_3^2 &= (-\gamma+\kappa+\mu+\nu)\;(\alpha-\kappa+\mu+\rho)\;;
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}_4 &= -\frac{i}{8\pi}\,\frac{\Gamma(\mu)\,\Gamma(\nu)\,\Gamma(\rho)}{\Gamma(2-\mu)\,\Gamma(1+\mu)\,\Gamma(2+\mu)\,\Gamma(2-\nu)\,\Gamma(2-\rho)}\,,\\
\mathcal{C}_4^1 &= (\alpha-\kappa+\mu+\rho)\;\Gamma(2+\mu)\,\Gamma(2-\rho)\;[(-2+\gamma-\kappa-\mu+\nu)\\
&\qquad \Gamma(2-\mu)\,\Gamma(1-\nu)+(2+\gamma-\kappa-\mu+\nu)\;\Gamma(1-\mu)\,\Gamma(2-\nu)]\;,\\
\mathcal{C}_4^2 &= (-\alpha+\kappa+\mu-\rho)\;\Gamma(2-\mu)\,\Gamma(2-\rho)\;[(-2+\gamma-\kappa+\mu+\nu)\\
&\qquad \Gamma(2+\mu)\,\Gamma(1-\nu)+(2+\gamma-\kappa+\mu+\nu)\;\Gamma(1+\mu)\,\Gamma(2-\nu)]\;,\\
\mathcal{C}_4^3 &= (\gamma-\kappa+\mu+\nu)\;\Gamma(2+\mu)\,\Gamma(2-\nu)\;[(2-\alpha+\kappa+\mu-\rho)\\
&\qquad \Gamma(2-\mu)\,\Gamma(1-\rho)+(-2-\alpha+\kappa+\mu-\rho)\;\Gamma(1-\mu)\,\Gamma(2-\rho)]\;,\\
\mathcal{C}_4^4 &= (\gamma-\kappa-\mu+\nu)\;\Gamma(2-\mu)\,\Gamma(2-\nu)\;[(-2+\alpha-\kappa+\mu+\rho)\\
&\qquad \Gamma(2+\mu)\,\Gamma(1-\rho)+(2+\alpha-\kappa+\mu+\rho)\;\Gamma(1+\mu)\,\Gamma(2-\rho)]\;;
\end{aligned}
$$

while for fluctuations in the frequency region $f_1 \ll f \ll f_s$ we obtain:

$$
\begin{aligned}
\mathcal{D}_1 &= -\frac{i}{8}\,e^{-i\pi/2\,(-\mu+\rho)}\;(-\gamma+\kappa+\mu-\nu)\;\pi^{-1}\,2\Gamma(\mu)\,\Gamma(\nu)\,,\\
\mathcal{D}_2 &= -\frac{i}{8}\,e^{-i\pi/2\,(\mu+\rho)}\,(-\gamma+\kappa-\mu-\nu)\;\pi^{-1}\,2\Gamma(-\mu)\,\Gamma(\nu)\,,\\
\mathcal{D}_3 &= -\frac{i}{8}\,e^{-i\pi/2\,(2\nu-\mu+\rho)}\,(-\gamma+\kappa+\mu+\nu)\;\pi^{-1}\,2\Gamma(\mu)\,\Gamma(-\nu)\,,\\
\mathcal{D}_4 &= \frac{i}{8}\,e^{-i\pi/2\,(-\mu+\rho)}\,\pi\;[(2-\gamma+\kappa+\mu-\nu)\;(1-\mu)+\\
&\qquad (-2-\gamma+\kappa+\mu-\nu)\;(1-\nu)]\;\frac{2\Gamma(\mu)\,\Gamma(\nu)}{(1-\mu)\,(1-\nu)}\,.
\end{aligned}
$$

REFERENCES

[1] See, for instance, E.W.Kolb and M.S.Turner, *The Early Universe* (Addison-Wesley, Redwood City, 1990) and references therein.

[2] G. Veneziano, Phys. Lett. **B265**, 287 (1991);
M. Gasperini and G. Veneziano, Astropart. Phys. **1** (1993) 317; Mod. Phys. Lett. **A8** (1993) 3701; Phys. Rev. **D50** (1994) 2519;
R. Brustein and G. Veneziano, Phys. Lett. **B329** (1994)429;
N. Kaloper, R. Madden and K. A. Olive, Nucl. Phys. **B452** (1995) 677; Phys. Lett. **B371** (1996) 34;
R. Easther, K. Maeda and D. Wands, Phys. Rev. **D53** (1996) 4247;
E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. **D50** (1994) 4868, **D51** (1995) 223;
J. D. Barrow and K. E. Kunze, Phys. Rev. **D56** (1997) 741.

[3] A continuously updated collection of papers on the PBB scenario is available at `http://www.to.infn.it/teorici/gasperini/`.

[4] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. **B494** (1997) 315.

[5] R. Brustein and R. Madden, *Graceful exit and energy conditions in string cosmology*, BGU-PH-97/06 (hep-th/9702043); *A model for graceful exit in string cosmology*, BGU-PH-97/11 (hep-th/9708046).

[6] G. Veneziano, ref. [2];
A.A. Tseytlin, Mod. Phys. Lett **A6** (1991) 1721;
K.A. Meissner and G. Veneziano, Phys. Lett. **B267** (1991) 33; Mod. Phys. Lett. **A6** (1991) 3397;
A.A. Tseytlin and C. Vafa, Nucl. Phys. **B372** (1992) 443.
A. Sen, Phys. Lett. **B271** (1991) 295;
S.F. Hassan and A. Sen, Nucl. Phys. **B375** (1992) 103;
M. Gasperini and G. Veneziano, Phys. Lett. **B277** (1992) 256;
K.A. Meissner, Phys. Lett. **B392** (1997) 298;
N. Kaloper and K.A. Meissner, *Duality beyond the first loop*, CERN-TH-97-113 (hep-th/9705193), to appear in Phys. Rev. **D56**.

[7] G. Veneziano, Phys. Lett. **B406** (1997) 297.

[8] A. Buonanno, K.A. Meissner, C. Ungarelli and G. Veneziano, *Classical inhomogeneities in string cosmology*, CERN-TH/97-124 (hep-th/9706221).

[9] M.S. Turner and E.J. Weinberg, Phys. Rev. **D56** (1997) 4604.

[10] M. Maggiore and R. Sturani, *The fine-tuning problem in pre-big bang inflation*, IFUP-TH-22-97 (gr-qc/9706053).

[11] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. **B361** (1995) 45;
R. Brustein, M. Gasperini and G. Veneziano, Phys. Rev. **D55** (1997) 3882;
A. Buonanno, M. Maggiore, C. Ungarelli, Phys. Rev. **D55** (1997) 3330;
B. Allen and R. Brustein, Phys. Rev. **D55** (1997) 3260;
M. Gasperini, Phys. Rev. **D56** (1997) 4815.

[12] R. Brustein, M. Gasperini, M. Giovannini, V.F. Mukhanov and G. Veneziano, Phys. Rev. **D51** (1995) 6744.

[13] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. **75** (1995) 3796;
D. Lemoine and M. Lemoine, Phys. Rev. **D52** (1995) 1955.

[14] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. **D52** (1995) 6651.

[15] E. J. Copeland, R. Easther and D. Wands, Phys. Rev. **D56** (1997) 874.

[16] R. Brustein, M. Gasperini and G. Veneziano, to appear.

[17] J. Maharana and J.H. Schwarz, Nucl. Phys. **B390** (1993) 3.

[18] M. Abramowicz and I. A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1972).

[19] E. N. Parker, *Cosmical magnetic fields*, (Clarendon Press, Oxford, 1979);
M.S. Turner and L.M. Widrow, Phys. Rev. **D37** (1988) 2743.

[20] K.M. Gorski et al., *Power spectrum of primordial inhomogeneity determined from the 4-year COBE DMR sky maps*, COBE-PREPRINT-96-03 (astro-ph/9601063).

[21] S.W. Hawking, talk presented at COSMO-97, Ambleside, England (September 1997).

[22] M. Gasperini, ref. [11].

[23] R. Brustein and M. Hadad, *Particle production in string cosmology models*, BGU-PH-97-12 (hep-th/9709022).

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CERN-TH/2001-149
hep-th/0106019

Localization of Scalar Fluctuations in a Dilatonic Brane-World Scenario

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Abstract

We derive and solve the full set of scalar perturbation equations for a class of Z_2-symmetric five-dimensional geometries generated by a bulk cosmological constant and by a 3-brane non-minimally coupled to a bulk dilaton field. The massless scalar modes, like their tensor analogues, are localized on the brane, and provide long-range four-dimensional dilatonic interactions, which are generically present even when matter on the brane carries no dilatonic charge. The shorter-range corrections induced by the continuum of massive scalar modes are always present: they persist even in the case of a trivial dilaton background (the standard Randall–Sundrum configuration) and vanishing dilatonic charges.

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1 Introduction

The possibility, first discovered in the context of Horava–Witten (HW) heterotic M-theory [1, 2], that our Universe could lie on a hypersurface (a "brane") embedded in some higher-dimensional "bulk" space-time –the so-called brane-world scenario– has recently attracted considerable attention. In the original HW paper, our world is a 9-brane sitting at one of the boundaries of an eleven-dimensional bulk, while in a large class of M-theory models [3, 4], in which six dimensions are compactified in a more traditional Kaluza–Klein (KK) way, one can envisage constructing fully consistent four-dimensional brane-world scenarios with, effectively, a five-dimensional bulk. Unfortunately, finding brane configurations which are consistent with all stringy constraints has proved to be a very hard, if not impossible, task.

At a more phenomenological level, i.e. when problems with quantization of the higher-dimensional gravity theory are ignored, one can consider the dimensions orthogonal to the 3-brane as either compact and large [5, 6, 7], or as having infinite proper size [8, 9, 10]. In the latter (so-called Randall–Sundrum (RS)) case, the bulk geometry is bent by an appropriate "warp-factor", providing a crucial difference with respect to the "old" KK scenario, where the bulk geometry is simply the direct product of an "internal" and an "external" manifold. In particular, unlike the KK models, RS models are able to reproduce the four-dimensional Newton law at large distances on the brane by *dynamically* binding the massless gravitons to it [10].

As a consequence of the non-factorized structure of the metric in RS-type scenarios, previous approaches to the study of metric fluctuations in higher-dimensional backgrounds [11]–[16], based on the isometries of a factorizable geometry, cannot be applied directly to the brane-world scenario. A new gauge-invariant formalism is required, like the one developed in [17]. The classical and quantum analysis of metric fluctuations is of primary importance for understanding the possible localization of massless modes on the brane, as well as the nature of the orrections to the long-range interaction due to the continuum of massive modes that are typically living in the bulk. Until now, the study of this problem has been mainly focused on the structure of tensor (i.e. transverse-traceless) perturbations of the bulk geometry (see [18] for a general discussion).

In all string/M-theory models, however, the graviton enjoys the company of (perturbatively) massless scalar partners (the dilaton, compactification moduli, etc.). These typically induce long-range interactions of gravitational strength [19], and are therefore dangerous in view of the existing experimental tests (see for instance [20]). The standard way to solve this problem is to assume that these scalars get a SUSY-breaking non-perturbative mass

(for alternatives see e.g. [21]). However, one may ask whether RS-type scenarios can offer an alternative solution to (or an alleviation of) this problem, e.g. by *not* confining scalar fluctuations on the brane. If this were the case, scalar interactions on the brane would be suppressed, or possibly become short-ranged, and brane-world scenarios would naturally solve one of the most serious potential problems with higher-dimensional or stringy extensions of the Standard Model and General Relativity. This particular aspect of brane-world scenarios has never been completely addressed, to the best of our knowledge, in spite of many studies recently appeared in the literature, and devoted to the perturbations of a brane-world background [22]–[30].

In this paper we present a detailed discussion of the localization of scalar metric fluctuations in a typical example of brane-world scenario. Unlike gravitons, which are decoupled from matter fluctuations, the scalar fluctuations of the bulk geometry are in general non-trivially coupled to the matter sources. We shall thus consider a non-compact, Z_2-symmetric, five-dimensional background, generated by a positive tension 3-brane and by a bulk dilaton field coupled to the brane and to the (negative) bulk energy density. We shall restrict ourselves, in particular, to the gravi-dilaton solutions discussed in [31] (hereafter called CLP backgrounds, for short), which generalize the AdS_5 RS scenario [10] in the presence of a bulk scalar field, and which are already known to guarantee the localization of tensor metric fluctuations [31].

By extending the analysis of the perturbations, and by using an appropriate gauge-invariant formalism [17], we find that the same class of CLP backgrounds that localize massless tensor perturbations on the four-dimensional brane also localize massless scalar perturbations (i.e. the dilatonic interaction). However, the short-range corrections to the scalar interactions, due to the massive modes propagating in the extra-dimension, are in general different from the higher-dimensional corrections affecting the pure tensor part of the gravitational interaction. We also find that, in general, scalar metric fluctuations exhibit a non-trivial, "self-sustained" spectrum of solutions, even for a trivial dilaton background. This implies that long- and short-range gravitational interactions on the brane are effectively of the scalar-tensor type, in agreement with previous results [32], and are therefore subject to strong phenomenological constraints.

The paper is organized as follows. In Section 2 we present the action and the classical equations of motion for a 3-brane non-minimally coupled to a five-dimensional gravi-dilaton background, and we retrieve the whole class of CLP solutions [31]. In Section 3 we give the full set of scalar perturbation equations in the so-called "generalized longitudinal gauge" [17], and we find the four independent canonical variables diagonalizing them. In Section 4 we discuss the localization of massless modes, and determine the class of backgrounds

admitting long-range dilatonic interactions confined on the brane. In Section 5 we present the general spectrum of solutions for the massive modes that propagate throughout the bulk, and determine the relative magnitude of their amplitudes. In Section 6 we evaluate, in the weak field limit, the leading-order corrections to the effective gravitational potential generated by a static source with a point-like mass and dilatonic charge, confined on the brane. The main results of this paper are finally summarized in Section 7.

2 Background equations

We shall consider a five-dimensional scalar-tensor background $\{g_{AB}, \phi\}$, possibly arising from the bosonic sector of a dimensionally reduced string/supergravity theory, and non-trivially coupled to a negative cosmological constant Λ and to a 3-brane of positive tension T_3:

$$\begin{aligned} S = S_{\rm bulk} + S_{\rm brane} &= M_5^3 \int d^5x \sqrt{|g|} \left(-R + \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - 2\Lambda e^{\alpha_1 \phi} \right) \\ &- \frac{T_3}{2} \int d^4\xi \sqrt{|\gamma|} \left[\gamma^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B g_{AB} e^{\alpha_2 \phi} - 2 \right]. \end{aligned} \tag{2.1}$$

Here M_5 is the fundamental mass scale of the five-dimensional bulk space-time, and the parameters α_1, α_2 control the coupling of the bulk dilaton to Λ and to the brane (particular values of these parameters may simply correspond to the rescaling of the minimally coupled σ-model action in the canonical Einstein frame [33], but here we allow in general for non-minimal couplings). The brane action (see for instance [34]) is parametrized by the coordinates $X^A(\xi)$ describing the embedding of the brane in the bulk manifold, and by the auxiliary metric tensor $\gamma_{\alpha\beta}(\xi)$ defined on the four-dimensional world-volume of the brane, spanned by the coordinates ξ^α. Consequently, $\partial_\alpha X^A$ is a short-cut notation for $\partial X^A(\xi)/\partial \xi^\alpha$.

Conventions: Greek indices run from 0 to 3, capital Latin indices from 0 to 4, lower-case Latin indices from 1 to 3. For the bulk coordinates we use the notation $x^A = (t, x^i, z)$. The metric signature is $(+,-,-,-,-)$, and the curvature tensor is defined by $R_{MNA}{}^B = \partial_M \Gamma_{NA}{}^B + \Gamma_{MP}{}^B \Gamma_{NA}{}^P - (M \longleftrightarrow N)$, $R_{NA} = R_{MNA}{}^M$.

The variation of the action with respect to g_{AB}, ϕ, X^A and $\gamma_{\alpha\beta}$ gives, respectively, the Einstein equations (in units such that $M_5^3 = 1$):

$$\begin{aligned} G^A{}_B &= \frac{1}{2} \left(\partial^A \phi \partial_B \phi - \frac{1}{2} \delta^A_B \partial_C \phi \partial^C \phi \right) + \Lambda e^{\alpha_1 \phi} \delta^A_B + \\ &+ \frac{T_3}{2} \frac{1}{\sqrt{|g|}} g_{BC} \int d^4\xi \sqrt{|\gamma|} \delta^5(x - X) \gamma^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^C e^{\alpha_2 \phi}, \end{aligned} \tag{2.2}$$

the dilaton equation:

$$\nabla_M \nabla^M \phi + 2\alpha_1 \Lambda e^{\alpha_1 \phi} + \frac{\alpha_2 T_3}{2} \frac{1}{\sqrt{|g|}} \int d^4\xi \sqrt{|\gamma|} \delta^5(x - X) \gamma^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B g_{AB} e^{\alpha_2 \phi} = 0, \quad (2.3)$$

the equation governing the evolution of the brane in the bulk space-time:

$$\partial_\alpha \left(\sqrt{|\gamma|} \gamma^{\alpha\beta} \partial_\beta X^B g_{AB} e^{\alpha_2 \phi} \right) = \frac{1}{2} \sqrt{|\gamma|} \gamma^{\alpha\beta} \partial_\alpha X^B \partial_\beta X^C \, \partial_A \left(g_{BC} e^{\alpha_2 \phi} \right) \Big|_{x = X(\xi)} , \quad (2.4)$$

and the induced metric on the brane:

$$\gamma_{\alpha\beta} = \partial_\alpha X^A \partial_\beta X^B g_{AB} e^{\alpha_2 \phi}. \quad (2.5)$$

We now specialize these equations to the case of a conformally flat bulk metric, with a warp factor a and a dilaton ϕ, both of which only depend on the fifth coordinate z. Also, we shall look for Z_2-symmetric solutions, describing a flat brane rigidly located at $z = 0$, and we set

$$g_{AB} = a^2(z) \eta_{AB}, \qquad \phi = \phi(z), \qquad X^A = \delta^A_\mu \xi^\mu. \quad (2.6)$$

where η_{AB} is the five-dimensional Minkowski metric. The induced metric thus reduces to

$$\gamma_{\alpha\beta} = \delta^A_\alpha \delta^B_\beta g_{AB} \; e^{\alpha_2 \phi}, \quad (2.7)$$

while the brane equations (2.4) are identically satisfied thanks to the Z_2 symmetry.

The dynamical equations are obtained from the dilaton equation (2.3), which becomes

$$3 \frac{a'}{a} \phi' + \phi'' - 2\alpha_1 \Lambda a^2 e^{\alpha_1 \phi} - 2\alpha_2 T_3 a e^{2\alpha_2 \phi} \delta(z) = 0, \quad (2.8)$$

and from the (α, β) and $(4, 4)$ components of the Einstein equations (2.2), which give, respectively,

$$-3 \frac{a''}{a} = \frac{\phi'^2}{4} + \Lambda a^2 e^{\alpha_1 \phi} + \frac{T_3}{2} a e^{2\alpha_2 \phi} \delta(z), \quad (2.9)$$

$$-6 \frac{a'^2}{a^2} = -\frac{\phi'^2}{4} + \Lambda a^2 e^{\alpha_1 \phi} \quad (2.10)$$

(a prime denotes differentiation with respect to z). The last three equations are not independent: the dilaton equation, for instance, can be obtained by differentiating eqs. (2.9), (2.10), as a consequence of the Bianchi identities.

No general solution is known for arbitrary values of α_1, α_2, T_3 and Λ. However, if we fine-tune these parameters by choosing:

$$\alpha_1 = 4\alpha_2 \, , \quad T_3 = 8\sqrt{\Lambda/\Delta} \, , \quad \alpha_1^2 = \Delta + \frac{8}{3} \, , \quad (2.11)$$

where the last equation defines Δ, we recover the four-dimensional sector of a known, one-parameter family of exact domain wall solutions [35], which can be written in an explicitly Z_2-symmetric form. For $\Delta = -2$ the solution is:

$$a(z) = e^{-\frac{k|z|}{3}}, \qquad \phi(z) = \sqrt{\frac{2}{3}} k|z|, \qquad k^2 = -2\Lambda, \tag{2.12}$$

otherwise ($\Delta \neq -2$):

$$a(z) = (1 + k|z|)^{\frac{2}{3(\Delta+2)}}, \qquad e^{\phi(z)} = (1 + k|z|)^{-\frac{2\alpha_1}{\Delta+2}}, \qquad k^2 = \frac{(\Delta+2)^2\Lambda}{\Delta}. \tag{2.13}$$

We shall choose $k > 0$ so that the z coordinate, transverse to the brane, may run from $-\infty$ to $+\infty$ (the proper size of the transverse dimension is finite, however, unless $\Delta = -8/3$). In that case, the solution corresponds to a brane of positive tension, $T_3 > 0$, provided $\Delta \leq -2$. This range of Δ guarantees a positive tension and also avoids the presence of naked singularities [31]. On the other hand, the reality of α_1 requires $\Delta \geq -8/3$. In the rest of this paper we shall thus assume

$$k > 0, \qquad\qquad -\frac{8}{3} \leq \Delta \leq -2. \tag{2.14}$$

We may note that in the limit $\Delta = -8/3$ the dilaton decouples and becomes trivial, and the solution reduces to the well studied pure AdS_5 background , originally introduced to localize gravity on a 3-brane [10].

In the following section we will obtain the canonical equations governing the evolution of scalar (metric + dilaton) fluctuations around the above CLP background solutions.

3 Scalar perturbations

We now perturb to first order the full set of bulk equations (2.2)–(2.5), keeping the position of the brane fixed, $\delta X^A = 0$. It is known, indeed, that the brane location can be consistently assumed to remain unchanged, to the relevant linear order, when perturbing a background of the type we are considering [17] (see [25] for a study that includes, instead, a non-vanishing deformation of the brane). We thus set

$$\delta g_{AB} = h_{AB}, \qquad \delta g^{AB} = -h^{AB}, \qquad \delta\phi = \chi, \qquad \delta X^A = 0, \tag{3.1}$$

where the indices of the perturbed fields are raised and lowered by the unperturbed metric, and the background fluctuations h_{AB}, χ are assumed to be inhomogeneous.

The perturbation of the background equations (2.2)–(2.5) gives, respectively, the linearized equations for the Einstein tensor:

$$\delta G^A{}_B = \frac{1}{2}\left(-h^{AC}\partial_C\phi\partial_B\phi + \partial^A\phi\partial_B\chi + \partial^A\chi\partial_B\phi\right) - \frac{1}{4}\delta^A_B\left(2\partial^M\partial_M\chi - h^{MN}\partial_M\phi\partial_N\phi\right)$$
$$+\Lambda\alpha_1\chi e^{\alpha_1\phi}\delta^A_B + \frac{T_3}{2}\frac{1}{\sqrt{|g|}}\int d^4\xi\sqrt{|\gamma|}\delta^5(x-X)\gamma^{\alpha\beta}\partial_\alpha X^A\partial_\beta X^C e^{\alpha_2\phi}$$
$$\times\left[h_{BC} + g_{BC}\left(-\frac{1}{2}g^{MN}h_{MN} + \frac{1}{2}\gamma^{\mu\nu}\delta\gamma_{\mu\nu} + \alpha_2\chi\right)\right]$$
$$+\frac{T_3}{2}\frac{1}{\sqrt{|g|}}g_{BC}\int d^4\xi\sqrt{|\gamma|}\delta^5(x-X)\partial_\alpha X^A\partial_\beta X^C e^{\alpha_2\phi}\delta\gamma^{\alpha\beta}, \tag{3.2}$$

for the dilaton:

$$\nabla_M\nabla^M\chi - h^{MN}\nabla_M\nabla_N\phi - g^{MN}\delta\Gamma^B_{MN}\partial_B\phi + 2\alpha_1^2\Lambda\chi e^{\alpha_1\phi}$$
$$+\frac{\alpha_2 T_3}{2}\frac{1}{\sqrt{|g|}}\int d^4\xi\sqrt{|\gamma|}\delta^5(x-X)\gamma^{\alpha\beta}\partial_\alpha X^A\partial_\beta X^B g_{AB}e^{\alpha_2\phi}$$
$$\times\left(-\frac{1}{2}g^{MN}h_{MN} + \frac{1}{2}\gamma^{\mu\nu}\delta\gamma_{\mu\nu} + \alpha_2\chi\right)$$
$$+\frac{\alpha_2 T_3}{2}\frac{1}{\sqrt{|g|}}\int d^4\xi\sqrt{|\gamma|}\delta^5(x-X)\partial_\alpha X^A\partial_\beta X^B e^{\alpha_2\phi}\left(g_{AB}\delta\gamma^{\alpha\beta} + h_{AB}\gamma^{\alpha\beta}\right) = 0, \tag{3.3}$$

for the brane:

$$\partial_\alpha\left\{\sqrt{|\gamma|}\partial_\beta X^B e^{\alpha_2\phi}\left[\delta\gamma^{\alpha\beta}g_{AB} + \gamma^{\alpha\beta}h_{AB} + \gamma^{\alpha\beta}g_{AB}\left(\alpha_2\chi + \frac{1}{2}\gamma^{\mu\nu}\delta\gamma_{\mu\nu}\right)\right]\right\} =$$
$$= \frac{1}{2}\sqrt{|\gamma|}\partial_\alpha X^B\partial_\beta X^C\left[\left(\delta\gamma^{\alpha\beta} + \frac{1}{2}\gamma^{\alpha\beta}\gamma^{\mu\nu}\delta\gamma_{\mu\nu}\right)\frac{\partial}{\partial x^A}\Big|_{x=X(\xi)}\left(g_{BC}e^{\alpha_2\phi}\right)\right.$$
$$\left.+\gamma^{\alpha\beta}\,\partial_A\left(h_{BC}e^{\alpha_2\phi} + \alpha_2\chi g_{BC}e^{\alpha_2\phi}\right)\Big|_{x=X(\xi)}\right], \tag{3.4}$$

and for the induced metric:

$$\delta\gamma_{\alpha\beta} = \partial_\alpha X^A\partial_\beta X^B e^{\alpha_2\phi}\left(h_{AB} + \alpha_2\chi g_{AB}\right). \tag{3.5}$$

Here all geometrical quantities, such as the perturbed connection $\delta\Gamma_{AB}{}^C$, the perturbed scalar curvature $\delta R = -h^{AB}R_{AB} + g^{AB}\delta R_{AB}$, and so on, are computed to first order in h_{AB}.

By expanding around the background (2.6), it is now easy to study the propagation of the spin-2 physical degrees of freedom on the brane, represented by the transverse and traceless perturbations $\overline{h}_{ij}$,

$$h_{AB} = a^2\delta^i_A\delta^j_B\overline{h}_{ij}, \qquad \overline{h}^i{}_i = 0, \qquad \nabla^i\overline{h}_{ij} = 0. \tag{3.6}$$

In the linear approximation, the tensor fluctuations $\overline{h}_{ij}$ are decoupled from the scalar and matter fluctuations. We can consistently set $\chi = 0$, and find that the dilaton and the

brane equations are trivially satisfied; in addition, the right-hand side of the Einstein equations (3.2) is identically vanishing, and the linearized Ricci tensor leads to the well known covariant wave equation for gravitons:

$$\Box_5 \overline{h}_{ij} \equiv \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^{i2}} - \frac{\partial^2}{\partial z^2} - \frac{3a'}{a}\frac{\partial}{\partial z} \right) \overline{h}_{ij} = 0, \tag{3.7}$$

where $\Box_5 \equiv \nabla_M \nabla^M$ is the five-dimensional covariant d'Alembert operator, describing free propagation in the warped bulk geometry.

In this paper (also in preparation of future cosmological applications) we are primarily interested in the scalar fluctuations of the bulk metric, which are coupled to the dilaton fluctuations. Thus, we shall keep $\delta\phi = \chi \neq 0$, and we shall expand around the background (2.6) in the so-called "generalized longitudinal gauge" [17], which extends the longitudinal gauge of standard cosmology [36] to the brane-world scenario. As discussed in [17], in five dimensions there are four independent degrees of freedom for the scalar metric fluctuations: in the generalized longitudinal gauge they are described by the four variables $\{\varphi, \psi, \Gamma, W\}$, defined by

$$\begin{aligned} h_{00} &= 2\varphi a^2, & h_{ij} &= 2\psi a^2 \delta_{ij}, \\ h_{44} &= 2\Gamma a^2, & h_{04} &= -W a^2. \end{aligned} \tag{3.8}$$

(Off-diagonal metric fluctuations have been taken into account also in a recent study of linearized gravity in a brane-world background [37]: in that case, however, there are no scalar sources in the bulk, and no long-range scalar interactions).

By inserting the explicit form of the background (2.6) and of the metric fluctuations (3.8) into the perturbed equations (3.2)–(3.5), we obtain the full set of constraints and dynamical equations governing the linearized evolution of the five scalar variables $\{\varphi, \psi, \Gamma, W, \chi\}$. Let us give them in components, starting from the Einstein equations, and using eq. (3.5) for the perturbations of the induced metric.

Equation $(0,0)$ gives:

$$\begin{aligned} &2\nabla^2\psi + 3\psi'' + \nabla^2\Gamma + 9\frac{a'}{a}\psi' - 3\frac{a'}{a}\Gamma' - \frac{\phi'}{2}\chi' - 6\frac{a''}{a}\Gamma - \frac{\phi'^2}{2}\Gamma \\ &-a^2 e^{\alpha_1\phi}\Lambda\alpha_1\chi - \frac{1}{2}ae^{2\alpha_2\phi}T_3\left(\Gamma + 2\alpha_2\chi\right)\delta(z) = 0. \end{aligned} \tag{3.9}$$

Equation (i,i) gives:

$$\begin{aligned} &-\nabla^2\varphi - \varphi'' - 2\ddot{\psi} + \nabla^2\psi + 2\psi'' - \ddot{\Gamma} + \nabla^2\Gamma \\ &-3\frac{a'}{a}\varphi' - \dot{W}' - 3\frac{a'}{a}\dot{W} + 6\frac{a'}{a}\psi' - 3\frac{a'}{a}\Gamma' - \frac{\phi'}{2}\chi' - 6\frac{a''}{a}\Gamma - \frac{\phi'^2}{2}\Gamma \\ &-a^2 e^{\alpha_1\phi}\Lambda\alpha_1\chi - \frac{1}{2}ae^{2\alpha_2\phi}T_3\left(\Gamma + 2\alpha_2\chi\right)\delta(z) = 0. \end{aligned} \tag{3.10}$$

Equation (i,j), with $i \neq j$, gives:

$$\partial_i \partial_j \left(\varphi - \psi - \Gamma\right) = 0. \tag{3.11}$$

Equation $(4,4)$ gives:

$$\begin{aligned} &-\nabla^2\varphi - 3\ddot{\psi} + 2\nabla^2\psi - 3\frac{a'}{a}\varphi' - 3\frac{a'}{a}\dot{W} + 9\frac{a'}{a}\psi' + \frac{\phi'}{2}\chi' \\ &-12\frac{a'^2}{a^2}\Gamma + \frac{\phi'^2}{2}\Gamma - a^2 e^{\alpha_1\phi}\Lambda\alpha_1\chi = 0. \end{aligned} \tag{3.12}$$

Equation $(i,0)$ gives:

$$\partial_i\left(\frac{W'}{2} + \frac{3}{2}\frac{a'}{a}W + 2\dot{\psi} + \dot{\Gamma}\right) = 0. \tag{3.13}$$

Equation $(4,0)$ gives:

$$\frac{1}{2}\nabla^2 W - 3\dot{\psi}' + 3\frac{a'}{a}\dot{\Gamma} + \frac{\phi'}{2}\dot{\chi} = 0. \tag{3.14}$$

Equation $(4,i)$ gives:

$$\partial_i\left(\frac{\dot{W}}{2} + \varphi' - 2\psi' + 3\frac{a'}{a}\Gamma + \frac{\phi'}{2}\chi\right) = 0. \tag{3.15}$$

The dilaton equation gives:

$$\begin{aligned} &\Box_5\chi - \phi'\varphi' - \phi'\dot{W} + 3\phi'\psi' - \phi'\Gamma' - 6\frac{a'\phi'}{a}\Gamma - 2\phi''\Gamma \\ &+2a^2 e^{\alpha_1\phi}\Lambda\alpha_1^2\chi + 2ae^{2\alpha_2\phi}T_3\alpha_2\left(\Gamma + 2\alpha_2\chi\right)\delta(z) = 0 \end{aligned} \tag{3.16}$$

(the dots denote differentiation with respect to Minkowski time on the brane). Finally, the brane perturbation equation gives a constraint at $z = 0$ which is always satisfied because of the Z_2 symmetry. A similar set of equations was already derived in [17]. We have no contributions from the time derivatives of the background, since our background is static.

In the absence of bulk sources with anisotropic stresses, we can now eliminate φ from eq. $(i \neq j)$, thus reducing to four scalar degrees of freedom, by setting:

$$\varphi = \psi + \Gamma. \tag{3.17}$$

As a consequence, we immediately find that the variable W decouples from the other fluctuations. The combination of the time derivative of eq. $(4,i)$ with the z-derivative of eq. $(i,0)$ and with eq. $(4,0)$ leads in fact to the equation

$$\Box_5 W = 3\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right)W. \tag{3.18}$$

Thus W is decoupled but, because of the non-trivial self-interactions, it does not freely propagate in the background geometry like the graviton, eq. (3.7).

In order to discuss the dynamics of the remaining variables ψ, Γ and χ, it is now convenient to recombine their differential equations in an explicitly covariant way, to obtain a canonical evolution equation. To this aim, we can use eq. (3.17) and eq. $(4,i)$ to eliminate φ and $\dot{W}$ in eqs. (i,i), $(4,4)$ and in the dilaton equation (3.16). We then combine the simplified version of eqs. $(4,4)$, (i,i) with eq. $(0,0)$, and obtain the following system of coupled equations, where the source terms depend only on Γ and χ:

$$\begin{aligned}
\Box_5\psi &= -2\frac{a''}{a}\Gamma + 2\frac{a'^2}{a^2}\Gamma + \frac{a'\phi'}{a}\chi - \frac{2}{3}a^2 e^{\alpha_1\phi}\Lambda\alpha_1\chi \\
&- \frac{1}{6}ae^{2\alpha_2\phi}T_3\left(\Gamma + 2\alpha_2\chi\right)\delta(z), \qquad (3.19)\\
\Box_5\Gamma &= -2\frac{a''}{a}\Gamma + 8\frac{a'^2}{a^2}\Gamma - \phi'^2\Gamma + \phi''\chi + \frac{a'\phi'}{a}\chi - \frac{2}{3}a^2 e^{\alpha_1\phi}\Lambda\alpha_1\chi \\
&- \frac{2}{3}ae^{2\alpha_2\phi}T_3\left(\Gamma + 2\alpha_2\chi\right)\delta(z), \qquad (3.20)\\
\Box_5\chi &= 2\phi''\Gamma - \phi'^2\chi - 2a^2 e^{\alpha_1\phi}\Lambda\alpha_1^2\chi \\
&- 2ae^{2\alpha_2\phi}T_3\alpha_2\left(\Gamma + 2\alpha_2\chi\right)\delta(z). \qquad (3.21)
\end{aligned}$$

This system can be diagonalized by introducing the fields

$$\omega_1 = 2\psi + \Gamma, \qquad \omega_2 = 6\alpha_2\Gamma + \chi, \qquad \omega_3 = \Gamma - 2\alpha_2\chi, \qquad (3.22)$$

relations that can be inverted as:

$$\begin{aligned}
\psi &= \frac{\omega_1}{2} - \frac{2\alpha_2\omega_2 + \omega_3}{2\left(1 + 12\alpha_2^2\right)}, \qquad \Gamma = \frac{2\alpha_2\omega_2 + \omega_3}{\left(1 + 12\alpha_2^2\right)} \\
\chi &= \frac{\omega_2 - 6\alpha_2\omega_3}{\left(1 + 12\alpha_2^2\right)}. \qquad (3.23)
\end{aligned}$$

In terms of these new variables, the perturbation equations (3.19)–(3.21) reduce to

$$\begin{aligned}
\Box_5\omega_1 &= 0, \qquad (3.24)\\
\Box_5\omega_2 &= 0, \qquad (3.25)\\
\Box_5\omega_3 &= \left[-\frac{\phi''}{2\alpha_2} - \phi'^2 - \frac{a'\phi'}{2\alpha_2 a} + \left(\frac{\alpha_1}{3\alpha_2} - 2\alpha_1^2\right)a^2 e^{\alpha_1\phi}\Lambda\right]\omega_3 \\
&+ \left[\left(\frac{2}{3} - 4\alpha_2^2\right)ae^{2\alpha_2\phi}T\delta(z)\right]\omega_3. \qquad (3.26)
\end{aligned}$$

Together with eq. (3.18), and the constraints (3.13)–(3.15), the above decoupled equations describe the complete evolution of the scalar (metric + dilaton) fluctuations in the CLP brane-world background (2.12), (2.13). Two variables (ω_1, ω_2) are (covariantly) free

on the background like the graviton, while the other two variables (ω_3, W) have non-trivial self-interactions.

In all cases, we can introduce the corresponding "canonical variables" $\widehat{W}$, $\widehat{\omega}_i$ $(i = 1, 2, 3)$, which have canonically normalized kinetic terms [36] in the action, simply by absorbing the geometric warp factor as follows:

$$W = \widehat{W} a^{-3/2}, \qquad\qquad \omega_i = \widehat{\omega}_i a^{-3/2}. \tag{3.27}$$

The variables $\widehat{W}$, $\widehat{\omega}_i$ are required for a correct normalization of the scalar perturbations to a quantum fluctuation spectrum, as they satisfy canonical Poisson (or commutation) brackets. When the general solution is written as a superposition of free, factorized plane-waves modes on the brane,

$$\widehat{W} = \Psi_w(z) e^{-ip_\mu x^\mu}, \qquad\qquad \widehat{\omega}_i = \Psi_i(z) e^{-ip_\mu x^\mu}, \tag{3.28}$$

they define the inner product of states with measure dz [18], as in conventional one-dimensional quantum mechanics, $\int dz |\Psi(z)|^2$.

The allowed mass spectrum of $m^2 = \eta^{\mu\nu} p_\mu p_\nu$, for the scalar fluctuations on the brane, can then be obtained by solving an eigenvalue problem in the Hilbert space $L^2(R)$ for the canonical variables Ψ_w, Ψ_i, satisfying a Schrödinger-like equation in z, which is obtained from the equations (3.18), (3.24)–(3.26) for W and ω_i, and which can be written in the conventional form as:

$$\Psi_w'' + \left(m^2 - \frac{\xi_w''}{\xi_w} \right) \Psi_w = 0, \qquad\qquad \Psi_i'' + \left(m^2 - \frac{\xi_i''}{\xi_i} \right) \Psi_i = 0. \tag{3.29}$$

Here, by analogy with cosmological perturbation theory [36], we have introduced four "pump fields" ξ_w, ξ_i, defined as follows:

$$\begin{aligned} &\xi_w = a^{\beta_w}, \qquad\qquad \xi_i = a^{\beta_i}, \\ &\beta_w = -\frac{3}{2}, \qquad \beta_1 = \beta_2 = \frac{3}{2}, \qquad \beta_3 = -\frac{1}{2}(1 + 3\alpha_1^2) = -\frac{3}{2}(\Delta + 3). \end{aligned} \tag{3.30}$$

The effective potential generated by the derivatives of the pump fields depends on β_w, β_i, and contains in general a smooth part, peaked at $z = 0$, plus a positive or negative δ-function contribution at the origin. We may have, in principle, not only volcano-like potentials, which correspond to the free covariant d'Alembert equation with $\beta = 3/2$ [31] (and which are known to localize gravity [10, 18]), but also potentials that are positive everywhere and admit no bound states. The possible localization of scalar interactions on the 3-brane, for the given background and perturbation equations, will be discussed in the next section.

4 Localization of the massless modes

The general solutions of the canonical perturbation equations (3.29) are labelled by the mass eigenvalue m, by their parity with respect to z-reflections, and by the parameters β_w, β_i, which depend on the type of perturbation. The massless scalar modes corresponding to bound states of the effective potential [10, 18, 31] will describe long-range scalar interactions confined to the four-dimensional brane. Concerning z-reflections, we shall follow the perturbative formalism developed in [17] consistently, restricting ourselves to a perturbed background that is still Z_2-symmetric, namely to Z_2-odd solutions for W, and to Z_2-even solutions for ω_i. To describe a bound state, we shall further restrict such solutions to those with a normalizable canonical variable w.r.t. the measure dz, namely to $\Psi(z) \in L^2(R)$. Among the acceptable solutions, we shall finally select those satisfying the constraints (3.13)–(3.15) derived in the previous section. The above set of conditions will determine the class of brane-world backgrounds allowing the four-dimensional localization of long-range scalar interactions.

For $m = 0$ the exact solution of the canonical equations (3.29), for a generic pump field $\xi(z)$, can be separated into an even and an odd part, Ψ^+ and Ψ^-, as follows:

$$\Psi_0^+(z) = c_+\xi(z), \qquad \Psi_0^-(z) = c_-\xi(z)\int dz'\xi^{-2}(z'), \tag{4.1}$$

where c_+ and c_- are integration constants. In order to parametrize the solutions for different values of Δ and β (and also in view of subsequent applications to the massive mode solutions), it is now convenient to introduce the two indices ν and ν_0, defined by

$$\nu_0 = \frac{\Delta}{2(\Delta+2)}, \qquad \nu = \frac{1}{2} - \frac{2\beta}{3(\Delta+2)}. \tag{4.2}$$

For the three possible values of β_w, β_i (see eq. (3.30)), they are related by

$$\begin{aligned} &\nu = 1 - \nu_0, && \beta = \beta_w = -\frac{3}{2}, \\ &\nu = \nu_0, && \beta = \beta_{1,2} = \frac{3}{2}, \\ &\nu = 2 - \nu_0, && \beta = \beta_3 = -\frac{3}{2}(\Delta+3). \end{aligned} \tag{4.3}$$

We recall that, in the class of backgrounds we are considering, the parameter Δ is constrained in the range $-8/3 \le \Delta \le -2$ (because of the conditions of positive tension, absence of naked singularities and reality of the dilaton couplings). As a consequence, the index ν_0 can range from 2 to $+\infty$.

Let us first discuss the limiting case $\nu = \nu_0 = \infty$, corresponding to $\Delta = -2$. For a generic pump field, with exponent β, the massless solutions are:

$$\Psi_{0,\infty}^+(z) = c_{0,\infty}^+ e^{-\frac{\beta}{3}k|z|}, \qquad \Psi_{0,\infty}^-(z) = c_{0,\infty}^- \mathrm{sgn}\{z\}\left(e^{\frac{\beta}{3}k|z|} - e^{-\frac{\beta}{3}k|z|}\right). \tag{4.4}$$

In our background $k > 0$, so that the fluctuations W and ω_3, both with $\beta = -3/2$, are not normalizable. The even solutions of the free d'Alembert equation, with $\beta = 3/2$, are instead normalizable, so we have acceptable solutions for ω_1 and ω_2. However, from the constraint $(i, 0)$ of eq. (3.13), ω_1 is forced to vanish when $W = 0$, unless the fluctuations are static, $\dot{\omega}_1 = 0$. All the other constraints (using also the background equations) are instead identically satisfied by ω_2. It follows that, for $\Delta = -2$, there are two independent massless modes localized on the brane: one, ω_2, is propagating and the other, ω_1, is static.

The same is true for the case $\Delta < -2$, i.e. finite ν_0. In that case the massless solutions can be written in the form

$$\begin{aligned}\Psi^+_{0,\nu}(z) &= c^+_{0,\nu}\,(1+k|z|)^{\frac{1}{2}-\nu}\,,\\ \Psi^-_{0,\nu}(z) &= c^-_{0,\nu}\mathrm{sgn}\{z\}\left[(1+k|z|)^{\frac{1}{2}+\nu}-(1+k|z|)^{\frac{1}{2}-\nu}\right].\end{aligned} \tag{4.5}$$

The solutions for W and ω_3 always correspond to $\nu < 0$, and are not normalizable. The even d'Alembert modes, with $\nu = \nu_0$, are normalizable for $\nu > 1$, i.e. $\Delta > -4$, so that ω_1 and ω_2 are always acceptable in our class of backgrounds, for which $\Delta \geq -8/3$. Again, however, the constraint (3.13) implies $\omega_1 = 0$, unless $\dot{\omega}_1 = 0$, while the other constraints are identically satisfied by ω_2, so we are left with non-trivial massless solutions only for a propagating fluctuation, ω_2, and for a static one, ω_1.

We may thus conclude that all backgrounds of the class defined by the conditions (2.14) localize on the brane not only the massless spin-2 degrees of freedom [31] (long-range tensor interactions), but also one propagating massless scalar degree of freedom (ω_2), corresponding to a long-range scalar interaction generated by the dilaton field. In the longitudinal gauge, the canonical representation ω_2 of such a scalar interaction is associated not only with the dilaton fluctuation χ, but also with the four-dimensional "Bardeen potential" ψ, and with the "breathing mode" Γ of the dimension orthogonal to the brane (see eqs. (3.23)). In addition, we have a second independent massless degree of freedom localized on the brane (ω_1), which is not propagating ($\dot{\omega}_1 = 0$), but is essential to reproduce the standard long-range gravitational interaction in the static limit, as we shall discuss in Section 6.

The localization of the scalar interactions does not impose any further constraints on the background, besides those of eq. (2.14). Also, in the limiting case of a pure AdS_5 solution ($\Delta = -8/3$, $\nu_0 = 2$, $\alpha_1 = 0 = \alpha_2$) the dilaton background disappears, and the dilaton fluctuation $\chi = \omega_2$ decouples from the others. The only (static) contribution to the scalar sector of metric fluctuations comes from ω_1, which generates the long-range Newton potential $\varphi = \psi$ on the brane (see Section 6). However, even in the case $\Delta = -8/3$, the propagating dilaton flucutuation ω_2 is non-vanishing, and remains there to describe a (possibly dangerous) long-range scalar interaction. By contrast, no propagating massless

scalar modes appear in the "pure-gravity" models without bulk scalar fields in the action, as the AdS_5 brane-world scenario discussed in [10].

5 The massive mode spectrum

The massive part of the spectrum of the canonical equations (3.29) is not localized on the brane; it may induce higher-dimensional (and short-range) corrections to the long-range scalar forces (just as in the case of the pure tensor part of the gravitational interaction [10, 18, 31]). Such corrections thus provide physical effects from the fifth dimension on our brane.

In order to determine the massive modes able to survive the constraints of Section 3, and to evaluate the corresponding short-range corrections, we shall first present the exact solutions of the massive canonical equations (3.29), distinguishing, as before, the even and odd parts under z-reflections. For $m \neq 0$, $\Delta = -2$ (i.e. $\nu_0 = \infty$), and a generic pump field with power β, such solutions can be written as follows

$$\begin{aligned} \Psi^+_{m,\infty}(z) &= \frac{1}{\sqrt{\pi m q}} \left(q \cos q|z| - \frac{\beta}{3} k \sin q|z| \right), \\ \Psi^-_{m,\infty}(z) &= -\sqrt{\frac{m}{\pi q}} \sin qz, \end{aligned} \tag{5.1}$$

where

$$q = \left(m^2 - \frac{k^2}{4} \right)^{1/2}. \tag{5.2}$$

For $\Delta < -2$, i.e. finite ν_0, the solution can be written as a combination of first- and second-kind Bessel functions J_ν and Y_ν [38], of index ν given by eq. (4.2):

$$\begin{aligned} \Psi^+_{m,\nu}(z) &= c_{m,\nu-1} \sqrt{1+k|z|} \left[Y_{\nu-1}\left(\frac{m}{k}\right) J_\nu(y) - J_{\nu-1}\left(\frac{m}{k}\right) Y_\nu(y) \right], \\ \Psi^-_{m,\nu}(z) &= c_{m,\nu}\ \mathrm{sgn}\{z\} \sqrt{1+k|z|} \left[Y_{\nu}\left(\frac{m}{k}\right) J_\nu(y) - J_{\nu}\left(\frac{m}{k}\right) Y_\nu(y) \right], \end{aligned} \tag{5.3}$$

where

$$c_{m,\nu} = \sqrt{\frac{m}{2k}} \left[J^2_\nu\left(\frac{m}{k}\right) + Y^2_\nu\left(\frac{m}{k}\right) \right]^{-\frac{1}{2}}, \qquad y = \frac{m}{k}(1+k|z|). \tag{5.4}$$

Note that in the above equations we have adopted the δ-function normalization of the continuum modes, as for plane waves in one-dimensional quantum mechanics. As a consequence, Ψ_m is dimensionless (unlike in [31], where a different normalization is adopted).

It is important to note that modes with negative squared mass (tachyons) are not included in the spectrum, as they would not correspond to a normalizable canonical variable

(Ψ would blow up in z). This can be regarded as a direct check of the stability of the given class of CLP backgrounds against scalar perturbations, since tachyonic modes would also blow up in time, and would destroy the assumed homogeneity of the four-dimensional brane. Another consequence of the normalization condition is the mass gap ($m^2 > k^2/4$, see eq. (5.2)) between the localized massless mode and the massive corrections, in the limiting background with $\nu_0 = \infty$ (already noticed in [31] for the case of pure tensor interactions).

We shall now impose the constraints (3.13)–(3.15) following from the perturbed background equations. It is convenient to introduce the four amplitudes A_i, A_w, defined by

$$\omega_1 = \frac{a^{-3/2}A_1}{c_{m,\nu_0-1}}\Psi^+_{m,\nu_0}(z)e^{-ip_\mu x^\mu}, \qquad \omega_2 = \frac{a^{-3/2}A_2}{c_{m,\nu_0-1}}\Psi^+_{m,\nu_0}(z)e^{-ip_\mu x^\mu}$$
$$\omega_3 = \frac{a^{-3/2}A_3}{c_{m,1-\nu_0}}\Psi^+_{m,2-\nu_0}(z)e^{-ip_\mu x^\mu}, \qquad W = \frac{a^{-3/2}A_w}{c_{m,1-\nu_0}}\Psi^-_{m,1-\nu_0}(z)e^{-ip_\mu x^\mu}, \quad (5.5)$$

where $\Psi^\pm$ are the above normalized solutions. Plugging this ansatz into the constraint equations (3.13)–(3.15), we find that, in contrast with the massless case, none of the four scalar fluctuations is forced to vanish. However, only two amplitudes are independent. By taking, for instance, ω_2 and ω_3 as independent variables, we can indeed express A_1 and A_w in terms of A_2 and A_3, for all values of ν_0, in such a way that all the constraints are identically satisfied. For a generic mode of mass m and momentum p, we find, in particular,

$$A_1 = \frac{m^2}{9m^2+6p^2}\frac{\sqrt{2\nu_0-1}}{\nu_0-1}\left(\sqrt{3\,(\nu_0-2)}A_2 + 3\sqrt{2\nu_0-1}A_3\right),$$
$$A_w = \frac{2im\sqrt{m^2+p^2}}{9m^2+6p^2}\frac{\sqrt{2\nu_0-1}}{\nu_0-1}\left(\sqrt{3\,(\nu_0-2)}A_2 + 3\sqrt{2\nu_0-1}A_3\right), \quad (5.6)$$

which also hold when $\nu_0 \to \infty$. Note that a single combination of A_2 and A_3 determines both A_1 and A_w.

For such backgrounds we thus have four types of higher-dimensional contributions to the scalar interactions on the brane, arising from the massive spectrum of ω_i and W. The exchange of such massive modes generates corrections to the four-dimensional scalar forces. The corrections are in general different from those of tensor interactions, arising from the massive spectrum of a variable satisfying the free d'Alembert equation, like ω_2. In the weak field limit, however, the leading-order contributions to the non-relativistic potential generated by a static scalar sources have the same qualitative behaviour as in the case of tensor interactions, as will be illustrated in the next section.

6 Static limit and leading-order corrections

To make contact with previous works, and for future phenomenological applications, we shall now compute the effective scalar-tensor interaction induced on the brane, in the weak field limit, by a static and point-like source of mass M and dilatonic charge Q.

In our longitudinal gauge (3.6), (3.8), in which the decomposition of the metric fluctuations is based on the $O(3)$ symmetry of the spatial hypersurfaces of the brane, the energy density of a point-like particle only contributes to the scalar part of the perturbed matter stress tensor (with T_{00} as the only non-vanishing component), and provides a δ-function source to the $(0,0)$ scalar perturbation equation (3.9). Similarly, the charge Q acts as a point-like source in the dilaton perturbation equation (3.16).

As a consequence, we obtain three δ-function sources in the equations for the three ω_i fluctuations, $S_i\delta^3(x-x')\delta(z)$, with three scalar charges S_i, which are "mixtures" of M and Q, while no source term is obtained in the static limit for the W fluctuation (in agreement with the fact that $W=0$, in the static limit, according to the constraint $(i,0)$). The effective sources S_i for the massless and massive ω_i fluctuations are defined by the combination of eqs. $(4,4),(i,i),(0,0)$, and by the dilaton equation (3.16), as follows:

$$S_1 = M, \qquad S_2 = Q + 2\alpha_2 M, \qquad S_3 = \frac{M}{3} - 2\alpha_2 Q. \tag{6.1}$$

The exact static solutions of eqs. (3.24)–(3.26), with the above point-like sources, can be easily obtained using the static limit of the retarded Green function evaluated on the brane ($z=0$), i.e.

$$\omega_i(\nu, x, x') = -S_i G_i(\nu, x, x'), \tag{6.2}$$

where

$$G_i(\nu, \vec{x}, \vec{x}', z=z'=0) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \left\{ \frac{\left[\psi^+_{0,\nu}(0)\right]^2}{p^2} + \int_{m_0}^{\infty} dm \frac{\left[\psi^+_{m,\nu}(0)\right]^2}{p^2+m^2} \right\} \tag{6.3}$$

(see also [31, 32, 39]). Here $\nu = \nu_0$ for $i=1,2$, while $\nu = 2-\nu_0$ for $i=3$, and $\psi^+_{0,\nu}(z), \psi^+_{m,\nu}(z)$ (including the case $\nu=\infty$) are the exact solutions (4.4), (4.5), (5.1), (5.3), obtained in the previous sections. The first term in the integrand corresponds to the long-range forces generated by the massless modes, the second term to the "short-range" corrections due to the massive modes, and m_0 is the lower bound for the massive spectrum ($m_0 = k/2$ if $\nu_0 = \infty$, while $m_o = 0$ if $\nu_0 < \infty$, see eqs. (5.1), (5.2)).

We should note that in the ω_1, ω_2 case we have to include both the massless and massive contributions, while in the ω_3 case only the massive ones survive (indeed, we recall that

for $m = 0$ the ω_3 modes are not normalizable, and that the massless sector of ω_1, in the static limit, is not eliminated by the constraint $(i, 0)$, see Section 4). We also note that the amplitude of the massless solutions (4.4), (4.5) has to be fixed by the correct normalization, i.e.

$$c^+_{0,\infty} = (k\beta/3)^{1/2}\,, \qquad\qquad c^+_{0,\nu} = [k(\nu-1)]^{1/2}\,. \tag{6.4}$$

Let us start with ω_1, for which $\nu = \nu_0$, and with the limiting background $\Delta = -2$, i.e. $\nu = \infty$. By setting $\beta = 3/2$, and using eqs. (4.4) and (5.1) for $\psi^+_{0,\infty}$ and $\psi^+_{m,\infty}$, respectively, we obtain

$$\omega_1(\nu=\infty) = -S_1 \frac{k}{8\pi r}\left[1 + \frac{2}{k\pi}\int_{k/2}^{\infty}\frac{dm}{m}\left(m^2 - \frac{k^2}{4}\right)^{1/2} e^{-mr}\right], \tag{6.5}$$

where $r = |\vec{x} - \vec{x}'|$. The same integral had already been obtained in [31] when discussing the localization of tensor fluctuations, and the associated leading-order corrections (in the large-distance limit) are known to be of the form $e^{-kr/2}(kr)^{-3/2}$.

In the case $\Delta < -2$, i.e. $\nu = \nu_o < \infty$, we shall use instead eqs. (4.5) and (5.3) for ψ^+_{0,ν_0} and ψ^+_{m,ν_0}, respectively. Also, we shall estimate the contribution of the massive modes by expanding the Bessel functions in the small-m regime (which is relevant at the long distances typical of the weak field limit). The small argument limit [38] of J_ν, Y_ν then gives

$$\left[\psi^+_{m,\nu_0}(0)\right]^2 = \frac{1}{\Gamma^2(\nu_0 - 1)}\left(\frac{m}{2k}\right)^{2\nu_0 - 3}, \tag{6.6}$$

and we are led to

$$\omega_1(\nu<\infty) = -S_1\frac{k(\nu_0-1)}{4\pi r}\left[1 + \frac{\Gamma(\nu_0 - 1/2)}{\sqrt{\pi}\Gamma(\nu_0)}\left(\frac{1}{kr}\right)^{2\nu_0-2}\right] \tag{6.7}$$

(here Γ obviously represents the Euler function, not to be confused with the g_{44} component of the metric fluctuations).

Exactly the same results (6.5), (6.7) are obtained for ω_2, which satisfies the same free d'Alembert equation as ω_1, with the only difference that S_1 has to be replaced by S_2.

Let us then compute ω_3, for which $\nu = 2 - \nu_0$, and which has only the massive mode contribution to the Green function. For $\Delta = -2$, i.e $\nu = \infty$, we shall use eq. (5.1) for $\psi^+_{m,\infty}$. At $z = 0$ the solution however is β-independent, and we obtain the (massive part) of the result already given in eq. (6.5) (with S_1 replaced by S_3).

In the case $\Delta < -2$, i.e $\nu < \infty$, we shall use $\psi^+_{m,2-\nu_0}$ from eq. (5.3). In the large-distance (small-m) regime, however, the contribution at $z = 0$ to the Green function is exactly the same as that of eq. (6.6), so that we obtain

$$\omega_3(\nu<\infty) = -S_3\frac{k(\nu_0-1)}{4\pi r}\frac{\Gamma(\nu_0-1/2)}{\sqrt{\pi}\Gamma(\nu_0)}\left(\frac{1}{kr}\right)^{2\nu_0-2}. \tag{6.8}$$

We are now ready to estimate, in the static limit, the dilaton and metric fluctuations $\chi, \Gamma, \psi, \varphi$. We shall explicitly consider the case $\nu_0 < \infty$, for an easy comparison with previous results relative to an AdS_5 background, for which $\nu_0 = 2$. Defining

$$A_{\nu_0} = \frac{k(\nu_0 - 1)}{4\pi}, \qquad B_{\nu_0} = \frac{\Gamma(\nu_0 - 1/2)}{\sqrt{\pi}\Gamma(\nu_0)}, \tag{6.9}$$

we first rewrite the ω_i solutions in compact form as:

$$\begin{aligned}
\omega_1 &= -\frac{S_1 A_{\nu_0}}{r}\left[1 + B_{\nu_0}\left(\frac{1}{kr}\right)^{2\nu_0 - 2}\right], \\
\omega_2 &= -\frac{S_2 A_{\nu_0}}{r}\left[1 + B_{\nu_0}\left(\frac{1}{kr}\right)^{2\nu_0 - 2}\right], \\
\omega_3 &= -\frac{S_3 A_{\nu_0}}{r} B_{\nu_0}\left(\frac{1}{kr}\right)^{2\nu_0 - 2}.
\end{aligned} \tag{6.10}$$

We may note, as a check of our previous computations, that the relative amplitude of the massive corrections, in the weak, static limit, is controlled by the correspondig scalar charges, which satisfy the relation (from eq. (6.1)):

$$S_3 + 2\alpha_2 S_2 = \left(\frac{1}{3} + 4\alpha_2^2\right) S_1. \tag{6.11}$$

By eliminating α_2 in favour of Δ according to eq. (2.11), and Δ in terms of ν_0, according to eq. (4.2), we obtain

$$S_1 = \frac{1}{\nu_0 - 1}\left[\frac{1}{3}\sqrt{3(\nu_0 - 2)(2\nu_0 - 1)} S_2 + (2\nu_0 - 1) S_3\right], \tag{6.12}$$

which exactly reproduces, in the static limit $m^2 + p^2 \to 0$, the general relation (5.6) between the massive amplitudes.

It is convenient, at this point, to explicitly introduce the four-dimensional gravitational constant M_p^{-2}, by noting that

$$A_{\nu_0} = \frac{k(\nu_0 - 1)}{4\pi} = \frac{1}{4\pi}\left[\psi^+_{0,\nu_0}(0)\right]^2 = \frac{1}{4\pi}\left[\int dz\ a^3(z)\right]^{-1}. \tag{6.13}$$

The above normalization integral, when expressed in terms of a new bulk coordinate y, with $dy = a(z)dz$, represents the warped extra-dimensional volume that controls the ratio between the four- and five-dimensional gravitational constants [9, 10], i.e.

$$M_p^2 = M_5^3 \int dy\ a^2(y) = M_5^3 \int dz\ a^3(z). \tag{6.14}$$

It follows that, in units such that $M_5^3 = 1$,

$$A_{\nu_0} = (4\pi M_p^2)^{-1} = 2G, \tag{6.15}$$

where G is the Newton coupling constant. The scalar and dilaton fluctuations (3.23), using the static solutions (6.10), can then be written in the form:

$$\begin{aligned}
\varphi &= -\frac{GM}{r}\left[1+\frac{2\alpha_2}{1+12\alpha_2^2}\left(\frac{Q}{M}+2\alpha_2\right)+\frac{4}{3}B_{\nu_0}\left(\frac{1}{kr}\right)^{2\nu_0-2}\right],\\
\psi &= -\frac{GM}{r}\left[1-\frac{2\alpha_2}{1+12\alpha_2^2}\left(\frac{Q}{M}+2\alpha_2\right)+\frac{2}{3}B_{\nu_0}\left(\frac{1}{kr}\right)^{2\nu_0-2}\right],\\
\Gamma &= -\frac{GM}{r}\left[\frac{4\alpha_2}{1+12\alpha_2^2}\left(\frac{Q}{M}+2\alpha_2\right)+\frac{2}{3}B_{\nu_0}\left(\frac{1}{kr}\right)^{2\nu_0-2}\right],\\
\chi &= -\frac{GQ}{r}\left[\frac{2}{1+12\alpha_2^2}\left(1+2\alpha_2\frac{M}{Q}\right)+2B_{\nu_0}\left(\frac{1}{kr}\right)^{2\nu_0-2}\right].
\end{aligned} \tag{6.16}$$

It should be noted that the short-range corrections induced by the massive scalar modes have the same qualitative behaviour as in the tensor case, discussed in [31], in spite of the fact that the massive scalar modes have different spectra. The dimensional decoupling (i.e., the suppression of the higher-dimensional corrections) is thus effective for all scales of distance r such that $kr \gg 1$, where $k = (M_5^3/M_p^2)(\nu_0-1)^{-1}$ is the mass scale relating the five- and four-dimensional gravitational constants, in our class of backgrounds.

The limiting case $\Delta = -8/3$, i.e. $\alpha_2 = 0$ and $\nu_0 = 2$, corresponds to a pure AdS_5 background, if there are no scalar charges on the brane. In that case ω_2 exactly corresponds to the dilaton fluctuation χ (see eq. (3.22)), and can be consistently set to zero (toghether with the dilaton background) if we want to match, in particular, the "standard" brane-world configuration originally considered by Randall and Sundrum [10]. In this limit, $B_2 = 1/2$, and we exactly recover previous results for the effective gravitational interaction on the brane [32], i.e.

$$\varphi = -\frac{GM}{r}\left(1+\frac{2}{3k^2r^2}\right), \qquad \psi = -\frac{GM}{r}\left(1+\frac{1}{3k^2r^2}\right). \tag{6.17}$$

The massless-mode truncation reproduces in this case the static, weak field limit of linearized general relativity (note, however, that for $Q \neq 0$ there is no way to get rid of the long-range scalar interactions). The massive tower of scalar fluctuations, however, induces deviations from Einstein gravity already in the static limit (as noted in [32]), and is the source of a short-range force due to the "breathing" of the fifth dimension,

$$\Gamma = -\frac{GM}{3r}\frac{1}{(kr)^2}, \tag{6.18}$$

even in the absence of bulk scalar fields, and of scalar charges for the matter on the brane.

In a more general gravi-dilaton background ($\Delta \neq -8/3$, $\alpha_2 \neq 0$), the static expansion (6.16) describes an effective scalar-tensor interaction on the brane, which is potentially

dangerous for the brane-world scenario, as it contains not only short-range corrections, but also long-range scalar deviations from general relativity (and, possibly, violations of the Einstein equivalence principle), even in the interaction of ordinary masses, i.e. for $Q = 0$ (similar results have been recently obtained also in the context of a multibrane scenario [40]). This seems to offer an interesting window to investigate the effects of the bulk geometry on the four-dimensional physics of the brane.

7 Conclusions

In this paper we have analysed the full set of coupled equations governing the evolution of scalar fluctuations in a dilatonic brane-world background, supporting a flat 3-brane rigidly located at the fixed point of Z_2 symmetry. We have diagonalized the system of dynamical equations, and found four decoupled but self-interacting variables representing, in a five-dimensional bulk, the four independent degrees of freedom of scalar excitations of the gravi-dilaton background.

We have then restricted our discussion to the class of background solutions characterized by a brane of positive tension, by a decreasing warp-factor as we move away from the brane, and by the absence of naked singularities [31]. Such a class of backgrounds can be characterized by a real parameter Δ ranging from $-8/3$ to -2 or, alternatively, by a real parameter ν_0 ranging from 2 to $+\infty$. The limiting case $\Delta = -8/3$, $\nu_0 = 2$, corresponds to the "pure-gravity" AdS_5 background [10].

We have presented the exact solutions of the canonical perturbation equation for all the scalar degrees of freedom, and we have discussed, in this class of backgrounds, the effects of their massless and massive spectrum for the scalar interactions on the brane, taking into account the appropriate parity under Z_2 symmetry, the normalization condition for the bound states of the spectrum, and the first-order differential constraints arising from the dynamics of scalar perturbations.

We have found that, for all backgrounds, there is one propagating massless mode localized on the brane, associated with a long-range dilatonic interaction in four dimensions. Only very exceptionally (i.e. for a RS background and vanishing dilatonic charges) this "fifth force" disappears. This interaction is always affected by higher-dimensional corrections, due to the scalar massive modes that are not confined on the brane and can freely propagate in the bulk space-time. The amplitudes of such massive modes are constrained by the scalar perturbation equations and, in general, only two amplitudes can be independently assigned. The scalar fluctuation spectra are in general different from the corresponding spectra of the spin-2 degrees of freedom. In the weak and static limit, however, the leading-order short-

range corrections to the scalar force have the same radial dependence as in the case of pure tensor interactions.

In particular, we have found a non-trivial massive spectrum of scalar metric fluctuations even in the pure Randall–Sundrum scenario with AdS_5 metric [10]. In more general backgrounds we have found that there are also scalar contributions to the long-range interactions of two massive bodies, even in the absence of specific "dilatonic" charges, with a resulting effective scalar-tensor interaction on the brane. In this sense, the bulk geometry seems to affect not only the radial dependence, but also the spin content of the effective gravitational forces. We believe that this effect is potentially interesting for further applications of (and constraints on) the brane-world scenario.

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References

[1] P. Horava and E. Witten, Nucl. Phys. B **460**, 506 (1996).

[2] P. Horava and E. Witten, Nucl. Phys. B **475**, 96 (1996).

[3] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Phys. Rev. D**59**, 086001 (1999).

[4] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Nucl. Phys. B**552**, 246 (1999).

[5] I. Antoniadis, Phys. Lett. B **246**, 377 (1990).

[6] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B **429**, 263 (1998).

[7] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B **436**, 257 (1998).

[8] V. Rubakov and M. E. Shaposhnikov, Phys. Lett. **125B**, 139 (1983).

[9] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 and 4690 (1999).

[10] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).

[11] R. B. Abbott, B. Bednarz and S. D. Ellis, Phys. Rev. D **33**, 2147 (1986).

[12] J. Garriga and E. Verdaguer, Phys. Rev. D **39**, 1072 (1989).

[13] L. Amendola, M. Litterio and F. Occhionero, Phys. Lett. B **237**, 348 (1990).

[14] M. Gasperini and M. Giovannini, Phys. Rev. D **47**, 1519 (1993).

[15] M. Gasperini and M. Giovannini, Class. Quantum Grav. **14**, 735 (1997).

[16] M. Giovannini, Phys. Rev. D **55**, 595 (1997).

[17] C. van de Bruck, M. Dorca, R. H. Brandenberger and A. Lukas, Phys. Rev. D **62**, 123515 (2000).

[18] C. Csaki, J. Erlich, T. J. Hollowood and T. Shirman, Nucl. Phys. B **581**, 309 (2000).

[19] T. R. Taylor and G. Veneziano, Phys Lett. B **213**, 459 (1988).

[20] E. Fischbach and C. Talmadge, Nature **356**, 207 (1992).

[21] T. Damour and A. M. Polyakov, Nucl. Phys. B **423**, 532 (1994).

[22] D. Langlois, Phys. Rev. D **62**, 126012 (2000).

[23] D. Langlois, R. Maartens, M. Sasaki and D. Wands, Phys. Rev. D **63**, 084009 (2001).

[24] B. Grinstein, D. R. Nolte and W. Skiba, hep-th/0005001.

[25] N. Deruelle, T. Dolezel and J. Katz, hep-th/0010215.

[26] U. Gen and M. Sasaki, gr-qc/0011078.

[27] A. Neronov and I. Sachs, hep-th/0011254.

[28] C. van de Bruck and M. Dorca, hep-th/0012073.

[29] R. Maartens, gr-qc/0101059.

[30] T. Boehm, R. Durrer and C. van de Bruck, hep-th/0102144.

[31] M. Cvetic, H. Lu and C.N. Pope, Phys. Rev. D **63**, 086004 (2001).

[32] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000).

[33] M. J. Duff, in *Second Conference on Gauge Theories, Applied Supersymmetry and Quantum Gravity* (Imperial Coll. Press, London, 1996).

[34] M. J. Duff, Class. Quantum Grav. **5**, 189 (1988).

[35] M. Cvetic, S. Griffies and H. H. Soleng, Phys. Rev. D **48**, 2613 (1993).

[36] V. F. Mukhanov, H. A. Feldman and R. Brandenberger, Phys. Rep. **215**, 203 (1992).

[37] I. Y. Aref'eva *et al*, Nucl. Phys. B **590**, 273 (2000).

[38] M. Abramovicz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

[39] A. Brandhuber and K. Sfetsos, JHEP **9910**, 013 (1999).

[40] I. I. Kogan, S. Mouslopoulos, A. Papazoglou and L. Pilo, hep-th/0105255.

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Assisting pre-big bang phenomenology through short-lived axions

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Abstract

We present the results of a detailed study of how isocurvature axion fluctuations are converted into adiabatic metric perturbations through axion decay, and discuss the constraints on the parameters of pre-big bang cosmology needed for consistency with present CMB-anisotropy data. The large-scale normalization of temperature fluctuations has a non-trivial dependence both on the mass and on the initial value of the axion. In the simplest, minimal models of pre-big bang inflation, consistency with the COBE normalization requires a slightly tilted (blue) spectrum, while a strictly scale-invariant spectrum requires mild modifications of the minimal backgrounds at large curvature and/or string coupling.

It is well known that, in the framework of pre-big bang cosmology (see [1, 2] for recent reviews), the primordial spectrum of scalar (and tensor) metric perturbations is characterized by a steep positive slope [3]. Since the high-frequency normalization of the spectrum is fixed by the ratio of the string to the Planck mass, the amplitude of metric fluctuations turns out to be strongly suppressed at large scales, and thus unable to account for the CMB anisotropies observed by COBE [4] and by other satellite experiments [5] (unless one accepts rather drastic modifications of pre-big bang kinematics, as recently suggested in [6]).

A possible solution to this problem could be provided, a priori, by the fluctuations of another background field of string theory, in particular of the so-called Kalb–Ramond axion σ (the dual of the NS-NS two-form appearing in the dimensionally reduced string effective action [7]). As first pointed out in [8], axionic quantum fluctuations of the vacuum are amplified by pre-big bang inflation, yielding a final spectrum whose index n_σ can vary, depending on the evolution of extra dimensions. The scale-invariant value of $n_\sigma = 1$ is attained, amusingly enough, for particularly symmetric evolutions of the nine spatial dimensions in which critical superstrings consistently propagate.

Indeed, even if no axion potential is present in the post-big bang era, a (generally non-Gaussian) spectrum of temperature anisotropies can be induced by the fluctuations of the massless [9, 10] axion field, at second order, through the so-called "seed" mechanism [11]. The same is true for a massive light axion that has not decayed yet [12]. Unfortunately, while the model is capable of reproducing the low-multipole COBE data [4], it clearly appears [13] to be disfavoured with respect to standard inflationary models when it comes to fitting data in the acoustic-peaks region [5].

An interesting alternative possibility, first suggested in [1], and recently discussed in detail (and not exclusively within a string cosmology framework) in [14]-[18], uses a general mechanism originally pointed out in [19]. It is based on two basic assumptions: i) the constant value of the axion background after the pre-big bang phase is displaced from the minimum (conventionally defined as $\sigma = 0$) of the non-perturbative potential $V(\sigma)$ generated in the post-big bang epoch; ii) the axion potential is strong enough to induce a phase of axion dominance before its decay into radiation. Under these two (rather plausible) assumptions, the initially amplified isocurvature axion fluctuations can be converted, without appreciable change of the spectrum, into adiabatic (and Gaussian) scalar curvature perturbations until the time of horizon re-entry: these can then possibly produce the observed CMB anisotropies.

Various aspects of this new mechanism have already been discussed in [14] for the string theory axion, and in [15]-[17] (mostly in the context of conventional inflationary models) for the case of a generic scalar field, dubbed the "curvaton" in [15] (see also [18] for a possible application of this mechanism to the ekpyrotic scenario). Here, after providing an explicit derivation and computation of the conversion of axion fluctuations into scalar curvature perturbations, we shall discuss the constraints imposed by the CMB data, and

its possible consistency with the small-scale normalization and tilts typical of pre-big bang models. It will be argued, in particular, that a strictly flat spectrum is only compatible with non-minimal models of pre-big bang inflation. A detailed account of this work, including numerical checks of the analytic arguments and estimates given here, will be presented in a forthcoming paper [20].

The conversion of the axionic isocurvature modes (amplified during the pre-big-bang phase) into adiabatic curvature inhomogeneities takes place in the post-big-bang phase, where we assume the dilaton to be frozen and the axion to be displaced from the minimum of its potential. The relaxation of the axionic field towards the minimum of its potential is determined by the following evolution equations (units where $16\pi G = 1$ are used)

$$R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = \frac{1}{2}T_\mu^\nu + \frac{1}{2}\partial_\mu\sigma\partial^\nu\sigma + \frac{1}{2}\delta_\mu^\nu\left[V - \frac{1}{2}(\nabla_\mu\sigma)^2\right],$$
$$\nabla_\mu\nabla^\mu\sigma + \frac{\partial V}{\partial\sigma} = 0, \tag{1}$$

where $T_{\mu\nu}$ is the stress tensor of the matter sources, which we assume to be dominated by the radiation fluid. In the case of a conformally flat metric, $g_{\mu\nu} = a^2\eta_{\mu\nu}$, the time and space components of such equations, together with the axion evolution equation, can be written (in conformal time and in three spatial dimensions) respectively as

$$6\mathcal{H}^2 = a^2\left(\rho_r + \rho_\sigma\right), \qquad 4\mathcal{H}' + 2\mathcal{H}^2 = -a^2\left(p_r + p_\sigma\right), \tag{2}$$
$$\sigma'' + 2\mathcal{H}\sigma' + a^2\frac{\partial V}{\partial\sigma} = 0, \tag{3}$$

where $\mathcal{H} = a'/a = d(\ln a)/d\eta$, $\rho_r = 3p_r$ is the energy density of the radiation fluid, and

$$\rho_\sigma = \frac{1}{2a^2}\sigma'^2 + V(\sigma), \qquad p_\sigma = \frac{1}{2a^2}\sigma'^2 - V(\sigma). \tag{4}$$

The combination of Eqs. (2) and (3) leads to the conservation equation for the radiation fluid, i.e. $\rho_r' + 4\mathcal{H}\rho_r = 0$.

While the background is radiation-dominated, at least at the onset of the post-big-bang phase, the initial large-scale inhomogeneities are dominated by the (isocurvature) perturbations coming from the pre-big bang amplification of the quantum fluctuations of the axion. In order to study the conversion of isocurvature into scalar curvature (adiabatic) modes, the background Eqs. (2) and (3) should be supplemented by the evolution equations of the scalar inhomogeneities, following from the perturbation of the Einstein equations (1).

Thanks to the absence of anisotropic stresses, the $i \neq j$ components of the perturbed Einstein equations imply that the scalar metric fluctuations can be parametrized in terms of a single gauge-invariant variable, the Bardeen potential Φ [21]. The full system of perturbed Einstein equations can then be written as

$$\Phi' + \mathcal{H}\Phi = \frac{1}{4}\chi\sigma' + \frac{1}{3}a^2\rho_r v_r, \tag{5}$$

$$\nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) = \frac{1}{4}a^2\left(\rho_r\delta_r + \rho_\sigma\delta_\sigma\right), \tag{6}$$

$$\Phi'' + 3\mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi = \frac{1}{4}a^2\left(\frac{1}{3}\rho_r\delta_r + \delta p_\sigma\right), \tag{7}$$

$$\chi'' + 2\mathcal{H}\chi' - \nabla^2\chi + a^2\frac{\partial^2 V}{\partial\sigma^2}\chi = 4\sigma'\Phi' - 2a^2\frac{\partial V}{\partial\sigma}\Phi, \tag{8}$$

where the gauge-invariant variables χ, $\delta\rho_r$, v_r are, respectively, the axion, radiation density and velocity potential fluctuations [with our conventions, in the longitudinal gauge the velocity potential is defined by $\delta T^0_i = (\rho_r + p_r)\partial_i v_r$], and where the following variables

$$\begin{aligned}
&\delta_r = \delta\rho_r/\rho_r, \qquad\qquad \delta_\sigma = \delta\rho_\sigma/\rho_\sigma,\\
&\delta\rho_\sigma = -\Phi\left(\rho_\sigma + p_\sigma\right) + \frac{\sigma'\chi'}{a^2} + \frac{\partial V}{\partial\sigma}\chi,\\
&\delta p_\sigma = -\Phi\left(\rho_\sigma + p_\sigma\right) + \frac{\sigma'\chi'}{a^2} - \frac{\partial V}{\partial\sigma}\chi
\end{aligned} \tag{9}$$

have been defined (we have also assumed $\delta p_r = \delta\rho_r/3$). By using the above perturbation equations, together with the background relations (2) and (3), two useful equations for the evolution of the radiation density contrast and of the velocity potential can be finally obtained:

$$\delta_r' = 4\Phi' + \frac{4}{3}\nabla^2 v_r, \qquad\qquad v_r' = \frac{1}{4}\delta_r + \Phi. \tag{10}$$

We now suppose to start at $t = t_i$ with a radiation-dominated phase in which the homogeneous axion background is initially constant and non-vanishing, $\sigma(t_i) = \sigma_i \neq 0$, $\sigma'(t_i) = 0$, providing a subdominant (potential) energy density, $\rho_\sigma(t_i) = -p_\sigma(t_i) = V_i \ll H_i^2 \sim \rho_r(t_i)$. The initial conditions of Eqs. (5)–(8) are imposed by assuming a given spectrum of isocurvature axion fluctuations, $\chi_k(t_i) \neq 0$, and a total absence of perturbations for the metric and the radiation fluid, $\Phi(t_i) = \delta_r(t_i) = v_r(t_i) = 0$. The initial values of the first derivatives of the perturbation variables are then fixed by enforcing the momentum and Hamiltonian constraints, i.e. Eqs. (5) and (6).

Before discussing the origin of curvature fluctuations we must specify the details of the background evolution. The axion, initially constant and subdominant, starts oscillating at a curvature scale $H_{\rm osc} \sim m$ (as can be argued from Eq. (3)), and eventually decays (with gravitational strength) into radiation, at a scale $H_d \sim m^3/M_{\rm P}^2 < H_{\rm osc}$ (a process that must occur early enough, not to disturb the subsequent standard evolution). When the axion is constant, ρ_σ behaves like an effective cosmological constant, while during the oscillatory phase its kinetic and potential energy density are equal on the average, so that $\langle p_\sigma\rangle = 0$, and $\langle\rho_\sigma\rangle \sim a^{-3}$ behaves like dust matter. Thus the radiation energy is always diluted faster, $\rho_r \sim a^{-4}$, and the axion background tends to become dominant at a scale $H_\sigma(t) \sim \sqrt{V[\sigma(t)]}$.

For an efficient conversion of the initial χ and δ_σ fluctuations into Φ and δ_r fluctuations it is further required [14]-[16], as we shall see, that the decay occur after the beginning of

the axion-dominated phase, i.e. when $H_\sigma > H_d$. Depending upon the relative values of H_σ and $H_{\rm osc}$ (i.e. depending upon the value of σ_i in Planck units) we have two different options which will now be discussed separately. In order to perform explicit analytical estimates, we shall assume here that $V(\sigma)$ can be approximated by the quadratic form $m^2\sigma^2/2$. This is certainly true for $\sigma_i \ll 1$, but it may be expected to be a realistic approximation also for the range of values of σ_i not much larger than 1 (which, as we shall see, is the appropriate range for a normalization of the spectrum compatible with present data). Actually, for the periodic potential expected for an axion the value of $|\sigma_i|$ is effectively bounded from above [14].

(1) If $\sigma_i < 1$ then $H_\sigma < H_{\rm osc}$, and the axion starts oscillating (at a scale $H \sim m$) when the Universe is still radiation-dominated. During the oscillations the average potential energy density decreases like a^{-3}, i.e. the typical amplitude of oscillation decreases, following an $a^{-3/2}$ law, from its initial value σ_i to the value $\sigma_{\rm dom}$ at which $H = H_\sigma \sim m\sigma_{\rm dom}$. During this period $a \sim H^{-1/2}$ (as the background is radiation-dominated), so that $\sigma_{\rm dom} \sim \sigma_i^4$, and $H_\sigma \sim m\sigma_i^4$. Finally, the background remains axion-dominated until the decay scale $H_d \sim m^3/M_{\rm P}^2$. This model of background is thus consistent for $H_i > H_{\rm osc} > H_\sigma > H_d$, namely for

$$1 > \sigma_i > (m/M_{\rm P})^{1/2}, \tag{11}$$

which allows for a wide range for σ_i, if we recall the cosmological bounds on the mass following from the decay of a gravitationally coupled scalar [22] (typically, $m > 10$ TeV to avoid disturbing standard nucleosynthesis).

(2) If $\sigma_i > 1$, and then $H_\sigma > H_{\rm osc}$, the axion starts dominating at the scale $H_\sigma \sim m\sigma_i$, which marks the beginning of a phase of slow-roll inflation, lasting until the curvature drops below the oscillation scale $H_{\rm osc} \sim m$. Such a model of background is consistent for $H_\sigma < H_1$, namely for

$$H_1/m > \sigma_i > 1, \tag{12}$$

where H_1 (fixed around the string scale) corresponds to the beginning of the radiation-dominated, post-big bang evolution. During the inflationary phase the slow decrease of the Hubble scale can be approximated (according to the background equations (2) and (3)) by $H(t) = \alpha m\sigma_i - \beta m^2(t - t_\sigma)$, where α and β are dimensionless coefficients of order 1. Inflation thus begins at the epoch $t = t_\sigma \sim 1/m\sigma_i$, and lasts until the epoch $t = t_m \sim (\sigma_i - 1)/m \sim \sigma_i/m$.

Finally, if $\sigma_i \sim 1$, $H_\sigma \sim H_{\rm osc} \sim m$, and the beginning of the oscillating and of the axion-dominated phase are nearly simultaneous. Let us now estimate, for these classes of backgrounds, the evolution of the Bardeen potential generated by the primordial axion fluctuations.

It is convenient, for this purpose, to introduce the gauge-invariant variable ζ representing the spatial curvature perturbation on uniform density (or equivalently, at large scales, on comoving) hypersurfaces. For purely adiabatic perturbations ζ is conserved (outside the

horizon), and can be written for a general background as [21]:

$$\zeta = -\Phi - \frac{\mathcal{H}\Phi' + \mathcal{H}^2\Phi}{\mathcal{H}^2 - \mathcal{H}'}. \tag{13}$$

Outside the horizon, Eq. (10) gives $4\Phi = \delta_r$; the sum of the two background equations (2) for the denominator $\mathcal{H}^2 - \mathcal{H}'$ and the Hamiltonian constraint (6) for the numerator $\mathcal{H}\Phi' + \mathcal{H}^2\Phi$, allow ζ to be rewritten in the convenient form

$$\zeta_k = \frac{\rho_\sigma \delta_\sigma(k) - (3/4)(\rho_\sigma + p_\sigma)\delta_r(k)}{4\rho_r + 3(\rho_\sigma + p_\sigma)}. \tag{14}$$

This expression has been obtained by neglecting the contribution of spatial gradients in Eqs. (5)–(8). Numerical integration shows [20] that the corrections coming from these terms are indeed negligible for the large-scale modes leading to the anisotropies in the CMB.

Consider now the beginning of the post-big bang phase, when the radiation dominates the background while the axion dominates the fluctuations. In this case Eq. (14) gives immediately:

$$\zeta_k = \frac{1}{4}\,\frac{\rho_\sigma \delta_\sigma(k)}{\rho_r} = \frac{1}{4\rho_r}\frac{\partial V}{\partial \sigma}\chi_k = \frac{1}{24}\frac{a^2}{\mathcal{H}^2}\frac{\partial V}{\partial \sigma}\chi_k \propto a^4, \tag{15}$$

where we have used the fact that, in the initial phase, σ is approximately constant. Since also Φ_k will behave like a^4, it is easy to find its relation to ζ using, inside (13), $\Phi' = 4\mathcal{H}\Phi$ and $\mathcal{H}' = -\mathcal{H}^2$, with the result:

$$\Phi_k = -\frac{2}{7}\zeta_k = -\frac{1}{14}\,\frac{\rho_\sigma \delta_\sigma(k)}{\rho_r}. \tag{16}$$

In order to proceed further, two alternatives (already discussed in the context of the background evolution) should now be separately examined:

(1) If $\sigma_i < 1$, during the oscillating (but still radiation-dominated) phase, ζ can still be obtained from Eq. (14), but now $\rho_\sigma \sim \delta\rho_\sigma \sim a^{-3}$, and ζ will evolve like $a \sim \eta \sim t^{1/2}$. Since a changes by a factor $(m/H_\sigma)^{1/2} \simeq (\sigma_i)^{-2}$, we end up with a value of ζ_k at t_σ given by:

$$\zeta_k(t_\sigma) \sim \frac{\chi_k(t_i)}{\sigma_i}, \qquad\qquad \sigma_i \leq 1. \tag{17}$$

On the other hand, using again Eq. (13) and the appropriate relations in the oscillating, radiation-dominated phase, we find $\Phi_k(t_\sigma) = -\zeta_k(t_\sigma)/2$. In the final phase, dominated by an oscillating axion, ρ_r is negligible, the (average) axion pressure is zero, and (the average of) Φ_k is constant, as well as the average of ζ_k, which oscillates around a final amplitude of the same order as $\zeta_k(t_\sigma)$ given in eq. (17). This implies, through Eqs. (5), (6) and (10), $\delta_\sigma(k) = -2\Phi_k = -(1/2)\delta_r(k)$, so that, from Eq. (13) we are led to

$$\langle \zeta_k \rangle = -\frac{5}{3}\langle \Phi_k \rangle = \frac{5}{6}\langle \delta_\sigma(k) \rangle, \tag{18}$$

where $\langle ... \rangle$ refers to averages over one oscillation period. We have checked the validity of this result by an explicit numerical integration (the same result has already been presented in [15], using different notations).

(2) If $\sigma_i > 1$, then Eq. (14) can still be used until $t_\sigma = 1/m\sigma_i$, where we find:

$$\zeta_k(t_\sigma) = \frac{1}{4\rho_\sigma} \frac{\partial V}{\partial \sigma} \chi_k \sim \frac{\chi_k(t_i)}{\sigma_i}. \tag{19}$$

During the period of axion-dominated slow-roll inflation, Eq. (14) is still valid. However, since ρ_r soon becomes subdominant with respect to $\rho_\sigma + p_\sigma$, it should be appreciated that at the end of the slow-roll period the latter term is of order m^2, and the resulting estimate will thus be:

$$\zeta_k(t_m) = \frac{1}{4} \frac{\partial V}{\partial \sigma} \frac{\chi_k}{m^2} \sim \chi_k(t_i)\sigma_i, \qquad \sigma_i > 1. \tag{20}$$

Note that this formula is in (qualitative) agreement with Eq. (18), if we use $\delta\rho_\sigma \sim m^2 \chi_k(t_i)\sigma_i$ and $\rho_\sigma \sim m^2$. No further amplification is expected in the course of the subsequent cosmological evolution. Similar expressions hold for the amplitude of Φ_k, related to ζ_k by Eq. (18).

It is amusing to observe that the results (17), (20), which determine the amplitude of the Bardeen potential in the oscillating (axion-dominated) phase preceding the moment ($t = t_d$) at which the decay occurs, can be summarized by an equation that holds in all cases, namely

$$\langle \Phi_k(t_d) \rangle = -\chi_k(t_i) f(\sigma_i), \qquad f(\sigma_i) = \left(c_1 \sigma_i + \frac{c_2}{\sigma_i} + c_3 \right), \tag{21}$$

where c_1, c_2, c_3 are numerical coefficients of the order of unity. A preliminary fit based on numerical and analytical integrations of the perturbation equations gives $c_1 = 0.129, c_2 = 0.183, c_3 = 0.019$ (see [20] for further details). The function $f(\sigma_i)$ has the interesting feature that it is approximately invariant under the transformation $\sigma_i \to \sigma_i^{-1}$ and, as a consequence, has a *minimal* value around $\sigma_i = 1$, a result we shall use later on.

The generated spectrum of super-horizon curvature perturbations is thus directly determined by the primordial spectrum of isocurvature axion fluctuations χ_k, according to Eqs. (17) and (20). The axion fluctuations, on the other hand, are solutions (with pre-big bang initial conditions) of Eq. (8) in the radiation era (no additional amplification is expected, for super-horizon modes, in the axion-dominated phase), computed for negligible curvature perturbations ($\Phi = 0 = \Phi'$), evaluated in the massive, non-relativistic limit (where we are, eventually, in the oscillating regime) and outside the horizon. The exact solution for χ_k, normalized to a relativistic spectrum of quantum fluctuations (amplified with the Bogoliubov coefficient c_k) has already been computed in [10]. Setting $x = \eta\sqrt{2\alpha}$, $\alpha = mH_1 a_1^2$, $b = -k^2/2\alpha$, it can be written in the form

$$\chi_k = \frac{c_k}{a} \left(\frac{k}{2\alpha} \right)^{1/2} y_2(b, x), \tag{22}$$

where y_2 is the odd part of the parabolic cylinder functions [23]. Outside the horizon ($-bx^2 \ll 1$) and for non-relativistic modes ($-b \ll x^2$), the solution can be expanded, to leading order, as $y_2 \sim x = \eta\sqrt{2\alpha}$. By inserting a generic power-law spectrum, with cut-off scale $k_1 = H_1 a_1$ and spectral index n, i.e. $|c_k| = (k/k_1)^{(n-5)/2}$, we finally obtain the generated spectrum of curvature perturbations:

$$k^3 |\Phi_k|^2 = f^2(\sigma_i) k^3 |\chi_k|^2 = f^2(\sigma_i) \left(\frac{H_1}{M_{\rm P}}\right)^2 \left(\frac{k}{k_1}\right)^{n-1}, \qquad k < k_1 \tag{23}$$

(we have absorbed into the definition of k_1 possible numerical factors of order one connecting the cut-off scale to the string mass).

Note that we have re-inserted the appropriate Planck mass factors, keeping σ_i dimensionless. It may be useful to recall that the spectral index n depends upon the pre big-bang dynamics [8], and that for an isotropic 6-dimensional subspace it can be written in the form [13]

$$n = \frac{4 + 6r^2 - 2\sqrt{3 + 6r^2}}{1 + 3r^2}, \tag{24}$$

where $r = (\dot{V}_6 V_3)/(2V_6 \dot{V}_3)$ accounts for the relative rate of variation of the six-dimensional internal volume V_6 and of the "external" (usual) volume V_3. As already mentioned, the case of a flat spectrum (i.e. $n = 1$) corresponds to $r = \pm 1$. Otherwise, n increases monotonically with r^2 from the value $n = 4 - 2\sqrt{3} \simeq 0.53$ when internal dimensions are static ($r = 0$), to $n = 2$ for the case of a static external manifold ($r \to \infty$).

The result (23) is valid during the axion-dominated phase, and has to be transferred to the phase of standard evolution, by matching the (well-known [21]) solution for the Bardeen potential in the radiation era (subsequent to axion decay) to the solution prior to decay, which is in general oscillating. The matching of Φ and Φ', conventionally performed at the fixed scale $H = H_d$, shows that the constant asymptotic value (21) of super-horizon modes is preserved (to leading order) by the decay process, modulo a random, mass-dependent correction which typically takes the form $1 + \epsilon \sin(m/H_d)$, with ϵ a numerical coefficient of order 1, and $m/H_d \gg 1$. Such a random factor, however, is a consequence of the sudden approximation adopted to describe the decay process, and disappears in a more realistic treatment in which the axion equation (3) is supplemented by the friction term $+\Gamma\sigma'/a$ (leading to the term $+\Gamma\sigma'^2/a^2$ in the equation for ρ_σ), and a corresponding antifriction term $-\Gamma\sigma'^2/a^2$ in the radiation equation. The axion fluctuations will follow the background and decay with a similar term, $+\Gamma\chi'/a$, in the perturbation equation (8).

The previous analysis performed up to $t = t_d$ remains valid for the modified equations, since for $\Gamma \ll H$ the decay terms are negligible. We have checked with a numerical integration [20] that the decay process preserves the value of the Bardeen potential prior to decay, damping the residual oscillations; ζ itself follows the same behaviour and is finally exactly a constant. When the axion has completely decayed, and the Universe is again dominated by radiation, we can properly match the standard evolution of Φ in the radiation phase

to the constant asymptotic value of Eq. (21). The expression we obtain for the (oscillating) Bardeen potential, valid until the epoch of matter–radiation equality (denoted in the following by $\eta_{\rm eq}$), can be written in the form

$$\Phi_k(\eta) = -3\Phi_k(\eta_d)\Big[\frac{\cos{(kc_s\eta)}}{(kc_s\eta)^2} - \frac{\sin{(kc_s\eta)}}{(kc_s\eta)^3}\Big], \qquad \eta_d < \eta < \eta_{\rm eq}, \tag{25}$$

where $c_s = 1/\sqrt{3}$ and $\Phi_k(\eta_d)$ is given in Eq. (21).

The above expression for the Bardeen potential provides the initial condition for the evolution of the CMB-temperature fluctuations, and the formation of their oscillatory pattern. Standard results [24] (see also [25]) imply that the patterns of the CMB anisotropies (and, in particular, the position of the first Doppler peak) are related to the sum of two oscillating contributions, with a relative phase of $\pi/2$. Denoting by $\eta_{\rm dec}$ the decoupling time, the first contribution oscillates like $A\cos{[kr_s(\eta_{\rm dec})]}$, while the second one oscillates like $B\sin{[kr_s(\eta_{\rm dec})]}$, where $r_s(\eta_{\rm dec})$ is the sound-horizon at $\eta = \eta_{\rm dec}$. The value of Φ_k for $\eta \ll \eta_{\rm eq} < \eta_{\rm dec}$ determines, in particular, the relative phase of oscillation of the two terms. In our case, from Eq. (25), $\Phi_k(\eta_i) =$ constant and $\Phi_k'(\eta_i) \simeq 0$, where $\eta_d < \eta_i < \eta_{\rm eq}$, and $k\eta_i \ll 1$. This implies $B = 0$, so that the temperature anisotropies $(\Delta T/T)_k$ will oscillate like [24] $\Phi_k(\eta_i)\cos{[kr_s(\eta_{\rm dec})]}$, as is generally the case for adiabatic fluctuations. The opposite case, $\Phi_k(\eta_i) \simeq 0$ and $\Phi_k'(\eta_i) =$ constant, corresponds instead to isocurvature initial conditions [26], producing a peak structure that is clearly distinguishable from the adiabatic case and, at present, observationally disfavoured.

After checking that the above scenario leads to the standard adiabatic mode, producing the observed peak structure of the CMB anisotropies, we still have to discuss the possibility of a correct large-scale normalization of the spectrum, compatible with the COBE data. We start from the observation that the final amplitude of the super-horizon perturbations (23), just like the spectral slope, is not at all affected by the non-relativistic corrections to the axion spectrum [12], in spite of the crucial role played by the mass in the decay process (see also [14]). The mass dependence reappears, however, when computing the amplitude of the spectrum at the present horizon scale ω_0, in order to impose the corrected normalization to the quadrupole coefficient C_2 determined by COBE, namely [27]

$$C_2 = \alpha_n^2 f^2(\sigma_i)\left(\frac{H_1}{M_{\rm P}}\right)^2\left(\frac{\omega_0}{\omega_1}\right)^{n-1}, \qquad \alpha_n^2 = \frac{2^n}{72}\frac{\Gamma(3-n)\Gamma\left(\frac{3+n}{2}\right)}{\Gamma^2\left(\frac{4-n}{2}\right)\Gamma\left(\frac{9-n}{2}\right)}, \tag{26}$$

where [28] $C_2 = (1.9 \pm 0.23)\times 10^{-10}$.

The present value of the cut-off frequency, $\omega_1(t_0) = H_1 a_1/a_0$, depends in fact on the kinematics as well as on the duration of the axion-dominated phase (and thus on the axion mass), as follows:

$$\omega_1(t_0) \;=\; H_1\left(\frac{a_1}{a_\sigma}\right)_{\rm rad}\left(\frac{a_\sigma}{a_d}\right)_{\rm mat}\left(\frac{a_d}{a_{\rm eq}}\right)_{\rm rad}\left(\frac{a_{\rm eq}}{a_0}\right)_{\rm mat}, \qquad \sigma_i < 1, \tag{27}$$

$$= H_1 \left(\frac{a_1}{a_\sigma}\right)_{\rm rad} \left(\frac{a_\sigma}{a_{\rm osc}}\right)_{\rm inf} \left(\frac{a_{\rm osc}}{a_d}\right)_{\rm mat} \left(\frac{a_d}{a_{\rm eq}}\right)_{\rm rad} \left(\frac{a_{\rm eq}}{a_0}\right)_{\rm mat}, \quad \sigma_i > 1. \tag{28}$$

Using $H_0 \simeq 10^{-6} H_{\rm eq} \simeq 10^{-61} M_{\rm P}$ we find

$$\frac{\omega_0}{\omega_1} \simeq 10^{-29} \left(\frac{H_1}{M_{\rm P}}\right)^{-1/2} \left(\frac{m}{\sigma_i^2 M_{\rm P}}\right)^{-1/3}, \qquad \sigma_i < 1, \tag{29}$$

$$\simeq 10^{-29} \left(\frac{\sigma_i H_1}{M_{\rm P}}\right)^{-1/2} \left(\frac{m}{M_{\rm P}}\right)^{-1/3} Z_\sigma, \qquad \sigma_i > 1, \tag{30}$$

where $Z_\sigma = (a_{\rm osc}/a_\sigma)$ denotes the amplification of the scale factor during the phase of axion-dominated, slow-roll inflation. The COBE normalization thus imposes

$$c_2^2 \alpha_n^2 \sigma_i^{2(n-4)/3} \left(\frac{H_1}{M_{\rm P}}\right)^{(5-n)/2} \left(\frac{m}{M_{\rm P}}\right)^{-(n-1)/3} 10^{-29(n-1)} \simeq 10^{-10}, \qquad \sigma_i < 1, \tag{31}$$

$$c_1^2 \alpha_n^2 Z_\sigma^{n-1} \sigma_i^{(5-n)/2} \left(\frac{H_1}{M_{\rm P}}\right)^{(5-n)/2} \left(\frac{m}{M_{\rm P}}\right)^{-(n-1)/3} 10^{-29(n-1)} \simeq 10^{-10}, \quad \sigma_i > 1. \tag{32}$$

We can notice, as a side remark, that the contribution of the gradients appearing in Eqs. (5)–(8) follows the same hierarchy of scales as provided by Eqs. (29), (30) and this is the reason why, ultimately, the contribution of the gradients can be neglected as far as the evolution of large-scale modes is concerned.

The condition (31) is to be combined with the constraint (11), the condition (32) with the constraint (12), which are required for the consistency of the corresponding classes of background evolution. Also, both conditions are to be intersected with the experimentally allowed range of the spectral index. Finally, in the case $\sigma_i > 1$ we are also implicitly assuming that the axion-driven inflation is short enough to avoid a possible contribution to C_2 arising from the metric fluctuations directly amplified from the vacuum, during the phase of axionic inflation. This requires that the smallest amplified frequency mode ω_σ, crossing the horizon at the beginning of inflation, be today still larger than the present horizon scale ω_0. This imposes the condition $\omega_\sigma(t_0) = H_\sigma(a_\sigma/a_0) > \omega_0$, namely

$$Z_\sigma \lesssim 10^{29} \sigma_i \left(\frac{m}{M_{\rm P}}\right)^{5/6}, \tag{33}$$

to be added to the constraint (12) for $\sigma_i > 1$. It turns out, however, that this condition is always automatically satisfied for the range of spectral indices we are interested in (in particular, for $n \leq 1.7$).

The allowed range of parameters compatible with all constraints is rather strongly sensitive to the values of the pre-big bang inflation scale H_1. In the context of minimal models of pre-big bang inflation [3] we have $H_1 \sim M_s$, and a flat spectrum ($n = 1$) is inconsistent with the normalization (31), (32). A growing ("blue") spectrum is instead allowed, and by setting for instance $c_2 \alpha_n H_1/M_{\rm P} = 10^{-2}$, using (as a reference value) the upper bound [29]

$n < 1.4$, and considering the case $\sigma_i \leq 1$, we find a wide range of allowed axion masses, but a rather narrow range of allowed values for σ_i, namely $1 \gtrsim \sigma_i \gtrsim 10^{-5/2}$, and of allowed values for the spectral index, $n \simeq 1.22$–1.4. In the case $\sigma_i > 1$ the results are complementary for the spectral index, but there are much more stringent bounds for σ_i, because the inflationary redshift factor Z_σ grows exponentially with σ_i^2, in such a way that the COBE normalization (32) cannot be satisfied, unless the upper value of σ_i is strongly bounded. This means that the apparent symmetry between the $\sigma_i < 1$ and the $\sigma_i > 1$ cases is broken by the requirement of the CMB normalization, which forbids too large values of σ_i.

The allowed region may be further extended if the inflation scale H_1 is lowered, and a flat ($n = 1$) spectrum may become possible if $c_2 \alpha_n H_1 \lesssim 10^{-5} M_{\rm P} \sigma_i$, for $\sigma_i < 1$, and if $c_1 \alpha_n H_1 \lesssim 10^{-5} M_{\rm P}/\sigma_i$, for $\sigma_i > 1$ (see Eqs. (31) and (32)). This possibility could arise in a recently proposed framework [30] according to which, at strong bare coupling e^ϕ, loop effects renormalize downwards the ratio $M_s/M_{\rm P}$ and allow M_s to approach the unification scale. In addition, a flat spectrum may be allowed even keeping pre-big bang inflation at a high-curvature scale, provided the relativistic branch of the primordial axion fluctuations is characterized by a frequency-dependent slope, which is flat enough at low frequency (to agree with large-scale observations) and much steeper at high frequencies (to match the string normalization at the end-point of the spectrum).

A typical example of such a spectrum can be parametrized by a Bogoliubov coefficient with a break at the intermediate scale k_s,

$$\begin{aligned} |c_k|^2 &= \left(\frac{k}{k_1}\right)^{n-5+\delta}, \qquad\qquad k_s < k < k_1, \\ &= \left(\frac{k_s}{k_1}\right)^{n-5+\delta} \left(\frac{k}{k_s}\right)^{n-5}, \qquad k < k_s, \end{aligned} \tag{34}$$

where $\delta > 0$ parametrizes the slope of the break at high frequency. Examples of realistic pre-big bang backgrounds producing such a spectrum of axion fluctuations have been already presented in [12]. Furthermore, a steeper axion spectrum at high frequency could also emerge if the exit from pre-big bang inflation occurred at relatively strong bare coupling, where various quantities may become dilaton-independent as argued in [30], and the renormalized axion pump field should approach the canonical pump field of metric perturbations. Quite independently of the effective mechanism, it is clear that the steeper and/or the longer the high-frequency branch of the spectrum, the larger the suppression at low-frequency scales, and the easier the matching of the amplitude to the measured anisotropies (in spite of possible σ_i-dependent enhancements).

Using the generalized input (34) for the spectrum of χ_k, the amplitude of the low-frequency ($k < k_s$) Bardeen spectrum (23) is to be multiplied by the suppression factor $\Delta = (k_s/k_1)^\delta \ll 1$, and the normalization condition at the COBE scale becomes

$$\alpha_n^2 c_2^2 \left(\frac{H_1}{\sigma_i M_{\rm P}}\right)^2 \left(\frac{\omega_0}{\omega_1}\right)^{n-1} \simeq C_2 \Delta^{-1}, \qquad \sigma_i < 1, \tag{35}$$

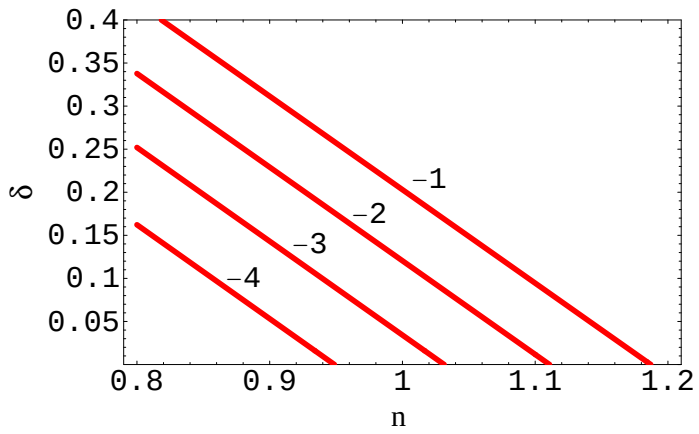

Figure 1: *Plot of the COBE normalization condition for* $\sigma_i = 1$, $f(1) = 0.33$, $m = 10^{10}$ *GeV*, $k_1/k_s = k_1/k_{\rm eq} \simeq 10^{27}(H_1/M_{\rm P})^{1/2}(m/M_{\rm P})^{1/3}$, *and for various values of the inflation scale* H_1. *The four curves correspond, from left to right respectively, to* $\log(H_1/M_{\rm P}) = -4, -3, -2, -1$.

$$\alpha_n^2 c_1^2 \left(\frac{\sigma_i H_1}{M_{\rm P}}\right)^2 \left(\frac{\omega_0}{\omega_1}\right)^{n-1} \simeq C_2 \Delta^{-1}, \qquad \sigma_i > 1. \tag{36}$$

A strictly flat spectrum is now possible, even for $\alpha_1 H_1 = \alpha_1 M_s \simeq 10^{-2} M_{\rm P}$, provided

$$\Delta \left(c_1^2 \sigma_i^2 + c_2^2 \sigma_i^{-2}\right) \lesssim 10^{-6}. \tag{37}$$

It thus becomes possible, in this context, to satisfy the stringent limits imposed by the most recent analyses of the peak and dip structure of the spectrum at small scales [31], which imply $0.87 \leq n \leq 1.06$ (see also [32]).

In order to illustrate this possibility, let us specify further Eq. (34) by identifying k_s with the scale $k_{\rm eq}$ of matter–radiation equivalence, in such a way that n will denote the value of the axion spectral index for the scales relevant to CMB anisotropies, while $n+\delta$ provides the (average) axion spectral index in the remaining range of scales, up to the cutoff k_1. Then, after imposing the COBE normalization condition $\alpha_n^2 f^2(\sigma_i)(H_1/M_{\rm P})^2(\omega_0/\omega_1)^2 = C_2$, we plot in Fig. 1 curves corresponding to some given values of the ratio $H_1/M_{\rm P} \sim M_s/M_{\rm P}$. We have done this choosing the values $\sigma_i = 1$ and $m = 10^{10}$ GeV, but for n around 1 the curves are very stable, even if we change m by many orders of magnitude, provided we stay at σ_i of order 1 (i.e. near the minimum of f). A look at the figure shows immediately that the phenomenologically allowed range for n is theoretically consistent even for $M_s/M_{\rm P}$ as large as 0.1, provided we allow for a small break in the spectrum, $\delta \simeq 0.2$. Conversely, we can allow having no break at all in the spectrum ($\delta = 0$), if we are willing to take $M_s/M_{\rm P} \sim 10^{-3}$, i.e. a string mass close to the GUT scale.

We conclude that, in the context of the pre-big bang scenario, a "curvaton" model based on the Kalb–Ramond axion is able to produce the adiabatic curvature perturbation needed

to explain the observed large-scale anisotropies. The simplest, minimal model of pre-big bang inflation seems to prefer blue spectra. A strictly scale-invariant (or even slightly red, $n < 1$) spectrum is not excluded but requires, for normalization purposes, non-minimal models of pre-big bang evolution leading to axion fluctuations with a sufficiently steep slope at high frequencies.

References

[1] J. E. Lidsey, D. Wands and E. J. Copeland, Phys. Rep. **337**, 343 (2000).

[2] M. Gasperini and G. Veneziano, CERN-TH/2002-104, to appear.

[3] M. Gasperini and G. Veneziano, Phys. Rev. **D50**, 2519 (1994); R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov and G. Veneziano, Phys. Rev. **D51**, 6744 (1995).

[4] G. F. Smooth et al., Astrophys. J. **396**, 1 (1992); C. L. Bennet et al., Astrophys. J. **430**, 423 (1994).

[5] P. de Bernardis et al., Nature **404**, 955 (2000); S. Hanay et al., Astrophys. Lett. **545**, 5 (2000).

[6] F. Finelli and R. Brandenberger, hep-th/0112249; R. Durrer and F. Vernizzi, hep-ph/0203275.

[7] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory* (Cambridge Univ. Press, Cambridge, 1987).

[8] E. J. Copeland, R. Easther and D. Wands, Phys. Rev. **D56**, 874 (1997); E. J. Copeland, J. E. Lidsey and D. Wands, Nucl. Phys. **B506**, 407 (1997).

[9] R. Durrer, M. Gasperini, M. Sakellariadou and G. Veneziano, Phys. Lett. **B436**, 66 (1998).

[10] R. Durrer, M. Gasperini, M. Sakellariadou and G. Veneziano, Phys. Rev. **D59**, 43511 (1999).

[11] R. Durrer, Phys. Rev. **D42**, 2533 (1990).

[12] M. Gasperini and G. Veneziano, Phys. Rev. **D59**, 43503 (1999).

[13] A. Melchiorri, F. Vernizzi, R. Durrer and G. Veneziano, Phys. Rev. Lett. **83**, 4464 (1999); F. Vernizzi, A. Melchiorri and R. Durrer, Phys. Rev. **D63**, 063501 (2001).

[14] K. Enqvist and M. S. Sloth, Nucl. Phys. **B626**, 395 (2002).

[15] D. H. Lyth and D. Wands, Phys. Lett. **B524**, 5 (2002).

[16] T. Moroi and T. Takahashi, Phys. Lett. **B522**, 215 (2001); T. Moroi and T. Takahashi, hep-ph/0206026.

[17] N. Bartolo and A. R. Liddle, astro-ph/0203076.

[18] A. Notari and A. Riotto, hep-th/0205019.

[19] S. Mollerach, Phys. Rev. **D42**, 313 (1990).

[20] V. Bozza, M. Gasperini, M. Giovannini and G. Veneziano, to appear.

[21] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

[22] J. Ellis, D. V. Nanopoulos and M. Quiros, Phys. Lett. **B174**, 176 (1986); J. Ellis, C. Tsamish and M. Voloshin, Phys. Lett. **B194**, 291 (1987).

[23] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions* (Dover, New York, 1972).

[24] W. Hu and N. Sugiyama, Astrophys. J. **444**, 489 (1995); *ibid.* **471**, 542 (1996); Phys. Rev. **D51**, 2599 (1995).

[25] S. Weinberg, Phys. Rev. **D64**, 123511 and 123512 (2001).

[26] H. Kodama and M. Sasaki, Int. J. Mod. Phys. **A1**, 265 (1986).

[27] R. Durrer, astro-ph/0109522.

[28] A. J. Banday et al., Astrophys. J. **475**, 393 (1997).

[29] C. L. Bennet et al., Astrophys. Lett. **464**, L1 (1996).

[30] G. Veneziano, hep-th/0110129.

[31] P. de Bernardis et al., Astrophys. J. **564**, 559 (2002).

[32] X. Wang, M. Tegmark and M. Zaldarriaga, Phys. Rev. **D65**, 123001 (2002).

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General solution for scalar perturbations in bouncing cosmologies

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Abstract. Bouncing cosmologies, suggested by String/M-theory, may provide an alternative to standard inflation to account for the origin of inhomogeneities in our universe. The fundamental question regards the correct way to evolve the scalar perturbations through the bounce. In this work, we determine the evolution of perturbations and the final spectrum for an arbitrary (spatially flat) bouncing cosmology, with the only assumption that the bounce is governed by a single physical scale. In particular, we find that the spectrum of the pre-bounce growing mode of the Bardeen potential (which is scale-invariant in some limit, and thus compatible with observations) survives unaltered in the post-bounce only if the comoving pressure perturbation is directly proportional to the Bardeen potential rather than its Laplacian, as for any known form of ordinary matter. If some new physics acting at the bounce justifies such relation, then bouncing cosmologies are entitled to become a real viable alternative for the generation of the observed inhomogeneities. Our treatment also includes some class of models with extra-dimensions, whereas we show that bounces induced by positive spatial curvature are structurally different from all bounces in spatially flat universes, requiring a distinct analysis.

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1. Introduction

The occurrence of singularities in classical General Relativity (GR) may be the symptom of the existence of a more general physical frame, where GR is embedded as a particular low curvature limit. In such a frame, that we will generically call "new physics", one may expect that singularities are cured by the existence of a natural physical cutoff, i.e. a scale where the new physics comes into play, replacing GR. Presently, the only consistent example of such a general theory is String theory, or its 11-dimensional generalization M-theory, where this cutoff is naturally provided by the string length l_s.

Following this reasoning line, in a cosmological context, it is natural to expect that the new physics would prevent the Big Bang singularity predicted by GR, allowing a continuation of the cosmological solutions before the Big Bang. This idea, firstly proposed by Veneziano and Gasperini [1], has found several alternative implementations and recurrently emerges in independent attempts to build consistent cosmological models from String-M/theories [2, 3, 4, 5]. All these models, when formulated in the Einstein frame, share the same general behavior: the universe starts from an asymptotically flat state and undergoes a contraction that pumps the space-time curvature up to the new physics scale. At this point, the new physics masters the transition to an expanding phase and the standard Friedman-Robertson-Walker (FRW) universe begins. In the present literature, such models are generically called *bouncing cosmologies*.

The interest in bouncing cosmological models comes from the fact that an accelerated contraction seems to solve the flatness and horizon problems as efficiently as a standard inflationary phase. Then the question opens whether a bouncing cosmology may replace standard inflation once and for all. In order to achieve this fascinating objective, a bouncing cosmology should provide its own mechanism of generation of the inhomogeneities of the universe, in such a way that all present constraints are satisfied at the same accuracy level of standard inflation. The most stringent observational constraint comes from CMB anisotropies, which require the initial power spectrum of the Bardeen potential (encoding the information on all scalar cosmological perturbations) to be (nearly) scale-invariant. Is this obtainable or not in a bouncing cosmology? The first studies in the Pre-Big Bang scenario gave a negative answer [6], indicating that the inhomogeneities must have an independent origin. For example, they can be generated in an axion-dominated era through the curvaton mechanism [7].

On the other hand, the authors of the Ekpyrotic/cyclic scenario [3] claimed that a scale-invariant spectrum can be generated in a slow contraction era and transmitted across the bounce. This has risen a considerable debate [8, 9] about the correct way to describe the evolution of perturbations through a bounce, but, after several years of investigation, it seems quite difficult to draw a general conclusion, valid for all models of bouncing cosmologies.

The evolution of perturbations in a bouncing cosmology can be summarized very quickly as follows: in the pre-bounce phase, the Bardeen potential has a growing

mode, endowed with a red spectrum (becoming scale-invariant in the limit of very slow contraction [3]) and a decaying mode with a blue spectrum (becoming scale-invariant in the limit of dust-dominated contraction [4]). After the bounce, we have two alternatives: either the spectrum of the growing mode is inherited by a constant mode in the post-bounce or the growing mode simply decays in the post-bounce and some other constant mode with a more blue spectrum dominates at horizon re-entry.

If the first possibility occurs, then the Ekpyrotic/cyclic models would become a complete alternative to standard inflation. Otherwise, a curvaton mechanism or some other one must be implemented to make such models compatible with observations. All studies of specific (spatially flat) toy models, where perturbations are explicitly calculable, show that the pre-bounce growing mode decays in the post-bounce and the Bardeen potential is finally dominated by a blue spectrum, incompatible with observations [10, 11, 12, 13] (the incorrect conclusions reached in Refs. [14, 15] are revised in Ref. [13] and reconciled with all other studies; also the model presented in Ref. [16] is a particular case of the general model presented in Ref. [13], with two perfect fluids with $w_a = 1/3$ and $w_b = 5/3$). However, the investigation of specific models does not allow to infer any general conclusions, since the possibility that some still unknown new physics may intervene at the bounce, producing different effects, remains open. Indeed, bounces induced by positive spatial curvature show a different behavior, with a generic mixing between growing and decaying modes [17, 18].

The description of the evolution of scalar perturbations through a bounce has acquired an independent interest that goes beyond the Ekpyrotic/cyclic models, thanks to the mathematical complexity of the problem, which poses several questions and issues of general theoretical interest.

In this work we present a completely analytical study of scalar perturbations in spatially flat bouncing cosmologies, making the only assumption that the bounce is governed by a single physical scale. In this way, we are able to establish a univocal relation between the final spectrum of perturbations and the form of the total energy-momentum tensor. We thus introduce a sort of golden rule for scalar perturbations in bouncing cosmologies.

In Sect. 2 we define the generic bouncing cosmology, stating the hypotheses that we need to deduce our general result. In Sect. 3 we elaborate our hypotheses, describing the background cosmology. In Sect. 4 we write the linear perturbation equations. In Sect. 5 we derive the general form of the effective energy-momentum tensor, including the high energy corrections driving the bounce. In Sect. 6 we compute the evolution of perturbations in the pre-bounce, in the bounce and in the post-bounce. In Sect. 7 we illustrate the general results of Sect. 6 by some numerical examples. In Sect. 8 we consider possible extensions to models with extra-dimensions and analyze the differences between bounces in spatially flat universes and bounces in spatially closed universes. Sect. 9 contains the conclusions.

2. The generic bouncing cosmology

Let us start by enumerating our hypotheses:

(i) We assume that it makes sense to define a four-dimensional metric tensor, both in the GR regime and in the new physics regime. Then it is possible to write the four-dimensional Einstein equations with an effective energy-momentum tensor on the right hand side, encoding all the corrections dictated by the new physics.

(ii) The universe is homogeneous, isotropic and spatially flat, thus being described by the FRW metric.

(iii) There is a unique physical scale η_B governing the bounce. We will set our units in such a way that $c = 1$, $8\pi G = 1$, $\eta_B = 1$.

(iv) In the GR regime far from the bounce ($|\eta| \gg 1$), the energy momentum tensor is characterized by a constant equation of state and a constant speed of sound, with $p/\rho > -1/3$.

In Sect. 8, we will comment on models with extra-dimensions which may fit into hypothesis (i) and discuss the differences with positive curvature bounces.

Now let us work out these hypotheses. The metric, including scalar perturbations, is

$$ds^2 = a^2(\eta)\left\{(1+2\phi)d\eta^2 - 2B_{,i}d\eta dx^i - [(1-2\psi)\delta_{ij} + 2E_{,ij}]\,dx^i dx^j\right\}, \quad (1)$$

where we have used the conformal time η. The scale factor is $a(\eta)$, while B, E, ψ and ϕ are scalar perturbations.

The total energy–momentum tensor reads

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho + \delta\rho & -(\rho+p)\mathcal{V}_{,i} \\ (\rho+p)\mathcal{V}_{,i} & -(p+\delta p)\delta_{ij} - \xi_{,ij} \end{pmatrix}, \quad (2)$$

where $\delta\rho$ is the energy density perturbation, δp is the pressure perturbation, $\mathcal{V}$ is the velocity potential and ξ is the anisotropic stress. Note that these expressions do not require GR, but are just dictated by the symmetry of the problem (hypothesis (ii)). Thus, also in the bounce regime, the new physics corrections encoded in $T^{\mu}{}_{\nu}$ will respect this symmetry and contribute to its components, both at the background and at the perturbative level. Since this tensor stays on the right hand side of the Einstein equations, it must be compatible with the Bianchi identities, thus it must be covariantly conserved: $T^{\mu}{}_{\nu;\mu} = 0$.

The independent background Einstein equations and the energy-momentum conservation give the Friedman equations

$$3\mathcal{H}^2 = a^2\rho \quad (3)$$

$$\mathcal{H}^2 + 2\mathcal{H}' = -a^2 p \quad (4)$$

$$\rho' + 3\mathcal{H}(\rho + p) = 0, \quad (5)$$

where the prime denotes derivative w.r.t. η and $\mathcal{H} = a'/a$ is the conformal Hubble rate.

In a spatially flat universe the Null Energy Condition (NEC), which amounts to require $\rho + p > 0$, must be necessarily violated during the bounce. Of course, this violation cannot be achieved by ordinary matter, but requires the intervention of the new physics. One may object that this implies the presence of ghosts at the quantum level. While the majority of explicit bouncing cosmologies in the literature indeed contain ghosts, there are at least two counterexamples. The first is the model by Gasperini, Giovannini and Veneziano [11], where the NEC violation is performed by a potential for the scalar field involving integration on the spatial volume of the universe. The second is the model by Biswas, Mazumdar and Siegel [5], where the generalized gravitational action is an analytic function of the D'Alembert operator acting on the curvature scalar without poles. In both cases there are no ghosts in the theory. In any case, the possible existence of ghosts for a finite time interval may not be a problem at all [19].

3. Background

Now let us determine the background evolution far from the bounce, using hypothesis (iv). In the pre-bounce we have $p = w_-\rho$ and in the post-bounce $p = w_+\rho$, with constant w_- and w_+. The restriction $w_+ > -1/3$ excludes the possibility of a post-bounce inflation, which would spoil the results of the bounce. On the other hand, a pre-bounce deflation, excluded by the restriction $w_- > -1/3$, would make the setting of initial conditions in the asymptotic past problematic.

We set $\eta = 0$ at the bounce, defined as the time where $a'(\eta) = 0$. The bounce duration scale must be of the same order of the curvature scale, according to hypothesis (iii). Otherwise, there would be two independent scales. So, the bounce will last from $\eta_- \simeq -1$ to $\eta_+ \simeq 1$.

Then, the pre-bounce background functions have the power law behavior

$$a(\eta) \simeq |\eta|^{q_-} \tag{6}$$

$$\mathcal{H} \simeq \frac{q_-}{\eta} \tag{7}$$

$$\rho_a \simeq \frac{3q_-^2}{|\eta|^{2+2q_-}} \tag{8}$$

$$q_- = \frac{2}{1+3w_-}, \tag{9}$$

where we have chosen all constant of integration in such a way that the space-time curvature at the onset of the bounce $\eta \sim -1$ is of order 1.

In the same way, we can write the post-bounce behavior of the background functions, which will be analogous to Eqs. (6)-(9) with all "$-$" in the subscripts replaced by "$+$".

4. Perturbations equations

We will write the perturbations equations in terms of the following gauge invariant variables

$$\Psi = \psi + \mathcal{H}(E' - B) \tag{10}$$
$$\zeta = \psi + \mathcal{H}\mathcal{V} \tag{11}$$
$$\delta\rho_{\mathcal{V}} = \delta\rho - \rho'\mathcal{V} \tag{12}$$
$$\delta p_{\mathcal{V}} = \delta p - p'\mathcal{V}, \tag{13}$$

plus the anisotropic stress ξ. Ψ is the Bardeen potential, ζ, $\delta\rho_{\mathcal{V}}$ and $\delta p_{\mathcal{V}}$ are respectively the curvature, energy density and pressure perturbations on comoving hypersurfaces. In the following we will drop the subscript "$\mathcal{V}$" from all variables. Then, from now on, $\delta\rho$ and δp must be intended as comoving energy density and pressure perturbations.

Using the component $(0i)$ of the Einstein equations to eliminate ϕ, the (00), (ij) and (ii) components become

$$\delta\rho = \frac{2}{a^2}\nabla^2\Psi \tag{14}$$
$$\delta p - \nabla^2\xi = -\frac{2(\mathcal{H}^2 - \mathcal{H}')}{a^2\mathcal{H}}\zeta' \tag{15}$$
$$a^2\mathcal{H}\xi = -\frac{\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}}\zeta + \frac{2\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}}\Psi + \Psi'. \tag{16}$$

Some of the coefficients of these equations become singular at the bounce ($\mathcal{H} = 0$) or at the NEC violation time ($\mathcal{H}^2 - \mathcal{H}' = 0$). In order to check the regularity of our variables, we may resort to the regular gauge argument used in Ref. [13]. If a gauge exists where all components of the metric and energy-momentum tensor stay finite, then we can explicitly check the regularity of our gauge invariant variables, expressing them in this gauge. As shown in [13], Ψ is regular while ζ is not. However, we may construct $\tilde{\zeta} = (\mathcal{H}^2 - \mathcal{H}')\zeta$, where the extra factor compensates the possible divergence of $\mathcal{V}$ at the NEC violation, allowed by the fact that it is the combination $(\rho + p)\mathcal{V}$ that appears in the energy-momentum tensor and is guaranteed to be regular by the regular gauge hypothesis. $\delta\rho_{\mathcal{V}}$ has no divergences (since $\rho' = 0$ at the NEC violation time, thanks to the continuity equation), while for $\delta p_{\mathcal{V}}$ we should use the regularized variable $\delta\tilde{p}_{\mathcal{V}} = (\mathcal{H}^2 - \mathcal{H}')\delta p_{\mathcal{V}}$. Dealing with regularized variables is absolutely necessary in all numerical applications, because no integrator can step through a divergence. However, for our analytical arguments, it is completely equivalent to discuss the equations (14)-(16) or their regularized counterparts. We choose the unregularized variables only because they have an immediate physical interpretation, but we would get exactly the same results with the regularized variables.

Going back to our equations, Eq. (14) represents the analogous of the Poisson equation, relating the Laplacian of the gravitational potential to the perturbations in the energy density (this is not a dynamical equation). Eq. (15) is a first order equation

in ζ, which depends on the sources δp and ξ. The anisotropic stress also enters as a source in the first order equation for Ψ (16). It is easy to verify that the perturbations of the continuity equations give no new independent equations.

All the dynamics of scalar perturbations is contained in these three equations. In order to solve them, we need to specify the sources δp and ξ for the energy-momentum content of our universe. This is easy for the pre-bounce and post-bounce phases. In fact, if they are dominated by a perfect fluid or a scalar field with some potential, the sources are

$$\delta p = c_s^2 \delta\rho; \qquad \xi = 0, \tag{17}$$

with s being "$-$" or "$+$" in the pre-bounce and post-bounce respectively. If the squared speed of sound c_s^2 is equal to w_s, we have a perfect fluid, otherwise we have a fluid with some intrinsic isocurvature mode [13]. In particular, a scalar field with a potential has always $c_s^2 = 1$, whatever the equation of state induced by the presence of the potential. In any case, with these sources, Eqs. (14)-(16) can be combined into a second order equation for ζ or Ψ, suitable for vacuum normalization. Expanding the variables in Fourier modes of wavenumber k as usual, in the asymptotic past we have

$$\zeta = C \frac{\mathcal{H}}{a\sqrt{\mathcal{H}^2 - \mathcal{H}'}} \sqrt{|\eta|} H_\nu^{(1)}(c_- k|\eta|), \tag{18}$$

where C is a normalization constant of order 1, $H_\nu^{(1)}$ is the Hankel function of the first kind and $\nu = 1/2 - q_-$.

We are interested in modes that are well outside the horizon at the bounce, thus having $k \ll 1$. For such modes, in the range $-1/k < \eta < -1$, we can use the expansion of the Hankel functions for small arguments and Eq. (15) to determine Ψ. Discarding all numerical factors of order one, we have two modes for each variable

$$\zeta \sim k^\nu |\eta|^{1-2q_-} + k^{-\nu} \tag{19}$$

$$\Psi \sim k^{\nu-2} |\eta|^{-1-2q_-} + k^{-\nu}. \tag{20}$$

Ψ is characterized by a growing mode with a red spectrum (becoming scale-invariant in the limit $q_- \to 0$) and a constant mode with a blue spectrum. ζ is characterized by a mode that is growing for $q_- > 1/2$ and decaying otherwise. This mode has a spectrum that is blue for $q_- < 2$ and red otherwise, being scale-invariant in the limit of dust-dominated pre-bounce [4]. The second mode is a blue constant mode.

5. General form of the energy-momentum tensor of perturbations

In order to discuss the evolution of perturbations through the bounce, we need to specify the form of the sources. In the ignorance of the new physics governing the bounce, we can still select all admissible forms of the pressure perturbation and the anisotropic stress and make a complete discussion of what can be expected at the end of the bounce.

Firstly, let us deal with the pressure perturbation, leaving aside the anisotropic stress for the moment.

The pressure perturbation can be a generic linear combination of $\delta\rho$, Ψ, ζ and a certain number n of additional internal variables $\delta\varphi_i$, with time depending coefficients. Of course, the pressure perturbation may also depend on the time and space derivatives of these variables. Now, through Eq. (14), any dependence on $\delta\rho$ can be read as an additional dependence on Ψ and its derivatives.

For each internal variable it is necessary to specify an independent linear differential equation in order to close the system. We are not interested in any new spectral dependence whose origin is different from vacuum normalization of the original scalar degree of freedom. Indeed, if such spectral dependences are carried by these internal variables, they may play an important role, giving rise to additional isocurvature modes in the post-bounce, as in the model discussed in Ref. [13]. They depend on the specific model and must be carefully considered in any complete treatment. However, our question is more general and only involves the fate of the original spectrum of Ψ and ζ. The internal variables may acquire this spectrum only if ζ, Ψ and their derivatives explicitly appear in their equations of motion. But then their feedback on δp would be an effective renormalization of the coefficients of ζ, Ψ and their derivatives in the expression of δp.

δp is then just a linear combination of Ψ, ζ and their derivatives with time-dependent coefficients. However, δp is scalar under spatial rotations and thus it must contain only scalars. The time derivatives of scalars are still scalars under spatial rotation, but the spatial derivatives must be combined with themselves or other vectors of the theory in order to produce scalars. More specifically, the spatial derivatives can only appear as ∇^2 or $A^i\partial_i$, where A^i is a generic 3-vector of the theory. However, since the background is homogeneous and isotropic, all 3-vectors in the background must be zero. So, spatial derivatives can only appear in a Laplacian or a function of the Laplacian.

Summing up, the most general expression for δp is

$$\delta p = \frac{2}{a^2}\sum_{n=0}^{\infty} F_n(\eta,\partial_\eta)k^{2n}\zeta + \frac{2}{a^2}\sum_{n=0}^{\infty} G_n(\eta,\partial_\eta)k^{2n}\Psi, \tag{21}$$

where we have supposed that the dependence on the Laplacians is analytic and allows a regular power expansion and we have introduced the factor $2/a^2$ for later convenience. Using the Fourier mode expansion, the Laplacians have been written as $-k^2$. With the same arguments, a similar expression should also hold for the anisotropic stress

$$\xi = \frac{2}{a^2}\sum_{n=0}^{\infty} J_n(\eta,\partial_\eta)k^{2n}\zeta + \frac{2}{a^2}\sum_{n=0}^{\infty} K_n(\eta,\partial_\eta)k^{2n}\Psi. \tag{22}$$

Of course, any possible crossed dependence between δp and ξ can be easily solved at the linear level.

Then the physics of the model is completely encoded in the time-dependent operators $F_n(\eta,\partial_\eta)$, $G_n(\eta,\partial_\eta)$, $J_n(\eta,\partial_\eta)$, $K_n(\eta,\partial_\eta)$. Let us make some examples.

For a single perfect fluid or a scalar field, as mentioned before, the only non-vanishing operator is $G_1 = -c_s^2$. In the bounce induced by two perfect fluids or scalar

fields, studied in Ref. [13] and containing the models by Peter & Pinto-Neto [14], Finelli [15], and Allen & Wands [12], two internal variables are present. Taking into account their dependence on ζ and Ψ, the final expression for the effective energy-momentum tensor only contains two non-vanishing coefficients: F_0, and G_1, i.e. the pressure perturbation only depends on ζ and $k^2\Psi$, while the anisotropic stress is zero. The model by Gasperini, Giovannini and Veneziano [11] is simpler, since it contains no internal variables and the only non-vanishing operator is $G_1 = 4(\mathcal{H}' - \mathcal{H}^2)/\varphi'^2$, where φ is the dilaton field in their model. Cartier's model [10] has a very involved effective energy-momentum tensor with a non-vanishing anisotropic stress. Using the expressions in the appendix of his work, we find that the covariant conservation is fulfilled only in the trivial case $\mathcal{F} = 1/2$, $\alpha' = 0$, indicating that some misprint may be present in the expressions published therein. So, we renounced to identify our operators F_n, G_n, J_n and K_n in this otherwise interesting model.

6. Evolution of perturbations

In this section we shall study the evolution of scalar perturbations through all cosmological eras, making no assumptions on the specific form of the energy-momentum tensor, thus retaining all terms in the general expressions (21), (22).

In Ref. [13] an easy way to find the evolution of perturbations outside the horizon was proposed. It amounts to write the first order differential equations in their integral form and write the solution as a recursive series. The integral form of Eqs. (15)-(16) is

$$\Psi = \frac{\mathcal{H}}{a^2}\left[\frac{c_1(k)}{k^2} + \int \frac{a^2(\mathcal{H}^2 - \mathcal{H}')}{\mathcal{H}^2}\zeta d\eta + \int a^2\mathcal{H}\xi d\eta\right], \tag{23}$$

$$\zeta = \left[c_2(k) - \int \frac{a^2\mathcal{H}}{2(\mathcal{H}^2 - \mathcal{H}')}\delta p d\eta - \int \frac{a^2\mathcal{H}}{2(\mathcal{H}^2 - \mathcal{H}')}k^2\xi d\eta\right], \tag{24}$$

where $c_1(k)$ and $c_2(k)$ are two integration constants.

With these equations it is very easy to find the evolution of perturbations in any phase. In fact, using the general expressions of δp and ξ, we can recast the set of equations in the following form

$$X(\eta, k) = X^{(0)}(\eta, k) + \hat{A}(\eta, k)\int \hat{B}(\eta, \partial_\eta, k)X(\eta, k)d\eta, \tag{25}$$

where $X = (\Psi, \zeta)$ is the vector of variables, $X^{(0)} = (c_1(k)\mathcal{H}/a^2, c_2(k))$ is the vector containing the decoupled solutions and $\hat{A}(\eta, k)$ and $\hat{B}(\eta, \partial_\eta, k)$ are two coupling matrices that can be deduced from Eqs. (23) and (24), once we use the proper form of the sources as functions of ζ and Ψ. The general solution to Eq. (25) is

$$X = \sum_{i=0}^{\infty} X^{(i)}, \tag{26}$$

where $X^{(i)}$ is determined by $X^{(i-1)}$ as

$$X^{(i)} = \hat{A}(\eta, k)\int \hat{B}(\eta, \partial_\eta, k)X^{(i-1)}(\eta, k)d\eta. \tag{27}$$

So, starting from the decoupled solution $X^{(0)}$, we can construct the full solution of the system. Of course, this only works for modes outside the horizon, where we can stop to the first few terms of the series because the higher order terms contain extra-factors of $k^2 \ll 1$ or $k^2\eta^2 \ll 1$. Let us examine the solution in each phase.

6.1. Pre-bounce

In the pre-bounce, the soulution assumes a very simple form. In fact, there we have $\delta p = -2c_-^2 k^2\Psi/a^2$ and $\xi = 0$, while the explicit expression for all background functions in terms of powers of η is given by Eqs. (6)-(9). So, it is very easy to perform all integrations. Discarding all numerical factors of order one, we start from the decoupled solution, which assumes the following form in the power law regime

$$\Psi^{(0)} = k^{-2}c_1(k)|\eta|^{-1-2q_-} \tag{28}$$

$$\zeta^{(0)} = c_2(k); \tag{29}$$

the first recursion gives

$$\Psi^{(1)} = c_2(k) \tag{30}$$

$$\zeta^{(1)} = c_1(k)|\eta|^{1-2q_-}, \tag{31}$$

and the second recursion gives

$$\Psi^{(2)} = c_1(k)|\eta|^{1-2q_-} \tag{32}$$

$$\zeta^{(2)} = k^2 c_2(k)|\eta|^2. \tag{33}$$

We can see that the second recursion gives the same modes already present in the decoupled solution, multiplied by $k^2\eta^2$. Since we are considering the evolution outside the horizon, this factor is much smaller than one and we can safely neglect the second and all higher order recursions. We are thus left with

$$\Psi_- \sim k^{-2}c_1(k)|\eta|^{-1-2q_-} + c_2(k) \tag{34}$$

$$\zeta_- \sim c_1(k)\eta^{1-2q_-} + c_2(k). \tag{35}$$

Comparing this solution with the expansion of the Hankel functions for small arguments, given by Eqs. (19) and (20), we can make the identifications $c_1(k) = k^\nu$, $c_2(k) = k^{-\nu}$, recovering the two integration constants in terms of the vacuum normalization.

6.2. Bounce

Now let us turn to the core of our work. It must be said that the specific shape of the solution during the bounce is not our concern, since we just want to know the final spectrum of scalar perturbations when they reenter the horizon. However, as we shall see in the next subsection, in order to construct the post-bounce solution, we need to evaluate the integrals appearing in Eq. (27) in a domain covering the whole bounce phase, which ranges from $\eta \simeq -1$ to $\eta \simeq 1$ in our normalized units. It turns out that

the evaluation of the bounce integrals is very easy, since the k and η dependences are always factored. In fact, the generic integral has the form

$$\int_{-1}^{1} f(k,\eta)d\eta = f_k(k)\int_{-1}^{1} f_\eta(\eta)d\eta = f_k(k). \tag{36}$$

The last equality is crucial. It comes from the fact that the only scales in our problem are η, k and the bounce scale η_B that we have set to 1. Since we have factored the k dependence, the only scale left in f_η is η itself. But integrating over η from -1 to 1, whatever the dependence of the primitive function on η, the final result will lose the only scale it contained, and would thus be of order 1. It cannot depend on any other scale, unless the bounce is actually governed by two or more scales. In that case, we cannot solve the integrals in such a trivial way, except in some special limits.

Since we are going to use several integrals over the bounce phase in the next section, we give their explicit derivation in Appendix A.

6.3. Post-bounce

In order to build the solution after the bounce, at every recursion we must evaluate integrals of the form

$$\int_{-\infty}^{\eta} f(k,\eta)d\eta. \tag{37}$$

The integration domain can be split in three parts, covering the pre-bounce, the bounce and the post-bounce phase, respectively. We thus have

$$\int_{-\infty}^{\eta} f(k,\eta)d\eta = \int_{-\infty}^{-1} f(k,\eta)d\eta + \int_{-1}^{1} f(k,\eta)d\eta + \int_{1}^{\eta} f(k,\eta)d\eta. \tag{38}$$

For the pre-bounce integrals, it is sufficient to use the pre-bounce solutions evaluated at $\eta \simeq -1$. For the bounce integrals, we will exploit the explicit forms calculated in Appendix A according to the prescription given in the previous subsection. The post-bounce integrals can be easily evaluated using the asymptotic background solution in the post-bounce phase, where all background functions are replaced by powers of η. For a generic function $f(k,\eta) = f_k(k)|\eta|^s$, we have

$$\int_{1}^{\eta} f_k(k)|\eta|^s d\eta \simeq f_k(k)\left[|\eta|^{s+1} + 1\right], \tag{39}$$

where we have discarded all numerical factors of order 1, as usual.

We are now ready to construct the post-bounce solution. Once more, we start from the decoupled solution, that in the post-bounce era again assumes the explicit power-law form

$$\Psi^{(0)} = k^{\nu-2}|\eta|^{-1-2q_+} \tag{40}$$

$$\zeta^{(0)} = k^{-\nu}. \tag{41}$$

Then, the first recursion gives

$$\Psi^{(1)} = |\eta|^{-1-2q_+}\left[k^{-\nu}(|\eta|^{1+2q_+}+1) + \int_{-1}^{1} f_\eta(\eta)\zeta^{(0)} + \int_{-1}^{1} f_\eta(\eta)\xi^{(0)}d\eta\right] \tag{42}$$

$$\zeta^{(1)} = k^{\nu}(|\eta|^{1-2q_+}+1) + \int_{-1}^{1} f_\eta(\eta)\delta p^{(0)}d\eta + k^2\int_{-1}^{1} f_\eta(\eta)\xi^{(0)}d\eta, \tag{43}$$

where the bounce integrals can be read from the appendix.

The second recursion gives

$$\begin{aligned}\Psi^{(2)} &= |\eta|^{-1-2q_+}\Big[k^{\nu}(|\eta|^{1+2q_+}+|\eta|^2+1)\\ &+\left(\int_{-1}^{1} f_\eta(\eta)\delta p^{(0)}d\eta + k^2\int_{-1}^{1} f_\eta(\eta)\xi^{(0)}d\eta\right)(|\eta|^{1+2q_+}+1)\\ &+\int_{-1}^{1} f_\eta(\eta)\zeta^{(1)}d\eta + \int_{-1}^{1} f_\eta(\eta)\xi^{(1)}d\eta\Big]\end{aligned} \tag{44}$$

$$\begin{aligned}\zeta^{(2)} &= k^{-\nu+2}(|\eta|^{1-2q_+}+|\eta|^2+1)\\ &+k^2\left(\int_{-1}^{1} f_\eta(\eta)\zeta^{(0)} + \int_{-1}^{1} f_\eta(\eta)\xi^{(0)}d\eta\right)(|\eta|^{1-2q_+}+1)\\ &+\int_{-1}^{1} f_\eta(\eta)\delta p^{(1)}d\eta + k^2\int_{-1}^{1} f_\eta(\eta)\xi^{(1)}d\eta.\end{aligned} \tag{45}$$

No further recursions are needed, since all new terms are negligible for superhorizon perturbations. At the end, according to Eq. (26) we can sum up all contributions to build the full post-bounce solution. A careful analysis reveals that almost all terms coming from the bounce integrals are of the same order of other terms already present or are negligible for perturbations outside the horizon at the bounce. Then, the final post-bounce solution takes the amazingly simple form

$$\Psi_+ \sim k^{\nu-2}|\eta|^{-1-2q_+} + k^{-\nu} + k^{\nu} + G_0 k^{\nu-2} \tag{46}$$

$$\zeta_+ \sim k^{\nu}\eta^{1-2q_+} + k^{-\nu} + k^{\nu} + G_0 k^{\nu-2}, \tag{47}$$

where the constant G_0 is set to 1 if the corresponding operator defined in Eq. (21) is non-vanishing; otherwise we have $G_0 = 0$. In other words, this term is present only if δp is directly proportional to Ψ or its time derivatives, without k^2 factors, during the bounce.

6.4. Discussion

The whole history of perturbations in a bouncing cosmology is contained in Eqs. (19) and (20) for the pre-bounce phase, and (46), (47) for the post-bounce phase.

Let us start our discussion from the pre-bounce. The Bardeen potential in the pre-bounce is dominated by the growing mode $k^{\nu-2}|\eta|^{-1-2q_-}$, endowed with a red spectrum

becoming scale-invariant in the limit of slow contraction $q \to 0$. This mode comes from the decoupled solution of Eq. (16), going as $\mathcal{H}/a^2$ and is destined to decay in the post-bounce phase, where it takes the form $k^{\nu-2}|\eta|^{-1-2q_+}$ in Eq. (46). There is also a subdominant constant mode $k^{-\nu}$, which plays no role in this phase. ζ couples to Ψ through a Laplacian and thus has a mode going as $k^{\nu}|\eta|^{1-2q_+}$ in Eq. (19), where the potentially scale-invariant spectrum is multiplied by a k^2 factor. This mode dominates for a fast contraction $q_- > 1/2$ and is subdominant w.r.t. the constant mode $k^{-\nu}$, coming from the decoupled solution of Eq. (15), in the case of slow contraction $q_- < 1/2$.

Now let us examine the post-bounce phase. The role of the bounce is to upset the scalar perturbations for a finite time and to determine the initial conditions for the post-bounce phase. Immediately after the bounce, in Eq. (46) we have the decoupled mode $k^{\nu-2}|\eta|^{-1-2q_+}$, carrying the potentially scale-invariant spectrum, two constants modes k^{ν} and $k^{-\nu}$ that are always present with a blue spectrum, and possibly one more constant mode with the $k^{\nu-2}$ spectrum. *This additional mode is present only if the comoving pressure perturbation is proportional to Ψ or its time derivatives without Laplacians.* Recalling that we are in the limit $k \ll 1$, this constant mode will dominate undisturbed until horizon re-entry and will determine the spectrum of scalar perturbations seeding the inhomogeneities of the universe.

In the other case (i.e. if δp is related to Ψ only through Laplacians), the Bardeen potential is initially dominated by its decaying mode. But at horizon re-entry, this mode has decayed by a factor $\eta_{re}^{-1-2q_+}$, where η_{re} is the time of horizon re-entry. Setting $\eta_{re} \simeq k^{-1}$, we have different possibilities. After a slow contraction in the pre-bounce phase, the larger constant mode is $k^{-\nu}$. Then, the decaying mode will still dominate only if $q_+ < q_-$, i.e. if the expansion in the post-bounce is slower than the pre-bounce contraction. Otherwise, the constant mode dominates. However, in a realistic cosmology the relevant modes for CMB enter the horizon in the radiation or dust-dominated era, with $q_+ = 1$ or 2, respectively. Then, the spectrum would be totally determined by the blue constant mode $k^{-\nu}$, that in the very slow contraction limit $q_- \ll 1$ gives a spectral index $n_s = 3$. The final possibility is that the universe has undergone a fast pre-bounce contraction $q_- > 1/2$, so that the largest constant mode is k^{ν}. In this case, the decaying mode dominates when the expansion is slow ($q_+ < 1/2$), while is negligible otherwise.

The modes dominating at horizon re-entry in all cases are summarized by the following diagram

$$\begin{cases} G_0 = 1 \Longrightarrow \Psi \sim k^{\nu-2} \\ G_0 = 0 \begin{cases} q_- < 1/2 \begin{cases} q_+ < q_- \Rightarrow \Psi \sim k^{\nu-2}\eta^{-1-2q_+} \\ q_+ > q_- \Rightarrow \Psi \sim k^{-\nu} \end{cases} \\ q_- > 1/2 \begin{cases} q_+ < 1/2 \Rightarrow \Psi \sim k^{\nu-2}\eta^{-1-2q_+} \\ q_+ > 1/2 \Rightarrow \Psi \sim k^{\nu} \end{cases} \end{cases} \end{cases} \tag{48}$$

At the same time, we see that there is a direct correspondence between the modes present in Ψ and those in ζ, so that an analogous discussion is valid also for ζ. It is interesting to note that a dependence on Ψ or its time derivatives in the anisotropic stress

does not produce any sizeable effects on the scalar perturbations. The anisotropic stress enters without Laplacians in the equation for Ψ, where, at most, it may renormalize the amplitude of the decaying mode. In the equation for ζ it comes through a Laplacian, thus being completely ineffective.

A very important *caveat* to keep in mind is that the presence of a dependence on Ψ in δp is just a necessary condition for the persistence of the $k^{\nu-2}$ spectrum in a post-bounce constant mode. In fact, bounces with particular symmetries may bring to special cancellations in the integrals, erasing the lowest order contributions to the constant mode and leaving space to higher order (most blue) contributions.

Another possibility is that the internal variables of the new physics at the bounce may provide new modes characterized by alternative spectra with a different physical origin w.r.t. quantum fluctuations in the initial vacuum state. These modes may contribute to the constant mode at the end of the bounce, providing another possible alternative source for phenomenologically acceptable scalar perturbations. However, in such case, the phenomenology would completely rely on the high energy physics at the bounce, whereas it would be preferable that the final spectrum is as independent as possible of the unknown bounce physics.

Our conclusions seem quite similar to those reached by Durrer and Vernizzi in Ref. [9], where the bounce is replaced by a spacelike hypersurface and the evolution of perturbations through the bounce is determined by Israel conditions. In fact, they find that the persistence of the growing mode of the Bardeen potential in the post-bounce era is very general, while its decay occurs only if the bounce hypersurface coincides with the uniform energy density hypersurface, which corresponds to force the surface tension to be proportional to $k^2\Psi$. Reinterpreting the surface tension as an integral of the pressure perturbation during the bounce, this would correspond to tune $G_0 = 0$ in our expression for δp. Thus, the two approaches seem in complete agreement.

7. Numeric testing on toy models

Our golden rule for scalar perturbations through bouncing cosmologies is based on a fully analytical derivation. However, it is useful to visualize the main conclusions on some toy models of bouncing cosmologies, so that they can be touched "by hand". In order to make things as simple as possible, we have considered a bounce with two perfect fluids: the first dominating the pre-bounce and post-bounce eras and the second, being characterized by negative energy density, representing the new physics effective tensor inducing the bounce. For a bounce with two perfect fluids, the Friedman equations are also integrable in terms of hypergeometric functions [15], though we will not use the analytical solutions. So, in this section, ρ_a is the energy density of the first fluid and ρ_b is the energy density of the second fluid. The two fluids have equations of state $p_a = w_a\rho_a$, $p_b = w_b\rho_b$, with $w_a < w_b$, in order to let the secondary fluid become negligible far from the bounce. According to Eq. (9), we have $q_- = q_+ = q$. A slow contraction followed by a slow expansion ($q < 1/2$) takes place for $w_a > 1$, while a fast

contraction followed by a fast expansion ($q > 1/2$) takes place when $w_a < 1$. As for all other models already present in the literature, in this model the post-bounce phase is the time-reversal of the pre-bounce phase. We are thus unable to directly investigate a fast contraction followed by a slow expansion or a slow contraction followed by a fast expansion. Such asymmetric bouncing cosmologies require the implementation of decay of the pre-bounce matter and creation of new matter at the bounce, which would make the model much more complicated, though not impossible to treat.

After the background is determined, we have to specify the form of the energy-momentum tensor for perturbations. We choose

$$\delta p = -\frac{2}{a^2} w_a k^2 \Psi + 3(w_b - w_a)(\rho_b + p_b)\zeta + G_0(\rho_b + p_b)\Psi \tag{49}$$

$$\xi = 0. \tag{50}$$

In the comoving pressure perturbation δp we have included three terms. The first one is the classical adiabatic perturbation of the first fluid, which is necessary to reproduce the ordinary physics in the pre-bounce and post-bounce phases. The other two terms are multiplied by the density of the secondary fluid, which becomes negligible far from the bounce. The term proportional to ζ is the same that one would have by the requirement that the perturbations in the secondary fluid vanish in the gauge $\psi = 0$. This choice is not essential and has been made just for computational reasons. The last term is a "new physics" term directly proportional to Ψ. We will switch it on or off by setting $G_0 = 1$ or $G_0 = 0$, respectively. Here we only present a model with vanishing anisotropic stress, as this plays no role in the final result, apart from a renormalization of some modes. Of course, we have checked that this is actually the case, testing some models (not to be presented here) with non-vanishing anisotropic stress at the bounce.

With this choice for the energy-momentum tensor of perturbations, the equations of motion are simply

$$\tilde{\zeta}' + (1 + 3w_a)\mathcal{H}\tilde{\zeta} = -w_a \mathcal{H} k^2 \Psi + \frac{G_0}{2} a^2 \mathcal{H}(\rho_b + p_b)\Psi \tag{51}$$

$$\Psi' + \frac{2\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}}\Psi = \frac{\tilde{\zeta}}{\mathcal{H}}. \tag{52}$$

As anticipated before, it is safer to use $\tilde{\zeta} = (\mathcal{H}^2 - \mathcal{H}')\zeta$ in all numerical applications, since this variable is guaranteed to be regular throughout the bounce if any regular gauge exists.

Imposing the vacuum normalization as the initial condition, our toy model is well defined and we can choose different values of w_a, w_b and G_0 in order to test the conclusions reached analytically. The results are summarized in Fig. 1, where we have plotted $\log|\Psi|$ and $\log|\zeta|$ as functions of $\mathrm{Sign}[\eta]\log a$, which is now a common practice for the visualization of the behavior of perturbations in bouncing cosmologies [12, 13].

For Fig. 1a, we have chosen $w_a = 1.5$ (implying $q = 0.36$), $w_b = 2.5$ and $G_0 = 0$. For Fig. 1b, we have $w_a = 1/3$ (implying $q = 1$), $w_b = 1$ and $G_0 = 0$. For Fig. 1c, we have $w_a = 1.5$, $w_b = 2.5$ and $G_0 = 1$. These three cases illustrate almost everything we can expect in a bouncing cosmology. The spectral index n_s of the perturbations can be

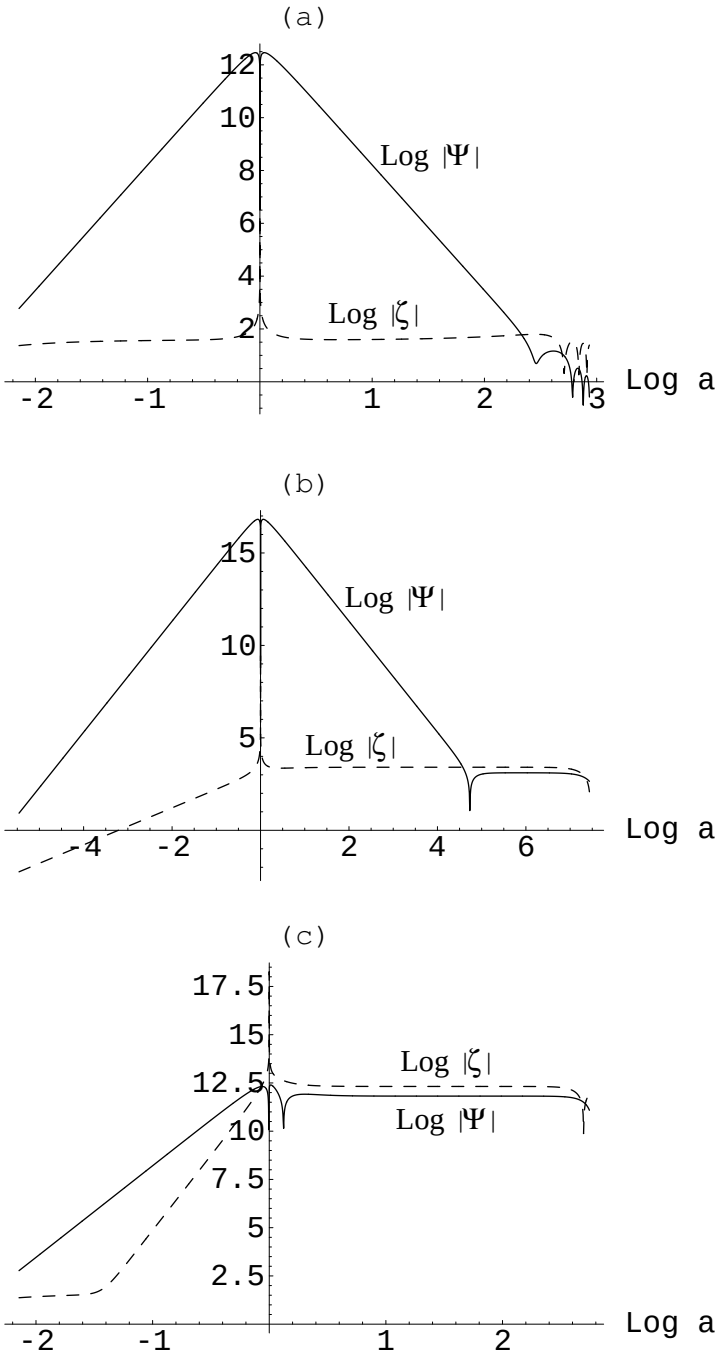

Figure 1. The three possible behaviors of scalar perturbations in bouncing cosmologies. (a) $\delta p \sim k^2\Psi$, with $q < 1/2$ (slow contraction and slow re-expansion); (b) $\delta p \sim k^2\Psi$, with $q > 1/2$ (fast contraction and fast re-expansion); (c) $\delta p \sim \Psi$, with any value of q.

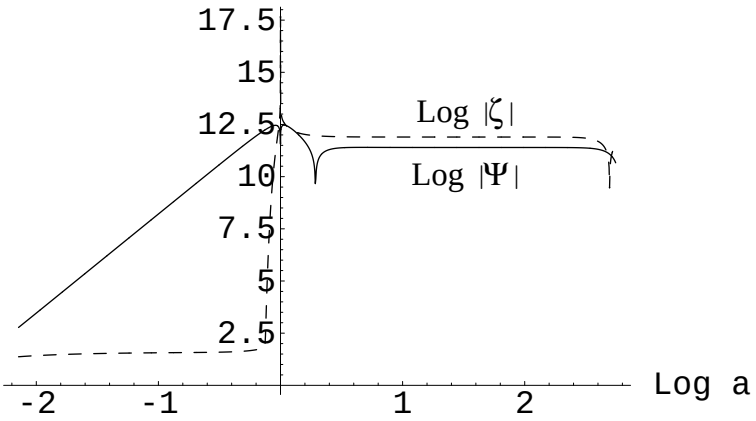

Figure 2. The same case as Fig. 1c with a different relation between δp and Ψ. Here we have chosen $\delta p \sim (\rho_b + p_b)e^{1/\rho_b}\Psi$.

evaluated at any time, evolving at least two Fourier modes with different k and fitting the power law k^{n_s-1}. The exponent of the η-dependence is extracted by the logarithmic time derivative of the modulus of the perturbation. In this way we can compare the analytical predictions for the k and η dependences with the numerical results in all phases.

For the pre-bounce, we just have to distinguish $q < 1/2$ (slow contraction) from $q > 1/2$ (fast contraction). In both cases, Ψ is dominated by the growing mode with the red spectrum $k^{\nu-2}|\eta|^{-1-2q_-}$, as can be seen in all three plots in Fig. 1. ζ is dominated by the constant mode $k^{-\nu}$ in a slow contraction (Fig. 1a) and by the growing mode $k^{\nu}|\eta|^{1-2q_-}$ in a fast contraction (Fig. 1b). However, when $G_0 = 1$, we see in Fig. 1c that the pre-bounce constant mode (we are in a slow contraction) is taken over by a mode going as $k^{\nu-2}|\eta|^{-2-3(1+w_b)q_-}$ at some time before the bounce. This mode is induced by the direct coupling of ζ to Ψ that starts to be effective during the approach to the bounce. In fact, in the model presented here, even if the term proportional to Ψ in δp is suppressed by $(\rho_b + p_b)$, Ψ grows so large that the G_0-mode starts to dominate already some time before the bounce. Indeed, choosing a function going to zero faster than $(\rho_b + p_b)$ far from the bounce, we can delay the growth of ζ as much as we like. For example, choosing $(\rho_b + p_b)e^{1/\rho_b}$, the growth of ζ practically starts right at the bounce, but the post-bounce result is essentially unchanged, as shown in Fig. 2.

The post-bounce scalar perturbations essentially depend on the presence of a direct proportionality between δp and Ψ. If we set $G_0 = 1$, we see from Fig. 1c that immediately after the bounce both ζ and Ψ are dominated by a constant mode carrying the pre-bounce spectrum of Ψ, i.e. $k^{\nu-2}$. We want to stress that this outcome is completely independent of the expansion and contraction rates and of the specific form of the background function that appears in the coupling of δp to Ψ. Choosing $(\rho_b + p_b)$, $(\rho_b + p_b)e^{1/\rho_b}$ or any other function gives exactly the same result, apart from numerical factors of order 1.

If the G_0-mode is completely absent, then the pre-bounce growing mode of the Bardeen potential is entirely converted into a decaying mode, leaving no trace in the constant modes. Then, at horizon re-entry it may still dominate or it may be negligible w.r.t. some constant mode, depending on the contraction and expansion rates. In Fig. 1a we have a slow contraction followed by an expansion with the same rate. We see that at horizon re-entry the decaying mode of Ψ is of the same order as the constant mode of ζ, in agreement with the diagram (48), where this case is marginal between the dominance of $k^{-\nu}$ and the dominance of the decaying mode. In Fig. 1b we have a fast contraction followed by an expansion with the same rate. In agreement with the diagram (48), the decaying mode is taken over by a constant mode with a spectrum k^{ν}.

8. Remarks on bounces induced by spatial curvature and extra-dimensions

In hypothesis (ii) we have excluded the presence of spatial curvature. Indeed, this would introduce another scale in our problem and in order to be compatible with hypothesis (iii), we have two possibilities.

The first is that the curvature scale is very low compared to the bounce scale. Then the universe is practically spatially flat and we can neglect the spatial curvature to follow the treatment of this paper.

The second is that the curvature itself determines the bounce so that the curvature scale at the bounce is actually what we have called $\eta_B = 1$. Spatial curvature may induce a cosmological bounce only if it is positive, because it enters the Friedman equation as a perfect fluid with negative energy density and equation of state $w = -1/3$. The remaining source-matter may keep satisfying $\rho+p > 0$, since the effective NEC violation is performed by the spatial curvature. In order to let the curvature become negligible before and after the bounce, we need $\ddot{a} > 0$, i.e. decelerated contraction and accelerated expansion. For example, the pre-bounce and the post-bounce phases may be dominated by sources with equations of state $w < -1/3$. This contradicts our hypothesis (iv) and the initial conditions would be problematic without another early transition from an accelerated contraction in the far pre-bounce. This possibility was considered by several authors [17] as a toy model for a bouncing cosmology.

However, even relaxing hypothesis (iv), there is a structural difference between a bounce induced by spatial curvature and all other bounces with negligible spatial curvature scale. In fact, the Laplace-Beltrami operator in a positively curved space has a discrete spectrum $k^2 = n(n+2)$ (recall that the curvature scale at the bounce has been set to 1). Then, all modes are in the situation $k > 1$, that means that all modes are *inside* the horizon at the bounce. Then our recursive solution of the integral equation would require an infinite number of terms and is completely useless. In practice, all modes re-enter the horizon during the pre-bounce deflation and oscillate in the approach to the bounce. During the inflationary phase after the bounce, they exit the horizon again, stopping the oscillations. This behavior has been put in evidence by Deruelle [18], who has described the oscillations analytically by hypergeometric functions. Then, it is not

surprising that mode mixing after the oscillations is a generic outcome of such scenarios.

We have to conclude that bounces with positive curvature have a substantially different physics and cannot be used as a step to investigate bounces in spatially flat universes. Then it is difficult to believe that they could be useful for any physically motivated models.

However, there is one remarkable feature of spatial curvature that deserves to be mentioned. We have already noted that spatial curvature behaves as a perfect fluid with equation of state $p = -\rho/3$ at the background level. At the linear perturbation level, we may try to recast all equations in such a way that they look like Eqs. (14)-(16) with some effective energy-momentum tensor containing the curvature terms. After some manipulation, we find the following form for the perturbations induced by the spatial curvature in the comoving gauge

$$\delta\rho = -\frac{6}{a^2}\Psi \tag{53}$$

$$\delta p = \frac{2}{a^2}\Psi + \frac{2}{a^2\mathcal{H}}(\Psi' - \zeta') - \frac{1}{a^2\mathcal{H}}k^2(\Psi - \zeta) \tag{54}$$

$$\xi = \frac{1}{a^2\mathcal{H}^2}(\zeta - \Psi). \tag{55}$$

Spatial curvature thus represents a very interesting example of some "new physics" with a pressure perturbation directly proportional to Ψ and its time derivative without the Laplacian. So, even if spatial curvature is not useful for realistic bouncing cosmologies by itself, it encourages the conjecture that the new physics driving the bounce may contain terms directly proportional to the Bardeen potential in the effective energy-momentum tensor of perturbations.

Perhaps an even more interesting surprise comes out from extra-dimensions. In some particular cases, in fact, it is possible to recast D-dimensional field equations in the form of 4-dimensional Friedman equations with some effective energy-momentum tensor containing the dynamics of the extra-dimensions. One example is the case in which the spacetime is factored into a 4-dimensional FRW metric and an internal homogeneous manifold, where none of the metric components depends on the coordinates spanning the internal space. For example, consider a five-dimensional spacetime with the metric

$$\begin{aligned} ds^2 = a^2(\eta)\left\{(1+2\phi)d\eta^2 - 2B_{,i}d\eta dx^i - \left[(1-2\psi)\delta_{ij} + 2E_{,ij}\right]dx^i dx^j\right\} \\ - b^2(\eta)(1+2\Gamma)dy^2 - 2W dt dy, \end{aligned} \tag{56}$$

with the assumption that all scalar perturbations do not depend on the internal coordinate y (i.e. we are only including zero modes and neglecting excitations of the extra-dimensions). This metric was used by Battefeld, Patil and Brandenberger [20] to study bounces with extra-dimensions. It is easy to show that the background equations have the form of 4-dimensional Friedman equations with an effective contribution to the total energy-momentum tensor coming from the extra-dimension

$$\rho_{extra} = -3\mathcal{H}a^{-2}\frac{b'}{b}, \qquad p_{extra} = a^{-2}\left(\frac{b''}{b} + \mathcal{H}\frac{b'}{b}\right), \tag{57}$$

where the (yy) component of the Einstein equations has been used to eliminate $T^5{}_5$.

In the same way, examining the perturbations equations, we can re-obtain Eqs. (14)-(16) with effective $\delta\rho_{extra}$, δp_{extra} and ξ_{extra} containing terms depending on Ψ, ζ Γ and their derivatives (W decouples from other perturbations). Thus, Γ plays the role of internal variable of the model. Its dynamical equation is given by the perturbed (yy) component, with $\delta T^5{}_5$ as a source.

We will not present the complete expressions of the perturbed effective energy-momentum tensor here, but the important lesson is that the presence of terms directly proportional to Ψ in δp is generic in this class of models as well. Then, bounces with extra-dimensions may be able to transfer the pre-bounce spectrum of Ψ to the post-bounce. This is in agreement with the conclusions of Ref. [20].

The fact that extra-dimensions perturbations contribute to the pressure perturbation was already noted by Giovannini [21]. Coherently with the behavior of scalar perturbations, he also finds that vector perturbations in multi-dimensional bouncing universes preserve the red spectrum acquired in the contraction phase, in contrast with 4-dimensional bouncing universes, where the red spectrum is only transferred to the decaying mode [22].

Nevertheless, a complete analysis of scalar perturbations in multi-dimensional bouncing universes that also takes into account non-zero modes is still missing. Their relevance certainly deserves some investigation, but they cannot fit into the effective four-dimensional picture presented in this paper. Further extensions of our methodology are certainly possible.

9. Conclusions

In this paper, we have presented the general solution for scalar perturbations in bouncing cosmologies, i.e. models of the universe where the present expansion is preceded by a contraction phase. This solution has been derived solving integral equations where the only assumption on the bounce is that it depends on a single physical scale. The "new physics" ruling the bounce has been encoded in an effective energy-momentum tensor, for which we have made no *a priori* assumption.

The final spectrum of the Bardeen potential in the expansion phase depends on the presence or absence in the new physics energy-momentum tensor of a direct relation between the comoving pressure perturbation and the Bardeen potential, not mediated by a Laplacian, as in all known forms of ordinary matter. If such term is present, the pre-bounce growing mode of the Bardeen potential persists in the post-bounce constant mode and it is possible to have a phenomenologically acceptable scale-invariant spectrum in Ekpyrotic-like cosmologies (i.e. models with very slow contraction in the pre-bounce). If such term is absent, the pre-bounce growing mode of the Bardeen potential decays in the post-bounce and becomes negligible w.r.t. a blue constant mode, unless the post-bounce expansion is slower than the pre-bounce contraction.

The negative conclusion reached by all regular models of 4-dimensional spatially

flat bounces studied up to now is due to the absence of such dependence in the energy-momentum tensor. Then, in order to support our conclusion, we have presented a numerical study of a toy model with $\delta p \propto \Psi$ which transfers the spectrum of the pre-bounce growing mode of Ψ to the post-bounce growing mode.

Bounces induced by spatial curvature are structurally different from spatially flat bounces, since all modes are inside the horizon for some time interval centered at the bounce. Our treatment cannot be applied to this distinct category of bounces, whose study is anyway useless for the understanding of bounces in spatially flat cosmologies. However, spatial curvature provides a remarkable example of an effective energy-momentum tensor where the pressure perturbation is proportional to the Bardeen potential without Laplacians. The same key property is also possessed by models of bounces with extra-dimensions where the dependence on the coordinates spanning the internal space is neglected. These multi-dimensional models can fit into our effective four-dimensional scheme. However, before drawing a positive conclusion, a more rigorous and complete treatment (taking into account non-zero modes) should be envisaged for multi-dimensional bounces.

It thus seems not impossible to conjecture a form for the new physics driving the bounce that realizes the Ekpyrotic dream. The problem now is not in matching conditions or any technical issues about the evolution of perturbations, which are now definitively settled by this work. The missing ring of the chain is just the existence of some explicit model with the required property of the effective energy-momentum tensor and that is physically motivated by a consistent high energy theory like string or M-theory. When this explicit model is found, then it will be possible to say that a real alternative to standard inflation exists.

On the other hand, a more prudent attitude suggests the possibility that there might exist some important physical reasons precluding the possibility that an effective energy-momentum tensor of the required form is allowed at the bounce and then the way for bouncing cosmologies would be definitively barred. However, it seems very hard to us to find any physical reason to forbid an effective pressure perturbation directly proportional to the Bardeen potential. Moreover, the explicit examples of spatial curvature and extra-dimensions seem to indicate that this possibility is actually open for high energy theories.

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Appendix A. Bounce integrals

The calculation of all bounce integrals according to the prescription of Sect. 6.2 is straightforward, though a bit boring. Though we need to calculate a relatively large number of terms for the second recursion, luckily most of them are similar to other terms coming from the pre-bounce or post-bounce phases, so that they are irrelevant for the final post-bounce spectrum.

The starting point is the decoupled solution

$$\Psi^{(0)} = k^{\nu-2}\frac{\mathcal{H}}{a^2} \tag{A.1}$$

$$\zeta^{(0)} = k^{-\nu}. \tag{A.2}$$

Note that we do not specify the shape of a and $\mathcal{H}$ during the bounce phase. Then, indicating by $f_\eta(\eta)$ a generic function of η, the integrals involved in the first recursion are

$$\int_{-1}^{1} f_\eta(\eta)\zeta^{(0)}d\eta = k^{-\nu} \tag{A.3}$$

$$\int_{-1}^{1} f_\eta(\eta)\xi^{(0)}d\eta = \sum_{n=0}^{\infty}\left(J_n k^{-\nu+2n} + K_n k^{\nu-2+2n}\right) \tag{A.4}$$

$$\int_{-1}^{1} f_\eta(\eta)\delta p^{(0)}d\eta = \sum_{n=0}^{\infty}\left(F_n k^{-\nu+2n} + G_n k^{\nu-2+2n}\right), \tag{A.5}$$

where the first and the second integrals enter $\Psi^{(1)}$, while $\zeta^{(1)}$ needs the second (multiplied by k^2) and the third. We have used the expressions (21) and (22) for $\delta p^{(0)}$ and $\xi^{(0)}$. By the symbols F_n, G_n, J_n, K_n we generically indicate the results of all integrals involving the corresponding operators defined in Eqs. (21) and (22). Of course, these integrals, if present, are of order 1, according to the argument in Eq. (36). We will keep the symbols F_n, G_n, J_n, K_n just to remember that these terms are present only if the corresponding operators are not zero. At the end it will be sufficient to put these symbols to 1 or 0 to consider models where these operators are present or absent, respectively.

In the second recursion we need the integrals

$$\int_{-1}^{1} f_\eta(\eta)\zeta^{(1)}d\eta = \sum_{n=0}^{\infty}\left(F_n k^{-\nu+2n} + G_n k^{\nu-2+2n} + J_n k^{-\nu+2n+2} + K_n k^{\nu+2n}\right) \tag{A.6}$$

$$\begin{aligned}\int_{-1}^{1} f_\eta(\eta)\xi^{(1)}d\eta = &\sum_{n=0}^{\infty} K_n k^{-\nu+2n} + \sum_{n,m=0}^{\infty}\Big(J_m F_n k^{-\nu+2m+2n} + J_m G_n k^{\nu-2+2m+2n} \\ &+ J_m J_n k^{-\nu+2+2m+2n} + J_m K_n k^{\nu+2m+2n} \\ &+ K_m J_n k^{-\nu+2m+2n} + K_m K_n k^{\nu-2+2m+2n}\Big)\end{aligned} \tag{A.7}$$

$$\int_{-1}^{1} f_\eta(\eta)\delta p^{(1)}d\eta = \sum_{n=0}^{\infty} G_n j^{-\nu+2n} + \sum_{n,m=0}^{\infty}\Big(F_m F_n k^{-\nu+2m+2n} + F_m G_n k^{\nu-2+2m+2n}$$

$$+F_m J_n k^{-\nu+2+2m+2n} + F_m K_n k^{\nu+2m+2n}$$
$$+G_m J_n k^{-\nu+2m+2n} + G_m K_n k^{\nu-2+2m+2n}\Big) . \quad \text{(A.8)}$$

These integrals are those used in Sect. 6.3 to derive the post-bounce solution for scalar perturbations.

References

[1] Veneziano G, 1991 *Phys. Lett.* B **265**, 287
Gasperini M and Veneziano G, 1993 *Astropart. Phys.* **1**, 317
Gasperini M and Veneziano G, 2003 *Phys. Rep.* **373**, 1

[2] Constantinidis C P, Fabris J C, Furtado R G and Picco M, 2000 *Phys. Rev.* D **61**, 043503
Kogan I I, Mouslopoulos S, Papazoglou A and Ross G G, 2001 *Phys. Rev.* D **64**, 124014
Mukherji S and Peloso M, 2002 *Phys. Lett.* B bf 547, 297
Kachru S and McAllister L, 2003 *JHEP* **0303**, 018
Shtanov Y and Sahni V, 2003 *Phys. Lett.* B **557**, 1
Foffa S, 2003 *Phys. Rev.* D **68**, 043511
Veneziano G, 2004 *JCAP* **0403**, 004
Setare M R, 2004 *Phys. Lett.* B bf 602, 1
Date G and Mortuza Hossain G, 2005 *Phys. Rev. Lett.* **94**, 011302
Rinaldi M and Watts P, 2005 *JCAP* **0503**, 006; Herdeiro C A R and Sampaio M, 2005 *Preprint* hep-th/0510052
Carloni S, Dunsby P K S and Solomons D, 2005 *Preprint* gr-qc/0510130

[3] Khouri J, Ovrut B A, Steinhardt P J and Turok N, 2002 *Phys. Rev.* D **66**, 046005
Gratton S, Khoury J, Steinhardt P J and Turok N, 2004 *Phys. Rev.* D **69**, 103505
Tolley A J, Turok N and Steinhardt P J, 2004 *Phys. Rev.* D **69**, 106005

[4] Finelli F and Brandenberger R, 2002 *Phys. Rev.* D **65**, 103522

[5] Biswas T, Mazumdar A and Siegel W, 2005 *Preprint* hep-th/0508194

[6] Brustein R, Gasperini M, Giovannini M, Mukhanov V and Veneziano G, 1995 *Phys. Rev.* D **51**, 6744

[7] Enqvist K and Sloth M S, 2002 *Nucl. Phys.* B **626**, 395
Lyth D H and Wands D, 2002 *Phys. Lett.* B **524**, 5
Bozza V, Gasperini M, Giovannini M and Veneziano G, 2002 *Phys. Lett.* B **543**, 14
Bozza V, Gasperini M, Giovannini M and Veneziano G, 2003 *Phys. Rev.* D **67**, 063514

[8] Lyth D, 2002 *Phys. Lett.* B **524**, 1
Brandenberger R and Finelli F, 2001 *JHEP* **0111**, 056
Lyth D, 2002 *Phys. Lett.* B **526**, 173
Hwang J, 2002 *Phys. Rev.* D **65**, 063514; Tsujikawa S, 2002 *Phys. Lett.* B **526**, 179
Martin J, Peter P, Pinto-Neto N and Schwarz D J, 2002 *Phys. Rev.* D **65**, 123513
Hwang J and Noh H, 2002 *Phys. Lett.* B **545**, 207
Martin J, Peter P, Pinto-Neto N and Schwarz D J, 2003 *Phys. Rev.* D **67**, 028301
Tsujikawa S, Brandenberger R and Finelli F, 2002 *Phys. Rev.* D **66**, 083513
Cartier C, Durrer R and Copeland E, 2003 *Phys. Rev.* D **67**, 103517
Peter P, Pinto-Neto N and Gonzalez D A, 2003 *JCAP* **0312**, 003
Creminelli P, Nicolis A and Zaldarriaga M, 2005 *Phys. Rev.* D **71**, 063505 (2005)

[9] Durrer R and Vernizzi F, 2002 *Phys. Rev.* D **66**, 083503

[10] C. Cartier, 2004 *Preprint* hep-th/0401036

[11] Gasperini M, Giovannini M and Veneziano G, 2003 *Phys. Lett.* B **569**, 113
Gasperini M, Giovannini M and Veneziano G, 2004 *Nucl. Phys.* B **694**, 206

[12] Allen L and Wands D, 2004 *Phys. Rev.* D **70**, 063515

[13] Bozza V and Veneziano G, 2005 *Phys. Lett.* B **625**, 177
Bozza V and Veneziano G, 2005 *JCAP* **0509**, 007
[14] Peter P and Pinto-Neto N, 2002 *Phys. Rev.* D **66**, 063509
[15] Finelli F, 2003 *JCAP* **0310**, 011
[16] Battefeld T J and Geshnizjani G, 2005 *Preprint* hep-th/0503160
[17] Hwang J and Noh H, 2002 *Phys. Rev.* D **65**, 124010
Gordon C and Turok N, 2003 *Phys. Rev.* D **67**, 123508
Martin J and Peter P, 2003 *Phys. Rev.* D **68**, 103517
Martin J and Peter P, 2004 *Phys. Rev. Lett.* **92**, 061301
Martin J and Peter P, 2004 *Preprint* gr-qc/0406062
[18] Deruelle N, 2004 *Preprint* gr-qc/0404126
Deruelle N and Streich A, 2004 *Phys. Rev.* D **70**, 103504
[19] Tsujikawa S, *private communication, reporting A. Starobinsky's opinion*
[20] Battefeld T J, Patil S P and Brandenberger R H, 2005 *Preprint* hep-th/0509043
[21] Giovannini M, 2005, *Class. Quant. Grav.* **22**, 2201
[22] Giovannini M, 2004, *Phys. Rev.* D **70**, 103509

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PLANCKIAN SCATTERING BEYOND THE SEMICLASSICAL APPROXIMATION

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ABSTRACT

Motivated by recent investigations of near-forward Planckian energy collisions, we compare various approaches giving the same leading eikonal approximation. We stress the need of going beyond this approximation in order to control impact parameters $b \simeq GE$ which are crucial for understanding S-matrix singularities ('t Hooft poles) and the physics of gravitational collapse (black hole formation). We point out a few problems that alternative methods have to tackle in order to improve on the direct calculations that we have discussed elsewhere.

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Introduction

A revival of interest has recently arisen in the study of small angle gravitational scattering at Planckian energies $\sqrt{s} \equiv 2E \geq M_p$. This regime is characterized by a strong effective coupling $\alpha_G \equiv \frac{Gs}{\hbar} \geq 1$ which makes any simple perturbative expansion unwarranted.

Different approaches have been proposed in the last few years [1-5] which take into account all orders in α_G. In this note we will show how and why they all coincide in the leading approximation, which has a semiclassical effective metric interpretation, while most of them fail in providing the non-leading terms under which new classical and quantum gravity effects are hiding.

To be more precise, leading contributions to the scattering at a given impact parameter $b = J/E$ contain all powers of α_G. The non-leading ones contain additional powers of the Newton constant G in the two dimensionless combinations*)

$$\frac{\hbar G}{b^2} = \frac{\lambda_p^2}{b^2}, \quad \frac{G^2 s}{b^2} = \frac{R^2}{b^2} = \alpha_G \frac{\lambda_p^2}{b^2},$$

which are small as long as b is much larger than the Planck length λ_p and the gravitational radius $R = 2GE$.

The leading result for the S matrix $S(b, E)$ – valid for $b \gg \lambda_p$, R – is the same in all approaches and has the semiclassical form

$$\begin{aligned} S(b, E) &= e^{2i\delta_0(b,E)}, \\ \delta_0(b, E) &= -\frac{Gs}{\hbar} \log b, \end{aligned} \tag{1}$$

where we have omitted, as usual, the unobservable, infrared-singular Coulomb phase.

Let us comment briefly on each one of these approaches**)

't Hooft[1] obtained this result by solving for the scattering amplitude of one particle, seen as a test body, in the background metric of the other particle, an Aichelburg–Sexl[6] metric.

Muzinich and Soldate[2] obtained an equivalent result by an eikonal approximation to reggeized graviton exchange and recognized that, for b approaching R, the b–representation fails in general to diagonalize and fulfill unitarity.

---

*) In the case of string theory, quantum string corrections of relative order $\frac{\hbar\alpha'}{b^2} \equiv \frac{\lambda_s^2}{b^2}$ are also present, and have been computed[3,4].

**) We follow the order of submission, noting that the first three papers were submitted within ten days in July 1987!

Our derivation[3] was based on the resummation of the string loop expansion at high energies. This gave the leading result, Eq.(1), and allowed the identification and computation of corrections in λ_s/b [3] and in R/b [4]. The semiclassical relation between impact parameter and scattering angle also suggested, at leading level, the interpretation in terms of an effective Aichelburg–Sexl metric.

In a recent paper[5], E. and H. Verlinde reobtained the leading result of Eq.(1) by studying the Einstein action in a gauge that separates $g_{\mu\nu}$ in longitudinal and transverse components with respect to the beam direction. In this gauge, they split the action in two terms: the first one can be treated classically at high energies, while the second, although exhibiting a strong coupling, can be dealt with thanks to its topological structure in the two longitudinal directions.

The eikonal amplitude

In order to understand better the leading approximation, one would like to rewrite it in the momentum transfer $t = -q^2$ representation. In principle, this requires knowledge of the amplitude $S(\underline{b}, E)$ for all impact parameters, including the dangerous region $b \leq R$.

Fortunately, the latter region is irrelevant at small angles $\theta^2 = \frac{q^2}{s} \ll 1$, for which the most important contribution comes from $b \simeq \frac{R}{\theta} \gg R$. Therefore, the small angle amplitude can be safely obtained either by a saddle point method[3] or, more simply, by extending[1] up to $b = 0$ the eikonal expression (1). In this last way one obtains

$$\frac{1}{s} A^{eik}(s, -\mathbf{q}^2) = 4 \int d^2b e^{i\mathbf{b}\cdot\mathbf{q}} \frac{e^{2i\delta_0(b,E)}}{2i} =$$

$$= \frac{8\pi\alpha_G}{\mathbf{q}^2} \left(\frac{4}{\mathbf{q}^2}\right)^{-i\alpha_G} \frac{\Gamma(1 - i\alpha_G)}{\Gamma(1 + i\alpha_G)} \qquad (q = 2E \sin\frac{\theta}{2}). \tag{2}$$

The saddle point evaluation yields the same result, apart from the Γ functions ratio, which for real s is a q-independent, hence irrelevant, phase factor.

It is interesting to notice that, for pure graviton exhange in four dimensions, the b transform – with the above definition of q – diagonalizes unitarity, as proven in Appendix B of ref.4. The alternative definition of q used in ref.2 explains the different unitarity properties of low partial waves in these two approaches to the leading eikonal. Thus, if the dynamics only comes from generalized ladders of pure gravitons, Eq. (2) is valid at all angles, a property of Coulomb-like interactions known to experts[7].

This observation is confirmed by the partial wave projection of Eq.(2), which yields:

$$\frac{i}{8\pi} A_J^{eik}(s) = e^{2i\delta_J^{eik}} = \left(\frac{s}{4}\right)^{i\alpha_G} \frac{\Gamma(J + 1 - i\alpha_G)}{\Gamma(J + 1 + i\alpha_G)}, \tag{3}$$

expression that coincides, up to a J-independent phase, with the ultra-relativistic limit of the relativistic Coulomb amplitude (without fine structure corrections)

$$e^{2i\delta_J^{Coul}} = \frac{\Gamma(J+1-i\frac{E}{k}\alpha)}{\Gamma(J+1+i\frac{E}{k}\alpha)}, \qquad k=\sqrt{E^2-m^2}, \tag{4}$$

after replacing α by α_G.

The above coincidence allows an interpretation of the 't Hooft poles of A^{eik} at $i\alpha_G = i\frac{Gs}{\hbar} = N = 1,2,....$ They correspond to the J-plane poles of Eq.(3) at $J = i\alpha_G - n - 1$ and are thus nothing but the $k\to\infty$ limit of the Regge trajectories $J_n(k) = -n-1+i\alpha\frac{E}{k}$, interpolating the Coulomb bound states at $k_n = \frac{i\alpha m}{J+n+1}(1+0(\frac{\alpha^2}{J^2}))$.

Note that the 't Hooft poles originate from the $b\to 0$ singularity of $A^{eik}(b,E)$ in Eq.(1), and thus from the short distance behaviour of the Coulomb interaction. If this is softened, as for instance in the string approach, the poles disappear[3]. A simple illustration of this fact is given by the replacement

$$\delta_0 \to \widetilde{\delta}_0 = -\frac{1}{2}\alpha_G \log(b^2+b_0^2), \tag{5}$$

which yields

$$\frac{1}{s}\widetilde{A}^{eik}(s,-\mathbf{q}^2) = \frac{16\pi\alpha_G}{\Gamma(1+i\alpha_G)}\left(\frac{q}{2b_0}\right)^{i\alpha_G-1} K_{1-i\alpha_G}(b_0 q)\ , \tag{6}$$

which is an entire function of α_G. Eqs.(6) and (2) have the same behaviour in the small angle region $\alpha_G \gg b_0 q$, i.e. if the saddle point value of b exceeds b_0. By contrast, for $i\alpha_G = 1$, $\widetilde{A}^{eik}$ is finite at finite qb_0.

In conclusion, we stress the crucial importance of understanding the $b = 0(R)$ region in order to reach finite angles and to study the singularity structure of the amplitude at large energies.

Beyond the eikonal

In our previous work[3,4] we have computed the first nonleading contributions, which could be embodied in a generalized eikonal representation for the elastic S matrix

$$S(b,E) = e^{2i(\delta_0+\delta_1+\delta_2+...)}, \tag{7}$$

where $\frac{\delta_1}{\delta_0} = 0\left(\frac{\lambda_p^2}{b^2}\right)$ and $\frac{\delta_2}{\delta_0} = 0\left(\frac{R^2}{b^2}\right)$. We found[4], for pure gravity,

$$\delta_1 = \frac{6}{\pi}\frac{G^2 s}{b^2}\log s, \quad Re\delta_2 = 2\frac{G^3 s^2}{\hbar b^2}. \tag{8}$$

As discussed before, δ_0 represents the physics of pure graviton exchange (i.e. the generalized ladders of Fig.1). On the other hand δ_1 originates from the one-loop radiative corrections of the type shown in Fig.2 and δ_2 from the "H diagram" of Fig.3 and other two-loop contributions. This last correction is important in the "classical" regime $R \gg \lambda_p$: it shows an increase of the Einstein deflection, and thus of the gravitational interaction, for $b \sim R$. This may be the first sign of a gravitational instability that could build-up from the resummation of higher orders in $\frac{R^2}{b^2}$.

From the metric point of view, δ_0 comes from semiclassical scattering in the Aichelburg–Sexl metric[6]

$$ds^2 = dx_+ dx_- - d\mathbf{x}^2 + 4GE\delta(x_+)(dx_+)^2 \log \mathbf{x}^2 \tag{9}$$

where $\mathbf{x} = (x_1, x_2)$ are the transverse coordinates and $x_\pm = x_0 \pm x_3$ the longitudinal ones.

On the other hand, δ_1 and δ_2 are related to gravitational emission and thus cannot be directly interpreted in terms of an external metric. Rather, they provide a rough estimate of the quantum fluctuations around the metric (9), as follows

$$\langle (\delta g)^2 \rangle \simeq \frac{\delta_1}{\delta_0} \simeq \frac{\delta_2}{\delta_0^2} \simeq \frac{\lambda_p^2}{b^2} \ . \tag{10}$$

A more precise description of graviton emission associated with gravitational scattering (Fig.3) is obtained by a suitable generalization[4,8,9] of Weinberg's soft graviton formula. Choosing the two physical (transverse, traceless) polarizations

$$\varepsilon^{(a)}_{\mu\nu} = \frac{1}{\sqrt{2}}(\varepsilon^L_\mu \varepsilon^T_\nu + \varepsilon^T_\mu \varepsilon^L_\nu) \ , \quad \varepsilon^{(b)}_{\mu\nu} = \frac{1}{\sqrt{2}}(\varepsilon^L_\mu \varepsilon^L_\nu - \varepsilon^T_\mu \varepsilon^T_\nu) \ , \tag{11}$$

where the vectors $\varepsilon^L_\mu (\varepsilon^T_\mu)$ are longitudinal (transverse) to the beam and orthogonal to the graviton mementum q_μ, we find that both are emitted, with amplitudes in the ratio

$$A^{(b)}/A^{(a)} = \tan\theta_{12} \ . \tag{12}$$

Here $\mathbf{q}_1, \mathbf{q}_2$ are the momentum transfers defined in Fig. 3, with $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$. In the soft limit $\mathbf{q} \to 0$ the ratio (12) vanishes, showing that polarization (a) contains the dominant infrared singularity, in agreement with Weinberg's results. We have seen[4] how such a singularity has a Bloch–Nordsieck interpretation and cancels in computing $Re\delta_2$, which gets finite contributions from both polarizations.

Admittedly, our method for computing non-leading terms is as tedious and cumbersome as a typical higher order field theory calculation. This makes it unlikely that we will

be able to go beyond the present two-loop stage. It is thus natural to ask whether the other approaches can be extended in any simple way in order to investigate non-leading corrections. We shall illustrate why, unfortunately, this is not the case, at least for the most naive extensions.

We have already discussed the inadequacy of the external metric approach[1] in dealing with the non-linear graviton interactions, which are responsible for corrections involving powers of $\frac{R^2}{b^2} = \left(\frac{Gs}{J}\right)^2$.

At first sight, Verlinde's[5] approach of separating longitudinal and transverse degrees of freedom through a gauge choice appears more promising. Unfortunately, a careful analysis shows that the salient features of the quantum fluctuations cannot be handled without modifying that framework.

In fact, the leading IR polarization $\varepsilon^{(a)}$, which is crucial for determining both $Im\delta_2$ and $Re\delta_2$ cannot be described in the Verlinde gauge $g_{LT} = 0$. Metric fluctuations of the type $\varepsilon^{(a)}_{\mu\nu}$, which are necessary to describe a real emission process, cannot be brought to block-diagonal form by any diffeomorphism $\delta g_{\mu\nu} = \xi_\mu q_\nu + \xi_\nu q_\mu$ because $\sum_{LT} \delta g_{LT} \varepsilon^{(a)}_{LT} = 0$.

This observation can be traced back to the fact that, at least around flat space time, the $g_{LT} = 0$ gauge has a non-trivial zero mode $\xi_L = q_L, \xi_T = -q_T$. Thus diffeomorphisms have three essential parameters, instead of four. The fact that this gauge is not appropriate for describing fluctuations around flat space-time has already been noted in Ref.(5). Our point here is that precisely these fluctuations determine the value of non-leading corrections.

Furthermore, the second physical polarization $\varepsilon^{(b)}$, which is already block-diagonal, has equal LL and TT amplitudes in contrast with the idea that the corresponding quantum fluctuations have different scales.

Finally, even if one could reduce the quantum problem to an effective action with two couplings of different magnitudes $g^2_{//} = \frac{\lambda^2_p}{\ell^2_{//}} \simeq \alpha_G$, $g^2_\perp = \frac{\lambda^2_p}{\ell^2_\perp} \simeq \frac{\lambda^2_p}{b^2}$, the two dynamics could not be disentangled in order to produce the subleading correction $\frac{\delta_2}{\delta_0} \simeq \frac{R^2}{b^2} \simeq g^2_{//} g^2_\perp$.

In conclusion, while we think that the crucial difference between longitudinal and transverse modes is present in the semiclassical (leading) approximation – only longitudinal gravitons coupled to the fast external sources – we see that all modes contribute on an equal footing to the higher order interactions which are responsible for the physics at $b \sim O(R)$.

We still hope that an effective eikonal approach could be constructed also for the nonleading contributions. Perhaps, the shock-wave nature of the classical field may be used to define an effective action for the longitudinal modes, in which quantum fluctuations

of the other components are integrated out (for a step in this direction, see Ref.10). To do so, however, we first have to cope with an all-order definition of the asymptotic states (infrared gravitons) and we also have to treat all metric fluctuations on the same footing.

References

1. G.'t Hooft, Phys. Lett. 198B(1987)61.
2. I.Muzinich and M.Soldate, Phys. Rev. D37(1988)353.
3. D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. 197B(1987)81; Int. J. Mod. Phys. A3(1988)1615.
4. D. Amati, M. Ciafaloni and G. Veneziano,Nucl. Phys. B347(1990)550.
5. E. Verlinde and H. Verlinde, Scattering at Planckian Energies, Princeton preprint PUTP-1279 (1991); see also R. Kallosh, Geometry of Scattering at Planckian Energies, Stanford preprint SU-ITP 903 (1991).
6. P. C. Aichelburg and R. U. Sexl, Gen. Rel. Grav. 2(1971)303.
7. J. D. Jackson, Phys. Lett. 215B(1988)151.
8. L. N. Lipatov, Sov. Phys. JETP. 55(1982)582; Nucl. Phys. B307(1988)705.
9. M. Ademollo, A. Bellini and M. Ciafaloni, Phys. Lett. B223(1989)318; Nucl. Phys. B338(1990)114.
10. L. N. Lipatov, Nucl. Phys. B365(1991)614.

Figure Captions

Fig.1 Graviton-exchange diagrams providing the leading eikonal approximation.

Fig.2 A typical diagram giving a quantum correction to the eikonal.

Fig.3 Diagrams giving the lowest order classical correction to the eikonal.

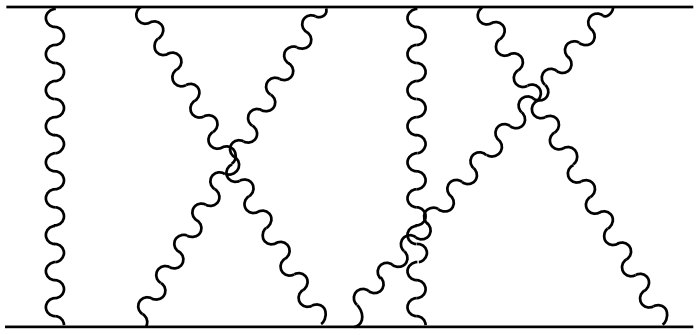

Fig 1

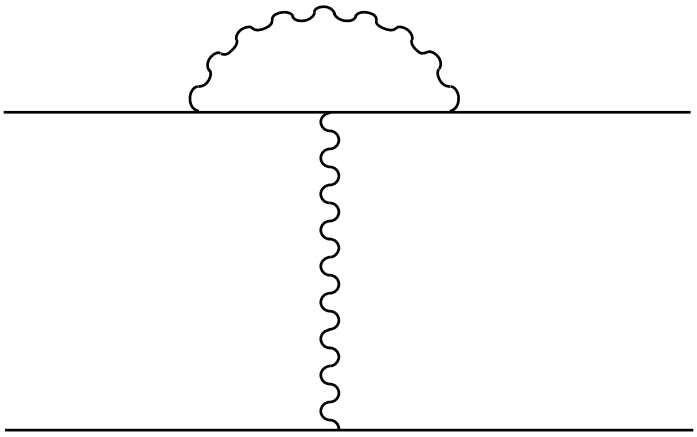

Fig 2

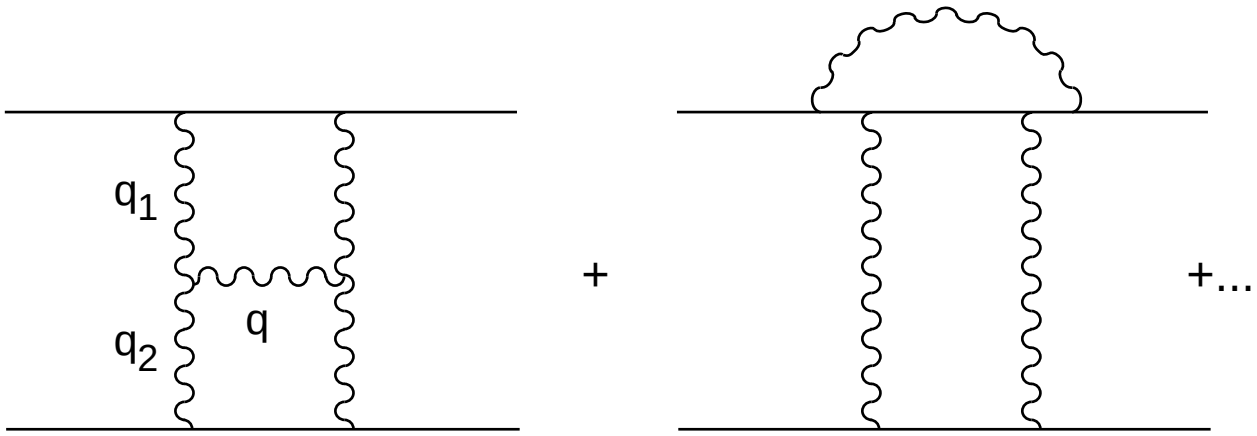

Fig 3

BA-TH/650-12, CERN-PH-TH/2012-018, LPTENS-11/46

Backreaction on the luminosity-redshift relation from gauge invariant light-cone averaging

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Using a recently proposed gauge invariant formulation of light-cone averaging, together with adapted "geodesic light-cone" coordinates, we show how an "induced backreaction" effect emerges, in general, from correlated fluctuations in the luminosity distance and covariant integration measure. Considering a realistic stochastic spectrum of inhomogeneities of primordial (inflationary) origin we find that both the induced backreaction on the luminosity-redshift relation and the dispersion are larger than naïvely expected. On the other hand the former, at least to leading order and in the linear perturbative regime, cannot account by itself for the observed effects of dark energy at large-redshifts. A full second-order calculation, or even better a reliable estimate of contributions from the non-linear regime, appears to be necessary before firm conclusions on the correct interpretation of the data can be drawn.

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I. INTRODUCTION

The so-called concordance (or ΛCDM) model, based on a suitable combination of dark matter, dark energy and baryons for an overall critical density, has become the reference paradigm for the late – i.e. post-equality epoch – evolution of our Universe (see e.g. [1]). It accounts equally well for the CMB data, the Large Scale Structure and, even more significantly, for the supernovae data in terms of a cosmic acceleration [2].

Strictly speaking these three tests of the concordance model are not at the same level of theoretical rigor. While the first two have to do, by definition, with the inhomogeneities present in our Universe, the third is based on an ideal homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) geometry. It is clear that a better treatment of cosmic acceleration should take inhomogeneities into account, at least in an average statistical sense. Only when this is done we can establish in a convincing way whether ΛCDM gives a simultaneous consistent description of the above-mentioned body of cosmological data.

This realization has led to a vast literature about averaging cosmological observables in realistic inhomogeneous cosmologies (see e.g. [3] for recent reviews). The conclusions, however, are still rather controversial: according to some authors [4] present inhomogeneities might explain, by themselves, cosmic acceleration without any need for dark-energy contributions; according to others [5] the effect of inhomogeneities is, instead, completely negligible. The truth may lie somewhere in between, in the sense that a quantitative understanding of inhomogeneities effects could be important in order to put precise constraints on dark-energy parameters, such as the critical fraction of dark-energy density, Ω_Λ, and the time evolution of its effective equation of state, $w_\Lambda(z)$.

In the first papers studying the dynamical effects of averaging, the problem was approached mainly following Buchert's prescriptions [6], namely averaging inhomogeneities over spacelike hypersurfaces and computing the ensuing "backreaction" on the averaged geometry. Nonetheless it is clear that a proper treatment of cosmic acceleration – which is indeed revealed through the experimental study of the luminosity-distance to redshift relation [2] – needs to consider the backreaction of averaged inhomogeneities along our past light cone, which is a null three-dimensional hypersurface.

While a number of papers have emphasized the importance of light-cone averaging (see e.g. [7–9]), a fully covariant

and gauge invariant formulation of such a procedure was only given recently [10], by generalizing to null hypersurfaces an analogous prescription previously derived for spacelike hypersurfaces [11, 12]. Actually, it turns out that the physically meaningful (covariant and gauge invariant) average of a scalar over a null hypersurface reduces to averaging over an appropriate two-dimensional surface a scalar object which is non local, as the integrand itself contains integrals along lightlike geodesic curves lying on the given null hypersurface. Nonetheless, we shall simply refer to this procedure as "light-cone averaging" [10].

The aim of this paper is to apply such an averaging procedure to the luminosity distance of a light source lying on our past light-cone, in order to compute the possible backreaction of inhomogeneities on the luminosity-redshift relation. We will show the emergence of an effect – called hereafter *"induced backreaction"* – which arises from a generic correlation between the inhomogeneities present in the variable we want to average (e.g. the luminosity distance) and those appearing in the covariant integration measure[1]. We stress immediately that our induced backreaction accounts only for a part of the total effect of inhomogeneities. As we shall discuss in detail below, the leading-order induced backreaction can be computed in terms of linear perturbation theory, while a complete leading-order calculation (left to future work) would require perturbation theory up to second order.

We will use a simple phenomenological model of inhomogeneous geometry based on a spatially flat FLRW metric which includes, to first order, scalar perturbations of primordial (inflationary) origin. Such perturbations are often conveniently parametrized in the longitudinal (or Newtonian) gauge [13]. However, we will take advantage of the gauge invariance of our formalism to use an adapted system of coordinates defining the so-called "geodesic light-cone" (GLC) gauge introduced in [10]. In that gauge the light-cone averages of luminosity distance and redshift take very simple, exact expressions that keep all the required degrees of freedom for being applicable to general geometries.

By further taking the *ensemble* average of the inhomogeneous terms, and using the stochastic properties of our model of perturbations, we will compute to leading order the induced backreaction on the luminosity distance, and the corresponding variance. If we limit ourselves to first order computations in the range of scales where linear perturbation theory is reliable, we find that the induced (second-order) corrections to the luminosity-redshift relation of the homogeneous CDM model, although much bigger than one could have naïvely expected, are not large enough to mimic a sizable fraction of Ω_Λ. However, our formalism suggests that other second-order corrections could be even larger, thus confirming the importance of performing a full second-order calculation or, even better, a reliable calculation in the non-linear (short scale) regime.

The paper is organized as follows. In Sect. II we recall definition and basic properties of GLC coordinates and give exact, non-perturbative expressions for the redshift, the luminosity distance and the light-cone average of the latter at constant redshift. In Sect. III we give the linear order transformations expressing the metric of a perturbed FLRW geometry in GLC form, starting from the more commonly used Newtonian gauge. We then express the luminosity distance as a function of the redshift and of the angular position of the source, to first order in the given perturbed geometry. In Sect. IV we take the light-cone average of the computed luminosity distance and we show how backreaction effects automatically emerge, in general, from correlations between the averaged variable and the covariant integration measure. We explicitly define such an induced backreaction and the cosmic variance, to leading order, taking into account the stochastic properties of our model of perturbations and thus including the appropriate *ensemble* average of the inhomogeneous corrections. In Sect. V we implement analytical calculations of the induced backreaction terms and of the dispersion, and in Sect. VI we present the corresponding numerical results for a specific spectrum of primordial scalar perturbations and for a given model of transfer function. Our main conclusions are finally summarized in Sect. VII. We also present, in Appendix A, a sample of the analytic manipulations needed for reducing our (light-cone plus stochastic) averages to the form of a one-dimensional integral over the wavenumber k labelling the scalar perturbation spectrum.

II. THE GLC GAUGE AND THE AVERAGED LUMINOSITY-REDSHIFT RELATION

A. Reminder of the GLC gauge and of its main properties

Our purpose, in this paper, is to compute the light-cone average of the luminosity distance as a function of the redshift by applying the gauge invariant procedure introduced in [10]. One of the main virtues of using a gauge invariant formalism is the freedom of choosing a gauge particularly adapted to the problem at hand. In the case of spacelike averaging, for instance, a convenient coordinate system corresponds to a gauge where the averaging

[1] As discussed in [10], this integration measure will also induce "backreaction" terms of the type usually discussed in the literature [6], namely terms that arise from (generalized) commutation rules between differential operators and averaging integrals.

hypersurfaces are identified with constant-time hypersurfaces. In many applications the averaging hypersurfaces are chosen indeed as the ones associated to a class of geodesic observers corresponding to constant values of the time parameter t of the synchronous gauge (see e.g. [14]).

Similarly, for light-cone averages, it is convenient to identify the null hypersurfaces with those on which a null coordinate takes constant values. For this reason we have introduced in [10] an adapted system of coordinates – defining what we have called a "geodesic light-cone" (GLC) gauge – where the averaging prescription greatly simplifies, while keeping all the required degrees of freedom for applications to general geometries. Furthermore we are also able to identify the timelike coordinate of the GLC gauge with the cosmic time t of the synchronous gauge. As a consequence we can easily introduce a family of geodesic reference observers which exactly coincide with the static ones of the synchronous gauge.

Let us first recall, as discussed in [10], that the coordinates $x^\mu = (\tau, w, \widetilde{\theta}^a)$, $a = 1, 2$, specifying the metric in the GLC gauge correspond to a complete gauge fixing of the so-called observational coordinates, defined e.g in [7, 15, 16]. The GLC metric depends on six arbitrary functions (Υ, a two-dimensional "vector" U^a and a symmetric matrix γ_{ab}), and its line element takes the form

$$ds^2_{GLC} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\widetilde{\theta}^a - U^a dw)(d\widetilde{\theta}^b - U^b dw). \tag{2.1}$$

In matrix form, the metric and its inverse are then given by:

$$g^{GLC}_{\mu\nu} = \begin{pmatrix} 0 & -\Upsilon & \vec{0} \\ -\Upsilon & \Upsilon^2 + U^2 & -U_b \\ \vec{0}^T & -U^T_a & \gamma_{ab} \end{pmatrix}, \qquad g^{\mu\nu}_{GLC} = \begin{pmatrix} -1 & -\Upsilon^{-1} & -U^b/\Upsilon \\ -\Upsilon^{-1} & 0 & \vec{0} \\ -(U^a)^T/\Upsilon & \vec{0}^T & \gamma^{ab} \end{pmatrix}, \tag{2.2}$$

where $\vec{0} = (0, 0)$, $U_b = (U_1, U_2)$, while the 2×2 matrices γ_{ab} and γ^{ab} lower and raise the two-dimensional indices. Clearly w is a null coordinate (i.e. $\partial_\mu w \partial^\mu w = 0$), and a past light-cone hypersurface is specified by the condition $w =$ const. We can also easily check that $\partial_\mu \tau$ defines a geodesic flow, i.e. that $(\partial^\nu \tau)\, \nabla_\nu\, (\partial_\mu \tau) = 0$ (as a consequence of the relation $g^{\tau\tau} = -1$).

In the limiting case of a spatially flat FLRW geometry, with scale factor a, cosmic time t, and conformal time parameter η such that $d\eta = dt/a$, the transformations to the GLC coordinates and the meaning of the new metric components are easily found as follows [10]:

$$\begin{aligned} &\tau = t, \qquad && w = r + \eta, \qquad && \Upsilon = a(t), \\ &U^a = 0, \qquad && \gamma_{ab} d\theta^a d\theta^b = a^2(t) r^2 (d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \tag{2.3}$$

Even though we will be mainly using the GLC gauge for a perturbed FLRW metric in the Newtonian gauge, it is important to stress that the equality between the coordinate τ and the proper time t of the synchronous gauge holds at the exact, non perturbative level: it is always possible, in fact, to choose the GLC coordinates in such a way that τ and t are identified like in the above FLRW limit.

In order to illustrate this point let us consider an arbitrary space-time metric written in the synchronous gauge (with coordinates $X^\mu = (t, X^i)$, $i, j = 1, 2, 3$), where the line element takes the form:

$$ds^2_{SG} = -dt^2 + h_{ij} dX^i dX^j. \tag{2.4}$$

Let us impose the condition $t = \tau$, and check whether we run into any contradiction with the exact metric transformation

$$g^{\rho\sigma}_{SG}(X) = \frac{\partial X^\rho}{\partial x^\mu} \frac{\partial X^\sigma}{\partial x^\nu} g^{\mu\nu}_{GLC}(x). \tag{2.5}$$

Using Eq. (2.2) for the GLC metric, and considering the transformation for the $g^{t\mu}_{SG}$ components, we then obtain the conditions

$$g^{t\mu}_{SG} = \{-1, \vec{0}\} = -\left[\partial_\tau + \Upsilon^{-1}(\partial_w + U^a \partial_a)\right] X^\mu = u^\nu \partial_\nu X^\mu = \frac{dX^\mu}{d\lambda}, \tag{2.6}$$

where $u_\mu = -\partial_\mu \tau$ is the four-velocity of the geodesic GLC observer, and where λ denotes an affine parameter along the observer world-line. So the requirement $\tau = t$ boils down to the statement that along the geodesic flow of the vector field u^μ the SG coordinates X^i are constant. This clearly defines the coordinate transformation in a non-perturbative way, and also shows that the geodesic observer $u_\mu = -\partial_\mu \tau$ of the GLC gauge corresponds to a static (and geodesic) observer in the synchronous gauge. It follows that the identification $t = \tau$ can always be taken for any space-time

metric, and that this simple connection between GLC and synchronous gauge has validity far beyond the particular FLRW case or its perturbed generalizations.

We also remark that, in GLC coordinates, the null geodesics connecting sources and observer are characterized by the simple tangent vector $k^\mu = g^{\mu\nu}\partial_\nu w = g^{\mu w} = -\delta^\mu_\tau \Upsilon^{-1}$, which means that photons reach the observer travelling at constant w and $\widetilde{\theta}^a$. This makes the calculation of the area distance and of the redshift particularly easy. Consider, for instance, a light ray emitted by a static geodesic source at the intersection between the past light-cone of our observer, $w = w_0$, and the spatial hypersurface $\tau = \tau_s$, and received by such static geodesic observer at $\tau = \tau_0 > \tau_s$. The associated redshift z_s is then given by [10]:

$$(1+z_s) = \frac{(k^\mu u_\mu)_s}{(k^\mu u_\mu)_o} = \frac{(\partial^\mu w \partial_\mu \tau)_s}{(\partial^\mu w \partial_\mu \tau)_o} = \frac{\Upsilon(w_0,\tau_0,\widetilde{\theta}^a)}{\Upsilon(w_0,\tau_s,\widetilde{\theta}^a)}\,, \tag{2.7}$$

where the subscripts "o" and "s" denote, respectively, a quantity evaluated at the observer and source space-time position. The expression for the angular distance will be explicitly derived in Sect. II B.

Let us finally recall that, in GLC coordinates, the covariant average of a scalar quantity $S(\tau, w, \widetilde{\theta}^a)$ over the compact two-dimensional surface Σ, defined by the intersection of our past light-cone $w = w_0$ with the spacelike hypersurface $\tau = \tau_s$, is simply given by [10]:

$$\begin{aligned}\langle S\rangle_{w_0,\tau_s} &= \frac{\int_\Sigma d^4x\sqrt{-g}\,\delta(w-w_0)\delta(\tau-\tau_s)S(\tau,w,\widetilde{\theta}^a)\,|\partial_\mu\tau\partial^\mu w|}{\int_\Sigma d^4x\sqrt{-g}\,\delta(w-w_0)\delta(\tau-\tau_s)\,|\partial_\mu\tau\partial^\mu w|} \\ &= \frac{\int d^2\widetilde{\theta}\sqrt{\gamma(w_0,\tau_s,\widetilde{\theta}^a)}\,S(w_0,\tau_s,\widetilde{\theta}^a)}{\int d^2\widetilde{\theta}\sqrt{\gamma(w_0,\tau_s,\widetilde{\theta}^a)}}\,,\end{aligned} \tag{2.8}$$

where $\gamma = \det\gamma_{ab}$. In the case of interest for this paper, namely light-cone averages on surfaces of constant redshift $z = z_s$, one then obtains [10]

$$\langle S\rangle_{w_0,z_s} = \frac{\int d^2\widetilde{\theta}\sqrt{\gamma(w_0,\tau(z_s,w_0,\widetilde{\theta}^a),\widetilde{\theta}^b)}\,S(w_0,\tau(z_s,w_0,\widetilde{\theta}^a),\widetilde{\theta}^b)}{\int d^2\widetilde{\theta}\sqrt{\gamma(w_0,\tau(z_s,w_0,\widetilde{\theta}^a),\widetilde{\theta}^b)}}\,, \tag{2.9}$$

where $\tau(z_s, w_0, \widetilde{\theta}^a)$ has to be determined by solving the redshift equation (2.7) for τ_s as a function of w_0, τ_0, z_s and $\widetilde{\theta}^a$. This general result will now be applied to the case in which S is identified with the luminosity distance d_L.

B. Light-cone average of the luminosity distance

Let us first recall that the luminosity distance d_L of a source at redshift z is related in general to the angular distance d_A of the source (as seen from the observer) by the so-called Etherington (or reciprocity) law [17]:

$$d_L = (1+z)^2 d_A\,. \tag{2.10}$$

In the particular case of an unperturbed, spatially flat FLRW background, and for a source with redshift z_s, the distance d_A is simply given by

$$d_A^{FLRW}(z_s) = a_s r_s = a_s(\eta_0 - \eta_s), \tag{2.11}$$

where $a_s = a(\eta_s)$, while $\eta_0 - \eta_s$ denotes the conformal time interval between the emission and observation of the light signal. For the unperturbed metric, on the other hand, we have $1 + z = a_0/a(t)$, and $d\eta = dt/a = -a_0^{-1}dz/H$, where $H = d(\ln a)/dt$. Hence:

$$d_L^{FLRW}(z_s) = (1+z_s)a_0\int_{\eta_s}^{\eta_0} d\eta = (1+z_s)\int_0^{z_s}\frac{dz}{H(z)} = \frac{1+z_s}{H_0}\int_0^{z_s} dz\left[\sum_n \Omega_{n0}(1+z)^{3(1+w_n)}\right]^{-1/2}. \tag{2.12}$$

In the last equality we have used the standard (spatially flat) Friedmann equation for H (see e.g. [18]), assuming that the given homogeneous model has perfect fluid sources with present fractions of the critical density Ω_{n0} and barotropic parameters w_n. Expanding in the limit $z_s \to 0$ we also obtain the expression

$$d_L^{FLRW}(z_s) \simeq \frac{1}{H_0}\left[z_s + \frac{1}{4}\left(1 - 3\sum_n w_n\Omega_{n0}\right)z_s^2 + O(z_s^3)\right] \equiv \frac{1}{H_0}\left[z_s + \frac{1}{2}(1-q_0)z_s^2 + O(z_s^3)\right], \tag{2.13}$$

which shows the well known sensitivity of the term quadratic in z_s to the composition of the cosmic fluid through the deceleration parameter q_0.

Let us now discuss how this well known result for d_L is modified when including generic inhomogeneities. We recall, to this purpose, that in a generic metric background the angular distance d_A can be computed by considering the null vector $k^\mu = dx^\mu/d\lambda$ tangent to the null ray connecting source and observer (i.e. belonging to the congruence of null geodesics forming the observer's past light-cone). Here λ is an affine parameter along the ray trajectory, chosen in such a way that $\lambda = 0$ at the observer and $\lambda = \lambda_s$ at the source position. The expansion Θ of the congruence of null rays is then given by $\Theta = \nabla_\mu k^\mu$, and the corresponding angular distance d_A is defined by the differential equation [15, 19]:

$$\frac{d}{d\lambda}\left(\ln d_A\right) = \frac{\Theta}{2} = \frac{1}{2}\nabla_\mu k^\mu. \tag{2.14}$$

Consider now a generic metric in the GLC gauge, and the null vector $k_\mu = \partial_\mu w$ such that $k^\mu = dx^\mu/d\lambda = -\delta^\mu_\tau \Upsilon$. We have, in this case,

$$\nabla_\mu k^\mu = -\Upsilon^{-1}\partial_\tau\left(\ln\sqrt{\gamma}\right) = \frac{d}{d\lambda}\left(\ln\sqrt{\gamma}\right), \tag{2.15}$$

where $\gamma = \det\gamma_{ab}$, and the integration of Eq. (2.14) gives, in general,

$$d_A^2(\lambda) = c\sqrt{\gamma(\lambda)}, \tag{2.16}$$

where c is independent of λ. In order to fix the constant c we may recall the boundary conditions required to guarantee that the null rays generated by k^μ belong to the past light cones centered on the world line of our observer. In the limit $d_A \to 0$ (or $\lambda \to 0$), where $\widetilde{\theta}^a \to \theta^a$ (see in particular next section, Eq. (3.8)), such conditions require in particular that [15]

$$\lim_{\lambda\to 0}\frac{\sqrt{\gamma}}{d_A^2} = \sin\theta^1_o = \sin\widetilde{\theta}^1, \tag{2.17}$$

where, in the last equality, we have used the already mentioned fact that $\widetilde{\theta}^a$ is constant along the null geodesic. This clearly fixes $c = 1/\sin\tilde{\theta}^1$, and uniquely determines the angular distance as

$$d_A(\lambda) = \gamma^{1/4}(\lambda)\left(\sin\tilde{\theta}^1\right)^{-1/2}. \tag{2.18}$$

Inserting this result into Eq. (2.9) we finally arrive at an exact expression for the light-cone average of the luminosity distance, as a function of z_s, in the GLC gauge:

$$\langle d_L\rangle_{w_0,z_s} = (1+z_s)^2\frac{\int d^2\widetilde{\theta}\;\gamma^{1/2}(w_0,\tau(z_s,w_0,\widetilde{\theta}^a),\widetilde{\theta}^b)d_A(w_0,\tau(z_s,w_0,\widetilde{\theta}^a),\widetilde{\theta}^b)}{\int d^2\widetilde{\theta}\;\gamma^{1/2}(w_0,\tau(z_s,w_0,\widetilde{\theta}^a),\widetilde{\theta}^b)}. \tag{2.19}$$

This equation, being exact, can be applied in principle to any given highly inhomogeneous cosmology. Specifically, it can be applied to a Lemaître-Tolman-Bondi (LTB) model [20] even in the case of a generic observer shifted away from the symmetry centre of the isotropic geometry. In this paper, however, we shall be mainly interested in working out the consequences of light-cone averaging for a particular model of perturbed FLRW Universe. Since we know how to describe the latter, for example, in the Newtonian (longitudinal) gauge, we need in general to connect such a coordinate system to the GLC gauge.

III. LUMINOSITY DISTANCE IN A PERTURBED FLRW GEOMETRY

A. First-order coordinate transformation from the Newtonian to the GLC gauge

Let us consider the scalar perturbations of a conformally flat FLRW background to describe the particular model of inhomogeneous geometry we are interested in this paper. Choosing in particular the so-called longitudinal or (conformally) Newtonian gauge (NG), using spherical coordinates $(r,\theta^a) = (r,\theta,\phi)$, and going up to the first order in perturbation theory, it is well known that the model is parametrized by the following (inverse) metric tensor [13]:

$$g^{\mu\nu}_{NG} = a^{-2}(\eta)\,\mathrm{diag}\left(-1+2\Phi, 1+2\Psi, (1+2\Psi)\gamma_0^{ab}\right). \tag{3.1}$$

Here

$$\gamma_0^{ab} = \mathrm{diag}\left(r^{-2}, r^{-2}\sin^{-2}\theta\right), \tag{3.2}$$

and Φ, Ψ are the usual gauge invariant Bardeen potentials [13], general functions of η, r, θ and ϕ. We will assume, for simplicity, that the matter sources have vanishing (or negligible) anisotropic stress, so that $\Phi \equiv \Psi$.

For the subsequent computations – in particular, for the application of the simple average prescription given previously – we need to re-express this metric in GLC form, finding the transformations from the NG coordinates $y^\mu = (\eta, r, \theta, \phi)$ to the GLC coordinates $x^\nu = (\tau, w, \widetilde{\theta}^1, \widetilde{\theta}^2)$, and computing the reparametrized metric as

$$g^{\rho\sigma}_{GLC}(x) = \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial x^\sigma}{\partial y^\nu} g^{\mu\nu}_{NG}(y). \tag{3.3}$$

To this purpose we will introduce the useful (zeroth-order) light-cone variables $\eta_\pm = \eta \pm r$, such that

$$\eta = \frac{1}{2}\left(\eta_+ + \eta_-\right) \qquad , \qquad r = \frac{1}{2}\left(\eta_+ - \eta_-\right), \tag{3.4}$$

with corresponding partial derivatives

$$\partial_\eta = \partial_+ + \partial_- \quad , \qquad \partial_r = \partial_+ - \partial_- \quad , \qquad \partial_\pm = \frac{\partial}{\partial \eta_\pm} = \frac{1}{2}\left(\partial_\eta \pm \partial_r\right). \tag{3.5}$$

Using these variables we solve the three differential equations obtained from Eq. (3.3) for the components $g^{\tau\tau}_{GLC} = -1$, $g^{ww}_{GLC} = 0$, $g^{wa}_{GLC} = 0$, by imposing the boundary conditions that *i*) the transformation is non singular around the observer position at $r = 0$, and *ii*) that the two-dimensional spatial sections $r = \mathrm{const}$ are locally parametrized at the observer's position by standard spherical coordinates, i.e. $\widetilde{\theta}^a(0) = \theta^a = (\theta, \phi)$. The sought for transformation can then be written, to first order in Ψ, as follows,

$$\tau = \int_{\eta_{in}}^{\eta} d\eta' a(\eta')\left[1 + \Psi(\eta', r, \theta^a)\right], \tag{3.6}$$

$$w = \eta_+ + \int_{\eta_+}^{\eta_-} dx\, \hat{\Psi}(\eta_+, x, \theta^a), \tag{3.7}$$

$$\widetilde{\theta}^a = \theta^a + \frac{1}{2}\int_{\eta_+}^{\eta_-} dx\, \hat{\gamma}_0^{ab}(\eta_+, x, \theta^a)\int_{\eta_+}^{x} dy\, \partial_b \hat{\Psi}(\eta_+, y, \theta^a), \tag{3.8}$$

where $\hat{\Psi}(\eta_+, \eta_-, \theta^a) \equiv \Psi(\eta, r, \theta^a)$, $\hat{\gamma}^{ab}(\eta_+, \eta_-, \theta^a) \equiv \gamma^{ab}(\eta, r, \theta^a)$ and η_{in} represents an early enough time when the perturbation (or better the integrand) was negligible. We can easily check that, to zeroth order in Ψ, we recover the homogeneous transformation (2.3) as expected.

To first order in Ψ we can use again Eq. (3.3) to compute the non-trivial entries of the GLC metric (2.2), and obtain:

$$\Upsilon = a(\eta)\left[1 + \hat{\Psi}(\eta_+, \eta_+, \theta^a) - \int_{\eta_+}^{\eta_-} dx\, \partial_+ \hat{\Psi}(\eta_+, x, \theta^a)\right] + \int_{\eta_{in}}^{\eta} d\eta' a(\eta')\partial_r \Psi(\eta', r, \theta^a); \tag{3.9}$$

$$U^a = \frac{1}{2}\hat{\gamma}_0^{ab}\int_{\eta_+}^{\eta_-} dx\, \partial_b \hat{\Psi}(\eta_+, x, \theta^a) - \frac{1}{a(\eta)}\gamma_0^{ab}\int_{\eta_{in}}^{\eta} d\eta' a(\eta')\, \partial_b \Psi(\eta', r, \theta^a)$$
$$+\frac{1}{2}\int_{\eta_+}^{\eta_-} dx\ \partial_+\left[\hat{\gamma}_0^{ab}(\eta_+, x, \theta^a)\int_{\eta_+}^{x} dy\ \partial_b \hat{\Psi}(\eta_+, y, \theta^a)\right] - \frac{1}{2}\lim_{x\to\eta_+}\left[\hat{\gamma}_0^{ab}(\eta_+, x, \theta^a)\int_{\eta_+}^{x} dy\ \partial_b \hat{\Psi}(\eta_+, y, \theta^a)\right]; \tag{3.10}$$

$$\gamma^{ab} = \frac{1}{a(\eta)^2}\left\{\left[1 + 2\Psi(\eta, r, \theta^a)\right]\gamma_0^{ab} + \frac{1}{2}\left[\hat{\gamma}_0^{ac}\int_{\eta_+}^{\eta_-} dx\, \partial_c\left(\gamma_0^{bd}(\eta_+, x, \theta^a)\int_{\eta_+}^{x} dy\, \partial_d \hat{\Psi}(\eta_+, y, \theta^a)\right) + a \leftrightarrow b\right]\right\}. \tag{3.11}$$

The term $\hat{\Psi}(\eta_+, \eta_+, \theta^a)$, appearing in the general equation for Υ, denotes (at any η) the value of the perturbation potential evaluated at the tip of the light-cone connecting the origin ($r = 0$) to the point $y^\mu = (\eta, r, \theta^a)$ (namely, $\hat{\Psi}(\eta_+, \eta_+, \theta^a) = \Psi(\eta + r, 0, \theta^a)$).

The above transformations can be immediately applied to obtain an explicit expression for the redshift parameter z_s. By inserting the result (3.9) into Eq. (2.7) and considering that the source is located on the past light-cone of the observer (identified by the equation $w = w_0 = \eta_0$), we obtain:

$$1 + z_s = \frac{a(\eta_0)}{a(\eta_s)}\Big[1 + J(z_s, \theta^a)\Big], \tag{3.12}$$

where $J = I_+ - I_r$, and where:

$$I_+ = \int_{\eta_+^s}^{\eta_-^s} dx\, \partial_+ \hat{\Psi}(\eta_+^s, x, \theta^a) = \Psi_s - \Psi_o - 2\int_{\eta_s}^{\eta_0} d\eta'\, \partial_r \Psi(\eta', \eta_0 - \eta', \theta^a), \tag{3.13}$$

$$I_r = \int_{\eta_{in}}^{\eta_s} d\eta' \frac{a(\eta')}{a(\eta_s)} \partial_r \Psi(\eta', \eta_0 - \eta_s, \theta^a) - \int_{\eta_{in}}^{\eta_0} d\eta' \frac{a(\eta')}{a(\eta_0)} \partial_r \Psi(\eta', 0, \theta^a)\,. \tag{3.14}$$

We have defined $\eta_\pm^s = \eta_s \pm r_s$, $\Psi_s = \Psi(\eta_s, \eta_0 - \eta_s, \theta^a)$, $\Psi_o = \Psi(\eta_0, 0, \theta^a)$ and we have used the zeroth-order light-cone condition $\eta_+^s = \eta_+^0 = \eta_0$. It should be stressed, however, that while the integrals appearing in I_r are evaluated at constant r (namely along timelike geodesics), all the other integrals are evaluated at fixed η_+ (i.e. along null geodesics on the observer's past light-cone).

The contribution associated to I_r can also be rewritten as

$$I_r = (\vec{v}_s - \vec{v}_o) \cdot \hat{n}, \tag{3.15}$$

where $\hat{n}$ is the unit tangent vector along the null geodesic connecting source and observer, and where

$$\vec{v}_{s,o} = -\int_{\eta_{in}}^{\eta_{s,o}} d\eta' \frac{a(\eta')}{a(\eta_{s,0})} \vec{\nabla} \Psi(\eta', r, \theta^a) \tag{3.16}$$

are the "peculiar velocities" of source and observer associated to a geodesic configuration perturbed up to first order in the NG gauge. In a realistic situation one should add to Eq.(3.12) similar terms taking into account a possible intrinsic (non-perturbative) motion of source and observer, unless our theoretical predictions are compared with data already corrected for these latter Doppler contributions. Note also that our first-order expression for the redshift (3.12), valid in general for any given scale factor $a(\eta)$, is in full agreement with the expression obtained in Eq.(38) of [21], where z_s is computed for the particular case of a CDM-dominated model.

B. The luminosity distance including first-order scalar perturbations

We now apply the above coordinate transformation to find, in the Newtonian gauge and to first order in perturbation theory, the other relevant quantities for this paper. Let us start with the determinant γ appearing in the angular distance (2.18). For a source emitting light at time η_s and radial distance r_s we obtain from Eq. (3.11), to first order[2],

$$\gamma^{-1}(\lambda_s) \equiv \det \gamma^{ab}(\lambda_s) = (a_s^2 r_s^2 \sin\theta)^{-2}\Big[1 + 4\Psi_s + 4\widetilde{J}_2(z_s, \theta^a)\Big], \tag{3.17}$$

where:

$$\widetilde{J}_2 = \frac{1}{4}\int_{\eta_+^s}^{\eta_-^s} dx\, \hat{\gamma}_0^{ab}(\eta_+^s, x, \theta^a) \int_{\eta_+^s}^{x} dy\, \partial_a \partial_b \hat{\Psi}(\eta_+^s, y, \theta^a) = \frac{1}{\eta_0 - \eta_s}\int_{\eta_s}^{\eta_0} d\eta' \frac{\eta' - \eta_s}{\eta_0 - \eta'}\Big[\partial_\theta^2 + (\sin\theta)^{-2}\partial_\phi^2\Big]\Psi(\eta', \eta_0 - \eta', \theta^a) \tag{3.18}$$

(the latter equality follows upon a simple integration by parts). Hence, from (2.18):

$$d_A(\lambda_s) = a_s r_s\Big[1 - \Psi_s - \widetilde{J}_2\Big]\left(\frac{\sin\widetilde{\theta}^1}{\sin\theta}\right)^{-1/2}. \tag{3.19}$$

---

[2] Note that, to first order, we have $\gamma^{-1} = (\gamma_{11}\gamma_{22})^{-1}$, and that, for these diagonal matrix elements, the operator ∂_b commutes with γ_0^{db}.

The last factor can be easily computed, to first order, by using Eq. (3.8) and the fact that the $\widetilde{\theta}^a$ are constant along the null geodesic. It is easy to check that it amounts to a redefinition of $\widetilde{J}_2$, and that the above angular distance becomes

$$d_A(\lambda_s) = a_s r_s \left[1 - \Psi_s - J_2(z_s, \theta^a)\right] \ , \tag{3.20}$$

where:

$$J_2 = \frac{1}{\eta_0 - \eta_s} \int_{\eta_s}^{\eta_0} d\eta \, \frac{\eta - \eta_s}{\eta_0 - \eta} \left[\partial_\theta^2 + \cot\theta \, \partial_\theta + (\sin\theta)^{-2} \partial_\phi^2\right] \Psi(\eta', \eta 0 - \eta', \theta^a) \equiv \frac{1}{\eta_0 - \eta_s} \int_{\eta_s}^{\eta_0} d\eta \, \frac{\eta - \eta_s}{\eta_0 - \eta} \, \Delta_2 \Psi \tag{3.21}$$

(here Δ_2 is the two-dimensional Laplacian operator on the unit two-sphere).

For the full explicit expression of the luminosity distance d_L at constant redshift what we need, at this point, is the first-order expansion of the factor $a_s r_s \equiv a(\eta_s) r_s$ appearing in Eq. (3.20). To this purpose we start from Eq. (3.12), considering z_s as a constant parameter localising the given light source on the past light-cone ($w = w_0$) of our observer, and we look for approximate solutions for $\eta_s = \eta_s(z_s, \theta^a)$.

Let us first define the zero-order solution $\eta_s^{(0)}$ through the exact relation

$$\frac{a(\eta_s^{(0)})}{a_0} = \frac{1}{1 + z_s}, \tag{3.22}$$

where $a_0 \equiv a(\eta_0)$. Expanding (3.12) with respect to the parameter $\delta\eta = \eta_s - \eta_s^{(0)}$ we then find:

$$\frac{1}{1 + z_s} = \frac{a(\eta_s^{(0)})}{a_0} [1 + \mathcal{H}_s \ \delta\eta - J(z_s, \theta^a)] = \frac{1}{1 + z_s} [1 + \mathcal{H}_s \, \delta\eta - J(z_s, \theta^a)] \ , \tag{3.23}$$

where $\mathcal{H}_s = d(\ln a(\eta_s^{(0)}))/d\eta_s^{(0)}$, so that:

$$\mathcal{H}_s \, \delta\eta = J(z_s, \theta^a) \ . \tag{3.24}$$

On the other hand, by applying Eq. (3.7) to the light-cone $w = w_0$ at the source position, we readily obtain

$$w_0 = \eta_+^s - 2\Delta\eta \Psi_{\rm av} = \eta_0, \tag{3.25}$$

where we have introduced the zero-order quantity $\Delta\eta = \eta_0 - \eta_s^{(0)}$, and denoted by $\Psi_{\rm av}$ the average value of Ψ along the (unperturbed) null geodesic connecting source and observer:

$$\int_{\eta_+^s}^{\eta_-^s} dx \, \hat{\Psi}(\eta_+^s, x, \theta^a) = -2 \int_{\eta_s}^{\eta_0} d\eta' \, \Psi(\eta', \eta_0 - \eta', \theta^a) \equiv -2\Delta\eta \Psi_{\rm av}. \tag{3.26}$$

Combining this result with Eq. (3.24) we can then determine, to first order, the value of the radial coordinate $r_s(z_s, \theta^a)$ corresponding to the given redshift z_s:

$$r_s(z_s, \theta^a) = w_0 - \eta_s^{(0)}(z_s) - \delta\eta + 2\Delta\eta\Psi_{\rm av} = \Delta\eta \left[1 - \frac{J(z_s, \theta^a)}{\mathcal{H}_s \Delta\eta} + 2\Psi_{\rm av}\right] . \tag{3.27}$$

Proceeding in the same way for a_s we obtain, from Eqs. (3.12), (3.22),

$$a_s(z_s, \theta^a) = a(\eta_s^{(0)}) \left[1 + J(z_s, \theta^a)\right] , \tag{3.28}$$

which, in turn, allows us to compute the value of $a_s r_s$ on the constant-z_s 2-surface:

$$[a_s r_s](z_s, \theta^a) = a(\eta_s^{(0)}) \Delta\eta \left[1 + 2\Psi_{\rm av} + \left(1 - \frac{1}{\mathcal{H}_s \Delta\eta}\right) J(z_s, \theta^a)\right] . \tag{3.29}$$

The angular distance (3.19), for a source at redshift z_s, can now be written as[3]

$$d_A(z_s, \theta^a) = a(\eta_s^{(0)}) \Delta\eta \left[1 + 2\Psi_{\rm av} + \left(1 - \frac{1}{\mathcal{H}_s \Delta\eta}\right) J - \Psi(\eta_s, \eta_0 - \eta_s, \theta^a) - J_2\right] . \tag{3.30}$$

---

[3] Let us note that in the first order terms we can always safely identify η_s with its unperturbed value $\eta_s^{(0)}$.

The (first-order, non-homogeneous, non-averaged) expression of d_L in our perturbed background, referred to the unperturbed value (2.12), is finally given by:

$$\frac{d_L(z_s,\theta^a)}{(1+z_s)a_0\Delta\eta} \equiv \frac{d_L(z_s,\theta^a)}{d_L^{FLRW}(z_s)} = 1-\Psi(\eta_s,\eta_0-\eta_s,\theta^a)+2\Psi_{\rm av}+\left(1-\frac{1}{\mathcal{H}_s\Delta\eta}\right)J-J_2. \tag{3.31}$$

If we apply this general result to the particular case of a CDM-dominated Universe we find almost full agreement with the result for the luminosity distance at constant redshift computed in [21]. After several manipulations, in fact, it turns out that our d_L is equivalent to the one found in [21], modulo a term which can be written as $\vec{v}_0\cdot\hat{n}$ (in the notations of Eq. (3.15)). Such a term gives a subleading contribution to the backreaction and can be neglected with no impact on our final results.

It should be noted, however, that by expanding the above expression in the small z_s limit, and comparing the result with the analogous expansion of the homogeneous distance (2.13), we could easily introduce a redefined value of H_0, say $H_0^{\rm ren}$, such that its inverse corresponds to the coefficient of the linear term in z_s for the perturbed relation. With such a "renormalized" Hubble parameter we have that $H_0^{\rm ren}d_L$ tends smoothly to $H_0 d_L^{FLRW}$ for $z_s\to 0$, and we recover full agreement with [21] provided the same renormalization is applied there. Such a renormalization of H_0 is also suggested by (and closely related to) the first-order computation of the scalar expansion factor $\nabla_\mu u^\mu$ for the flow wordlines $u_\mu=\partial_\mu\tau$ of a local geodesic observer.

At this point we could go on by taking the average of $H_0^{\rm ren}d_L(z_s,\theta^a)$, computing the associated backreaction, and evaluating the corrections to the standard homogeneous relation given by $H_0 d_L^{FLRW}(z_s)$. We have performed that exercise, but we have found that the contribution of the renormalization terms give large z_s-independent contributions to the variance. Furthermore, we think that renormalizing H_0 at $z_s=0$ is physically incorrect since, at very small-z_s, the backreaction is dominated by short-scale inhomogeneities which are deeply inside the non-linear regime (where even the concept of a Hubble flow becomes inappropriate). We could instead try to renormalize H_0 at some small but finite z_s, e.g. at a redshift corresponding to the closest used supernovae (say $z_s\sim 0.015$, see e.g. [22]), but then the results (although much better behaved) would depend on the choice of the particular "renormalization point". Thus, it seems best to consider just the full "unrenormalized" expression (3.31) in a limited region of z_s where one can trust the approximations made, and use that expression for a phenomenological parametrization of the backreaction effects which could possibly include a redefinition of H_0.

IV. COMBINING SPACE-TIME AND *ENSEMBLE* AVERAGES

In the following sections the inhomogeneous deviations from the standard FLRW quantities are sourced by a stochastic background of primordial perturbations, satisfying $\overline{\Psi}=0$, $\overline{\Psi^2}\neq 0$, where the bar denotes statistical (or *ensemble*) average (see Sec. V). Hence, if we limit ourselves to a first-order computation of d_L, we immediately obtain $\overline{d_L}=d_L^{FLRW}$. Non-trivial effects can only be obtained from quadratic and higher-order perturbative corrections, or from the spectrum of the two-point correlation function $\overline{d_L(z,\theta^a)d_L(z',\theta'^a)}$ (discussed in detail in [21]).

In this paper we will consider the *ensemble* average not of d_L but of $\langle d_L\rangle$, where the angular brackets refer to the light-cone average defined in Eq. (2.19) (see e.g. [23–25] for previous attempts of combining *ensemble* average with averages over spacelike hypersurfaces). We will see that the light-cone average automatically induces quadratic (and higher-order) backreaction terms, due to the inhomogeneities present both in d_L and in the covariant integration measure, and since the *ensemble* average of such terms is non-vanishing we obtain, in general, $\overline{\langle d_L\rangle}\neq d_L^{FLRW}$.

We will start this section with some general considerations on how to combine space-time and *ensemble* averaging, and how to isolate those terms in $\overline{\langle d_L\rangle}$ that we may genuinely call "backreaction" effects, i.e. effects on averaged quantities due to inhomogeneities. We shall also discuss how to estimate the variance around mean values due to such fluctuations[4]. Many of these considerations can be certainly found elsewhere, but are nonetheless presented here for the sake of being self-contained.

Let us consider a typical average over the compact surface Σ (topologically equivalent to a two-sphere) embedded on the past light-cone $w=w_0$ at constant z_s. We simply denote such an average by:

$$\langle S\rangle_\Sigma = \frac{\int_\Sigma d^2\mu\, S}{\int_\Sigma d^2\mu}\ , \tag{4.1}$$

[4] The importance of the cosmic variance for a precise measurement of the cosmological parameters, taking into account backreaction effects from averaging on domains embedded in a spatial hypersurface (according to [6]), has been recently pointed out also in [26].

where $d^2\mu$ is the appropriate measure provided by our gauge invariant prescription (see Eq. (2.9)) and S is the (possibly non local) scalar observable (d_L in our case). We can conveniently extract, from both $d^2\mu$ and from S, a zeroth-order homogeneous contribution by defining:

$$d^2\mu = (d^2\mu)^{(0)}(1+\mu), \qquad S = S^{(0)}(1+\sigma), \tag{4.2}$$

and use the possibility of rescaling both integrals in Eq. (4.1) by the same constant, in order to normalize $\int (d^2\mu)^{(0)} = 1$. We then easily get:

$$\left\langle \frac{S}{S^{(0)}} \right\rangle = \frac{\int (d^2\mu)^{(0)}(1+\mu)(1+\sigma)}{\int (d^2\mu)^{(0)}(1+\mu)} = \frac{\langle (1+\mu)(1+\sigma)\rangle_0}{1+\langle\mu\rangle_0} \ , \tag{4.3}$$

where we have dropped, for simplicity, the subscript of the averaging region Σ, and where we have defined by $\langle \dots \rangle_0$ averages with respect to the unperturbed measure $(d^2\mu)^{(0)}$ (we shall drop the subscript 0 hereafter).

Let us now perform the *ensemble* average, denoted by an overbar, paying attention to the fact that *ensemble* averages do *not* factorize, i.e. $\overline{AB} \neq \overline{A}\ \overline{B}$. A simple calculation leads to:

$$\overline{\langle S/S^{(0)}\rangle} = 1 + \overline{(\langle\sigma\rangle + \langle\mu\sigma\rangle)(1+\langle\mu\rangle)^{-1}} \ . \tag{4.4}$$

This last equation is supposedly exact but, as such, pretty useless. It becomes an interesting equation, though, if we can expand the quantities μ and σ in a perturbative series:

$$\mu = \sum_i \mu_i, \qquad \sigma = \sum_i \sigma_i, \tag{4.5}$$

and we further assume that the first order quantities μ_1, σ_1 have vanishing *ensemble* averages (as it is the case for typical cosmological perturbations coming from inflation). In that case we can easily expand the result and obtain, for instance:

$$\overline{\langle S/S^{(0)}\rangle} = 1 + \overline{\langle\sigma_2\rangle} + \mathrm{IBR}_2 + \overline{\langle\sigma_3\rangle} + \mathrm{IBR}_3 + \dots \tag{4.6}$$

where

$$\mathrm{IBR}_2 = \overline{\langle\mu_1\sigma_1\rangle} - \overline{\langle\mu_1\rangle\langle\sigma_1\rangle}, \tag{4.7}$$

$$\mathrm{IBR}_3 = \overline{\langle\mu_2\sigma_1\rangle} - \overline{\langle\mu_2\rangle\langle\sigma_1\rangle} + \overline{\langle\mu_1\sigma_2\rangle} - \overline{\langle\mu_1\rangle\langle\sigma_2\rangle} - \overline{\langle\mu_1\rangle\langle\mu_1\sigma_1\rangle} + \overline{\langle\mu_1\rangle\langle\mu_1\rangle\langle\sigma_1\rangle}, \tag{4.8}$$

and where we have used again the non-factorization property, i.e. $\overline{\langle\mu_1\rangle\langle\sigma_1\rangle} \neq \overline{\langle\mu_1\rangle}\ \overline{\langle\sigma_1\rangle}$, and so on. We see that the result contains, to a given order, both terms that depend on expanding S to that order (but *not* on the precise averaging prescription), and "induced backreaction" (IBR) terms that depend on correlations between the fluctuations of S and those in the measure. These latter terms only depend on lower-order perturbations of S and the measure separately. In particular, our first-order calculation provides the full second-order IBR effect that comes from the above interplay of μ_1 and σ_1, although the full second-order result needs also the harder computation of σ_2 (but *not* of μ_2). Note also that, whenever the fluctuations of S and $d^2\mu$ are uncorrelated, all IBR effects drop out. We will see below how to apply the above general reasoning to the particular case of d_L.

Let us now discuss instead the issue of the variance, i.e. of how broad is the distribution of values for $S/S^{(0)}$ around its mean value $\overline{\langle S/S^{(0)}\rangle}$. This dispersion is due to both the fluctuation on the averaging surface and to those due to *ensemble* fluctuations. Let us thus define:

$$\mathrm{Var}[S/S^{(0)}] \equiv \overline{\left\langle \left(S/S^{(0)} - \overline{\langle S/S^{(0)}\rangle} \right)^2 \right\rangle} = \overline{\langle (S/S^{(0)})^2\rangle} - \left(\overline{\langle S/S^{(0)}\rangle} \right)^2 . \tag{4.9}$$

Inserting the definition (4.2) we get, after a little algebra:

$$\mathrm{Var}[S/S^{(0)}] = \overline{\langle\sigma^2(1+\mu)\rangle(\langle(1+\mu)\rangle)^{-1}} - \left(\overline{\langle\sigma(1+\mu)\rangle(1+\langle\mu\rangle)^{-1}} \right)^2 . \tag{4.10}$$

If we now make the same assumptions as before on expanding μ and σ, we find that the second term in (4.10) is at least of fourth order. Therefore, for the leading term in the variance we find the amazingly simple result (see also [26]):

$$\mathrm{Var}[S/S^{(0)}] = \overline{\langle\sigma_1^2\rangle}. \tag{4.11}$$

As in the case of IBR_2 we only need to know the first-order perturbation, but this time the effect is completely independent of the averaging measure. As we shall see, the dispersion (which is actually the square root of the variance) turns out to be larger than the averaging corrections due to IBR_2.

We may note, at this point, that we could have also considered the dispersion of the angular average $\langle S/S^{(0)}\rangle$ due to the stochastic fluctuations, namely:

$$\mathrm{Var}'[S/S^{(0)}] \equiv \overline{\left(\langle S/S^{(0)}\rangle - \overline{\langle S/S^{(0)}\rangle}\right)^2} = \overline{\left(\langle S/S^{(0)}\rangle\right)^2} - \left(\overline{\langle S/S^{(0)}\rangle}\right)^2 , \tag{4.12}$$

which, after calculations similar to the ones above, gives

$$\mathrm{Var}'[S/S^{(0)}] = \overline{\langle\sigma_1\rangle^2}\,. \tag{4.13}$$

Such a quantity is much smaller than the previous one, as can be inferred from Section VI, indicating that the main reason for the dispersion lies in the angular scatter of the data rather than in their stochastic distribution due to the *ensemble*.

Let us finally identify the quantities appearing in IBR_2 for our particular case. To this purpose let us stress that, to describe the real impact of inhomogeneties on the observational data, we should consider the scalar S corresponding to the true observed quantity. In the case of the supernovae data [2] this quantity should be the received flux of radiation, which is proportional to $\sim d_L^{-2}$. On the other hand, if we consider the average of $S = d_L$ instead of the average of the flux, the difference is only of second order in our perturbative expansion [5]. Since in this paper we will not evaluate such genuine second-order contribution to the average we can limit ourselves, for the sake of simplicity, to consider hereafter $S = d_L(z_s, \theta^a)$.

In such a case, from Eq. (3.31) we immediately find

$$\sigma_1 = A_1 + A_2 + A_3 + A_4 + A_5, \tag{4.14}$$

where we have defined:

$$A_1 = -\Psi_s\ ; \quad A_2 = 2\Psi_{\rm av}\ ; \quad A_3 = \left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right) I_+\ ; \quad A_4 = -\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right) I_r\ ; \quad A_5 = -J_2. \tag{4.15}$$

On the other hand, in our particular case, the general measure $d^2\mu$ will be given by Eq. (2.9):

$$d^2\mu = d^2\widetilde{\theta}\sqrt{\gamma(w_0, \tau(z_s, w_0, \tilde{\theta}^a), \tilde{\theta}^a)}\,. \tag{4.16}$$

We need to transform the integral in $d^2\widetilde{\theta}$ (over the "2-sphere" Σ) to the standard polar coordinates $d^2\theta = d\theta d\phi$ of the Newtonian gauge. To first order, it is easy to check (by taking into account the Jacobian determinant of the transformation $\theta^a \to \widetilde{\theta}^a$, see Eq. (3.8)) that:

$$\int d^2\widetilde{\theta}\sqrt{\gamma} = \int d\phi d\theta \sin\theta \left([a_s r_s](z_s, \theta^a)\right)^2 (1 - 2\Psi_s)\ . \tag{4.17}$$

The normalized unperturbed measure is then given by $(d^2\mu)^{(0)} = d^2\Omega/4\pi$, where $d^2\Omega = d\phi d\theta \sin\theta$. Proceeding as in the previous section, and using in particular Eq. (3.29), we also obtain:

$$\mu_1 = -2\Psi_s + 4\Psi_{\rm av} + 2\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right) J(z_s, \theta^a) = 2(A_1 + A_2 + A_3 + A_4)\,. \tag{4.18}$$

The second-order induced backreaction IBR_2, and the variance of $d_L/d_L^{\rm FLRW}$, can now be obtained by inserting the results (4.14), (4.18) into Eqs. (4.6) and (4.11), respectively. Their explicit evaluation, first analytic and then numerical, will be presented in the next two sections.

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[5] Since the flux is proportional to $d_L^{-2}(z_s, \theta^a)$ the distance modulus (see Sec. VI) should be computed as a function of $(\overline{\langle d_L^{-2}\rangle})^{-1/2}$ rather than of $\overline{\langle d_L\rangle}$. Up to the second order the difference between the two is shown in the following expressions:

$$(\overline{\langle d_L^{-2}\rangle})^{-1/2} = d_L^{FLRW}\left(1 + \overline{\langle\sigma_2\rangle} + IBR_2 - \frac{3}{2}\overline{\langle\sigma_1^2\rangle}\right)\ ; \qquad \overline{\langle d_L\rangle} = d_L^{FLRW}\left(1 + \overline{\langle\sigma_2\rangle} + IBR_2\right).$$

V. INDUCED BACKREACTION AND DISPERSION: ANALYTIC CONSIDERATIONS

This section will be devoted to a systematic evaluation of the various terms contributing to the induced backreaction IBR_2 defined in the previous section, as well as to the variance associated to the (angle and *ensemble*)-averaged value of $d_L(z_s,\theta^a)$.

Let us start by noting that the simplest way to implement the *ensemble* average of our stochastic background of scalar perturbations Ψ is to consider their Fourier decomposition in the form:

$$\Psi(\eta,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3k\, \mathrm{e}^{i\vec{k}\cdot\vec{x}}\Psi_k(\eta)E(\vec{k})\,, \tag{5.1}$$

where – assuming that the fluctuations are statistically homogeneous – E is a unit random variable satisfying $E^*(\vec{k}) = E(-\vec{k})$ as well as the *ensemble*-average condition:

$$\overline{E(\vec{k}_1)E(\vec{k}_2)} = \delta(\vec{k}_1+\vec{k}_2). \tag{5.2}$$

According to the results of Sec. IV, all corrections we need to compute are bilinear terms in the potential Ψ always occurring in the combination $\overline{\langle A_iA_j\rangle}$ or $\overline{\langle A_i\rangle\langle A_j\rangle}$, where the quantities A_i, A_j are defined in Eq. (4.15). Let us illustrate a typical computation with one of the simplest contributions associated to $A_1 = -\Psi_s$. Following the notations of Sect. IV we will denote with the angular brackets the integration over the two-surface Σ embedded on the light-cone (at $w=w_0$, $z=z_s$). The measure of integration is the unperturbed normalized one, i.e. $d^2\Omega/4\pi$. We then obtain:

$$\begin{aligned}
\overline{\langle\Psi_s\Psi_s\rangle} &= \int\frac{d^3k\;d^3k'}{(2\pi)^3}\overline{E(\vec{k})E(\vec{k'})}\int\frac{d^2\Omega}{4\pi}\left[\Psi_k(\eta_s)e^{ir\vec{k}\cdot\hat{x}}\right]_{r=\eta_0-\eta_s}\cdot\left[\Psi_{k'}(\eta_s)e^{ir\vec{k'}\cdot\hat{x}}\right]_{r=\eta_0-\eta_s}\\
&= \int\frac{d^3k}{(2\pi)^3}\,|\Psi_k(\eta_s)|^2\int_{-1}^{1}\frac{d(\cos\theta)}{2}\left[e^{ik\Delta\eta\cos\theta}\right]\cdot\left[e^{-ik\Delta\eta\cos\theta}\right]\\
&= \int\frac{d^3k}{(2\pi)^3}\,|\Psi_k(\eta_s)|^2 = \int_0^\infty\frac{dk}{k}\,P_\Psi(k,\eta_s),
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
\overline{\langle\Psi_s\rangle\langle\Psi_s\rangle} &= \int\frac{d^3k\;d^3k'}{(2\pi)^3}\overline{E(\vec{k})E(\vec{k'})}\left[\int\frac{d^2\Omega}{4\pi}\Psi_k(\eta_s)e^{ir\vec{k}\cdot\hat{x}}\right]_{r=\eta_0-\eta_s}\cdot\left[\int\frac{d^2\Omega'}{4\pi}\Psi_{k'}(\eta_s)e^{ir\vec{k'}\cdot\hat{x}'}\right]_{r=\eta_0-\eta_s}\\
&= \int\frac{d^3k}{(2\pi)^3}\,|\Psi_k(\eta_s)|^2\left[\int_{-1}^{1}\frac{d(\cos\theta)}{2}e^{ik\Delta\eta\cos\theta}\right]\cdot\left[\int_{-1}^{1}\frac{d(\cos\theta')}{2}e^{-ik\Delta\eta\cos\theta'}\right]\\
&= \int\frac{d^3k}{(2\pi)^3}\,|\Psi_k(\eta_s)|^2\left(\frac{\sin(k\Delta\eta)}{k\Delta\eta}\right)^2 = \int_0^\infty\frac{dk}{k}\,P_\Psi(k,\eta_s)\left(\frac{\sin(k\Delta\eta)}{k\Delta\eta}\right)^2,
\end{aligned} \tag{5.4}$$

where in the second line of both terms we made use of isotropy (i.e. Ψ_k only dependent on $k=|\vec{k}|$), and defined θ and θ' as the angles between $\vec{k}$ and $\vec{x}\equiv r\hat{x}$ and between $\vec{k}'$ and $\vec{x}'\equiv r\hat{x}'$. We have also introduced the (so-called dimensionless) power spectrum of Ψ, defined in general by:

$$P_\Psi(k,\eta)\equiv\frac{k^3}{2\pi^2}|\Psi_k(\eta)|^2. \tag{5.5}$$

More complicated examples, that contain almost all the subtleties of these computations, will be presented in Appendix A.

In general we have many contributions like the above ones, appearing in both the induced backreaction and the variance, and generated by all the A_i terms of Eq. (4.15). In this paper we will consider the particularly simple case of a CDM-dominated background geometry, with a time-independent spectral distribution of sub-horizon scalar perturbations, $\partial_\eta\Psi_k=0$. In such a case all the relevant contributions can be parameterized in the form

$$\overline{\langle A_iA_j\rangle} = \int_0^\infty\frac{dk}{k}\,P_\Psi(k)\mathcal{C}_{ij}(k,\eta_0,\eta_s), \tag{5.6}$$

$$\overline{\langle A_i\rangle\langle A_j\rangle} = \int_0^\infty\frac{dk}{k}\,P_\Psi(k)\mathcal{C}_i(k,\eta_0,\eta_s)\;\mathcal{C}_j(k,\eta_0,\eta_s) \tag{5.7}$$

(where $\mathcal{C}_{ij}, \mathcal{C}_i$ are constant spectral coefficients), valid for any given model of (time-independent) scalar perturbation spectrum. With such parametrization, the leading-order induced backreaction, called IBR_2 in the expansion (4.6) of $\overline{\langle d_L \rangle}/d_L^{FLRW}$, can be written as

$$\mathrm{IBR}_2 = 2\sum_{i=1}^{4}\sum_{j=1}^{5}\left[\overline{\langle A_i A_j\rangle} - \overline{\langle A_i\rangle\langle A_j\rangle}\right] = \int_0^\infty \frac{dk}{k}\, P_\Psi(k)\, 2\sum_{i=1}^{4}\sum_{j=1}^{5}\left[\mathcal{C}_{ij}(k,\eta_0,\eta_s) - \mathcal{C}_i(k,\eta_0,\eta_s)\,\mathcal{C}_j(k,\eta_0,\eta_s)\right]. \quad (5.8)$$

The dispersion (from Eq.(4.11)) takes instead the form:

$$\left(\mathrm{Var}\left[\frac{d_L}{d_L^{FLRW}}\right]\right)^{1/2} = \sqrt{\overline{\langle\sigma_1^2\rangle}} = \left[\sum_{i=1}^{5}\sum_{j=1}^{5}\overline{\langle A_i A_j\rangle}\right]^{1/2} = \left[\int_0^\infty \frac{dk}{k}\, P_\Psi(k)\sum_{i=1}^{5}\sum_{j=1}^{5}\mathcal{C}_{ij}(k,\eta_0,\eta_s)\right]^{1/2}. \quad (5.9)$$

We have analytically computed all the required coefficients $\mathcal{C}_{ij}, \mathcal{C}_i$, and their final values are given in Table I for $\mathcal{C}_{ij}(k,\eta_0,\eta_s)$, and in Table II for $\mathcal{C}_i(k,\eta_0,\eta_s)$. In Table I we also show the small-k limit ($k\Delta\eta \ll 1$) of $\mathcal{C}_{ij}$ and of the products $\mathcal{C}_i\mathcal{C}_j$. For notational convenience we have introduced in the tables the dimensionless variable $l = k\Delta\eta$, and we have used the definition:

$$\mathrm{SinInt}(l) = \int_0^l \frac{dx}{x}\sin x. \quad (5.10)$$

TABLE I: The spectral coefficients $\mathcal{C}_{ij}(k,\eta_0,\eta_s)$ for the $\overline{\langle A_i A_j\rangle}$ terms defined by Eq. (5.6). We also give the $k\Delta\eta \ll 1$ limit (up to leading order in $k\Delta\eta$) of $\mathcal{C}_{ij}$ and of the products $\mathcal{C}_i\mathcal{C}_j$ for the coefficients defined in Table II.

$\overline{\langle A_i A_j\rangle}$	$\mathcal{C}_{ij}(k,\eta_0,\eta_s)$	$\mathcal{C}_{ij}$ for $k\Delta\eta \ll 1$	$\mathcal{C}_i\,\mathcal{C}_j$ for $k\Delta\eta \ll 1$
$\overline{\langle A_1 A_1\rangle}$	1	1	$1 - \frac{l^2}{3}$
$\overline{\langle A_1 A_2\rangle}$	$-\frac{2}{l}\mathrm{SinInt}(l)$	$-2 + \frac{l^2}{9}$	$-2 + \frac{4}{9}l^2$
$\overline{\langle A_1 A_3\rangle}$	$\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\left[1 - \frac{\sin(l)}{l}\right]$	$\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{l^2}{6}$	$-\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{l^2}{6}$
$\overline{\langle A_1 A_4\rangle}$	$\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{f_0}{\Delta\eta}\left[\cos l - \frac{\sin(l)}{l}\right]$	$-\frac{f_0}{\Delta\eta}\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{l^2}{3}$	$-\frac{f_s}{\Delta\eta}\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{l^2}{3}$
$\overline{\langle A_1 A_5\rangle}$	$-2\left[1 - \frac{\sin(l)}{l}\right]$	$-\frac{l^2}{3}$	0
$\overline{\langle A_2 A_2\rangle}$	$\frac{8}{l^2}\left[-1 + \cos l + l\,\mathrm{SinInt}(l)\right]$	$4 - \frac{l^2}{9}$	$4 - \frac{4}{9}l^2$
$\overline{\langle A_2 A_3\rangle}$	0	0	$\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{l^2}{3}$
$\overline{\langle A_2 A_4\rangle}$	$2\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{f_0+f_s}{\Delta\eta}\left[1 - \frac{\sin(l)}{l}\right]$	$\frac{f_0+f_s}{\Delta\eta}\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{l^2}{3}$	$\frac{f_s}{\Delta\eta}\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{2}{3}l^2$
$\overline{\langle A_2 A_5\rangle}$	$\frac{2}{3l^2}\left[-4 + (4 + l^2)\cos l + l\sin l + l^3\mathrm{SinInt}(l)\right]$	$\frac{l^2}{3}$	0
$\overline{\langle A_3 A_3\rangle}$	$2\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\left[1 - \frac{\sin(l)}{l}\right]$	$\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\frac{l^2}{3}$	$\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\frac{l^4}{36}$
$\overline{\langle A_3 A_4\rangle}$	$\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\frac{f_0-f_s}{\Delta\eta}\left[\cos l - \frac{\sin(l)}{l}\right]$	$-\frac{f_0-f_s}{\Delta\eta}\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\frac{l^2}{3}$	$\frac{f_s}{\Delta\eta}\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\frac{l^4}{18}$
$\overline{\langle A_3 A_5\rangle}$	$-2\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\left[1 - \frac{\sin(l)}{l}\right]$	$-\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{l^2}{3}$	0
$\overline{\langle A_4 A_4\rangle}$	$\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\left[\frac{f_0^2+f_s^2}{\Delta\eta^2}\frac{l^2}{3} - \frac{2f_0 f_s}{\Delta\eta^2}\left(2\cos l + (-2 + l^2)\frac{\sin l}{l}\right)\right]$	$\left(\frac{f_0-f_s}{\Delta\eta}\right)^2\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\frac{l^2}{3}$	$\left(\frac{f_s}{\Delta\eta}\right)^2\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\frac{l^4}{9}$
$\overline{\langle A_4 A_5\rangle}$	$\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\left[\frac{f_0+3f_s}{\Delta\eta}\cos l + \frac{f_0-f_s}{\Delta\eta}\frac{\sin l}{l} + \frac{(f_0+f_s)(-2+l\mathrm{SinInt}(l))}{\Delta\eta}\right]$	$\frac{f_0-f_s}{\Delta\eta}\left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{l^2}{3}$	0
$\overline{\langle A_5 A_5\rangle}$	$\frac{1}{15l^2}\left[-24 + 20l^2 + (24 - 2l^2 + l^4)\cos(l) + l(-6 + l^2)\sin(l) + l^5\mathrm{SinInt}(l)\right]$	$\frac{l^2}{3}$	0

It should be noted that the computation of some of the above terms requires the explicit expression of the scale factor $a(\eta)$. In those cases we have assumed a dust-dominated scale factor with $a(\eta) = a(\eta_0)(\eta/\eta_0)^2$, and we have defined

$$f_{0,s} \equiv \int_{\eta_{in}}^{\eta_{0,s}} d\eta \frac{a(\eta)}{a(\eta_{0,s})} = \frac{\eta_{0,s}^3 - \eta_{in}^3}{3\eta_{0,s}^2} \simeq \frac{1}{3}\eta_{0,s} \quad (5.11)$$

(recall that η_{in} satisfies, by definition, $\eta_{in} \ll \eta_{0,s}$). Using such a CDM-dominated model also imposes the following (zeroth-order) relation between z_s and $\Delta\eta$:

$$\Delta\eta = \frac{2}{a_0 H_0}\left[1 - (1 + z_s)^{-1/2}\right]. \quad (5.12)$$

TABLE II: The spectral coefficients $\mathcal{C}_i$ for the $\overline{\langle A_i\rangle\langle A_j\rangle}$ terms defined by Eq. (5.7).

A_i	$\mathcal{C}_i(k,\eta_0,\eta_s)$
A_1	$\frac{\sin l}{l}$
A_2	$-\frac{2}{l}\mathrm{SinInt}(l)$
A_3	$-\left(1-\frac{1}{\mathcal{H}_s\Delta\eta}\right)\left(1-\frac{\sin l}{l}\right)$
A_4	$\left(1-\frac{1}{\mathcal{H}_s\Delta\eta}\right)\frac{f_s}{\Delta\eta}\left(\cos l-\frac{\sin l}{l}\right)$
A_5	0

This relation will be used in section VI when performing the numerical integration over k as a function of z_s.

In order to proceed with some qualitative and quantitative considerations it is important, at this point, to specify the properties of the power spectrum $P_\Psi(k)$. Limiting ourselves to sub-horizon perturbations, and considering the standard CDM model, we can simply obtain Ψ_k by applying an appropriate, time-independent transfer function to the primordial (inflationary) spectral distribution (see e.g. [27]). The power spectrum of the Bardeen potential is then given by

$$P_\Psi(k)=\left(\frac{3}{5}\right)^2\Delta^2_{\mathcal{R}}T^2(k)\ ,\qquad\qquad \Delta^2_{\mathcal{R}}=A\left(\frac{k}{k_0}\right)^{n_s-1}\ , \tag{5.13}$$

where $T(k)$ is a constant transfer function which takes into account the sub-horizon evolution of the modes re-entering during the radiation-dominated era, and $\Delta^2_{\mathcal{R}}$ is the primordial power spectrum of curvature perturbations, amplified by inflation, outside the horizon. The typical parameters of such a spectrum, namely the amplitude A, the spectral index n_s and the scale k_0, are determined by the results of recent WMAP observations [28]. In our computations we will use, in particular, the following approximate values:

$$A=2.45\times10^{-9}\ ,\qquad n_s=0.96\ ,\qquad k_0/a_0=0.002\,\mathrm{Mpc}^{-1}\ . \tag{5.14}$$

Finally, since we are mainly interested in the overall magnitude of the transfer function, it will be enough for our needs to approximate $T(k)$ with the effective shape of the transfer function for density perturbations[6] with no baryon oscillations, namely $T(k)=T_0(k)$, where [29]:

$$T_0(q)=\frac{L_0}{L_0+q^2C_0(q)}\ ,\qquad L_0(q)=\ln(2e+1.8q)\ ,\qquad C_0(q)=14.2+\frac{731}{1+62.5q}\ ,\qquad q=\frac{k}{13.41k_{\mathrm{eq}}}\ , \tag{5.15}$$

and where k_{eq} is the scale corresponding to matter-radiation equality. We can easily see that the above transfer function goes to 1 for $k\ll k_{\mathrm{eq}}$, while it falls like $k^{-2}\log k$ for $k\gg k_{\mathrm{eq}}$. In the next section we will numerically evaluate the coefficients of Tables I and II, and we will discuss the corresponding backreaction effects on $\overline{\langle d_L\rangle}$, together with its dispersion, for the scalar perturbation spectrum described by Eqs. (5.13) and (5.15).

VI. INDUCED BACKREACTION AND DISPERSION: NUMERICAL RESULTS AND DISCUSSION

In the following computations we will set $a_0=1$, $\Omega_m=1$, and we will use $h\equiv H_0/(100\,\mathrm{km\,s^{-1}Mpc^{-1}})=0.7$. In that case we obtain [29] $k_{\mathrm{eq}}\simeq0.036\,\mathrm{Mpc}^{-1}$, and we can more precisely define the asymptotic regimes of our transfer function as $T_0\simeq1$ for $k\lesssim10^{-3}\,\mathrm{Mpc}^{-1}$, and $T_0\sim k^{-2}\log k$ for $k\gtrsim2.5\,\mathrm{Mpc}^{-1}$. This second scale is already deep inside the so-called non-linear regime, roughly corresponding to $k\gtrsim0.1\,h\,\mathrm{Mpc}^{-1}$ (see e.g. [30]) .

Let us start by comparing the behaviour of the power spectrum with the behaviour of the coefficients in Table I and II. As shown in Table I, all the coefficients $\mathcal{C}_{ij}$ and $\mathcal{C}_i\mathcal{C}_j$ will go at most as $\mathcal{O}(1)$ while their combination $\mathcal{C}_{ij}-\mathcal{C}_i\mathcal{C}_j$ will vanish at least as $\mathcal{O}(k^2\Delta\eta^2)$ in the IR limit $k\Delta\eta\ll1$. As a consequence, the infrared part of the terms in the induced backreaction and in the dispersion will give a subleading contribution, and we can safely fix our infrared cut-off to be $k=H_0$ – i.e. we can limit ourselves to sub-horizon modes – without affecting the final result.

[6] The relation between density perturbations and metric perturbations is of course fully under control in the linear perturbative regime, but could be unreliable in the high-k non-linear regime.

Furthermore, as can be seen in Table I and II, all the coefficients $\mathcal{C}_{ij}$ and $\mathcal{C}_i\mathcal{C}_j$ will go at most as $\mathcal{O}(k^3\Delta\eta^3)$ in the UV limit $k\Delta\eta \gg 1$. As a consequence, all the integrals involved in the induced backreaction and in the dispersion will be UV-convergent and their main contribution will come from the range $1/\Delta\eta \ll k \leq 2.5\,\mathrm{Mpc}^{-1}$. In this range, some of the backreaction coefficients grow as positive power of k while the transfer function is not yet decreasing as $\log k/k^2$. In particular, there are only two leading contributions corresponding to the integrals controlled by the coefficients $\mathcal{C}_{44}$ and $\mathcal{C}_{55}$ (this is the reason why the dispersion of the angular average of d_L, (4.13), is smaller than the one of d_L itself (4.11)). For such coefficients we have, in fact, the following behaviour for $k\Delta\eta \gg 1$:

$$\mathcal{C}_{44} \simeq \left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2 (f_0^2 + f_s^2)\frac{k^2}{3}, \tag{6.1}$$

$$\mathcal{C}_{55} \simeq \frac{k^3\Delta\eta^3}{15}\mathrm{SinInt}(k\Delta\eta)\,. \tag{6.2}$$

From the above expressions it is easy to understand that $\overline{\langle A_4A_4\rangle}$ will give the largest contribution for $z_s \ll 1$ (see Fig. 1), while $\overline{\langle A_5A_5\rangle}$ will give the largest contribution for $z_s \gg 1$ (see Fig. 2).

It is also clear that both the induced backreaction IBR_2 and the dispersion depend in principle on the UV cut-off k_{UV} eventually used to evaluate the integrals. On the other hand, when k_{UV} is taken inside the regime where the spectrum goes like $(\log k)^2/k^4$, the dependence on the cut-off will not be too strong. In particular the leading contribution to IBR_2, given by $2\overline{\langle A_4A_4\rangle}$, depends very weakly on the particular value of k_{UV}. The leading contribution to the dispersion (5.9), controlled by $\overline{\langle A_5A_5\rangle}$, has a somewhat stronger dependence on k_{UV} (because of the extra power of k in the integrand of $\overline{\langle A_5A_5\rangle}$). The numerical integrations of $\overline{\langle A_4A_4\rangle}$ and $\overline{\langle A_5A_5\rangle}$ are presented in Figs. 1 and 2, where we illustrate the magnitude of the backreaction effects as a function of the redshift (in the range of values relevant to supernovae observations), and as a function of its dependence on the cut-off (ranging from $k_{UV} = 0.1\,\mathrm{Mpc}^{-1}$ to $k_{UV} = +\infty$). We should emphasize that the two abovementioned contributions have a clear and distinct physical meaning in the Newtonian gauge. Going back to their explicit expressions it appears that $\overline{\langle A_4A_4\rangle}$ represents a Doppler effect while $\overline{\langle A_5A_5\rangle}$ is associated with lensing, both in qualitative agreement with previous claims in the literature [19, 21]. Note that the latter term can only appear in the variance, because it contributes to σ_1^2, while $\overline{\langle A_4A_4\rangle}$ appears

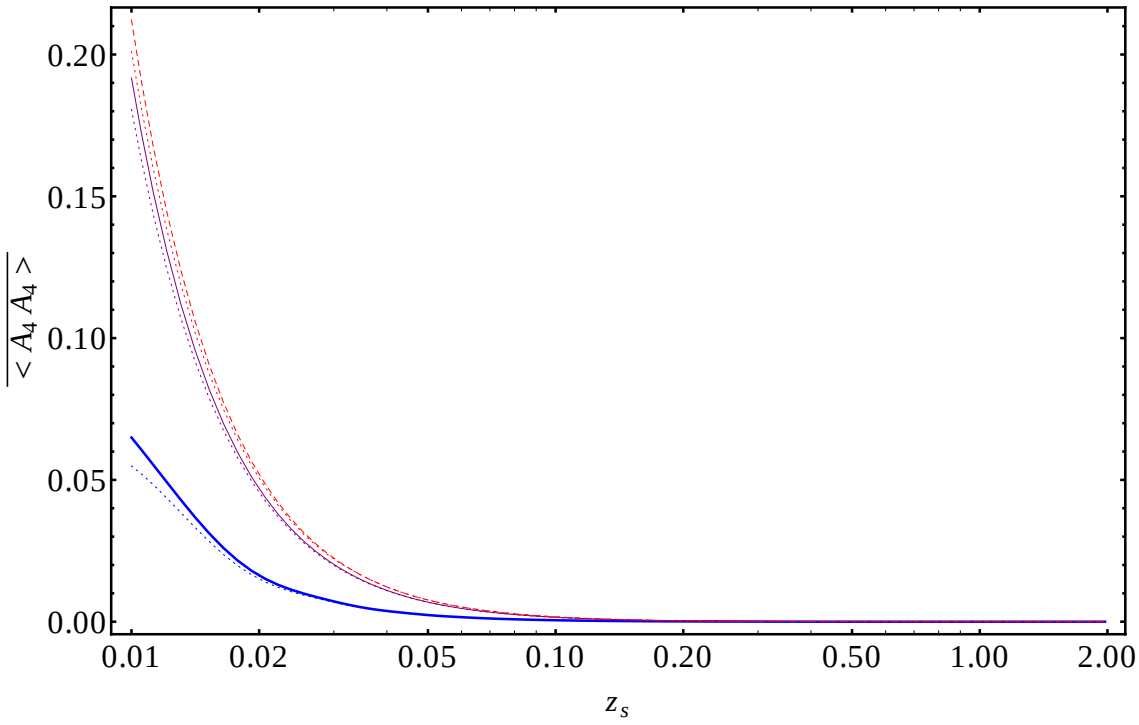

FIG. 1: The result of the numerical integration of Eq. (5.6) for $\overline{\langle A_4A_4\rangle}$ is plotted as a function of z_s for three different values of the UV cut-off: $k = 0.1\,\mathrm{Mpc}^{-1}$ (thick blue line), $k = 1\,\mathrm{Mpc}^{-1}$ (thin purple line), $k = +\infty$ (dashed red line). Also shown (by the corresponding dotted curves) is the plot of the contribution to IBR_2 $\overline{\langle A_4A_4\rangle} - \overline{\langle A_4\rangle\langle A_4\rangle}$.

also in IBR_2 since A_4 is present in both μ_1 and σ_1. In Fig. 1 we have also plotted the full contribution to IBR_2, namely $\overline{\langle A_4 A_4\rangle} - \overline{\langle A_4\rangle\langle A_4\rangle}$ (see Eq. (5.8)): we can see that the difference from the behavior of the $\overline{\langle A_4 A_4\rangle}$ term is very small, with no qualitative effect on our discussion.

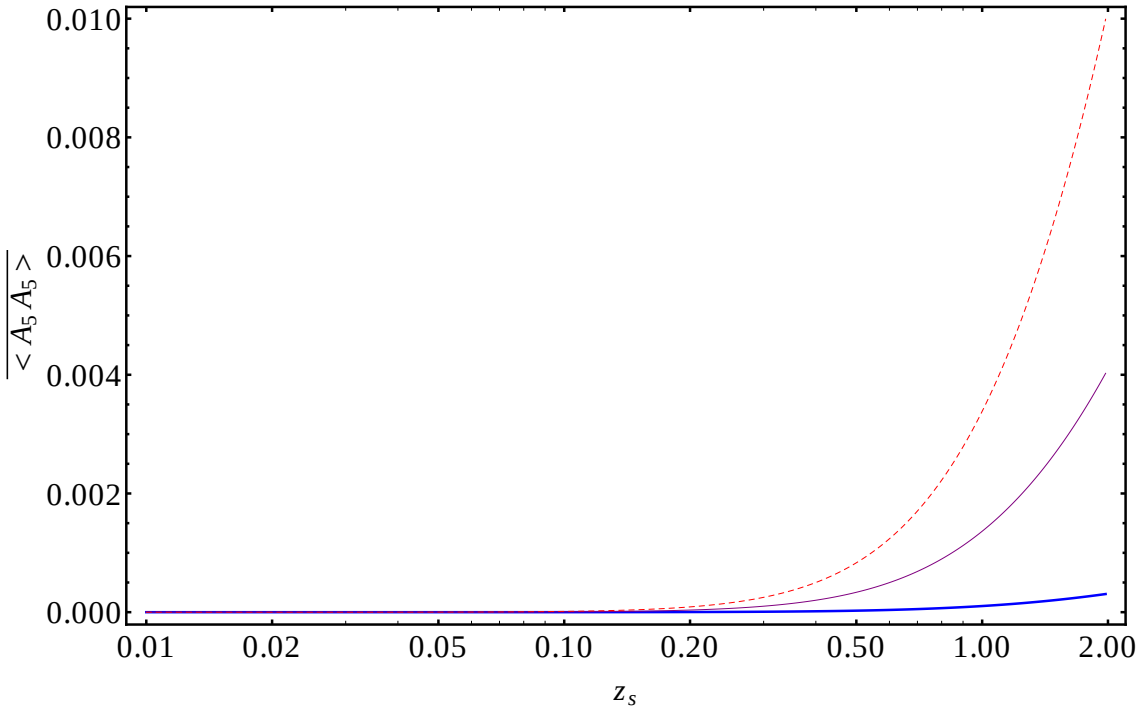

FIG. 2: The result of the numerical integration of Eq. (5.6) for $\overline{\langle A_5 A_5\rangle}$ is plotted as a function of z_s for three different values of the UV cut-off: $k = 0.1\,\mathrm{Mpc}^{-1}$ (thick blue line), $k = 1\,\mathrm{Mpc}^{-1}$ (thin purple line), $k = +\infty$ (dashed red line).

It is important to stress that, although all considered backreaction contributions are (at any redshift) UV-finite, they can induce relatively big effects at the distance scales relevant for supernovae observations (see Figs. 1 and 2), provided the given spectrum is extrapolated to sufficiently large values of k. We also stress that the decoupling of the small-distance (i.e. high-k) scales from the considered large-scale backreaction is due to the efficient suppression of the linear perturbation modes inside the horizon, an effect well described by the transfer function (5.15).

Let us now sum up all contributions to the induced backreaction (5.8) and to the dispersion (5.9), and compare the results for $\overline{\langle d_L\rangle} \pm d_L^{CDM}\sqrt{\overline{\langle\sigma_1^2\rangle}}$ with the homogeneous luminosity-distance of a pure CDM model and of a successful ΛCDM model. We will include into $\overline{\langle d_L\rangle}$ only the IBR_2 contribution (4.7), namely we will set $\overline{\langle d_L\rangle} = d_L^{CDM}(1+\mathrm{IBR}_2)$. It is clear that a full computation should include additional contributions, of the same order as IBR_2 or even larger, arising from second-order perturbations of d_L (we are referring to the term called $\overline{\langle\sigma_2\rangle}$ in Eq. (4.6)). Nonetheless, we believe that (modulo cancellations) our computation may estimate a reliable "lower limit" on the strength of the possible corrections to the luminosity-redshift relation in the context of our inhomogeneous geometry.

The comparison between the homogeneous and inhomogeneous (averaged) values of d_L can be conveniently illustrated by plotting the so-called distance modulus $m - M = 5\log_{10} d_L$ or, even better, by plotting the difference between the distance modulus of the considered model and that of a flat, linearly expanding Milne-type geometry, used as reference value (see e.g. [18]). In such a case we can plot, in particular, the following quantity:

$$\Delta(m-M) = 5\log_{10}\left[\overline{\langle d_L\rangle}\right] - 5\log_{10}\left[\frac{(2+z_s)z_s}{2H_0}\right]. \tag{6.3}$$

The results are illustrated in Fig. 3 for the case of a cut-off $k_{UV} = 0.1\,\mathrm{Mpc}^{-1}$, and in Fig. 4 for $k_{UV} = 1\,\mathrm{Mpc}^{-1}$. The averaged luminosity-redshift relation of our inhomogeneous model is compared, in particular, with that of a pure CDM model and with that of a ΛCDM model with $\Omega_\Lambda = 0.1$ and with $\Omega_\Lambda = 0.73$. We have also explicitly shown the expected dispersion around the averaged result, by plotting the curves corresponding to $\overline{\langle d_L\rangle} \pm d_L^{CDM}\sqrt{\overline{\langle\sigma_1^2\rangle}}$ (bounding the coloured areas appearing in the figures).

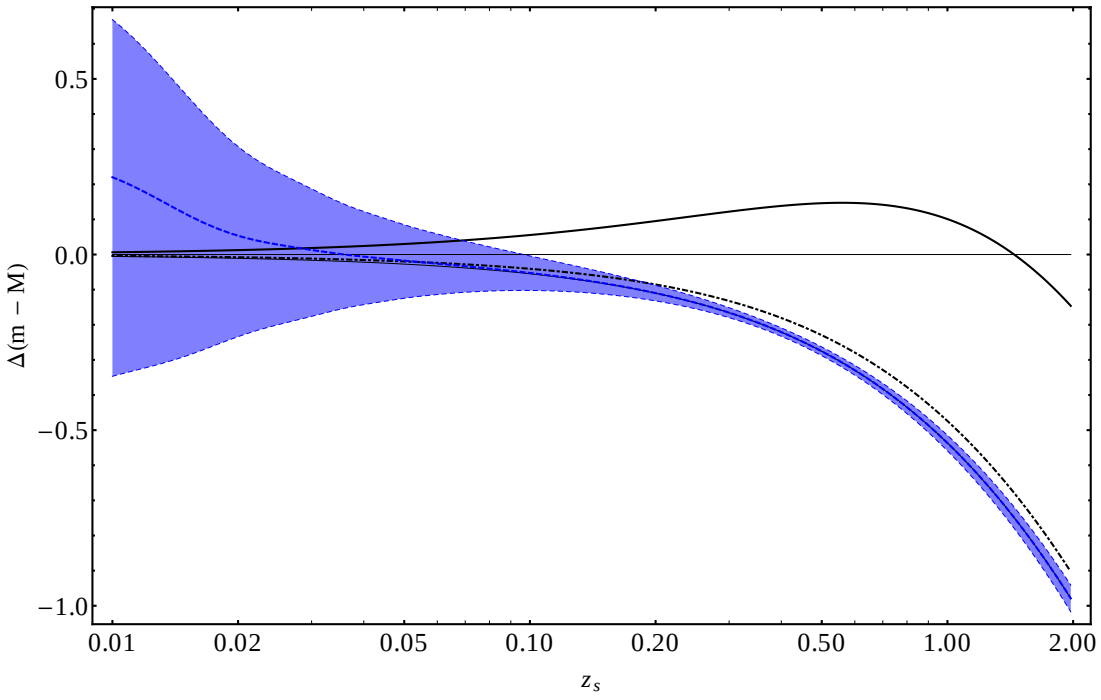

FIG. 3: The distance-modulus difference of Eq. (6.3) is plotted for a pure CDM model (thin line), for a CDM model including the contribution of IBR_2 (dashed blue line) plus/minus the dispersion (coloured region), and for a ΛCDM model with $\Omega_\Lambda = 0.73$ (thick line) and $\Omega_\Lambda = 0.1$ (dashed-dot thick line). We have used for all backreaction integrals the cut-off $k = 0.1\,\mathrm{Mpc}^{-1}$.

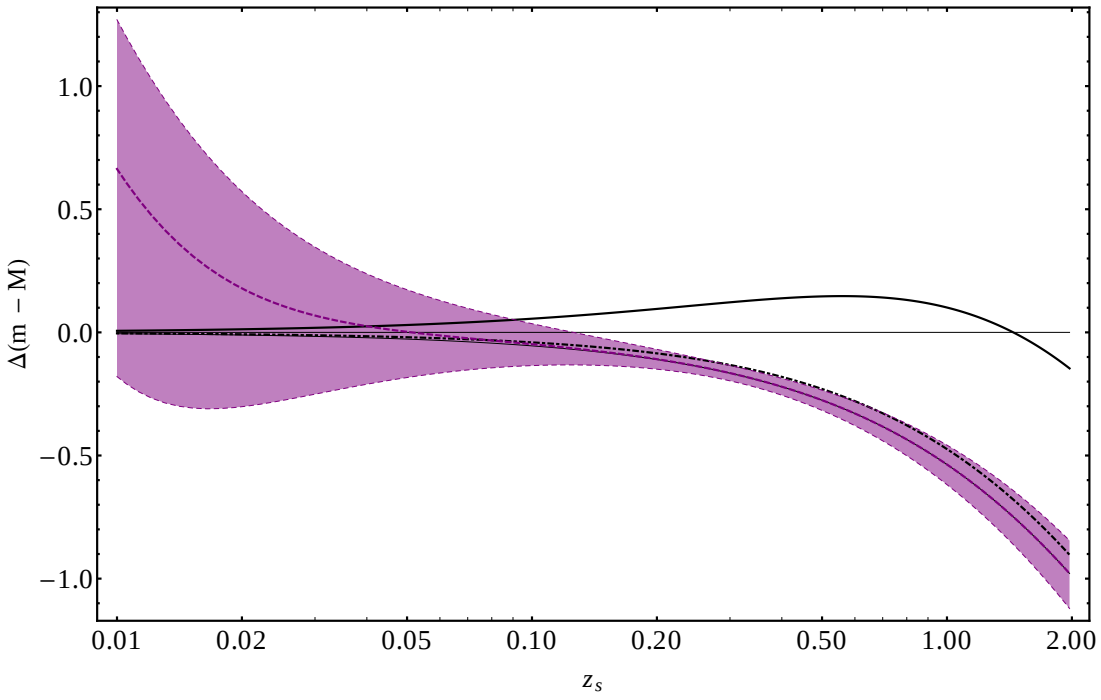

FIG. 4: The distance-modulus difference of Eq. (6.3) is plotted for a pure CDM model (thin line), for a CDM model including the contribution of IBR_2 (dashed blue line) plus/minus the dispersion (coloured region), and for a ΛCDM model with $\Omega_\Lambda = 0.73$ (thick line) and $\Omega_\Lambda = 0.1$ (dashed-dot thick line). We have used for all backreaction integrals the cut-off $k = 1\,\mathrm{Mpc}^{-1}$.

Let us stress again that the choice of the cut-off may affect (even if not dramatically) the final result when the values of k_{UV} are varying in the range $(0.1-1)\,\mathrm{Mpc}^{-1}$, while the precise choice becomes less important at higher values of k_{UV}. We should recall, however, that the inhomogeneous model adopted in this paper is fully under control only in the linear perturbative regime, and that the spectrum cannot be extrapolated at scales higher than about $k \sim 1\,\mathrm{Mpc}^{-1}$ without taking into account the complicated effects of its non-linear dynamical evolution. The approximate coincidence between the above limiting value of k_{UV} and the scale marking the beginning of the non-linear perturbative regime is not only a particular consequence of the transfer function adopted in this paper (see Eq. (5.15)), but also an avoidable property of realistic non-linear perturbations spectra (see e.g. [31]).

As clearly shown by the two figures, the corrections induced by IBR_2 on the luminosity distance of a homogeneous CDM model, even taking into account the expected dispersion of values around $\overline{\langle d_L \rangle}$, cannot be used to successfully simulate realistic dark-energy effects. In the figures we have also plotted, for illustrative purposes, an example of standard ΛCDM model with $\Omega_\Lambda = 0.1$, which seems to be compatible (within the allowed region defined by the dispersion) with the prediction of our simple CDM+IBR_2 model, at least for sufficiently high values of the cut-off scale (see in particular Fig. 4). However, we warn the reader that it would be wrong to conclude that our backreaction can mimic a fraction of dark energy of the order of $\Omega_\Lambda \sim 0.1$, because a similar conclusion might be reliably reached only by averaging inhomogeneities on a background which already includes a significant amount of dark energy from the beginning.

Let us conclude this section with some important comments. First, as already stressed, a consistent second-order computation of the backreaction should include the contribution of $\overline{\langle \sigma_2 \rangle}$. This requires a full treatment of metric perturbations and coordinate transformations up to second order (work is in progress on this point [32]). Nevertheless, we can easily see from Eqs.(2.18) and (2.19) that $\overline{\langle \sigma_2 \rangle}$ contains, among others, contributions of the type $\overline{\langle A_5 A_5 \rangle}$ already computed in this paper. The behaviour of this term, in the asymptotic regime $k\Delta\eta \gg 1$, is very different from the behaviour of terms like $\overline{\langle A_4 A_4 \rangle}$ which give the leading contribution to IBR_2: the contribution of $\overline{\langle A_5 A_5 \rangle}$, in particular, grows at large redshifts, as illustrated in Fig. 2. Using our results for $\overline{\langle A_5 A_5 \rangle}$ – and barring cancellations – we can obtain, for instance, the following numerical estimates: with an UV cutoff $k_{UV} = 1\,\mathrm{Mpc}^{-1}$ we expect $|\overline{\langle \sigma_2 \rangle}| \gtrsim 1.5 \times 10^{-3}$ at $z = 1$ and $|\overline{\langle \sigma_2 \rangle}| \gtrsim 4 \times 10^{-3}$ at $z = 2$. These expectations do not change much if we increase further the UV cutoff: they only go up by about a factor two even if we send k_{UV} arbitrarily high (at least within our form of the power spectrum).

These examples suggest that a full computation of $\overline{\langle \sigma_2 \rangle}$ could strongly enhance the overall backreaction effects at large z_s, with respect to the effects due to IBR_2 discussed previously. Hence a full second-order computation, possibly joined to a reliable estimate of contributions from the non-linear regime[7], appears to be necessary before firm conclusions on the correct interpretation of the data can be drawn. Also, the different behaviour of the different backreaction contributions, at small z_s and large z_s, could represent an important signature to distinguish the effects due to averaged inhomogeneities from the more conventional dynamical effects of homogeneous dark energy sources.

The second comment is that, although a reliable estimate of the full backreaction on the averaged luminosity distance requires a full second-order calculation, some suitable linear combinations of averages of different powers of d_L only depend on the first-order quantity σ_1 (defined by the expansion of d_L). As an example, one can show that the following equality holds at second order for any value of the real parameter α:

$$\overline{\left\langle \left(d_L/d_L^{FLRW}\right)^\alpha \right\rangle} - \alpha \overline{\left\langle d_L/d_L^{FLRW} \right\rangle} = 1 - \alpha + \frac{\alpha(\alpha-1)}{2}\overline{\langle \sigma_1^2 \rangle}. \tag{6.4}$$

This quantity can be plotted for a given inhomogeneous model, and compared with its (deterministic) value in a ΛCDM model, for various values of α. The result is that the two models disagree for generic α, leading to the conclusion that realistic inhomogeneities added to CDM lead to a model that can be distinguished, in principle, from the conventional ΛCDM scenario. In practice, however, we only have a single quantity measured by the supernovae experiments (basically the received flux of radiation), and one cannot exclude that the two models happen to give the same result for that particular observable.

VII. CONCLUSIONS

Let us briefly summarize the main ideas and results of this work. Using a previously introduced gauge invariant light-cone averaging prescription, as well as an adapted coordinate system, we were able to write down an exact

[7] The importance of such a non linear regime was also recently underlined in [33], following a different approach to describe the impact of inhomogeneities on the supernovae observations.

expression for the averaged luminosity distance $d_L(z)$ (or any function of it) as a function of the redshift z. In principle such an expression can be used to study the effect of inhomogeneities even outside the domain of cosmological perturbation theory. In this paper we have only attempted a first study of the leading-order effect generated by stochastic perturbations of the type produced by inflation on top of a pure CDM model.

We have then been able to separate two distinct contributions to the backreaction. The first, that we called induced backreaction (IBR), is sensitive to our specific averaging prescription and originates from correlations in the fluctuations of the luminosity distance and those in the integration measure. As a consequence, IBR can be computed to second order (giving what we have called IBR_2) by using results from linear perturbation theory. The second contribution is insensitive to the averaging prescription but needs a full computation of second order perturbations in the luminosity distance itself, something that we are leaving to future work. Linear perturbation theory also allows to compute the expected variance on the (angular and ensemble) average of $d_L(z)$.

Our final integrals for these quantities are nicely behaved both in the infrared (showing explicitly that perturbations on scales much larger than the source's distance do not contribute) and in the ultraviolet, provided short-scale perturbations behave roughly as in simple, realistic models for the matter power spectrum. This does not mean, however, that the final result is insensitive to the detailed structure of inhomogeneities in the non-linear (or even non-perturbative) regime.

Although the power spectrum has an overall normalization of order 10^{-9} (from CMB data), it induces a backreaction which is strongly enhanced by phase-space factors of order $(k_*/H_0)^p$ (with p a power that can be as big as 2 or 3), where k_* is some typical scale appearing in the power spectrum. These enhancement factors can bring the IBR effects many orders of magnitude higher. However, if we use a rather low momentum cutoff (by insisting on staying inside the linear regime), IBR_2 effects turn out to be way too small to mimic an appreciable cosmological constant (see for instance Fig. 3). The total backreaction could instead be larger, but a reliable determination of its magnitude depends on being able to carry out the full second-order calculation mentioned above, as well as on having some knowledge about the non-perturbative (short-distance) contribution to d_L.

We have also noticed that the expected (angular) dispersion of d_L is quite large (see Figs. 3 and 4), particularly at small and at high z_s (with a minimum in the region of intermediate z_s, i.e. $z_s = 0.3$-0.6), something that one should be able to check quite precisely once more data become available.

We note that the dominant contribution to the dispersion at small z comes from the $\overline{\langle A_4 A_4\rangle}$ term, and is therefore associated with "Doppler-type" contributions, since the quantity I_r is related to the source and observer velocities. This is in agreement with the claims made in [19]. By contrast, the main contribution at large z comes from $\overline{\langle A_5 A_5\rangle}$, i.e. from terms usually referred to as "lensing" contributions. Again, this is in agreement with the literature (see [21]).

The size and the nice IR and UV behaviour of our gauge invariant quantities can be contrasted with what happens if one approaches the backreaction problem using averaging on spacelike hypersurfaces, as in [25, 34]. In that case the calculation of the average expansion rate is UV convergent and practically cut-off independent. In a reasonable range of cut-off values it gives a backreaction of order 10^{-5} for the concordance model. However, for other quantities such as the variance of the expansion rate or the deceleration parameter, terms like $\sim(\partial^2\Psi)^2$ appear, giving UV-divergent integrals and a backreaction of $O(1)$ for relatively low cut-off. In our case, considering a UV cut-off equal to $1\,\mathrm{Mpc}^{-1}$ we obtain a backreaction effect (estimated from $\overline{\langle A_5 A_5\rangle}$) of the order of 10^{-3} for $z\sim 1$, namely 2 orders of magnitude bigger than the result presented in [25, 34]. Furthermore, there is no qualitative difference between the calculation of the induced backreaction and the one of the variance. They are all given by nice IR and UV-convergent integrals which never produce numbers of $O(1)$ even for very large momentum cutoffs.

Finally, by suitably combining averages of different functions of d_L, we can again obtain results that, like the variance, only depend on linear perturbation theory. They show that, in principle, an inhomogeneous CDM model can be neatly distinguished from a homogeneous ΛCDM model. It remains to be seen whether any such combination is accessible to observations. Nonetheless, in the light of this last observation, it would appear quite unlikely that one will be able to fully account for the supernovae data in terms of inhomogeneity effects. Rather, depending on the full contribution of the second-order terms and, possibly, on the one from the highly non-linear regime, the effect of inhomogeneities could be relevant for future precise determinations of the critical fraction and equation of state of dark energy. In this spirit we plan to extend the calculations presented here to the (only slightly more complicated) case of generic ΛCDM models.

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Appendix A. Other examples of backreaction integrals

Here we present a detailed computation of some of the terms contributing to the induced backreaction and to the dispersion associated to the light-cone average of $d_L(z_s,\theta^a)$. Such computations also define the corresponding spectral coefficients appearing in Table I and Table II. We will start with $\overline{\langle A_2A_5\rangle}$ and $\overline{\langle A_2\rangle\langle A_5\rangle}$ as illustrative examples. We will then consider the two leading contributions given by $\overline{\langle A_4A_4\rangle}$ and $\overline{\langle A_5A_5\rangle}$ (and, for completeness, the associated terms $\overline{\langle A_4\rangle\langle A_4\rangle}$ and $\overline{\langle A_5\rangle\langle A_5\rangle}$).

Using equations (3.21), (3.26), (4.15), and working in the hypothesis of time-independent Ψ_k, we obtain:

$$\begin{aligned}
\overline{\langle A_2A_5\rangle} &= \int \frac{d^3k\; d^3k'}{(2\pi)^3}\overline{E(\vec{k})E(\vec{k'})}\int\frac{d^2\Omega}{4\pi}\left[\frac{2}{\Delta\eta}\int_{\eta_s}^{\eta_0}d\eta'\;\Psi_k\,e^{i(\eta_0-\eta')\vec{k}\cdot\hat{x}}\right]\left[-\int_{\eta_s}^{\eta_0}\frac{d\eta''}{\Delta\eta}\frac{\eta''-\eta_s}{\eta_0-\eta''}\;\Delta_2\left(\Psi_{k'}\,e^{i(\eta_0-\eta'')\vec{k'}\cdot\hat{x}}\right)\right]\\
&= \int\frac{d^3k}{(2\pi)^3}\;|\Psi_k|^2\int_{-1}^{1}\frac{d(\cos\theta)}{2}\left[\frac{2}{\Delta\eta}\int_{\eta_s}^{\eta_0}d\eta'\;e^{ik(\eta_0-\eta')\cos\theta}\right]\\
&\quad\times\left[\frac{1}{\Delta\eta}\int_{\eta_s}^{\eta_0}d\eta''\frac{\eta''-\eta_s}{\eta_0-\eta''}\left(k^2(\eta_0-\eta'')^2\sin^2\theta-2ik(\eta_0-\eta'')\cos\theta\right)e^{-ik(\eta_0-\eta'')\cos\theta}\right]\\
&= \int_0^\infty\frac{dk}{k}\;P_\Psi(k)\cdot\frac{2}{3(k\Delta\eta)^2}\left[-4+(4+(k\Delta\eta)^2)\cos(k\Delta\eta)+k\Delta\eta\sin(k\Delta\eta)+(k\Delta\eta)^3\mathrm{SinInt}(k\Delta\eta)\right];
\end{aligned}\tag{A.1}$$

$$\begin{aligned}
\overline{\langle A_2\rangle\langle A_5\rangle} &= \int\frac{d^3k\; d^3k'}{(2\pi)^3}\overline{E(\vec{k})E(\vec{k'})}\left[\int\frac{d^2\Omega}{4\pi}\frac{2}{\Delta\eta}\int_{\eta_s}^{\eta_0}d\eta'\;\Psi_k\,e^{i(\eta0-\eta')\vec{k}\cdot\hat{x}}\right]\\
&\quad\times\left[-\int\frac{d^2\Omega'}{4\pi}\int_{\eta_s}^{\eta_0}\frac{d\eta''}{\Delta\eta}\frac{\eta''-\eta_s}{\eta_0-\eta''}\;\Delta_2\left(\Psi_{k'}\,e^{i(\eta_0-\eta'')\vec{k'}\cdot\hat{x}}\right)\right]\\
&= \int\frac{d^3k}{(2\pi)^3}\;|\Psi_k|^2\left[\int_{-1}^{1}\frac{d(\cos\theta)}{2}\frac{2}{\Delta\eta}\int_{\eta_s}^{\eta_0}d\eta'\;e^{ik(\eta_0-\eta')\cos\theta}\right]\\
&\quad\times\left[\int_{-1}^{1}\frac{d(\cos\theta')}{2}\frac{1}{\Delta\eta}\int_{\eta_s}^{\eta_0}d\eta''\frac{\eta''-\eta_s}{\eta_0-\eta''}\left(k^2(\eta_0-\eta'')^2\sin^2\theta'-2ik(\eta_0-\eta'')\cos\theta'\right)e^{-ik(\eta_0-\eta'')\cos\theta'}\right]\\
&= \int_0^\infty\frac{dk}{k}\;P_\Psi(k)\cdot\left[\frac{2}{k\Delta\eta}\mathrm{SinInt}(k\Delta\eta)\right]\times 0\;=\;0.
\end{aligned}\tag{A.2}$$

In the second lines of Eqs. (A.1) and (A.2) we have applied Eq. (5.2) to remove the integration over $\vec{k'}$, and we have used the isotropy of the scalar product $\vec{k}\cdot\hat{x}$.

We move now to the terms representing the two leading contributions to IBR_2 and to the dispersion. Starting from Eq. (3.14), and using the same hypotheses as in the previous calculations, we have

$$\begin{aligned}
I_r &= \int_{\eta_{in}}^{\eta_s}d\eta'\frac{a(\eta')}{a(\eta_s)}\partial_r\Psi(\eta',\eta_0-\eta_s,\theta^a)-\int_{\eta_{in}}^{\eta_0}d\eta'\frac{a(\eta')}{a(\eta_0)}\partial_r\Psi(\eta',0,\theta^a)\\
&= \int\frac{d^3k}{(2\pi)^{3/2}}\left\{\int_{\eta_{in}}^{\eta_s}d\eta'\frac{\eta'^2}{\eta_s^2}(ik\cos\theta)\Psi_k\,e^{i\Delta\eta\vec{k}\cdot\hat{x}}-\int_{\eta_{in}}^{\eta_0}d\eta'\frac{\eta'^2}{\eta_0^2}(ik\cos\theta)\Psi_k\right\}\\
&= \int\frac{d^3k}{(2\pi)^{3/2}}\Psi_k\cdot\left(f_s e^{ik\Delta\eta\cos\theta}-f_0\right)(ik\cos\theta).
\end{aligned}\tag{A.3}$$

Then, according to the definitions (4.15):

$$\overline{\langle A_4A_4\rangle}=\left(1-\frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\overline{\langle I_rI_r\rangle},\tag{A.4}$$

where

$$\overline{\langle I_r I_r\rangle} = \int \frac{d^3k}{(2\pi)^3}\,|\Psi_k|^2 \int_{-1}^{1}\frac{d(\cos\theta)}{2}\left[\left(f_s e^{ik\Delta\eta\cos\theta}-f_0\right)(ik\cos\theta)\right]\left[\left(f_s e^{-ik\Delta\eta\cos\theta}-f_0\right)(-ik\cos\theta)\right]$$
$$= \int_0^\infty \frac{dk}{k}\,P_\Psi(k)\cdot\frac{1}{3\Delta\eta^2}\left[(f_0^2+f_s^2)(k\Delta\eta)^2-12f_0f_s\cos(k\Delta\eta)-6f_0f_s(-2+(k\Delta\eta)^2)\frac{\sin(k\Delta\eta)}{k\Delta\eta}\right], \quad (A.5)$$

and where we have used the following exact integral result:

$$\int_{-1}^{1}\frac{d(\cos\theta)}{2}(\cos\theta)^2 e^{\pm ik\Delta\eta\cos\theta} = \frac{2k\Delta\eta\cos(k\Delta\eta)+(-2+(k\Delta\eta)^2)\sin(k\Delta\eta)}{(k\Delta\eta)^3}\,. \quad (A.6)$$

In the same way we obtain:

$$\overline{\langle A_4\rangle\langle A_4\rangle} = \left(1-\frac{1}{\mathcal{H}_s\Delta\eta}\right)^2\overline{\langle I_r\rangle\langle I_r\rangle}, \quad (A.7)$$

where

$$\overline{\langle I_r\rangle\langle I_r\rangle} = \int\frac{d^3k}{(2\pi)^3}\,|\Psi_k|^2\left[\int_{-1}^{1}\frac{d(\cos\theta)}{2}\left(f_s e^{ik\Delta\eta\cos\theta}-f_0\right)(ik\cos\theta)\right] \quad (A.8)$$
$$\times\left[\int_{-1}^{1}\frac{d(\cos\theta')}{2}\left(f_s e^{-ik\Delta\eta\cos\theta'}-f_0\right)(-ik\cos\theta')\right]$$
$$= \int_0^\infty\frac{dk}{k}\,P_\Psi(k)\cdot\left[\frac{f_s}{\Delta\eta}\left(\cos(k\Delta\eta)-\frac{\sin(k\Delta\eta)}{k\Delta\eta}\right)\right]^2, \quad (A.9)$$

and where we have used the integral

$$\int_{-1}^{1}\frac{d(\cos\theta)}{2}\,\cos\theta e^{\pm ik\Delta\eta\cos\theta} = \pm i\left(\frac{-k\Delta\eta\cos(k\Delta\eta)+\sin(k\Delta\eta)}{(k\Delta\eta)^2}\right). \quad (A.10)$$

Following a similar procedure Eq.(3.21) leads us to

$$\overline{\langle A_5 A_5\rangle} = \int\frac{d^3k}{(2\pi)^3}\,|\Psi_k|^2\int_{-1}^{1}\frac{d(\cos\theta)}{2}\left[\int_{\eta_s}^{\eta_0}\frac{d\eta'}{\Delta\eta}\frac{\eta'-\eta_s}{\eta_0-\eta'}\left(k^2(\eta_0-\eta')^2\sin^2\theta+2ik(\eta_0-\eta')\cos\theta\right)e^{ik(\eta_0-\eta')\cos\theta}\right]$$
$$\times\left[\int_{\eta_s}^{\eta_0}\frac{d\eta''}{\Delta\eta}\frac{\eta''-\eta_s}{\eta_0-\eta''}\left(k^2(\eta_0-\eta'')^2\sin^2\theta-2ik(\eta_0-\eta'')\cos\theta\right)e^{-ik(\eta_0-\eta'')\cos\theta}\right]$$
$$= \int_0^\infty\frac{dk}{k}\,P_\Psi(k)\cdot\frac{1}{15(k\Delta\eta)^2}\left[-24+20(k\Delta\eta)^2+(24-2(k\Delta\eta)^2+(k\Delta\eta)^4)\cos(k\Delta\eta)\right.$$
$$\left.-6k\Delta\eta\sin(k\Delta\eta)+(k\Delta\eta)^3\sin(k\Delta\eta)+(k\Delta\eta)^5\mathrm{SinInt}(k\Delta\eta)\right]. \quad (A.11)$$

Note that, for the same reason for which $\overline{\langle A_2\rangle\langle A_5\rangle}=0$, i.e. for the fact that the integrand of A_5 gives zero when averaged over the angles, we finally also obtain $\overline{\langle A_5\rangle\langle A_5\rangle}=0$.

[1] K. Nakamura et al., JPG **37**, 075021 (2010) (`http://pdg.lbl.gov`).

[2] A. G. Riess et al., Astron. J. **116**, 1009 (1998); S. Perlmutter et al., Astrophys. J **517**, 565 (1999).

[3] C. Clarkson, G. Ellis. J. Larena and O. Umeh, Rep. Prog. Phys. **74**, 112901 (2011); T. Buchert, Class. Quantum Grav. **28**, 164007 (2011); G. F. R. Ellis, Class. Quantum Grav. **28**, 164001 (2011). T. Buchert and S. Rasanen, arXiv:1112.5335 [astro-ph.CO].

[4] T. Buchert, Jap. Gen. Rel. Grav. **9**, 306 (2000); T. Buchert and M. Carfora, Phys. Rev. Lett. **90**, 031101 (2003); S. Rasanen, JCAP **02**, 003 (2004); JCAP **11**, 003 (2006); E. Barausse, S. Matarrese, and A. Riotto, Phys. Rev. **D 71** 063537 (2005); E. W. Kolb, S. Matarrese and A. Riotto, New. J. Phys. **8** (2006) 322.

[5] E. E. Flanagan, Phys. Rev. **D 71**, 103521 (2005); C. M. Hirata and U. Seljak, Phys. Rev. **D 72**, 083501 (2005); A. Ishibashi, and R. M. Wald, Class. Quant. Grav. **23**, 235 (2006); A. Paranjape and T. P. Singh, Phys. Rev. Lett. **101**, 181101 (2008); D. Baumann, A. Nicolis, L. Senatore, and M. Zaldarriaga, arXiv:1004.2488 [astro-ph.CO]; S. R. Green and R. M. Wald, Phys. Rev. D **83**, 084020 (2011).

[6] T. Buchert, Gen. Rel. Grav. **32**, 105 (2000); **33**, 1381 (2001).
[7] R. Maartens, PhD Thesis, University of Cape Town, South Africa (1980).
[8] R. Maartens and D. Matravers, Class. Quant. Grav. **11**, 2693 (1994).
[9] V. Marra, E. W. Kolb and S. Matarrese, Phys. Rev. **D 77**, 023003 (2008); N. Li, M. Seikel and D. J.Schwarz, Fortsch. Phys. **56**, 465 (2008); S. Rasanen, JCAP **02**, 011 (2009); E. W. Kolb and C. R. Lamb, arXiv:0911.3852.
[10] M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, JCAP **07**, 008 (2011).
[11] M. Gasperini, G. Marozzi and G. Veneziano, JCAP **03**, 011 (2009).
[12] M. Gasperini, G. Marozzi and G. Veneziano, JCAP **02**, 009 (2010).
[13] V. F. Mukhanov, H. A. Feldman and R. R. Brandenberger, Phys. Rep. **215**, 203 (1992).
[14] G. Marozzi, JCAP **01** (2011) 012.
[15] G. F. R. Ellis, S. D. Nel, R. Maartens, W. R. Stoeger and A. P. Whitman, Phys. Rep. **124**, 315 (1985).
[16] C. Clarkson and R. Maartens, Class. Quantum Grav. **27**, 12 (2010).
[17] I. M. H. Etherington, Phil. Mag. **15**, 761 (1933).
[18] M. Gasperini, *Elements of String Cosmology* (Cambridge University Press, Cambridge, UK 2007).
[19] R. A Vanderveld, E. E. Flanagan and I. Wasserman, Phys. Rev. **D 76**, 083504 (2007).
[20] G. Lemaître, Ann. Soc. Sci. Brux. **A 53**, 51 (1933). Reprint in G. Lemaître, Gen. Rel. Grav. **29**, 5 (1997); R.C. Tolman, Proc. Natl. Acad. Sci. **20**, 169 (1934); H. Bondi, Mon. Not. R. Astron. Soc. **107**, 410 (1947).
[21] C. Bonvin, R. Durrer and A. Gasparini, Phys. Rev. **D 73**, 023523 (2006) (see also the last revised version at arXiv:astro-ph/0511183v5)
[22] A. Amanullah et al. (The Supernova Cosmology Project), *Astrophys. J.* **716** 712 (2010).
[23] N. Li and D. J. Schwarz, Phys. Rev. D **78**, 083531 (2008).
[24] C. Clarkson, K. Ananda and J. Larena, Phys. Rev. **D 80**, 083525 (2009).
[25] C. Clarkson and O. Umeh, Class. Quantum Grav. P**28**, 164010 (2011).
[26] A. Wiegand and D. J. Schwarz, arXiv:1109.4142 [astro-ph.CO].
[27] R. Durrer, *The Cosmic Microwave Background* (Cambridge University Press, Cambridge, UK 2008).
[28] E. Komatsu *et al.*, Astrophys. J. Suppl. **192**, 18 (2011).
[29] D. J. Eisenstein, W. Hu, Astrophys. J. **496**, 605 (1998).
[30] R. E. Smith, R. K. Sheth and R. Scoccimarro, Phys.Rev. **D 78**, 023523 (2008).
[31] R. E. Smith et al., Mon. Not. R. Astron. Soc. **341**, 1311 (2003).
[32] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, *Second order light-cone backreaction*, in preparation.
[33] C. Clarkson, G. Ellis, A. Faltenbacher, R. Maartens, O. Umeh and J. P. Uzan, arXiv:1109.2484 [astro-ph.CO].
[34] E. W. Kolb, Class. Quantum Grav. P**28**, 164009 (2011).

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The Price of an Exact, Gauge-Invariant RG-Flow Equation

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Abstract

We combine old ideas about exact renormalization-group-flow (RGF) equations with the Vilkovisky-De Witt (VDW) approach to reparametrization invariant effective actions and arrive at a new, exact, gauge-invariant RGF equation. The price to be paid for such a result is that both the action and the RGF equation depend explicitly upon the base point (in field space) needed for the VDW construction. We briefly discuss the complications originating from this fact and possible ways to overcome them.

The idea of renormalization, originally introduced to remove infinities from perturbative calculations, has evolved into a powerful tool that helps understanding the global behaviour of quantum and statistical systems under changes of the observation scale[1, 2].

The search for new, non-perturbative methods to handle problems out of the reach of perturbation theory has prompted in recent years a renewed and growing interest [3, 4, 5, 6, 7] in the "old" subject [1, 8] of "exact" renormalization group (RG) equations. One typically defines a scale-dependent

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effective action, Γ_k, which interpolates between the classical (bare) action S at $k = \Lambda$ (the UV cutoff) and the effective action Γ at $k = 0$. The free term is modified by the introduction of a suitable (but largely arbitrary) cutoff function that effectively kills the contribution to the functional integral from momenta below the running scale k. The implementation of such a procedure for gauge theories poses however a major problem: the presence of the cutoff function prevents the possibility of defining a gauge invariant Γ_k (see [9] for earlier attempts to circumvent this problem).

In this letter we follow the spirit of Ref. [1, 8], that of direct integration over successive shells of degrees of freedom, and combine this idea with the geometrical approach pioneered by Vilkovisky and De Witt [10, 11] (see also [12]) in order to define a gauge-invariant (more generally a reparametrization-invariant) effective action. The final outocome will be an exact, gauge-invariant, RGF equation that, to the best of our knowledge, has never appeared before in the literature. It contains an explicit dependence upon the base-point (in field space) that enters the VDW construction, a price that we believe to be inevitable for achieving the goal. We shall comment at the end on the possible complications due to such dependence when one applies it to specific problems. As for the derivation of the equation itself, it will be oulined, for pedagogical reasons, in three steps. We first present a new (and in our opinion more transparent) derivation of basically known results for scalar theories. The main new results follow as we turn to the case of reparametrization-invariant effective actions a la VDW and their exact RGF equation. The final step, going over to the case of gauge theories, is then straightforward, as often emphasized by Vilkovisky.

We thus begin by defining Γ_k for a simple scalar field theory. If Λ is the UV cutoff, we introduce the notation ϕ_0^Λ for the field, to indicate that it contains "modes" in the range $[0, \Lambda]$, and write the classical (bare) action as $S[\phi_0^\Lambda]$. For any given scale k, we divide ϕ_0^Λ into the "low-frequency" and "high-frequency" components, ϕ_0^k and ϕ_k^Λ respectively, where ϕ_0^k contains the modes ϕ_p with $0 < p < k$, and ϕ_k^Λ those in the range $[k, \Lambda]$. Even though for the scalar theory it is always possible to define the RG flow in Fourier space, it is well known that the notion of RG flow is much more general. Neither k nor Λ must necessarily have the meaning of momenta (this observation is important for the following where we have to implement a gauge invariant flow for gauge theories).

Let us now introduce the notion of "shell", described by δk, denote

the fields $\phi_0^{k-\delta k}$, $\phi_{k-\delta k}^k$ and ϕ_k^Λ by $\phi_<$, ϕ_S and $\phi_>$, respectively, and use de Witt's [11] condensed notation whereby an index such as i denotes all indices (Fourier, Lorentz, spinor, space-time coordinate x, ...). Repeated indices will denote summation over internal indices as well as integration over space-time (or momenta). The components of ϕ_S and $\phi_>$ will be indicated by ϕ^s and ϕ^a (same for $\bar{\phi}$), and differentiation w.r.t. any ϕ^i ($\bar{\phi}^i$) by a comma followed by the index i. Later on we will also use $A, B, \ldots$ to denote fields with components in the slightly larger interval $[k-\delta k, \Lambda]$.

The effective action $\Gamma[\bar{\phi}]$, a functional of the "classical" (or "mean") fields $\bar{\phi}$, can be defined as the solution of the functional-integral equation:

$$e^{-\Gamma[\bar{\phi}]} = \int [D\phi] e^{-S[\phi]+(\phi^i-\bar{\phi}^i)\Gamma[\bar{\phi}]_{,i}} \,. \tag{1}$$

The scale-dependent generalization of (1) that we propose to use, and later generalize, is simply obtained from (1) after inserting under the integral a product of δ-functions, $\Pi_0^k \delta(\phi_p - \bar{\phi}_p)$, i.e.:

$$e^{-\Gamma_k[\bar{\phi}]} = \int [D\phi_>] e^{-S[\bar{\phi}_<,\phi_>]+(\phi^a-\bar{\phi}^a)\Gamma_k[\bar{\phi}]_{,a}} \,. \tag{2}$$

This very natural definition of a scale-dependent effective action clearly interpolates between the classical and the quantum action, $\Gamma_\Lambda[\bar{\phi}] = S[\bar{\phi}]$ and $\Gamma_0[\bar{\phi}] = \Gamma[\bar{\phi}]$, and can be obtained by a partial Legendre transform [13] of a functional $W_k[\bar{\phi}_0^k, J_k^\Lambda]$ in which the low-frequency fields $\bar{\phi}_0^k$ are kept as parameters, while the high frequency degrees of freedom are Legendre-transformed.

We now derive some identities that will be useful in the following. By differentiating Γ_k in Eq.(2) w.r.t. $\bar{\phi}^a$, we find (for a non-singular 2nd-derivative matrix of Γ_k)

$$< \phi^a >= \bar{\phi}^a \,, \tag{3}$$

where the average is computed with the weight in Eq. (2). Thus, as we expect, $\bar{\phi}^a$ is the mean value of ϕ^a. Differentiating Eq.(3) w.r.t. $\bar{\phi}^s$ we get:

$$\begin{aligned} & < \; S(\bar{\phi}_<, \bar{\phi}_S, \phi_>)_{,s}(\phi^a - \bar{\phi}^a) > \\ = \; & \Gamma_{k,\,sb} < (\phi^b - \bar{\phi}^b)(\phi^a - \bar{\phi}^a) >= \Gamma_{k,\,sb}(\Gamma_{k,ba})^{-1} \,, \end{aligned} \tag{4}$$

where $-(\Gamma_{k,ba})^{-1}$ is the propagator for modes above the shell. A second useful relation comes from differentiating Γ_k w.r.t. $\bar{\phi}^s$:

$$< S(\bar{\phi}_<, \bar{\phi}_S, \phi_>)_{,s} >= \Gamma_{k,\,s} \,. \tag{5}$$

Finally, differentiating Γ_k once more w.r.t. $\bar{\phi}^s$, and making use of Eq.(4), we obtain the relation:

$$\begin{aligned} &< \ S_{,ss'} > - < S_{,s} S_{,s'} > + < S_{,s} >< S_{,s'} > \\ &= \ \Gamma_{k,\,ss'} - \Gamma_{k,\,sa} (\Gamma_{k,ab})^{-1} \Gamma_{k,\,bs'} \ . \end{aligned} \tag{6}$$

Let us consider now the effective action Γ_k at a slightly lower scale $k - \delta k$. From Eq.(2) we have :

$$e^{-\Gamma_{k-\delta k}[\bar{\phi}_0^\Lambda]} = \int [D\phi_{_S}] e^{(\phi^s - \bar{\phi}^s)\Gamma_{k-\delta k,\,s}} e^Y \ , \tag{7}$$

where

$$e^Y = \int [D\phi_{_>}] e^{-S[\bar{\phi}_{_<}, \phi_{_S}, \phi_{_>}] + (\phi^a - \bar{\phi}^a)\cdot\Gamma_{k-\delta k, a}} \ . \tag{8}$$

We are interested in computing the difference between Γ_k and $\Gamma_{k-\delta k}$ to $O(\delta k)$ and thus start expanding to first order $\Gamma_{k-\delta k,a}$ around $\Gamma_{k,a}$ in Eq.(7). At the same time we expand $S[\bar{\phi}_{_\leq}, \phi_{_S}, \phi_{_>}]$ around $\phi_{_S} = \bar{\phi}_{_S}$. Denoting the fluctuations $(\phi^s - \bar{\phi}^s)$ and $(\phi^a - \bar{\phi}^a)$ by η^s and η^a respectively, we get :

$$e^Y = e^{-\Gamma_k} < e^{-[S_{,s}\eta^s + \frac{1}{2} S_{,ss'}\eta^s\eta^{s'} + \ldots + \delta k \frac{\partial \Gamma_{k,a}}{\partial k}\eta^a]} > \ , \tag{9}$$

where the (omitted) arguments of $S_{,s}$ and $S_{,ss'}$ are $[\bar{\phi}_{_<}, \bar{\phi}_{_S}, \phi_{_>}]$.

Following the classic arguments of [8], we know that, in order to collect all terms up to $O(\delta k)$, we only need to keep terms up to $O((\eta^s)^2)$, and thus we neglect the ellipses. The r.h.s. of Eq.(9) can be now computed using the identity

$$\left\langle e^{-f} \right\rangle = e^{-<f> + \frac{1}{2}(<f^2> - <f>^2) + O(f^3)} . \tag{10}$$

Thanks to (3), the last term in (9) can only contribute $O((\delta k)^2)$, so we also neglect this term. Then, with the help of the relations (5) and (6), we immediately compute the r.h.s. of Eq.(9) and find that (7) becomes :

$$e^{-\Gamma_{k-\delta k}} = e^{-\Gamma_k} \int [D\eta_{_S}] e^{\Delta\Gamma_{k,\,s}\eta^s - \frac{1}{2} K_{ss'}\eta^s\eta^{s'}} \ , \tag{11}$$

where $\Delta\Gamma_{k,\,s} = \Gamma_{k-\delta k,\,s} - \Gamma_{k,\,s}$ and $K_{ss'}$ is nothing but the r.h.s. of Eq.(6), i.e. :

$$K_{ss'} = \Gamma_{k,\,ss'} - \Gamma_{k,\,sa} (\Gamma_{k,ab})^{-1} \Gamma_{k,\,bs'} \ . \tag{12}$$

As $\Delta\Gamma_{k,\,s}$ is $O(\delta k)$, it would contribute an $O((\delta k)^2)$ term after performing the gaussian integral. Neglecting again this higher order term, we finally find that the difference between $\Gamma_{k-\delta k}$ and Γ_k (evaluated at the same values of their arguments) consists, to $O(\delta k)$, of just a determinant, i.e.

$$\Gamma_{k-\delta k} = \Gamma_k + \frac{1}{2}\mathrm{Tr}\ln K_{ss'}. \tag{13}$$

Using standard properties of determinants, Eq. (13) can be rewritten in a form that will be more useful for our subsequent generalizations, i.e.

$$\Gamma_{k-\delta k} - \Gamma_k = \frac{1}{2}\ln\left(\frac{\det\Gamma_{k,AB}}{\det\Gamma_{k,ab}}\right)\ , \tag{14}$$

where we recall that the capital indices (A, B) span the region $[k-\delta k, \Lambda]$, while (a, b) are for the region $[k, \Lambda]$.

Eq.(13) was already derived in [14] for the case of a spin hamiltonian $H(\sigma_{\mathbf{p}})$ (where $\sigma_{\mathbf{p}}$ is the Fourier component of the spin field) following a different, though equivalent, line of reasoning. The derivation presented above is new and, furthermore, is more suitable for extension to the more general cases we shall consider below. This is why we have presented the different steps in great detail[4].

Let us now discuss how one can extend our results to the general case, including gauge theories. It was first noted by Vilkovisky [10] that the usual definition of the effective action, Eq.(1), is in general not invariant under a reparametrization of the classical fields. Obviously this holds true also for our definition (2) of Γ_k at any scale k. He also pointed out that, in the case of gauge theories, the gauge dependence of the off-shell effective action is just a manifestation of such a reparametrization dependence.

The origin of the problem can be seen easily from the definition of the effective action (1). Let us think of the (field) configuration space as a manifold $\mathcal{M}$ endowed with a metric g_{ij} and assume that Γ, like S, is a scalar field on $\mathcal{M}$. While the functional integration measure can be made reparametrization invariant through the introduction of a $\sqrt{g}$, the second term in the exponential has bad transformation properties since the gradient is a covariant vector while the "coordinate difference" $(\phi - \bar{\phi})$ is a contravariant vector only if the

[4]Note also that a rederivation of Eq. (14), equation that already appeared in a previous version of the present paper, was given in [15].

ϕ's are euclidean coordinates in a trivial (flat) manifold. In the case of gauge theories there is an additional complication coming from the fact that the physical space is the quotient space $\mathcal{M}/\mathcal{G}$ ($\mathcal{G}$ is the gauge group) rather than $\mathcal{M}$. We'll came back on this point later.

Vilkovisky and De Witt have shown that a meaningful definition of the effective action can be given also in the general (curved) case in terms of geodesic normal fields, $\sigma^i[\varphi_*, \phi]$, based at a point φ_* in $\mathcal{M}$[10, 11]. The $\sigma^i[\varphi_*, \phi]$ are the components of a vector tangent at φ_* to the geodesic from φ_* to ϕ. Its length is the distance between the two points along the geodesic itself. Under coordinate transformations $\sigma^i[\varphi_*, \phi]$ transforms as a vector at φ_* and as a scalar at ϕ. A useful property of the σ fields is that any scalar function $A[\phi]$ can be expanded in a covariant Taylor series[10, 11] (the semicolon denotes covariant derivatives w.r.t. ϕ) :

$$A[\phi] \equiv A[\varphi_*, \sigma] = \sum_{n=0}^{\infty} \frac{1}{n!} A_{;\, a_1 \cdots a_n}[\varphi_*] \sigma^{a_1} \cdots \sigma^{a_n} \,. \tag{15}$$

As emphasized before, the definitions of the upper space, of the shell, and of the lower space are completely general and can be obtained with the help of any mode decomposition of the fields. From now on we denote by λ these generic modes. As before we introduce the notation $\sigma^i = (\sigma_{_<}, \sigma_{_S}, \sigma_{_>})$. The submanifold spanned by $\sigma_{_>}$ we denote by $\mathcal{M}_{_>}$ and the one spanned by $(\sigma_{_S}, \sigma_{_>})$ by $\mathcal{M}_{_\geq}$. The metric in σ coordinates is related to the original metric by

$$\hat{g}_{_{lm}}(\varphi_*, \sigma) = \frac{\partial \phi^i}{\partial \sigma^l} \frac{\partial \phi^j}{\partial \sigma^m} \, g_{_{ij}}(\phi) \,. \tag{16}$$

The induced metric on $\mathcal{M}_{_>}$ ($\mathcal{M}_{_\geq}$) is just the restriction of $\hat{g}_{_{lm}}$ to the appropriate set of indices, $\hat{g}_{_{ab}}$ ($\hat{g}_{_{AB}}$).

Given the arbitrary coordinates (fields) ϕ^i, the base point φ_*, and the gaussian normal coordinates σ^i in $\mathcal{M}$, we can now define, following [11], the scale (i.e. λ)-dependent effective action, $\hat{\Gamma}_\lambda$, as :

$$e^{-\hat{\Gamma}_\lambda[\varphi_*, \bar{\sigma}]} = \int [D\sigma_{_>}] \sqrt{\hat{g}} \; e^{-S + (\sigma^a - \bar{\sigma}^a) \hat{\Gamma}_\lambda[\varphi_*, \bar{\sigma}]_{,a}} \,, \tag{17}$$

where $\hat{g} = \det \hat{g}_{_{ab}}$. S is the classical action expanded as in (15), where, as in the analogous Eq.(2), the $\sigma_{_<}$ are replaced by the mean values $\bar{\sigma}_{_<}$: $S = S[\varphi_*; \bar{\sigma}_{_<}, \sigma_{_S}, \sigma_{_>}]$. Since φ_* is kept fixed, the steps that lead from Eq.(2)

to the RG equation (14) can be now repeated with almost no changes. The only modification is due to the presence in Eq.(17) of the non-trivial metric factor $\sqrt{\hat{g}}$ (compare with Eq.(2) where the metric is trivial). The impact of this term can be easily seen from the r.h.s. of Eq.(9), where it contributes the additional $O(\delta k)$ term:

$$-\frac{1}{2}(\ln\,\det\hat{g}_{_{AB}} - \ln\,\det\hat{g}_{_{ab}}) = \frac{1}{2}(\ln\,\det\hat{g}^{AB} - \ln\,\det\hat{g}^{ab}) \tag{18}$$

The final result is then:

$$\hat{\Gamma}_{_{\lambda-\delta\lambda}}[\varphi_*,\bar{\sigma}] = \hat{\Gamma}_{_\lambda}[\varphi_*,\bar{\sigma}] + \frac{1}{2}\ln\,\left(\frac{\det\hat{\Gamma}_{\lambda,A}{}^{B}}{\det\hat{\Gamma}_{\lambda,a}{}^{b}}\right)\,, \tag{19}$$

where the indices are raised with the help of the corresponding induced metrics on each submanifold that appear in Eq.(18)[5].

We now wish to rewrite Eq.(19) in general coordinates. Define:

$$\Gamma_{_\lambda}[\varphi_*,\bar{\phi}] = \hat{\Gamma}_{_\lambda}[\varphi_*,\sigma(\varphi_*,\bar{\phi})] = \hat{\Gamma}_{_\lambda}[\varphi_*,\bar{\sigma}]\ . \tag{20}$$

It is rather straightforward, though tedious, to connect the partial derivatives of $\hat{\Gamma}$ with respect to the $\bar{\sigma}$'s to the partial *covariant* derivatives of Γ with respect to the $\bar{\phi}$'s (both taken, of course, at fixed φ_*). Consider first these relations at the level of the full effective actions $\hat{\Gamma}$ and Γ.

For the first derivatives the result is simply:

$$\Gamma_{,i} = D_i^k\hat{\Gamma}_{,k}\ , \tag{21}$$

where, following [11], we have introduced:

$$D_i^k = \frac{\partial\bar{\sigma}^k}{\partial\bar{\phi}^i}\ . \tag{22}$$

The bi-vector D_i^k has the property that, once contracted with a covariant vector at φ_*, converts it into a covariant vector at $\bar{\phi}$, as exemplified indeed in (21).

[5]To be precise in Eq. (19) the determinants of the metrics appear under an expectation value sign rather than being computed at the expectation value of the field. We expect the difference to be insignificant.

The relation connecting second derivatives can be put in the form:

$$\hat{\Gamma}_{,kl} = (D^{-1})^i_k (D^{-1})^j_l \bar{\Gamma}_{ij} \ , \tag{23}$$

where

$$\bar{\Gamma}_{ij} \equiv \Gamma_{;ij} - \sigma^l_{;ij} (D^{-1})^k_l \Gamma_{,k} \tag{24}$$

is a second-rank tensor at $\bar{\phi}$. The quantity $\sigma^l_{;ij}$ has a covariant expansion [11] in the distance between φ_* and $\bar{\phi}$.

The above formulae can be easily generalized to the case in which the derivatives are restricted to lie on the $\mathcal{M}_>$ (or $\mathcal{M}_\geq$) manifold. Indeed the derivatives of $\hat{\Gamma}_\lambda$ with respect to $\bar{\sigma}^a$ will be related to the derivatives of Γ_λ with respect to generic coordinates ξ^a on $\mathcal{M}_>$ by exactly the same formulae (21), (23) where now:

$$D^a_b = \frac{\partial \bar{\sigma}^a}{\partial \xi^b} \ . \tag{25}$$

Using Eq. (16) we obtain our final result:

$$\Gamma_{\lambda - \delta\lambda}[\varphi_*, \bar{\phi}] = \Gamma_\lambda[\varphi_*, \bar{\phi}] + \frac{1}{2} \ln \left(\frac{\det \bar{\Gamma}_{\lambda\, A}{}^B}{\det \bar{\Gamma}_{\lambda\, a}{}^b} \right) , \tag{26}$$

where indices and covariant derivatives are all now defined in terms of the induced metrics g_{AB} and g_{ab} on the corresponding submanifolds.

Let us stress, already at this point, an important feature of (26): It was important, for our derivation, to carry out our differentiations at fixed φ_*. In other words, we have been forced to work with $\Gamma_\lambda[\varphi_*, \bar{\phi}]$. We believe, instead, that no closed RGF-equation holds for the original Vilkovisky-De Witt effective action $\Gamma_{VDW}[\bar{\phi}] \equiv \Gamma[\bar{\phi}, \bar{\phi}]$. This is probably related to the fact that, unlike $\Gamma[\varphi_*, \bar{\phi}]$, $\Gamma_{VDW}[\bar{\phi}]$ does not generate 1PI vertex functions [11, 16]. Since these vertex functions are related to operators that depend explicitly on φ_*, it is not surprising that the same is true for the RGF equation. This is indeed apparent through the second term appearing in the definition (24) of $\bar{\Gamma}_{ij}$. Note that the presence of this term, one of the main novelties of our paper, is *not* required by reparametrization invariance: both terms in (24) are fine from this point of view. It is required instead by φ_*-dependence and thus, we believe, it is a necessary price to pay for the whole procedure to work.

Let us see now how the previous steps can be repeated in the case of a gauge theory. As it was shown by Vilkovisky and DeWitt [10, 11], we first need to reduce the gauge theory to a "non-gauge" one. Let us indicate as before by $\mathcal{M}$ the field space, by ϕ^i the gauge fields, with g_{ij} the associated metric, by σ^m a complete set of gauge-invariant coordinates, and by R^i_α the generators of the gauge transformations:

$$\delta\phi^i = R^i_\alpha d\epsilon^\alpha \ , \tag{27}$$

where ϵ^α are coordinates on the gauge orbits. The metric decomposes into the block diagonal form [16]

$$ds^2 = h_{mn}d\sigma^m d\sigma^n + \gamma_{\alpha\beta}d\epsilon^\alpha d\epsilon^\beta \ , \gamma_{\alpha\beta} = R^i_\alpha g_{ij} R^j_\beta, \tag{28}$$

where

$$h_{mn} = \frac{\partial\phi^i}{\partial\sigma^m}\frac{\partial\phi^j}{\partial\sigma^n}\ \Pi_{ij} \ , \tag{29}$$

and we defined the projector on physical orbit space

$$\Pi^j_i = \delta^j_i - g_{ik}R^k_\alpha \gamma^{\alpha\beta} R^j_\beta \ . \tag{30}$$

Although the σ^m were so far arbitrary, we used an important result of [10] to take them as gaussian normal coordinates both in the induced metric h_{mn} and in the full space (provided geodesics are defined, in the latter, with respect to Vilkovisky's connection [10]).

Instead of using ϵ^α to parametrize points on orbits one can start with the "gauge fixing" coordinates χ^α and write the definition of the effective action a la Faddeev-Popov:

$$e^{-\Gamma[\varphi_*,\bar\phi]} = \int [D\phi^i]\sqrt{g}\delta(\chi^\alpha)\det\left(\frac{\partial\chi^\alpha}{\partial\epsilon^\beta}\right)\ e^{-S(\phi)+(\sigma^m-\bar\sigma^m)(D^{-1})^n_m\Gamma,_n} \ . \tag{31}$$

Changing integration variables to σ^m, ϵ^α we get

$$\begin{aligned} e^{-\hat\Gamma[\varphi_*,\bar\sigma]} &= \int [D\sigma^m][D\epsilon^\alpha]\sqrt{h}\sqrt{\gamma}\delta(\chi^\alpha)\det\left(\frac{\partial\chi^\alpha}{\partial\epsilon^\beta}\right) e^{-S(\phi_*,\sigma)+(\sigma^m-\bar\sigma^m)\hat\Gamma,_m} \\ &= \int [D\sigma^m]\sqrt{h}e^{-\tilde S(\phi_*,\sigma)+(\sigma^m-\bar\sigma^m)\hat\Gamma,_m} \ , \end{aligned} \tag{32}$$

where

$$\tilde{S} = S - \frac{1}{2}\ln\det(\gamma)\ . \tag{33}$$

With the gauge effective action written in this form we can directly apply the procedure followed from (17) to (19) and obtain, as before,

$$\hat{\Gamma}_{\lambda-\delta\lambda}[\varphi_*,\bar{\sigma}] = \hat{\Gamma}_{\lambda}[\varphi_*,\bar{\sigma}] + \frac{1}{2}\ln\ \left(\frac{\det\ \hat{\Gamma}_{\lambda,A}{}^{B}}{\det\ \hat{\Gamma}_{\lambda,a}{}^{b}}\right)\ . \tag{34}$$

We can now repeat the steps (20)-(26) and, following [10, 11, 12], write (34) in arbitrary coordinates $\bar{\phi}$ as

$$\Gamma_{\lambda-\delta\lambda}[\varphi_*,\bar{\phi}] = \Gamma_{\lambda}[\varphi_*,\bar{\phi}] + \frac{1}{2}\ln\left[\frac{\det(P_{\geq}\Pi\bar{\Gamma}\Pi P_{\geq})}{\det(P_{>}\Pi\bar{\Gamma}\Pi P_{>})}\right]\ , \tag{35}$$

where $\bar{\Gamma}$ is defined as in (24) in terms of the Vilkovisky connection, Π stands for the projector on the physical space (30), and $P_{>}$ ($P_{\geq}$) is a projector on $\mathcal{M}_{>}$ ($\mathcal{M}_{\geq}$). Eq.(35) is our desired gauge-invariant RG-flow equation for $\Gamma_\lambda[\varphi_*,\bar{\phi}]$. As we already stressed in the non-gauge case, no closed RGF-equation is expected to hold for the original VDW effective action.

As a check of (35) we can compute the one-loop effective action and compare it with [12]. Within this approximation we can set $\Gamma = S$ on the rhs of the definition (24) of $\bar{\Gamma}$. Noting that S is only a function of $\bar{\phi}$, we have the freedom to set $\varphi_* = \bar{\phi}$, yielding $\bar{\Gamma}_{ij} = S_{;ij}$ in (35). We finally integrate the evolution from $\lambda = \Lambda$ to $\lambda = 0$. Using $\Gamma_\Lambda = \tilde{S}$, together with (33), we get:

$$\Gamma_0 = S + \frac{1}{2}\ln\frac{\det(\Pi_i^k S^l_{;k}\Pi_l^j)}{\det\ \gamma}\,, \tag{36}$$

in agreement with the one-loop result of [12].

Beyond one-loop, our evolution equations should be useful in a variety of problems pertaining to non-abelian gauge theories and to quantum gravity. In practice, one will necessarily have to resort to some form of truncation of Γ_k, so that our exact equations become approximate RG-flow equations for a finite set of gauge-invariant low-energy parameters. A potential complication, at this stage, is represented by the explicit appearance, in the definition of $\bar{\Gamma}$, of the base point φ_* and of the geodesic coordinates built around it. It is not excluded, however, that this can be turned to one's advantage by a judicious

choice of φ_*. Work is now in progress in addressing this kind of questions within specific examples such as non-linear σ-models, gauge theories, and quantum gravity.

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References

[1] K. G. Wilson, Phys. Rev. **B4** (1971) 3184;
K. G. Wilson, J. Kogut, Phys. Rep. **12** (1974) 75.

[2] S. Weinberg, *Critical Phenomena for Field Theorists*, Erice Subnucl. Phys. 1 (1976).

[3] J. Polchinski, Nucl. Phys. **B231** (1984) 269.

[4] A. Hasenfratz, P. Hasenfratz, Nucl. Phys. **B270** (1986) 685.

[5] C. Wetterich, Phys. Lett. **B301** (1993) 90.

[6] M. Bonini, M. D'Attanasio, G. Marchesini, Nucl. Phys. **B409** (1993) 441.

[7] T.R. Morris, Int. J. Mod. Phys. **A9** (1994) 2411.

[8] F. Wegner, A. Houghton, Phys. Rev. **A8** (1973) 401.

[9] M. Reuter, C. Wetterich, Nucl. Phys. **B417** (1994) 181; M. Bonini, M. D'Attanasio, G. Marchesini, Nucl. Phys. **B421** (1994) 429.

[10] G. A. Vilkovisky, in *B. S. De Witt Sixtieth Aniversary Volume*, ed. S. Christensen (Adam Hilger, Bristol, 1983); Nucl. Phys. **B234** (1984) 125.

[11] B. De Witt, in *Quantum Field Theory and Quantum Statistics, Essays in Honor of the 60th Birthday of E.S. Fradkin* (I.A. Batalin, C.J. Isham and G.A. Vilkovisky editors), IOP publishing Ltd. (1987), p. 191; *The Global Approach to Quantum Field Theory*, Oxford University Press (2003).

[12] E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. **B234** (1984) 509.

[13] B. Zumino, Brandeis lectures 1970 (Vol. 2, p. 441, S. Deser ed., M.I.T. Press).

[14] J.F. Nicoll and T.S. Chang, Phys. Lett. **A62** (1977) 287.

[15] F.P. Correia, hep-th/0207282.

[16] G. Kunstatter, Class. Quant. Grav. **9** (1992) 157.

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CFT, Holography, and Causal Entropy Bound

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Abstract

The causal entropy bound (CEB) is confronted with recent explicit entropy calculations in weakly and strongly coupled conformal field theories (CFTs) in arbitrary dimension D. For CFT's with a large number of fields, N, the CEB is found to be valid for temperatures not exceeding a value of order $M_P/N^{\frac{1}{D-2}}$, in agreement with large N bounds in generic cut-off theories of gravity, and with the generalized second law. It is also shown that for a large class of models including high-temperature weakly coupled CFT's and strongly coupled CFT's with AdS duals, the CEB, despite the fact that it relates extensive quantities, is equivalent to (a generalization of) a purely holographic entropy bound proposed by E. Verlinde.

I. INTRODUCTION

Recently, there has been growing interest in, and proliferation of, various kinds of entropy bounds. Much of this interest stems from the idea of holography [1], a bold conjecture that some dynamical systems in D space-time dimensions can be completely described in terms of degrees of freedom living on their $(D-2)$-dimensional boundary. Maldacena's AdS/CFT correspondence [2] is the prototypical example realizing such a conjecture. A necessary condition for holography is that the number of degrees of freedom of the system does not exceed the area of the $(D-2)$-dimensional hypersurface surrounding it in units of some fundamental area, usually taken to be Planck's. Consequently, the validity of holography hinges upon, although it is by no means guaranteed by, a holographic bound on entropy.

Many systems seem to obey a holographic entropy bound. For instance, limited-gravity systems whose size R is larger than their gravitational radius $R > R_g \equiv 2G_N E$ (E is the total energy of the system), satisfying Bekenstein's bound [3] $S < S_B$, $S_B \sim ER$ automatically satisfy the holography bound since[1] $ER = R_g^{D-3}\ R\ l_P^{2-D} < (R/l_P)^{D-2} = S_{HOL}$. The real challenge to holography, therefore, is associated with its application to strong-gravity systems, such as the whole Universe.

Bekenstein himself [4] proposed an extension of his bound to cosmology by identifying the linear size R appearing in his bound with the particle horizon. For regions much larger than the particle horizon, or after reheating at the end of inflation, the cosmological Bekenstein bound becomes too loose. Instead, it could become too strong if one tried to apply it to sufficiently small regions. Fischler and Susskind (FS) [5] applied holography to cosmology, and proposed that the area of the particle horizon should holographically bound the entropy on the backward-looking light cone. It was soon realized, however, that the FS proposal requires modifications, since violations of it were found to occur in physically reasonable

[1]We will use units in which $c = k_B = \hbar = 1$, define Planck's length by $l_P^{D-2} = G_N$, and often ignore numerical factors of order unity.

situations, such as a closed, adiabatically contracting FRW universe. Several attempts were made to mend the FS proposal, which finally resulted in a covariant proposal by Bousso involving entropy on suitably constrained forward and/or backward-looking light-cones [6]. Bousso's proposal is defined such that it can be applied to more general space-times and not just to cosmological ones.

In parallel, several groups tried to modify the FS proposal by bounding entropy inside space-like regions. This resulted in various proposals [7], [8], [9], [10], all roughly identifying the maximal scale over which holography applies with a scale of about the Hubble radius H^{-1}. This line of reasoning resulted in the so-called Hubble Entropy Bound (HEB), bounding entropy density by Hl_P^{2-D}. Eventually, these ideas were synthesized in an improved covariant form through the introduction of a causal bound on entropy in a generic space-like region (CEB) [11].

Interestingly enough, both Bousso's proposal and CEB appear to follow from the same underlying bound (of the kind first proposed by Flanagan et al. [12]) on a local entropy current. Bousso's proposal is obtained by projecting along an arbitrary null vector and CEB by projecting on an arbitrary time-like vector. HEB, CEB, and the local bound on entropy current, all scale with the square root of the energy of the system and thus lie around the geometric mean between S_B, which scales as $S_B \sim E^1$, and S_{HOL}, scaling as $S_{HOL} \sim E^0$.

Recently, E. Verlinde [13] argued that the radiation in a closed, Radiation-Dominated(RD) Universe can be modeled by a CFT, and that its entropy can be evaluated using a generalized Cardy formula, which, in some cases, can be derived using the AdS/CFT correspondence [14]. On the basis of this entropy formula, Verlinde proposed a new, entirely holographic bound on entropy stating that the subextensive component of the entropy (the "Casimir entropy") of the entire closed universe has to be less than the entropy of a black hole of the same size. The well-known square-root appearing in Cardy's formula, reminiscent of the square-root occurring in the above-mentioned geometric mean of S_B and S_{HOL}, prompted Verlinde to point out a close connection between HEB and his new proposal. But,

in spite of their close similarity, Verlinde's new bound still holds for cases for which HEB appears to be violated.

Subsequently, Kutasov and Larsen (KL) [15] (see also [16]) have shown, by explicit weak-coupling, high-temperature CFT calculations, that Verlinde's generalization of Cardy's formula is not always correct and that, consequently, his proposed bound between two holographic quantities cannot be generally valid.

In this paper we try to shed some (hopefully bright!) light on this rather puzzling state of affairs. In Section II we give a generalization of CEB to arbitrary D and then, in Section III, we check it against the CFT calculations of Refs. [15], [16]. We find that CEB passes the CFT test provided temperatures are kept below a certain scale Λ which differs from M_P by a D-dependent factor scaling as an inverse power of the number of species N in the CFT. We also present, in Section IV, a modification of Verlinde's bound between holographic quantities which evades the KL criticism and show that the new bound, within the CFT framework, is exactly equivalent to CEB. We finally point out the reasons why the naive HEB is problematic, as pointed out in Ref. [13], while its CEB improvement is not.

II. CEB IN D DIMENSIONS

As mentioned in the introduction, CEB is an improved, covariant version of HEB, which is applicable, in principle, to any space-like region. Before extending CEB to any dimension D, let us briefly recall the basic ideas behind its predecessor, HEB. HEB was motivated by the following reasonable assumptions [7] (see also [8–10])

(i) entropy is maximized by the largest stable black hole that can fit in a given region of space. This is because the merging of two black holes into a larger one always results in an entropy increase.

(ii) the largest stable black hole in a cosmological background is typically of size comparable to that of the Hubble horizon. This assumption is qualitatively supported by previous calculations [20]

In cosmological backgrounds, CEB refines HEB by defining more precisely the "horizon" concept through the identification of a critical ("Jeans"-like) causal connection scale R_{CC}, above which perturbations are causally disconnected, so that black holes of larger size, very likely, cannot form, and by putting the resulting entropy bound in an explicitly covariant form.

The causal-connection scale R_{CC} is found by looking at perturbation equations in D dimensions. For gravitons, in the case of flat universe, one finds [21]

$$R_{CC}^{-2} = \frac{D-2}{2}\mathrm{Max}\left[\dot{H} + \frac{D}{2}H^2\,, -\dot{H} + \frac{D-4}{2}H^2\right]\,. \tag{1}$$

If $H \gg \dot{H}$, $R_C C \propto H^{-1}$ and one recovers HEB with a D-dependent prefactor scaling as$\sqrt{D(D-2)}$. The above result generalizes to the case of a spatially curved universe in the form [22,23]

$$R_{CC}^{-2} = \frac{D-2}{2}\mathrm{Max}\left[\dot{H} + \frac{D}{2}H^2 + \frac{D-2}{2}\frac{\kappa}{a^2}\,, -\dot{H} + \frac{D-4}{2}H^2 + \frac{D-2}{2}\frac{\kappa}{a^2}\right]\,. \tag{2}$$

A covariant definition of R_{CC} is obtained by expressing (2) in terms of the "00" components of curvature tensors. One easily finds:

$$R_{CC}^{-2} = \frac{D-2}{2(D-1)}\mathrm{Max}\left[G_{00} \mp R_{00}\right] = 4\pi G_N\left[\frac{1}{D-1}\rho - p\,, \frac{2D-5}{D-1}\rho + p\right]\,, \tag{3}$$

where, to derive the second equality, we have used Einstein's equations, $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$ and a perfect-fluid form for the energy-momentum tensor.

The Bekenstein-Hawking entropy of a Schwarzchild black hole of radius R_{BH} in D dimensions is given by $S = \mathcal{A}/4l_P^{D-2}$. The generalization of S_{CEB} for a region of proper volume V is therefore

$$S_{\mathrm{CEB}} = \beta n_H S^{BH} = \beta\frac{V}{V(R_{CC})}\frac{\mathcal{A}}{4l_P^{D-2}} \tag{4}$$

where $n_H \equiv \frac{V}{V(R_{CC})}$ is the number of causally connected regions in the volume considered, $V(x)$ denotes the volume of a region of size x, and β is a fudge factor reflecting current uncertainty on the actual limiting size for black-hole stability. For a spherical volume in flat

space we have $V(x) = \Omega_{D-2} x^{D-1}/(D-1)$, with $\Omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma\left(\frac{D-1}{2}\right)$, but in general the result is different and depends on the spatial-curvature radius.

Following Ref. [11], the expression for $S_{\rm CEB}$ in D dimensions can be rewritten in the explicitly covariant form

$$S_{\rm CEB} = B l_P^{-(D-2)} \int_{\sigma<0} {\rm d}^D\, x\sqrt{-g}\delta(\tau)\sqrt{{\rm Max}_\pm[(G_{\mu\nu} \pm R_{\mu\nu})\partial^\mu\tau\partial^\nu\tau]} =$$
$$B(8\pi)^{1/2} l_P^{-D/2+1} \int\limits_{\sigma<0} d^4x\sqrt{-g}\delta(\tau)\sqrt{{\rm Max}_\pm\left[(T_{\mu\nu} \pm T_{\mu\nu} \mp \frac{1}{2} g_{\mu\nu}\, T)\partial^\mu\tau\partial^\nu\tau\right]}\,, \qquad (5)$$

where $\sigma < 0$ defines the spatial region inside the $\tau = 0$ hypersurface whose entropy we are discussing, and T is the trace of the energy-momentum tensor.

The prefactor B can be fixed by comparing eqns. (4) and (5). In fact, consider the expression (4) in the limit $R_{CC} << a$, where a is the radius of the Universe: in this case, over a region of size R_{CC} we may neglect spatial curvature and write $V(R_{CC}) = \Omega_{D-2} R_{CC}^{D-1}/(D-1)$, and the area of the black hole horizon as $\mathcal{A} = \Omega_{D-2} R_{BH}^{D-2}$, thus giving (apart for negligible terms of order $(R_{CC}/a)^2$)

$$S_{\rm CEB} = \beta \frac{D-1}{4} V R_{CC}^{-1} l_P^{-(D-2)} = B\,\sqrt{\frac{2(D-1)}{D-2}}\, V R_{CC}^{-1} l_P^{-(D-2)}\,. \qquad (6)$$

This fixes $B = \sqrt{\frac{(D-1)(D-2)}{32}}\beta$.

Since (5) applies to any space-like region, it can be rewritten in a local rather than integrated form by introducing an entropy current s_μ such that $S = \int d^D x\sqrt{-g}\delta(\tau) s_\mu \partial^\mu\tau$. Then (5) becomes equivalent to (with λ^μ a arbitrary time-like vector):

$$s_\mu \lambda^\mu \leq l_P^{-D/2+1}(8\pi)^{1/2} B \sqrt{{\rm Max}_\pm\left[(T_{\mu\nu} \pm T_{\mu\nu} \mp \frac{1}{2} g_{\mu\nu}\, T)\lambda^\mu\lambda^\nu\right]}\,. \qquad (7)$$

In the limit of a light-like vector λ we get one of the conditions proposed by Flanagan et al. [12] in order to recover Bousso's proposal. Their bound corresponds (in $D = 4$) to $B = \frac{1}{4\pi}$ and could be used to fix β (assuming that it is D-independent).

Specializing now to the case of a RD universe, for which $\rho = (D-1)p$, the 00 equation for the scale factor becomes

$$H^2 + \frac{\kappa}{a^2} = \frac{16\pi G_N}{(D-1)(D-2)}\rho = \frac{16\pi G_N}{(D-1)(D-2)}\rho_0 R_0^D a^{-D}\,, \quad \kappa = \pm 1, 0, \tag{8}$$

and, in terms of the conveniently rescaled conformal time η, defined by $a(\eta)d\eta = (D-2)dt$, the solutions can be put in the simple form

$$a(\eta) = A^{\frac{1}{D-2}} \begin{cases} [\sin(\eta/2)]^\alpha & \kappa = 1 \\ (\eta/2)^\alpha & \kappa = 0 \\ [\sinh(\eta/2)]^\alpha & \kappa = -1 \end{cases}, \quad A = \frac{16\pi G_N \rho_0 R_0^D}{(D-1)(D-2)}\,, \quad \alpha = \frac{2}{D-2}\,. \tag{9}$$

As can be seen, the qualitative behavior of solutions does not depend strongly on D. In a (closed, open or flat) RD universe one always has $R_{00} = G_{00}$, therefore $R_{CC}^{-2} = \frac{D-2}{2}\left(-\dot{H} + \frac{D-4}{2}H^2 + \frac{D-2}{2}\frac{\kappa}{a^2}\right)$. The behaviour of S_{CEB} is easily derived from the explicit solution for the scale factor and R_{CC}. In the case D=4 it is shown in Fig. 1.

III. CEB VS. CFT

E. Verlinde proposed [13] that a radiation-dominated closed Universe in D space-time dimensions can be modeled by a D-dimensional CFT, and that its entropy is given by a generalization of Cardy's formula (we will denote it by S_{CV} for Cardy-Verlinde):

$$S = S_{CV} \equiv \frac{2\pi R}{D-1}\sqrt{2E_C E_E}\,, \tag{10}$$

where $R = a$ is the radius of the finite closed universe, and E_E and E_C are the extensive and sub-extensive components of the energy. The sub-extensive (Casimir) component, E_C, is conveniently normalized by

$$E_C = (D-1)(E - TS + pV) = DE - (D-1)TS \sim V/R^2\,, \tag{11}$$

so that the total energy E is given by $E = E_E + \frac{1}{2}E_C$, and E_E is purely extensive.

Verlinde motivates his proposal from the AdS/CFT correspondence and provides an example, taken from [14], of strongly coupled CFT's which have AdS duals and satisfy (10). Indeed, for such systems,

$$S = \frac{c}{12}\frac{V}{L^{D-1}} \tag{12}$$

$$E = \frac{c}{12}\frac{D-1}{4\pi L}\left(1+\frac{L^2}{R^2}\right)\frac{V}{L^{D-1}} \tag{13}$$

$$T = \frac{1}{4\pi L}\left(D+(D-2)\frac{L^2}{R^2}\right), \tag{14}$$

where c is the central charge of the CFT. The validity of the CV formula can be explicitly verified.

Next Verlinde proposes a new, purely holographic, entropy bound stating that the entropy associated with E_C, $S_C = 2\pi R E_C/(D-1)$, must be bounded by the entropy of a black-hole filling the whole Universe, $S_{BH} = (D-2)\frac{V}{4l_P^{D-2}R}$,

$$S_C < S_{BH}. \tag{15}$$

This bound is indeed satisfied in the specific cases he considers. We shall come back to Verlinde's bound in Section IV.

Kutasov and Larsen [15] pointed out that, in general, the CV formula (10) is not valid in weakly coupled CFTs. Instead, the free energy F, the entropy S, the total energy E, and the Casimir energy E_C can be expanded at weak coupling and large $x \equiv 2\pi RT$,

$$-FR = f(x) = \sum_{n\geq 0} a_{D-2n}x^{D-2n} + \dots \tag{16}$$

$$S = 2\pi f'(x)\,, \tag{17}$$

$$ER = (x\partial_x - 1)f(x), \tag{18}$$

$$E_C R = \sum_{n\geq 1} -2n a_{D-2n}x^{D-2n} + \dots \quad . \tag{19}$$

where the dots represent non-perturbative contributions. It is clear that, unless some special relation holds between a_D and a_{D-2}, the CV formula (10) cannot be generally valid.

We can explicitly check under which conditions the entropy of weakly coupled CFT's obeys CEB, $S < S_{\rm CEB} = 4B\sqrt{\pi}\sqrt{EV}l_P^{-(D-2)/2}$. In the limit $TR \gg 1$ we find

$$\frac{S^2}{S^2_{\rm CEB}} = \frac{\pi a_D D^2}{4B^2(D-1)\Omega_{D-1}} \left(2\pi l_P T\right)^{D-2} . \tag{20}$$

Thus, CEB is obeyed provided that

$$\left(\frac{T}{M_P}\right)^{D-2} < \frac{K(D)}{a_D}, \tag{21}$$

where $K(D)$ is a D-dependent (but CFT independent) constant. We conclude that CEB is obeyed as long as temperatures are below M_P by a factor $a_D^{-\frac{1}{D-2}}$ Since a_D is proportional to the number N of CFT-matter species, we obtain a bound on temperature which, in Planck units, scales as $N^{-\frac{1}{D-2}}$. We can also explicitly check under which conditions strongly coupled CFT's possessing AdS duals as considered by Verlinde obey CEB. In this case, in the limit $R/L \sim TR \gg 1$ we find

$$\frac{S^2}{S^2_{\rm CEB}} = \frac{1}{4(D-1)B^2}\frac{c}{12}\left(\frac{l_P}{L}\right)^{D-2} \tag{22}$$

and thus CEB is obeyed for

$$\frac{1}{4(D-1)B^2}\frac{c}{12}\left(\frac{4\pi T}{DM_P}\right)^{(D-2)} < 1\,. \tag{23}$$

Since the central charge c is proportional to the number of CFT fields N, we obtain a bound on temperature which, in Planck units, scales as $N^{-\frac{1}{D-2}}$, exactly as previously obtained for the weakly coupled case.

For the case $ER \sim a_D$ (which corresponds to $RT \sim 1$) KL have argued that the bound proposed by Verlinde has further problems, which they attributed to its relation with Bekenstein's bound. We notice that CEB is fine also in this case, its validity guaranteed by a condition similar to eq. (21).

Finally, we would like to show that CEB holds also when $ER \sim 1$. In this case $S_{CEB} \simeq 4B\sqrt{\pi}\sqrt{V/R}l_P^{-(D-2)/2}$ scales as $\left(\frac{R}{l_P}\right)^{\frac{D-2}{2}}$. As noted by KL the appropriate setup for calculating the entropy is the microcanonical ensemble with the result $S \sim \log a_D \sim \log N$; thus $S < S_{CEB}$ is guaranteed for a macroscopic Universe as long as

$$\left(\frac{R}{l_P}\right)^{\frac{D-2}{2}} > \log N\,. \tag{24}$$

In a quantum theory of gravity we expect the UV cut-off Λ to be finite and to represent an upper bound on T (Cf. the example of superstring theory and its Hagedorn temperature) and a lower bound on R (Cf. the minimal compactification radius). Thus conditions (21), (23) for the validity of CEB are satisfied as long as $\left(\frac{\Lambda}{M_P l}\right)^{D-2} < 1/N$. A bound of the same form was previously proposed in [4] and [17], and independent arguments in support of bounds of this sort have also been recently put forward [18], [19].

IV. CEB AND A NEW PURELY HOLOGRAPHIC BOUND

The entropy of all CFT's that we have considered so far could be expressed in terms of the superextensive entropy $S_B \equiv 2\pi RE/(D-1)$, and a subextensive entropy S_{SUB}

$$S_{CFT} = \sqrt{2S_B S_{SUB}}\,. \tag{25}$$

Since S_B is super-extensive and S_{CFT} is extensive, eq.(25) can be taken as a definition for sub-extensive entropy S_{SUB} scaling as V/R^2.

Our claim is then simply that in this context, CEB is equivalent to the following holographic bound:

$$S_{SUB} < \beta^2 \frac{(D-2)(D-1)^2}{8} \frac{V}{R l_P^{D-2}}\,, \tag{26}$$

which replaces Verlinde's bound for any CFT. Recall that β is a numerical factor introduced in the definition of CEB (4). The proof of our claim should be obvious by now, by writing S_{CEB} as

$$\begin{aligned} S_{CEB} &= B\sqrt{16\pi \frac{EV}{l_P^{D-2}}} \\ &= \beta(D-1)\sqrt{\frac{D-2}{8}}\sqrt{2S_B \frac{V}{R l_P^{D-2}}}\,. \end{aligned} \tag{27}$$

For systems obeying the CV formula, $S_{SUB} = S_C(1 - S_C/2S_B)$ so CEB coincides, up to a multiplicative factor, with the holographic entropy bound proposed by Verlinde (neglecting terms of order E_C/E). In this case the equivalence of the two bounds, which can also be

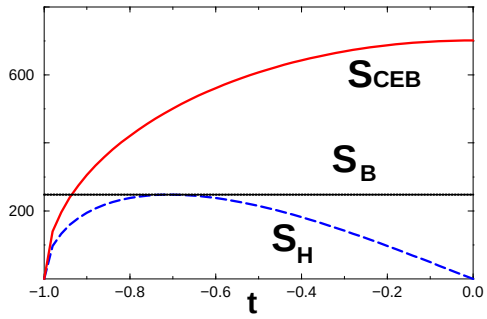

FIG. 1. S_{CEB} compared with S_H and S_B in the expanding phase of a closed $D = 4$, RD Universe. Here we set $\beta = \frac{D-2}{D-1}$.

visualized in $D = 4$ using the diagramatic representation of Verlinde (in this context $S_{\rm CEB}$ is proportional to the cord subtended by η), can be checked explicitly by looking at their evolution in a RD closed Universe if, following Verlinde, we write (15) as a combination of Bekenstein's and Hubble Entropy Bound, according to the value of the parameter HR:

$$\left.\begin{array}{l} S < S_B \\ S < S_H \end{array}\right\} \quad \text{for} \quad \left\{\begin{array}{l} HR < 1 \\ HR > 1 \end{array}\right. , \tag{28}$$

where $S_H = (D-2)\frac{HV}{4G_N}$ (here we set $\beta = \frac{D-2}{D-1}$, in such a way that the normalization is the same of Verlinde). As can be seen in Figure 1, CEB and bound (28) are parametrically equivalent throughout the whole evolution of the Universe.

Consider instead HEB; as noted in [13], S_H can be expressed as

$$S_H = \sqrt{S_{BH}(2S_B - S_{BH})}\,. \tag{29}$$

Clearly, CEB and HEB are of the same order of magnitude as long as $S_{BH} < 2S_B$. However, while in the regime we have considered $S_C < 2S_B$ (assuring the validity of CEB), $S_{BH} < 2S_B$ is not always true. When this happens (e.g. at the turning point), it is possible to violate HEB while respecting CEB. In retrospect we could have expected problems with HEB since it makes a non-covariant split between intrinsic and extrinsic curvature and uses just the latter for the bound. By contrast, CEB uses the full covariant curvature tensors which, through the Einstein equations, can be directly related to the energy-momentum tensor of the matter fields.

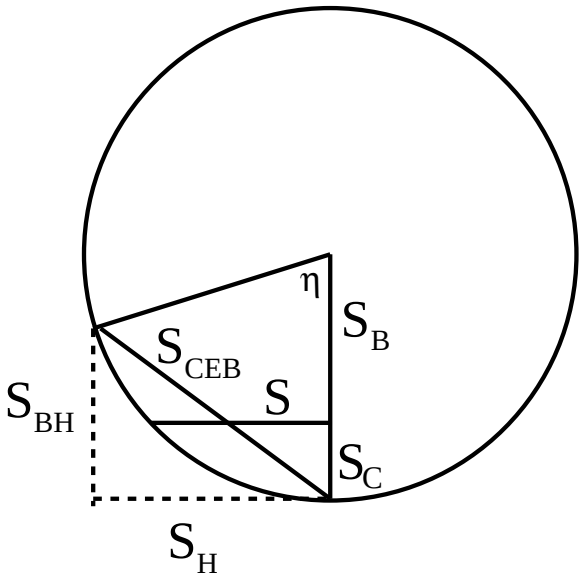

FIG. 2. Entropies in a closed RD Universe with $D = 4$.

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REFERENCES

[1] G. 't Hooft, *Abdus Salam Festschrift: a collection of talks*, eds. A. Ali, J. Ellis and S. Randjbar-Daemi (World Scientific, Singapore, 1993),gr-qc/9321026; L. Susskind, J. Math. Phys. **36** (1995) 6377. See also C. Thorn, hep-th/9405069.

[2] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).

[3] J. D. Bekenstein, Phys. Rev. **D23** (1981) 287; **D49** (1994) 1912, and references therein.

[4] J. D. Bekenstein, Int. J. Theor. Phys. **28** (1989) 967.

[5] W. Fischler and L. Susskind, " Holography and cosmology", hep-th/9806039.

[6] R. Bousso, JHEP **07** (1999) 004; JHEP **06** (1999) 028; "The holographic principle for general backgrounds," hep-th/9911002.

[7] G. Veneziano, Phys. Lett. **B454** (1999) 22; "Entropy bounds and string cosmology", hep-th/9907012.

[8] R. Easther and D.A. Lowe, Phys. Rev. Lett. **82** (1999) 4967.

[9] D. Bak and S. Rey, "Cosmic holography," hep-th/9902173.

[10] N. Kaloper and A. Linde, Phys. Rev. **D60** (1999) 103509.

[11] R. Brustein and G. Veneziano, Phys. Rev. Lett. **84** (2000) 5965.

[12] E.E. Flanagan, D. Marolf and R.M. Wald, "Proof of Classical Versions of the Bousso Entropy Bound and of the Generalized Second Law," hep-th/9908070.

[13] E. Verlinde, "On the holographic principle in a radiation dominated universe," hep-th/0008140.

[14] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 505.

[15] D. Kutasov and F. Larsen, "Partitions sums and entropy bounds in weakly coupled CFT," hep-th/0009244.

[16] F. Lin, "Casimir effect of graviton and the entropy bound," hep-th/0010127.

[17] R. Brustein, Phys. Rev. Lett. **84** (1999) 2072; see also R. Brustein, S. Foffa and R. Sturani, Phys. Lett. **B471** (2000) 352.

[18] R. Brustein, D. Eichler and S. Foffa, "The shortest scale of quantum field theory," hep-th/0009063.

[19] G. Veneziano, "Large N bound on the gravitational coupling", to appear.

[20] B. J. Carr and S. W. Hawking, MNRAS **168** (1974) 399; B. J. Carr, Astrophys. J. **201** (1975) 1; I. D. Novikov and A. G. Polnarev, Astron. Zh. **57** (1980) 250 [Sov. Astron. **24** (1980) 147].

[21] M. Gasperini and M. Giovannini, Phys. Rev. D **47** (1993) 1519.

[22] J. Garriga, et al., Nucl. Phys. B **513**, 343 (1998).

[23] A. Ghosh, G. Pollifrone and G. Veneziano, Phys. Lett. **B440** (1998) 20.

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Relic Gravitational Waves from String Cosmology

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Abstract

A large class of string-cosmology backgrounds leads to a spectrum of relic stochastic gravitational waves, strongly tilted towards high frequencies, and characterized by two basic parameters of the cosmological model. We estimate the required sensitivity for detection of the predicted gravitational radiation and show that a region of our parameter space is within reach for some of the planned gravitational-wave detectors.

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1 Introduction

It is notoriously difficult to find accessible experimental signatures of fundamental strings because of their small Planckian size. Possibly, an interesting exception to this rule is represented by the cosmological predictions of string theory, since these originate from physics of the Early-Universe, when space-time curvatures may have been of Planckian strength.

In order to arrive at some concrete, yet generic, predictions of string cosmology, we shall consider a large class of models in which a period of dilaton-driven inflation [1, 2, 3] is followed by a stringy epoch, during which the curvature remains of the order of the string scale λ_s^{-2}, and then finally, after possibly a short dilaton-relaxation era, by the standard (radiation then matter dominated) cosmology. As discussed in detail in [4], the presence of a high-curvature stringy epoch appears to be unavoidable for a viable inflationary string cosmology scenario. Additional support to this point of view was given in [5].

Recently, in collaboration with V. Mukhanov [6], we have discussed the main properties of metric perturbations in a dilaton-driven background. Particular attention was given to the correct treatment of scalar perturbations which, in a standard treatment, appear to grow too large for the applicability of linear perturbation theory. By carefully "gauging down" certain growing modes, we were able to show that both scalar and tensor perturbations can be treated perturbatively and that they exhibit very similar spectra. A characteristic feature of these spectra is that, unlike the spectra of the standard inflationary scenarios, they are not flat but strongly tilted towards higher frequencies, as originally noted in [7].

In this note we will concentrate on tensor perturbations, those associated with gravitational waves (GW), at frequencies which may be accessible to earth-based GW detectors. Quite amazingly, their spectrum turns out to be rather independent of most of the details of the above string cosmology scenario. As shown in the first part of this paper, the spectrum can be completely given in terms of two parameters, the value g_s of the string coupling parameter at the end of the dilaton-driven phase (equivalently, at the beginning of the string

epoch), and the total red-shift z_s occurring during the string era. The theoretical advantage of considering GW signals stems from the fact that, unlike the electromagnetic perturbations which underwent a complicated history until recombination, gravitons decoupled since right after the Planck era. As a consequence, their present spectrum should be a faithful portrait of the very early Universe. On the other hand, the detection of a cosmological background of GW requires extremely precise length measurements, typically at least of the order of $\delta L/L < 10^{-21}$ [8]. In the second part of this paper we will show that the expected range of g_s and z_s includes a region which should be accessible to future gravitational wave experiments.

Throughout this paper we shall be working in the so-called String-frame, in which weakly interacting strings move along geodesic surfaces. Identical results would follow by adopting the more conventional Einstein frame in which the curvature is canonically normalized, but we believe the physics to be more transparent in the former frame. In the String frame the string length parameter which is the short-distance cut-off of string theory, $\lambda_s = \sqrt{\alpha'\hbar}$, is constant, while the Planck length, $\lambda_P = \sqrt{G_N\hbar}$, evolves in time as $\lambda_P = e^{\phi/2}\lambda_s$ in a time-dependent dilaton background. In our scenario, the background evolution starts from the string perturbative vacuum [3], therefore λ_P grows from a very small initial value to a value $g_s\lambda_s$ reached at the beginning of the stringy era and, finally, to its present (very large!) value of about 10^{-33} cm at the beginning of the radiation dominated era.

2 Primordial gravitational-wave spectra

Let us consider, for the moment, an isotropic, $(3+1)$-dimensional, spatially flat cosmology. Following [9] (see also [6]) it is easy to show that the canonical variable ψ associated with tensor perturbations is related to the String-frame metric $g_{\mu\nu}$ via

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}) = a^2(\eta_{\mu\nu} + \frac{g}{a}\psi_{\mu\nu}), \tag{2.1}$$

where $a(t)$ is the isotropic scale factor, $\eta_{\mu\nu}$ is the flat Minkowski metric and $g = \exp(\phi/2)$. The Fourier modes of each of the two physical, transverse-traceless, polarizations satisfy the

following simple wave equation [9]

$$\psi_k'' + [k^2 - V(\eta)]\psi_k = 0, \qquad V(\eta) = (g/a)(a/g)'' \tag{2.2}$$

where a prime denotes differentiation with respect to conformal time η ($ad\eta \equiv dt$) and k is the comoving wave number, related to the physical one, ω, by $k = \omega a$. Note that, since a/g has power-like behavior in η during the dilaton-driven phase [1, 2, 3], $V(\eta)$ grows like η^{-2} during that epoch reaching a maximum at $\eta = \eta_s$, i.e. when $H^{-1} \sim \eta_s a_s \sim \lambda_s \equiv M_s^{-1}$. We expect $V(\eta)$ to keep growing during the stringy era and then to fall rapidly to zero at the onset ($\eta = \eta_1$) of the radiation-dominated era since, in that phase, $a/g \sim \eta$.

A given mode k will be well inside the horizon initially, then hit the potential barrier $V(\eta)$ at some "exit" time $\eta_{ex} \sim k^{-1}$, and leave the barrier at some later "reentry" time $\eta = \eta_{re} \sim \eta_1$. The approximate solutions of eq. (2.2) in these three regimes are given by

$$\psi_k = \frac{\lambda_s}{\sqrt{k}} e^{-ik\eta}\ , \qquad \eta < \eta_{ex} \tag{2.3}$$

$$\psi_k = \frac{a}{g}\left[A_k + B_k \int^{\eta} d\eta' \left(\frac{g}{a}\right)^2\right]\ , \quad \eta_{ex} < \eta < \eta_{re} \tag{2.4}$$

$$\psi_k = \frac{\lambda_s}{\sqrt{k}}\left[c_+(k)e^{-ik\eta} + c_-(k)e^{ik\eta}\right]\ , \quad \eta > \eta_{re} \tag{2.5}$$

Equation (2.3) enforces the proper normalization of the primordial vacuum fluctuations. In the regime described by eq. (2.4) the perturbation is frozen outside the horizon and the two terms appearing there correspond to the freezing of h and of its canonically conjugate momentum, respectively. Finally, in (2.5), the magnitude of c_- (so-called Bogoliubov coefficient) gives the amplification of the GW with respect to a minimal vacuum fluctuation. The actual value of $|c_-|$ can be easily obtained by matching the above solutions and their first derivatives at each transition time,

$$2k\eta_1|c_-(k)| \simeq \frac{g_{ex}/a_{ex}}{g_{re}/a_{re}}\left[1 + k\eta_1\left(\frac{g_{re}/a_{re}}{g_{ex}/a_{ex}}\right)^2 + k(g_{ex}/a_{ex})^{-2}\int_{\eta_{ex}}^{\eta_1} d\eta\ (g/a)^2\right] \tag{2.6}$$

At this point we have to insert some information about the background evolution. In the simple case at hand of a $D = 3 + 1$ isotropic cosmology with static extra dimensions, the

dilaton-driven inflationary background is simply given by [3]

$$a(\eta) = (-\eta)^{-\frac{1}{1+\sqrt{3}}}, \qquad \phi(\eta) = -\sqrt{3}\ln(-\eta)\ , \qquad a/g \sim (-\eta)^{1/2}, \quad -\infty < \eta < 0 \tag{2.7}$$

while, for the string era, we will assume that H and $\partial_t\phi$ are approximately constant and of order λ_s^{-1}. Limiting our attention for the time being to those scales which crossed the horizon during the dilaton-driven phase, we thus arrive at the following estimate

$$2k\eta_1|c_-(k)| \simeq (k/k_s)^{1/2} z_s (g_s/g_1)\left[1 + z_s^{-3}(g_1/g_s)^2 + \ln(k_s/k) + I\right], \qquad k < k_s \tag{2.8}$$

where we have denoted for convenience $a_{re}/a_s \simeq a_1/a_s = z_s$. In eq.(2.8) $k_s^{-1} \sim \eta_s \sim (H_s a_s)^{-1}$ is the last scale exiting during the dilaton-driven phase, $g_1 = \exp(\phi_1/2)$ is a number of order unity which may be determined in term of the (known) present value of the ratio λ_p/λ_s [10], and I is the k-independent quantity

$$I = \int_1^{\eta_1/\eta_s} \frac{d\eta}{\eta_s} \frac{(g/a)^2}{(g_s/a_s)^2} \sim 1 + z_s^{-3}(g_1/g_s)^2 \tag{2.9}$$

The r.m.s. perturbation amplitude over a comoving length scale k^{-1} (see for instance [6]) is given, in general, by $|\delta h_k(\eta)| \simeq k^{3/2}|h_k| = (g/a)k^{3/2}|\psi_k|$. For $\eta > \eta_1$, we then find

$$|\delta h_k(\eta)| \simeq \frac{H_1 a_1}{a(\eta) M_p} \left(\frac{k}{k_s}\right)^{1/2} \frac{g_s}{g_1} z_s \left[1 + \frac{1}{2}\ln\left(\frac{k_s}{k}\right) + z_s^{-3}\left(\frac{g_1}{g_s}\right)^2\right] \tag{2.10}$$

Equation (2.10) is the main result of this section and we will use it subsequently to estimate the required sensitivity for detection.

It is worth stressing that the leading term in $|c_-|$ comes from the integral appearing in eq. (2.4), associated with the freezing of the momentum variable conjugate to h. This unusual result is due to the fact [3] that, in the Einstein frame, the scale factor is $a_E = a/g$, and our background corresponds to a contracting, rather than expanding, Universe. The amplification of tensor perturbations in a contracting Universe was first considered long ago in [11, 12].

In spite of the presence of a high-curvature regime, we expect our estimates to be valid for scales that went out of the horizon in the dilaton-driven phase, since they follow from the general physical principle that a perturbation and its canonically conjugate momentum

should remain frozen while outside the horizon. The perturbations thus evolve in a purely kinematical way, giving rise to the logarithmic term in (2.8) from evolution during the dilaton-dominated phase and to the second term I from the stringy epoch.

Finally, let us digress a moment to show the stability of the result (2.10) with respect to $O(d,d)$ transformations which connect [13] different homogeneous string cosmologies. A simple derivation of this nice feature is obtained by working in cosmic time and by using directly the amplitude h [defined in eq. (2.1)] rather than the canonically normalized perturbation ψ. It is straightforward to check that, in arbitrary $O(d,d)$-related Bianchi I-type backgrounds (including possible dynamical internal dimensions and an antisymmetric tensor $B_{\mu\nu}$), the Fourier modes of h satisfy the following simple equation (see also [9])

$$\ddot{h_\omega} - \dot{\bar{\phi}}\dot{h_\omega} + \omega^2 h_\omega = 0 \tag{2.11}$$

where dots denote derivatives with respect to cosmic time and $\bar{\phi} = \phi - \ln|detg_{\mu\nu}|^{1/2}$ is the so-called shifted dilaton, invariant under $O(d,d)$ transformations. Using the fact that, in any dilaton-driven vacuum cosmology, $\bar{\phi} \sim -\ln t$, we easily obtain, for perturbations well outside the horizon,

$$h_\omega(t) \sim \ln\left|\frac{t}{t_{ex}}\right| \sim \ln|\omega t| \ , \tag{2.12}$$

showing that the spectrum of GW is independent of the chosen string-cosmology background, and increasing our confidence that eq. (2.10) is indeed the generic GW spectrum of a large class of string cosmology models. Note, incidentally, that this is not the case for the electromagnetic perturbations discussed in [14].

3 Observability

In order to discuss the observability of our signal it is useful to rewrite our main result (2.10) in terms of present, red-shifted proper frequencies $\omega = k/a$. One easily finds

$$|\delta h_\omega| \simeq \sqrt{\frac{H_0}{M_s}} z_{eq}^{-1/4} g_s z_s \left(\frac{\omega}{\omega_s}\right)^{1/2} \left[1 + \frac{1}{2}\ln\left(\frac{\omega_s}{\omega}\right) + z_s^{-3}\left(\frac{g_1}{g_s}\right)^2\right], \ \ \omega < \omega_s \tag{3.1}$$

$$\omega_s = k_s/a \simeq z_{eq}^{-1/4}\sqrt{H_0 M_s} z_s^{-1} \equiv z_s^{-1}\omega_1 \sim z_s^{-1} g_1^{1/2} 10^{11} Hz \tag{3.2}$$

where $z_{eq} = a/a_{eq} \sim 10^4$ takes into account the transition from radiation to matter dominance at $t = t_{eq}$, $\omega_1 = H_1 a_1/a \sim 10^{11}$Hz is the maximal frequency reached during the string phase, $M_s = \lambda_s^{-1} \sim H_1$, and $H_0 \sim 10^{-18}$Hz is the present value of the Hubble scale.

It is also convenient to rewrite our results in terms of another commonly used quantity, the fraction of critical density, $\Omega_{GW} = \rho_{GW}/\rho_c$, stored in our GW per logarithmic interval of ω. Defining $d\Omega_{GW}/(d\ln\omega) = \omega^4|c_-|^2/(M_p H)^2$ we have

$$\frac{d\Omega_{GW}}{d\ln\omega} = z_{eq}^{-1} g_s^2 \left(\frac{\omega}{\omega_s}\right)^3 \left[1 + \frac{1}{2}\ln\left(\frac{\omega_s}{\omega}\right) + z_s^{-3}\left(\frac{g_1}{g_s}\right)^2\right]^2 \sim \left(\frac{\omega}{H_0}\right)^2 |\delta h_\omega|^2, \;\; \omega < \omega_s \qquad (3.3)$$

It emerges from eq. (3.3) that ω_s plays the role of an effective temperature in the sense that, below ω_s, the spectrum is Planckian (up to logarithms of ω). The normalization of the spectrum, however, is different from Planck's because of the further amplification due to the stringy phase. Also, we do not expect the spectrum to stay Planckian above ω_s (see below), but rather to keep growing and to reach its maximum at ω_1 before falling exponentially.

Finally we give, with the appropriate caveats, the generalization of the above results to frequencies whose exit occurred during the stringy phase, i.e. to the high frequency part $\omega_s < \omega < \omega_1$. We are aware of the possible dangers in using field theoretic methods to discuss perturbations in this regime. However, in the absence of a full string theoretic calculation, we shall present our results as an indication of what a possible outcome might be. One finds, after straightforward calculations,

$$|\delta h_\omega| \simeq g_1 \sqrt{\frac{H_0}{M_s}} z_{eq}^{-1/4} \left[\left(\frac{\omega}{\omega_1}\right)^{2-\beta} + \left(\frac{\omega}{\omega_1}\right)^{\beta-1}\right] \qquad (3.4)$$

$$\frac{d\Omega_{GW}}{d\ln\omega} \simeq g_1^2 z_{eq}^{-1} \left[\left(\frac{\omega}{\omega_1}\right)^{6-2\beta} + \left(\frac{\omega}{\omega_1}\right)^{2\beta}\right], \quad \omega_s < \omega < \omega_1 \qquad (3.5)$$

where $\beta = -\log(g_s/g_1)/\log z_s$ is also the average value of $\dot{g}/(gH)$, which we have assumed to vary little during the string phase. We have also used the fact that during the string phase the curvature stays controlled by the string scale λ_s so that, in the String frame, the metric describes a de Sitter-like expansion with $z_s = a_1/a_s = \eta_s/\eta_1$ (see [15] for further details, and for a different derivation of the same spectrum in the Einstein frame).

We would like to discuss now the prospects of observing our spectrum in gravitational wave detectors. Our main emphasis will be on the planned large interferometers LIGO [16]

and VIRGO [17], which are expected [8] to start operating at sensitivities (for detection of a stochastic GW background) of $d\Omega_{GW}/d\ln\omega = 10^{-6}$ in a frequency band around a few hundred Hz, and have set the ambitious final sensitivity goals of $d\Omega_{GW}/d\ln\omega = 10^{-10}$ in a frequency band around $\omega_I = 100Hz$. It may well be, especially in the first stages of operation, that coincidence experiments between bars and interferometers [18] could also be able to reach similar sensitivities at frequencies around 1 KHz. We will mention later other possible devices which seem to have some good potential sensitivity, especially in the higher frequency range, but which have not yet matured into concrete operating systems. Detection of stochastic GW backgrounds at frequencies below 1 Hz does not seem accessible with current technologies, and we therefore limit our attention to the range above 1 Hz.

We are interested in finding the regions in our $\{z_s, g_s\}$ parameter space that may be accessible to experimental detection. From eq. (3.2) we can immediately see that the accessible region requires large values of z_s. We may distinguish values of z_s in the range $z_s < 10^9$ (i. e. $\omega_s > \omega_I$), in which the observable spectrum at ω_I comes mainly from perturbations that crossed the horizon during the dilaton-driven era, from those in the range $z_s > 10^9$ ($\omega_s < \omega_I$) in which the observable spectrum comes mainly from those perturbations that crossed the horizon during the stringy era. The predictions in the range $z_s > 10^9$ should be considered as less robust than those in the range $z_s < 10^9$. In addition, we may distinguish values of g_s in the range $g_s \lesssim g_1$ in which the dilaton does not change much during the stringy phase, from those in the range $g_s \ll g_1$ in which the dilaton changes by a large amount during the stringy epoch.

In all cases we have to impose the bound following from pulsar-timing measurements [19], which implies $d\Omega_{GW}/d\ln\omega \lesssim 10^{-6}$ at $\omega_P = 10^{-8}Hz$. We also accept the bound $\Omega_{GW} \lesssim 0.1$, imposed by standard nucleosynthesis [20]. Moreover, we have derived the spectrum in the linear approximation, expanding around a homogeneous background. We have thus to impose, for consistency, that the amplified perturbations have a negligible back-reaction on the metric, namely $d\Omega_{GW}/d\ln\omega < 1$ at all frequencies and times. This, together with the nucleosynthesis bound, requires a range of parameters corresponding to a spectrum which is growing also in the stringy phase, $0 < \beta < 3$, namely $(g_s/g_1) < 1$ and $(g_s/g_1) > z_s^{-3}$.

Inserting the appropriate numbers in eqs. (3.3), (3.5) we find, respectively, the following conditions for detectability in interferometers, i. e. $d\Omega_{GW}/d\ln\omega > 10^{-10}$ at $\omega_I = 10^2 Hz$ (assuming that the design goals would actually be achieved),

$$z_s^3 g_s^2 \left[11 - \frac{1}{2}\ln z_s + z_s^{-3}(g_s/g_1)^{-2}\right]^2 > 10^{21}, \tag{3.6}$$

for $z_s < 10^9$, and either

$$\log_{10}\frac{g_1}{g_s} < \left(\frac{1}{3} + \frac{1}{18}\log_{10} g_1^2\right)\log_{10} z_s, \qquad \beta < 3/2 \tag{3.7}$$

or

$$\log_{10}\frac{g_1}{g_s} > \left(\frac{8}{3} - \frac{1}{18}\log_{10} g_1^2\right)\log_{10} z_s, \qquad \beta > 3/2 \tag{3.8}$$

for $z_s > 10^9$.

It may be useful to list approximate forms of the GW spectral distribution, $d\Omega_{GW}/d\ln\omega$, in different regions of parameter space, which we do in **Table 1**,

	$z_s < 10^9$	$z_s > 10^9$
$\beta < 3/2$	$z_{eq}^{-1} g_s^2 \left(\frac{\omega}{\omega_s}\right)^3$	$z_{eq}^{-1} g_1^2 \left(\frac{\omega}{\omega_1}\right)^{2\beta}$
$\beta > 3/2$	$z_{eq}^{-1} z_s^{-6} g_1^4 g_s^{-2} \left(\frac{\omega}{\omega_s}\right)^3$	$z_{eq}^{-1} g_1^2 \left(\frac{\omega}{\omega_1}\right)^{6-2\beta}$

Table 1. Approximate forms of $(d\Omega_{GW}/d\ln\omega)$ in various regions of parameter space, $\omega_s = \omega_1/z_s$, $\beta = -\log(g_s/g_1)/\log z_s$.

Actually, if one considers also the amplification of electromagnetic (EM) perturbations in this scenario [14], one finds an even stronger bound following from the condition $\Omega_{EM} < 1$, i.e. $(g_1/g_s) < z_s^2$. Such a constraint washes out completely the allowed region of parameter space corresponding to the lower row of **Table 1**. The condition $\Omega < 1$ has to be imposed, however, for the validity of the linear approach, but it could be evaded in a more general, inhomogeneous model of background, in which the evolution of fluctuations is treated non-perturbatively.

In **Figure 1** the allowed region of parameter space, corresponding to the possible detection of the spectrum appearing in the upper row of **Table 1**, and compatible with the various bounds on the parameters, is plotted by taking $g_1 = 1$ as a reference value. Also shown are

the parameter intervals in which other detectors may be useful, for the range corresponding to the upper left corner of **Table 1**.

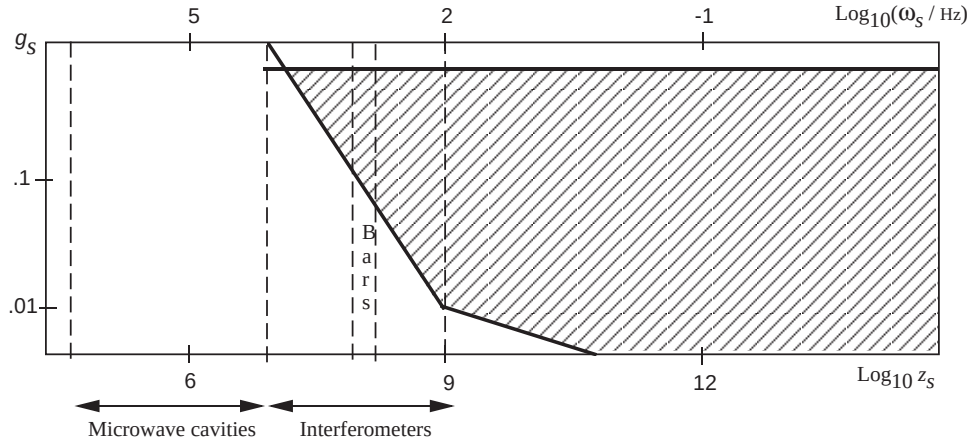

Figure 1. The allowed region in $\{z_s, g_s\}$ parameter space corresponding to the first row in **Table 1** is the shaded region defined by the various constraints. The dashed lines mark the regions in which various detectors may be useful, for the range $z_s < 10^9$ and $\beta < 3/2$.

We now turn to discuss in more detail the observability of our spectrum as function of frequency, limiting our attention, for sake of simplicity, to the frequencies leaving the horizon during the dilaton-driven phase [eqs.(3.1), (3.3)]. This spectrum is pictorially described in **Fig. 2** for the case $\beta < 3/2$, in terms of the quantity $|\delta h_\omega|$ (denoted h_c in [8]), which represents the characteristic amplitude of a stochastic background. The odd-shaped region in **Fig. 2** shows detection sensitivities for the so called "Advanced LIGO" project, in terms of the quantity $h_{3/yr}$ defined as the amplitude necessary for detection of a stochastic background at the 90% confidence level in a 1/3 of a year (see [8] for exact definitions). In **Fig. 2** we can observe clearly that larger amplification goes together with larger red-shift for this region of parameter space. For a given red-shift z_s, the higher amplitudes, for all regions of parameter space, are at higher frequencies.

In addition to interferometers and bars, microwave cavities may be operated as gravity wave detectors for the high frequency range $10^6 - 10^9$ Hz. For the MHz range specific suggestions [21, 22] were actually implemented [23]. As can be seen from **Fig. 2**, the required sensitivity for detection of gravity waves in the MHz region is $|\delta h_\omega| \sim 10^{-26}$, corresponding

to $h_{3/yr}$ of the same order. We leave to experts to study whether or not such a sensitivity is accessible with current technologies and may be reached in a near future, with microwave cavity detectors or with other experimental apparatus.

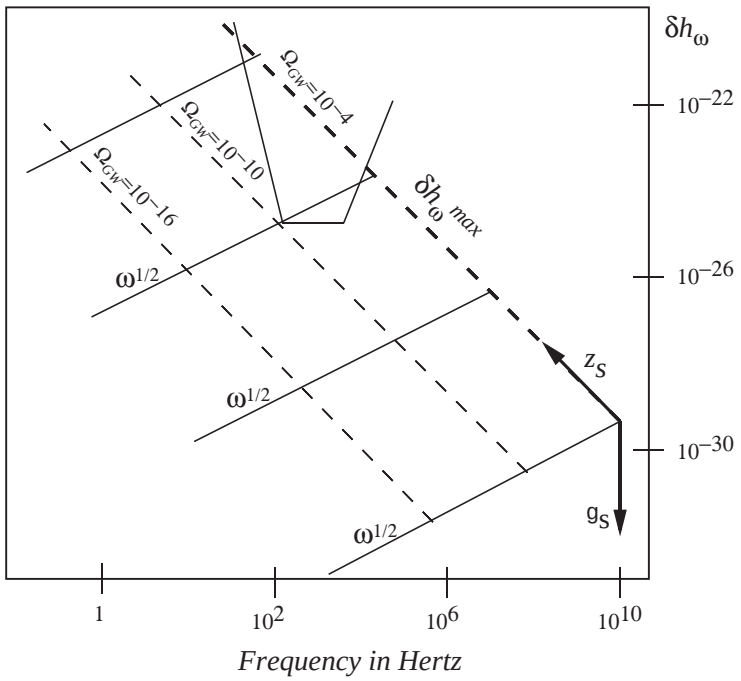

Figure 2. The characteristic spectral amplitude of gravitational waves $|\delta h_\omega|$. The solid lines show several individual spectra for different values of z_s and $g_s = 1$. The thick dashed line shows the maximum amplitude $|\delta h_\omega^{max}|$ as a function of z_s for $g_s = 1$. The dashed lines are lines of fixed g_s and therefore lines of constant energy density. Ω_{GW} is roughly the maximal amount of gravitational energy density at a given value of g_s. Also shown in the figure is the odd-shaped region marking the sensitivity goals for the detection of a stochastic background according to "Advanced LIGO" project.

4 Conclusion

We showed that a rather generic string cosmology scenario leads, naturally, to the production of an amplified quasi-thermal spectrum of gravitons during the dilaton-driven phase. This spectrum is very stable under modifications of the background and in particular under $O(d, d)$ transformations. The slope of the spectrum may change for modes crossing the horizon in the subsequent string phase, but remains in general characterized by an enhanced production

of high frequency gravitons, irrespective of the particular value of the spectral index.

We showed, in particular, that it may be possible to detect such a relic GW background with large interferometers for a range of the two parameters characterizing our class of models. We would like, however, to encourage the study and the developments of gravitational detectors with enhanced sensitivity in the high frequency, KHz - GHz, range. This frequency band should be in fact all but a "desert" of relic gravitational radiation that one may expect on the grounds of the standard inflationary scenario or from ordinary astrophysical sources. Our string cosmology scenario is unique in predicting a strong signal in this range of frequencies. In general, a sensitivity of $\Omega \sim 10^{-4} - 10^{-5}$ (which is not out of reach, in the KHz region, for coincidence experiments between bars and interferometers [18]), could be already enough to detect a signal, so even a null result at that level of sensitivity would already constrain in a significant way the parameters of the string background, while detection would provide a first glimpse at some new and exciting Planckian physics.

As we stressed, many of our results are independent of details of the string cosmology scenario. However, it would be worthwhile pointing out again that, although some ideas have been put forward [24], a solid string-theoretic treatment of the stringy phase, which we propose as the necessary bridge between the dilaton-driven and the standard decelerated era, does not yet exist. Recent progress [24] on the interpretation of singularities in string theory, as simply a failure to describe physics in terms of the original set of massless fields, may shed light on the resolution of this issue. The understanding of singularities in string theory would certainly help putting our string cosmology scenario on a firmer basis, and may even provide a framework for the calculation of the parameters g_s and z_s.

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References

[1] G. Veneziano, Phys. Lett. B265 (1991) 287.

[2] M. Gasperini and G. Veneziano, Astropart. Phys. 1 (1993) 317 ; Mod. Phys. Lett. A8 (1993) 3701 .

[3] M. Gasperini and G. Veneziano, Phys. Rev. D50 (1994) 2519.

[4] R. Brustein and G. Veneziano, Phys. Lett. B329 (1994) 429.

[5] N. Kaloper, R. Madden and K. Olive, "Towards a singularity-free inflationary Universe?", Preprint UMN-TH-1333/95 (hep-th/9506027).

[6] R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov and G. Veneziano, Phys. Rev. D51 (1995), in press.

[7] M. Gasperini and M. Giovannini, Phys. Lett. B282 (1992) 36.

[8] K. S. Thorne, in 300 Years of Gravitation, S. W. Hawking and W. Israel, Eds. (Cambridge Univ. Press, Cambridge, 1987).

[9] M. Gasperini and M. Giovannini, Phys. Rev. D47 (1993) 1519.

[10] V. Kaplunovsky, Phys. Rev. Lett. 55 (1985) 1036.

[11] L. P. Grishchuk, Ann. Acad. Sci. N.Y. 302 (1977) 439.

[12] A. A. Starobinski, Pis'ma Zh. Eksp. Teor. Fiz. 11 (1979) 719.

[13] K. A. Meissner and G. Veneziano, Phys. Lett. B267 (1991) 33; Mod. Phys. Lett. A6 (1991) 3397; M. Gasperini and G. Veneziano, Phys. Lett. B277 (1992) 256.

[14] M. Gasperini, M. Giovannini and G. Veneziano, "Primordial magnetic fields from string cosmology", Preprint CERN-TH/95-85 (hep-th/9504083);
"Electromagnetic origin of the CMB anisotropy in string cosmology", Preprint CERN-TH/95-102 (astro-ph/9505041).

[15] M. Gasperini, "Amplification of vacuum fluctuations in string cosmology backgrounds", in Proc. of the 3rd Colloque Cosmologie (Paris, June 1995), Preprint DFTT-38/95 (hep-th/9506140).

[16] A. Abramovici et al., Science 256 (1992) 325.

[17] B. Caron et. al., "Status of the VIRGO experiment", preprint Lapp-Exp-94-15.

[18] P. Astone, J. A. Lobo and B. F. Schutz, Class. Quantum Grav. 11 (1994) 2093.

[19] D. R. Stinebring et al., Phys. Rev. Lett. 65 (1990) 285.

[20] V. F. Schwartzmann, JEPT Lett. 9 (1969) 184;
T. Walker et al, Ap. J. 376 (1991) 51.

[21] F. Pegoraro, E. Picasso, L. Radicati, J. Phys. A11 (1978) 1949.

[22] C. M. Caves, Phys. Lett. B80 (1979) 323.

[23] C. E. Reece et al., Phys. Lett A104 (1984) 341.

[24] E. Kiritsis and K. Kounnas, Phys. Lett. B331 (1994) 51;
A. A. Tseytlin, Phys. Lett. B334 (1994) 315;
E. Martinec, Class.Quant.Grav.12 (1995) 941;
A. Strominger, "Massless black holes and conifolds in string theory", Preprint hep-th/9504090;
B. Greene, D. Morrison and A. Strominger, "Black hole condensation and the unification of string vacua", Preprint hep-th/9504145.

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THE GRACEFUL EXIT PROBLEM IN STRING COSMOLOGY

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ABSTRACT

Isotropic string cosmology solutions can be classified into two duality related branches: one can be connected smoothly to an expanding Friedman-Robertson-Walker Universe, the other describes accelerated inflation or contraction. We show that, if the dilaton potential has certain generic properties, the two branches can evolve smoothly into each other. We find, however, that just the effects of a potential do not allow for a graceful exit from accelerated inflation in the weak curvature regime. We explain how conformal field theory techniques could be used to give a decisive answer on whether graceful exit from accelerated inflation is possible at all.

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There has been growing interest, recently, in the question of whether or not inflationary cosmology can be accommodated naturally in superstring theory and arguments in both directions have been presented.

On one hand, dilaton interactions appear to inhibit inflationary evolution [1],[2]. This observation is based on the standard assumption that inflationary evolution is driven by a non-vanishing potential energy density. Since the dilaton couples necessarily to any kind of energy density, a dilaton potential is induced. The induced potential turns out to be so steep that the dilaton kinetic energy dominates over the potential energy leaving nothing to drive super-luminal expansion. Some attempts have been made to overcome this fact by assuming that inflation occurs after the dilaton has settled down to its minimum or invoking anthropic-like arguments about the initial conditions of the dilaton [3]. Neither of these seem to be completely satisfactory.

On the other hand, the duality-like symmetries that characterize the equations of string cosmology [4], [5] suggest a totally different mechanism [4], [6] for inflationary evolution. Rather than relying on potential energy as source for inflation it involves a novel and interesting interplay between the dilaton and the metric. This mechanism is based on the fact that cosmological solutions to string dilaton-gravity come in duality-related pairs (branches) connected to a sign ambiguity in solving a quadratic equation. We will thus call them the plus, $(+)$, and the minus, $(-)$, branch. The $(-)$ branch can be connected smoothly to a standard Friedman$-$Robertson$-$Walker (FRW) decelerated expansion of the universe with constant dilaton, while the $(+)$ branch, even in the absence of a dilaton potential, has solutions of the accelerated inflation type (so called superinflation [7]), in which the Hubble parameter increases with time. Duality related cosmologies are actually part of a larger, continuous family of $O(d,d)$ related solutions [8] which involve the antisymmetric tensor as well. For the purpose of this paper, this larger family does not appear to play an important role.

The mere existence of duality related solutions does not provide per se an answer to the questions raised in [1] and [2]. The real question, that we set out to answer, is whether or not the dynamics itself likes to pair duality-related solutions in a single, continuous history of the

universe taking place in cosmic time. In other words, we would like to understand whether string cosmology allows the inflationary phase to end and to connect smoothly to a FRW expanding universe[1]. So far such a picture was seen to emerge either by invoking an ad-hoc symmetry principle (self-duality) or by using quite contrived forms of dilaton potentials [6]. Neither of these appears to be completely satisfactory.

Here we use the low energy effective field theory approach to set up the problem. We assume therefore, throughout our analysis, that curvatures and derivatives are small in string units. We also assume that quantum corrections are not too large, i.e., we work in a region where string theory is weakly coupled. This is consistent with the standard view of string theory whereby the gauge coupling constant at the string scale is related, in our conventions, to the string-loop expansion parameter by $e^{\phi} = g^2_{string} \sim g^2_{gauge}$. The present value of the dilaton is therefore reasonably perturbative and we shall assume that it had never been much larger than that in the past. As we shall see below, this fits very well with the whole scenario.

The transition between inflationary evolution and a FRW expanding universe is usually referred to as "graceful exit" from inflation and it is a well known problem to be faced by any model of inflation. The possibility of graceful exit from accelerated inflation is closely related, in our setup, to the question of whether the two branches can be smoothly connected to one another. Incidentally, the notion that small curvature solutions with different characteristics (e.g. topology) can be smoothly connected to one another as some parameter is varied continuously is not new in string theory [10]. What is new in our context is that the parameter should represent physical (cosmic) time. If such "branch changes" can occur dynamically, the following appealing scenario can be easily imagined. The dilaton starts somewhere in the weak coupling region and, initially, the Hubble parameter, H, is also very small. Both then evolve as in the $(+)$ branch solution. The universe inflates and the dilaton evolves toward stronger coupling. This evolution is essentially unaffected by the dilaton potential or by matter or radiation sources. At some later time a "branch change" occurs. The potential and sources become important, the universe evolves as a regular FRW

[1] In pure Brans-Dicke theory a related analysis was carried out by Weinberg [9]

Universe, reheats and the dilaton settles down at a minimum of its potential.

We will find that smooth branch changes do occur if the dilaton potential has certain generic properties. We also find, however, that, it is not possible to find a solution which describes a complete, graceful exit from accelerated inflation. The analysis points to the possibility that strong curvature effects may play an essential role in allowing graceful exit from accelerated inflation. Analysis of exact (to all orders in α') stringy solutions should give a definite answer concerning the viability of the whole idea and encouraging results in this direction have been recently obtained [11]

There are different "frames" in which one can describe Brans-Dicke (BD) type theories. These are related by local field redefinitions which, supposedly, do not affect physical observables (see e.g. [12]). Here we perform the analysis in the BD (string) frame because the physics interpretation is simpler in such frame [13],[2] and the arguments are more transparent.

The relevant terms in the superstring effective action below the Planck scale are:

$$S_{eff} = \frac{1}{16\pi\alpha'}\int d^4x\sqrt{-g}\left\{e^{-\phi}\left[R + \partial_\mu\phi\partial^\mu\phi\right] - V(\phi) + \cdots\right\} \qquad (1)$$

where we have assumed to be in the critical number of dimensions (central charge $c = 10$ with the aid of some passive coordinates or internal degrees of freedom) while all the interesting dynamics takes place in ordinary 4-dimensional space-time.

Our conventions are those of Weinberg [9]. Note that the dilaton (ϕ) kinetic term has the "wrong" sign, corresponding to a BD ω parameter equal to -1, as required by duality [4]. The origin of the dilaton potential $V(\phi)$ is expected to be non-perturbative, related to supersymmetry breaking, and derived from the superpotential of the theory in the standard way. There will be also perturbative and non-perturbative corrections to the coefficients of the other terms in the action, but they are expected to be small because of the assumption that the theory remains weakly coupled throughout the cosmological evolution. As will become clear, the details of the shape of the potential are not particularly important for our analysis. Finally, the dots stand for terms involving other fields, more derivatives etc.

The equations of motion resulting from the action (1) are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi + g_{\mu\nu}D_\mu D^\mu\phi - D_\mu D_\nu\phi - \frac{1}{2}g_{\mu\nu}U(\phi) &= 0 \\ R + D_\mu\phi D^\mu\phi - 2D_\mu D_\mu\phi - U(\phi) &= 0 \end{aligned} \qquad (2)$$

where $U(\phi) \equiv e^\phi V(\phi)$. We look for solutions of the above equation in which the metric is of the isotropic, FRW type with vanishing spatial curvature (we will discuss later more general cases) and the dilaton depends only on time.

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)dx_i dx^i \\ \phi &= \phi(t) \end{aligned} \qquad (3)$$

The Hubble parameter, H, is related to the scale factor, a in the usual way, $H \equiv \frac{\dot{a}}{a}$.

After some algebra one obtains two independent first order equations for the dilaton and H [4]. The original dilaton equation is a consequence of these two equations, which read

$$\begin{aligned} \dot{H} &= \pm H\sqrt{3H^2 + U} - \frac{1}{2}U' \\ \dot{\phi} &= 3H \pm \sqrt{3H^2 + U} \end{aligned} \qquad (4)$$

The $(\pm)$ signifies that either a $(+)$ or $(-)$ is chosen for both equations simultaneously. These equations are the starting point of our analysis.

The solutions to the equations (4) belong to two branches, according to which sign is chosen. Another twofold ambiguity is related to the sign of H. For simplicity, we start by discussing the solutions in the very simple case in which there is no potential. The four solution one obtains in this case are related to one another by scale-factor-duality, time reversal, or both. The $(-)$ branch is a regular, negative feedback, branch. The solution $\{H^{(-)}, \phi^{(-)}\}$ as functions of time is simply given by

$$\begin{aligned} H^{(-)} &= \pm \frac{1}{\sqrt{3}}\frac{1}{t - t_0} \\ \phi^{(-)} &= \phi_0 + (\pm\sqrt{3} - 1)\ln(t - t_0) \quad , \quad t > t_0 \end{aligned} \qquad (5)$$

This solution describes decelerated expansion ($H > 0$, $\dot{H} < 0$) or decelerated contraction ($H > 0$, $\dot{H} < 0$) depending on the initial sign of H. Correspondingly, the evolution is towards

strong or weak coupling, respectively. As we show explicitly later (see Eqs.(12) below) this branch can be joined smoothly to an ordinary FRW radiation-dominated expanding universe. Provided the dilaton potential has a minimum with vanishing cosmological constant and suitable sources are added, the dilaton will eventually settle down at it vacuum expectation value and standard cosmology will follow.

The $(+)$ branch has instead some unusual and quite remarkable properties [4],[6]. In the absence of any potential the solution for $\{H, \phi\}$ is now given by

$$\begin{aligned} H^{(+)} &= \pm\frac{1}{\sqrt{3}}\frac{1}{t-t_0} \\ \phi^{(+)} &= \phi_0 + (\pm\sqrt{3}-1)\ln(t_0-t) \quad , \quad t < t_0 \end{aligned} \tag{6}$$

This solution describes either accelerated expansion and evolution from a cold, flat and weakly coupled universe towards a hot, curved and strongly coupled one [6] or accelerated contraction and evolution towards weak coupling. In general, the effects of a potential on this branch are quite mild. The dilaton zooms through potential minima. It is impossible, in this branch, for the dilaton to sit at a minimum of its potential. Inflation, in this solution, is driven, not hampered, by the dilaton's kinetic term, thanks to the negative value (-1) of the BD ω parameter.

The following table should help clarifying the difference between the two branches

	$(+)$ branch	$(-)$ branch
$H<0$	$\dot{H}<0, \quad \dot{\phi}<0$	$\dot{H}>0, \quad \dot{\phi}<0$
$H>0$	$\dot{H}>0, \quad \dot{\phi}>0$	$\dot{H}<0, \quad \dot{\phi}>0$

Table 1. Classification of two branches of solutions as functions of time.

The above analysis becomes quite complicated when one considers non-vanishing potential and matter contributions. Since Eqs.(4) are first order equations we found that it is best to represent their solutions as trajectories in the phase space $\{H, \phi\}$. In Figure 1. some $(+)$ and $(-)$ trajectories are shown. The units on the vertical H axis are arbitrary.

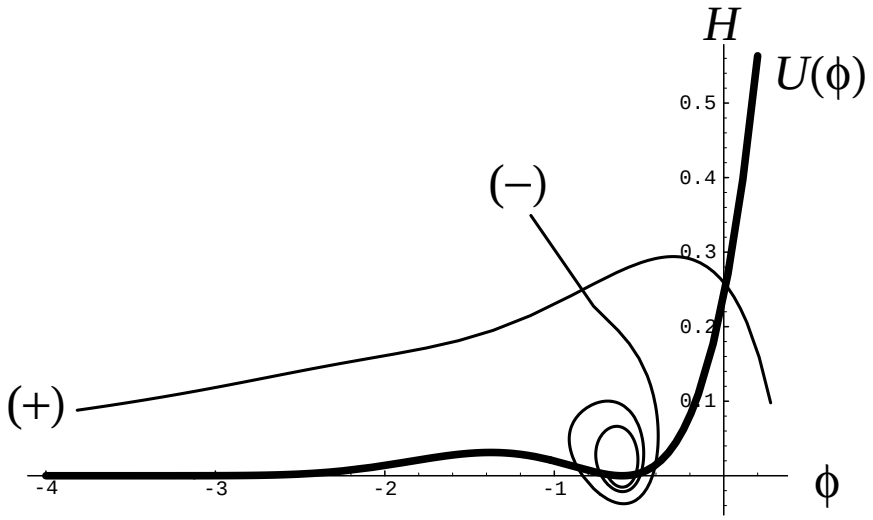

Figure 1. The potential $U(\phi)$ and trajectories of $(+)$ and $(-)$ branches. The $(-)$ branch settles down at a minimum of the potential.

From Eqs.(4) we see that a branch change can occur only if the potential $V(\phi)$ becomes negative for some range of ϕ. This is because the only points allowing a continuous transition from one branch to the other are those where

$$3H^2 + U(\phi) = 0 \quad , \tag{7}$$

or else there will be a discontinuity in $\dot{\phi}$. Similarly, continuity in H will imply that, of the four solutions of Table 1, only those with the same sign of H can turn into one another. It is clear that these correspond to pairs of solutions related by the product of scale-factor-duality and time reversal, since the combination of the two operations does not reverse the sign of H.

Since $H^2 > 0$ we have to have $U(\phi) < 0$ to allow for solutions to Eq.(7) to exist. The locus of points satisfying $3H^2 + U(\phi)$ is a curve in phase space. For potentials for which $V(\phi) < 0$ in some bounded interval the shape of the curve resembles an egg and we will therefore refer to it from now on as the "Egg". We assume that the "Egg" lies in the mild-coupling region. In Figure 2 a typical dilaton potential and the corresponding "Egg" are shown. In supergravity theories it is a generic property that the global minimum of the

potential is negative. Moreover, if one considers a generic potential induced by field theoretic non-perturbative effects in superstring theory, it does indeed have the property that it has an Egg and the Egg is completely in the weak (mild) coupling region (see e.g. [1]).

It turns out that a negative potential is not a sufficient condition for inducing a branch change. We turn now to explain the additional condition that ensures that a branch change does indeed occur. Consider a trajectory in phase space near the Egg. We can compute its direction in phase space, $\frac{dH}{d\phi}$ as compared to the direction of the Egg, $\frac{dH}{d\phi}\Big|_{Egg}$, for the two branches

$$\frac{dH}{d\phi} - \frac{dH}{d\phi}\Big|_{Egg} = \pm\frac{1}{3}\epsilon\Big[1 - \frac{1}{2}\frac{U'}{U}\Big] + \mathcal{O}(\epsilon^2) \tag{8}$$

where the prime denotes derivative with respect to ϕ and $\epsilon = \sqrt{3H^2 + U(\phi)}$ is a measure of the distance of the trajectory from the Egg. Note that $\epsilon > 0$ always. Consider the $(+)$ branch and $H > 0$ so the trajectory is above the Egg as in Figure 2. If U' is negative and large such that the quantity $\Big[1 - \frac{1}{2}\frac{U'}{U}\Big]$ is negative (recall $U < 0$) the trajectory is attracted to the Egg, touches it and then has to turn into the $(-)$ branch by continuity. If the quantity $\Big[1 - \frac{1}{2}\frac{U'}{U}\Big]$ is positive the trajectory is repelled from the Egg and a $(+) \to (-)$ branch change will not occur. Consider now the $(-)$ branch and $H > 0$ so the trajectory is above the Egg. If the quantity $\Big[1 - \frac{1}{2}\frac{U'}{U}\Big]$ is positive the trajectory is attracted to the Egg, then touches the Egg and then has to turn into the $(+)$ branch by continuity. If the quantity $\Big[1 - \frac{1}{2}\frac{U'}{U}\Big]$ is negative the trajectory is repelled from the Egg and a $(-) \to (+)$ branch change will not occur. When the trajectory is below the Egg, a similar analysis can be carried through with the result that branch changes indeed occur, depending on the value of $\Big[1 - \frac{1}{2}\frac{U'}{U}\Big]$.

In Figure 2. a $(+) \to (-)$ branch change induced by a negative potential is shown. The units on the vertical H axis are arbitrary.

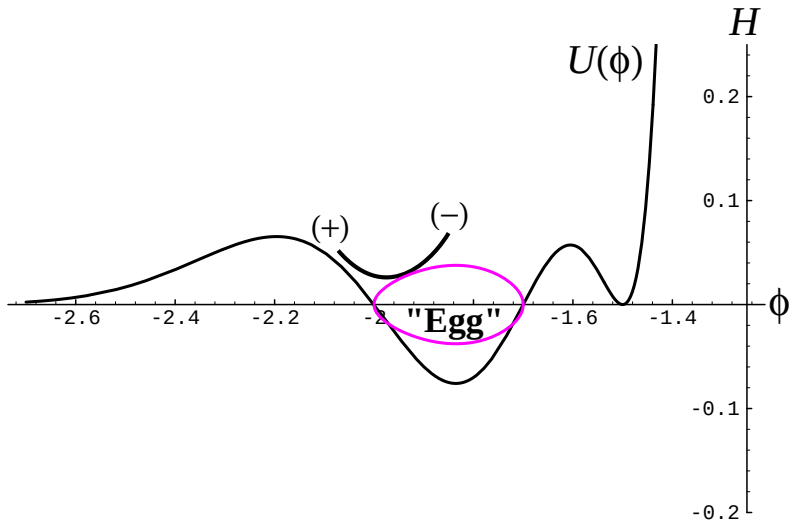

Figure 2. The dilaton potential and corresponding "Egg".
A branch change induced by negative potential.

To our surprise we discovered that, although branch changes can be induced by a negative potential, it does not provide a useful way of achieving a graceful exit from accelerated inflation because branch changes occur in pairs. Let us start explaining this point using the following example. Suppose that a branch change from $(+)$ to $(-)$ has occurred at a point $\phi = \phi_0$ for a positive value of $H(\phi_0)$ (as in Fig.2) and we are interested in finding out whether another one will occur again or not. Define $\Delta^{(-)}H(\phi) = H^{(-)}(\phi) - H_{Egg}(\phi)$. If, for any ϕ in the Egg, $\Delta H^{(-)}$ is positive, then there will be no second hit and the two $(-)$ branch evolutions shown in Fig. 1 and in Fig. 2 can join smoothly to provide a graceful exit. If instead, ΔH vanishes somewhere in the Egg region, there will be another branch change back to the $(+)$ branch and no exit occurs. It is possible to compute $\Delta H^{(-)}$ using Eqs.(4)

$$\Delta H^{(-)}(\phi) = \int d\phi [\frac{dH^{(-)}}{d\phi} - \frac{dH}{d\phi}\Big|_{Egg}] = -\frac{1}{3}\int d\phi \Big[\epsilon(1 - \frac{1}{2}\frac{U'}{U})\Big] + \mathcal{O}(\epsilon^2)] \qquad (9)$$

Assuming, as we observed to be the case for many potentials, that ϵ remains small for a trajectory that touched the Egg (recall that ϵ is always positive), we may estimate $\Delta^{(-)}H$ by assuming that ϵ is constant

$$\Delta^{(-)}H(\phi) \quad = \quad -\frac{1}{3}\epsilon \int d\phi \Big[(1 - \frac{1}{2}\frac{U'}{U})\Big]$$

$$= -\frac{1}{3}\epsilon\left[(\phi-\phi_0)-\frac{1}{2}\ln\left|\frac{U(\phi)}{U(\phi_0)}\right|\right] \tag{10}$$

At the rightmost point of the Egg, ϕ_1, $U(\phi_1)=0$. Therefore, for any point ϕ_0 there will be a another point, ϕ^*, for which $\Delta^{(-)}H(\phi^*)=0$ and also $U'>0$. In that case the trajectory will touch the Egg again and a branch change back to the $(+)$ branch will take place. There are of course variations on this theme, possibilities of multiple hits bellow and above the Egg. We have analysed the problem numerically for a wide range of potentials and possibilities. We have not been able to construct any explicit counterexample. Although this is not an absolute no-go theorem but only a a "very-hard-to-go theorem" we are convinced that this is not the way to proceed. Rather, we suggest later what we view as a more attractive possibility for finding a solution to the graceful exit problem.

We have tried, of course, to go beyond the simplest model just illustrated in order to search for possible ways out of this conclusion. We mention here, in particular,

1. Adding classical sources in the form of a perfect fluid.

This modifies the system of equations (4) into:

$$\begin{aligned}
\dot{H} &= \pm H\sqrt{3H^2+U+e^{\phi}\rho}-\frac{1}{2}U'+\frac{1}{2}e^{\phi}p \\
\dot{\phi} &= 3H\pm\sqrt{3H^2+U+e^{\phi}\rho} \\
\dot{\rho} &+ 3H(\rho+p)=0
\end{aligned} \tag{11}$$

These equations allow us to illustrate a point made earlier. If the potential has a minimum with vanishing cosmological constant (where $U=U'=0$) then such point acts as an attractor for the the dilaton in the $(-)$ branch (for radiation-like matter sources, $p=1/3\rho$). Indeed, in such case, the standard radiation-dominated cosmology

$$a=t^{1/2} \quad , \quad \phi=const. \quad , \quad \rho=a^{-4} \tag{12}$$

is a stable solution, as one can easily check by looking at small perturbations around it. By contrast, in the $(+)$ branch, nothing can keep ϕ from growing as long as the universe expands.

Coming back to our problem, since for a stringy fluid the ratio p/ρ can be anywhere between $-1/3$ and $+1/3$, we have taken it to be some constant in this range during the

Egg-hit epoch. Unfortunately, irrespectively of that ratio, equations similar to (9) and (10) still hold and force a second hit.

2. Adding extra compact dimensions.

This is a rather natural choice since we may expect that some compact dimensions evolve and eventually stabilize with string-size radii. In this case the potential is expected to depend both on the internal-dimension radii and on the effective 4-dimensional coupling, $\alpha_4 = \frac{e^{\phi}}{V} \equiv e^{\phi}$, where V is the volume of the compactified space. We assume, for simplicity, that there are three non-compact spatial dimensions with common scale factor a and n compact dimensions characterized by a common radius b. We can then define the internal analog of the Hubble parameter H, $K \equiv \frac{\dot{b}}{b}$. With $U = U(\varphi, b)$ and $\varphi = \phi - n \ln b$, the resulting generalization of Eqs.(4) now becomes:

$$\begin{aligned} \dot{H} &= \pm H\sqrt{3H^2 + nK^2 + U} - \frac{1}{2}\frac{\partial U}{\partial \varphi} \\ \dot{K} &= \pm K\sqrt{3H^2 + nK^2 + U} - \frac{1}{2}\frac{\partial U}{\partial log b} \\ \dot{\varphi} &= \dot{\phi} - nK = 3H \pm \sqrt{3H^2 + nK^2 + U} \end{aligned} \tag{13}$$

In this case, the Egg becomes a two-dimensional surface allowing intuitively more chances for avoiding a second hit. Alas, this turns out not to be the case (at least we have found no example where graceful exit does occur). We have even tried combinations of 1) and 2), again without success.

We thus appear to be forced to the conclusion that a graceful exit from an accelerated inflation era to a standard FRW cosmology is not possible in the weak curvature regime for any sort of (semi) realistic dilaton potential. The remaining possibilities lie therefore in the strong curvature (large derivatives) regime, in which the full extent of string corrections should be felt.

It is perfectly conceivable that the strong curvature regime inducing the branch change is encountered while the string coupling is still tiny, since a constant dilaton shift is always allowed as long as the potential is negligible. Within this hypothesis, our problem becomes that of searching for conformal field theories endowed with suitable duality symmetries for

describing an interesting cosmological scenario. In particular, one should select backgrounds which go to the $(+)$ and $(-)$ duality$\otimes$time-reversal-related branches at large negative and positive cosmic times, respectively, with an intermediate stringy regime at small times. It is quite easy to find ad hoc modifications of Eqs.(4) with terms $O(H^4)$ and to achieve a very gracious exit this way. The challenge is to find a truly conformal background that does the same job!

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References

[1] R. Brustein and P. J. Steinhardt, Phys. Lett. B302 (1993) 196.

[2] B. A. Campbell, A. Linde and K. A. Olive, Nucl. Phys. B335 (1991) 146.

[3] S. Shenker, Private communication;
H. Goldberg, Northeastern preprint, NUB-3082-93-TH (1993) ;
F. Adams, R. Bond, K. Freese, J. Frieman and A. Olinto, Phys. Rev. D47 (1993) 426.

[4] G. Veneziano, Phys. Lett. B265 (1991) 287.

[5] A. A. Tseytlin, Mod. Phys. Lett. A6 (1991) 1721;
A. A. Tseytlin and C. Vafa, Nucl. Phys. B372(1992) 443.

[6] M. Gasperini and G. Veneziano, Astropart. Phys. 1 (1993) 317.

[7] F. Lucchin and S. Matarrese, Phys. Lett. B164 (1985) 282.

[8] K. A. Meissner and G. Veneziano, Phys. Lett. B267 (1991) 33;
K. A. Meissner and G. Veneziano, Mod. Phys. Lett. A6(1991) 3397;
A. Sen, Phys. Lett B271 (1991) 295;
S. F. Hassan and A. Sen, Nucl. Phys. B375 (1992) 103;
M. Gasperini and G. Veneziano, Phys. Lett. B277 (1992) 256.

[9] S. Weinberg, Gravitation and Cosmology, John Wiley & Sons, Inc., 1972.

[10] E. Witten, Talk given at Strings 1993 conference , Berkeley, California, May 1993;
P. Aspinwall, D. Morrison and B. Greene, Phys. Lett. B303 (1993) 249;
A. Giveon and E. Kiritsis, Nucl. Phys. B411 (1994) 487.

[11] E. Kiritsis and C. Kounnas , private communication and to appear.

[12] M. Gasperini and G. Veneziano, Dilaton Production in String Cosmology, preprint, CERN TH. 7178/94.

[13] N. Sanchez and G. Veneziano, Nucl. Phys. B333 (1990) 253.